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AFWAL-TR-85-3113

BASIC RESEARCH ON ADAPTIVE MODEL ALGORITHMIC CONTROL

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December 1985

AD-A168 016

Final Report for Period February 1982 - June 1985

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS NONE
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for Public Release; distribtuion is unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFWAL-TR-85-3113
6a. NAME OF PERFORMING ORGANIZATION Scientific Systems, Inc 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Air Force Systems Command
6c. ADDRESS (City, State and ZIP Code) 54 Cambridge Park Drive Cambridge, MA 02140	7b. ADDRESS (City, State and ZIP Code) Wright-Patterson Air Force Base OH 45433-6553
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Flight Dynamics Laboratory 8b. OFFICE SYMBOL (If applicable) AFWAL/FIGC	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F33615-82-C-3600
8c. ADDRESS (City, State and ZIP Code) Wright-Patterson Air Force Base OH 45433-6553	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO.: 61102F PROJECT NO.: 2403 TASK NO.: N3 WORK UNIT NO.: 19
11. TITLE (Include Security Classification) Basic Research on Adaptive Model Algorithmic Control (U)	
12. PERSONAL AUTHOR(S) W.E. Larimore; S. Mahmood	
13a. TYPE OF REPORT Final 13b. TIME COVERED FROM 02/82 TO 06/85	14. DATE OF REPORT (Yr., Mo., Day) December 1985 15. PAGE COUNT 201
16. SUPPLEMENTARY NOTATION	

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR	Control, System Identification, Adaption, Robustness
23	09		
23	01		

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The Model Algorithmic Control (MAC) method is investigated in terms of robustness and adaption to unknown or changing plants. The adaption method used is Canonical Variate Analysis (CVA) system identification. CVA is shown to provide system identification accuracy comparable to maximum likelihood and to provide an optimal selection of instrumental variables. Computationally CVA is a noniterative procedure that gives a numerically and statistically well conditioned solution to the system identification problem. A one-step-ahead MAC is explained using the classical root locus technique. Conditions are developed for robutness of the controller to perturbations in the plant due to error in plant identification. Selection of an optimal sampling rate is based upon the controlability and observability matrices. Simulations illustrating the above theory are presented using a Multi-Input Multi-Output (MIMO) missile aerodynamic model.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL Lt Timothy E. McQuade	22b. TELEPHONE NUMBER (Include Area Code) (513) 255-8275 22c. OFFICE SYMBOL AFWAL/FIGC

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

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CHAPTER 1

INTRODUCTION & SUMMARY

MAC is a relatively new digital control design technique that can be implemented using dedicated microcomputers or microprocessors. In its simplest form, MAC consists of:

(i) an internal model of the system to be controlled

(ii) a reference trajectory description of the desired closed loop behavior

(iii) an on-line optimization of future control inputs to produce the desired performances.

This technique has been proven successful in many industries and aerospace applications. Although the methodology was originally developed by practicing engineers from heuristic arguments, single-input single-output MAC under some reasonable assumptions has been extensively analyzed in the previous report AFWAL-TR-80-3125. As a result of basic research questions arising in this previous study, the present work on adaptive MAC was undertaken.

The main objective of this project is to develop an adaptive MAC and an appropriate framework for robustness analysis particularly when the plant is compensated apriori by a fixed gain analog controller. Based on the objective of this project, this report is primarily divided into three parts: an adaptive estimation scheme for system identification of the unknown plant dynamics is developed and analyzed in Part 1; classical and modern robustness analysis techniques are applied to MAC in Part 2; and Part 3 contains the results on simulation.

The methods of Parts 1 and 2 are demonstrated on several examples by computer simulation in Part 3. Detailed derivations and proofs of a number of the results are contained in the Appendices in the form of published research papers or papers being submitted for publication.

In Chapter 2, the system identification procedure for adaptation to system changes is presented. The method used for identification is the canonical variate analysis (CVA) technique. This method has been developed in the last several years and overcomes the difficult problems in currently available methods which prevent their use in general real-time automated systems. Some of the difficulties of other methods are first discussed, and the attractive features of CVA are described including the statistical and computational robustness of the method as well as the inherent ability to determine the appropriate model state order from the observational data. The basic conceptual aspects of CVA are then developed which include the choice of a best set of reduced states of the past for prediction of the future evolution of the process. This is accomplished by a canonical variate analysis of the past and future. The details of such an analysis are given in two of the appendices. The computational aspects of the procedure involve a singular value decomposition which is a very accurate and numerically stable algorithm. The close relationship between the CVA method and the maximum likelihood and instrumental variable methods are described. To investigate the effect of external input excitations on the accuracy of the identified system model, simultaneous confidence bands on the identified plant transfer function and disturbance noise power spectrum are computed. The details of this computation are contained in an appendix. Using these results the output tracking error due to both control and identification errors is derived in the context of stochastic and dual control. The computational aspects of the algorithms are described including the basic steps and amount of computation with the detailed computational equations contained in the appendices.

Chapter 3 analyzes MAC when applied to a lightly damped plant that has been compensated apriori by constant gain output feedback. MAC software uses an impulse response description of the plant which has a large number of terms and is not suitable for analytical studies. Therefore in this chapter MAC has been described using a rational transfer function model (difference equation model) of the plant which shows that

one-step-ahead MAC can also be explained using the classical root locus technique. In chapter 4 an appropriate framework is developed for robustness analysis applying the perturbational argument to the Nyquist plot of the steady state MAC loop transfer function. It has been possible to apply the current robustness analysis technique to MAC under this framework. The analysis gives a set of sufficient conditions, and the perturbed closed-loop system remains stable if the additive or multiplicative modelling error of the plant satisfies these conditions. These conditions define the neighborhood of the identified model such that if the actual plant lies in this neighborhood then the MAC control law designed on the basis of the identified model also stabilizes the actual plant. Finally, in Chapter 5, new techniques are developed for selecting optimum (possibly unique) sampling rates, which play a crucial role in an adaptive control scheme. The sampling time interval is selected on the basis of a minimax approach and also satisfies the classical Nyquist sampling rate.

Finally, in Chapter 6, extensive simulation results have been presented and in Chapter 7 conclusions and summary are provided.

The major conclusion of this report is that MAC is a very effective and superior control technique for linear multivariable plants in a deterministic environment as well as in an uncertain environment where the plant is not exactly known. The adaptive MAC has also been found to be successful where the plant is slowly time varying and/or non-linear. The robustness properties of standard MAC and adaptive MAC have been verified by extensive simulations of the missile attitude control problem. A complete model of MAC for a multi-step-ahead optimization horizon and input-blocking is not yet available, and without this the theoretical properties of a real world MAC are not available in an analytical form. It is recommended that future studies of MAC concentrate on (i) developing a complete model of the MAC algorithm, (ii) comparison of MAC performances with other control design techniques, and (iii) applying an adaptive MAC to a full scale flight control problem.

Part 2

CHAPTER 2: SYSTEM IDENTIFICATION

2.1 Introduction

There has been considerable progress in system identification in recent years. The method of maximum likelihood has been established as the most accurate in theory, although the computational burden and numerical conditioning are serious problems particularly for general applications where the number of parameters can easily be dozens of even hundreds. A number of simplified schemes have been considered such as recursive ML and instrumental variable methods. While these methods have reduced computational requirements, there are difficulties with initialization and with accuracy in small samples which are of particular interest in tracking dynamical systems. Also these methods are not entirely reliable numerically since they depend upon the ARMA parameterization which is known to have global singularities (Gevers and Wertz, 1984). Also if the system order is over estimated, then the computations become ill-conditioned. This considerably complicates the task of determining the state order which is usually unknown. A number of more ad hoc schemes are available, but these have even less desirable statistical or computational properties.

Fortunately, in the last several years, a new method has been developed using the approaches of canonical variate analysis (CVA) method of mathematical statistics, stochastic realization concepts from system theory, and information or entropy methods for the statistical choice of model order and structure. This method has some highly desirable properties. The order of the state is determined statistically. The computation is based upon a singular value decomposition which is one of the most stable and accurate numerical procedures available. The model fitting and state order selection is always numerically well conditioned. The model fitting accuracy has been found to be very close to maximum likelihood in moderate and large samples sizes. The canonical variate analysis method for system identification has been used as the primary procedure in this study

because it is the only method currently available with the above properties. Furthermore, it handles with no additional complication the difficult multi-input multi-output system identification problem. In the development, the CVA method is discussed in Section 2.2, and the close relationship of CVA to the instrumental variable and maximum likelihood methods are discussed in Sections 2.3 and 2.4 respectively. The topics of input design and sampling for identifiability are described in Section 2.5, while the approaches of stochastic and dual control for input design are discussed in Section 2.6. Finally the computational aspects of the CVA method are discussed in Section 2.7. The detailed derivations supporting these sections are contained the various appendices.

2.2 Canonical Variate Analysis of Time Series

The canonical variate analysis method of system identification was first proposed by Akaike (1975). In this fundamental contribution, a stochastic realization algorithm was proposed by using the statistical method of canonical correlation analysis on the Hankel covariance matrix to choose a basis for the state space and to statistically determine the rank of the state space. This provided a fundamentally new and statistical approach to the determination of a dynamical system on the basis of noisy and finite length data. The statistical determination of state order was based upon the Akaike information criterion (AIC). This initial work did not consider the case of an input to the system, but considered only the case of an output.

Later work (Larimore, 1983, in Appendix B) includes the more general case of a multi-input multi-output system. The computational procedure of this method is more efficient in requiring only one canonical correlation analysis, and can also be used to solve the reduced order modeling problem using a general quadratic weighting on the prediction error of the future. Furthermore, a more exact computation of the AIC is used for order determination than that used in the original work of Akaike.

The approach to system identification using generalized canonical variables is described in some detail in Larimore (1983, in Appendix B).

That approach involves consideration of the past $p(t)$ and future $f(t)$ of a vector process at a time t defined as

$$p^T(t)=(y^T(t),u^T(t),y^T(t-1),u^T(t-1),\ldots)^T \tag{2.1}$$

$$f^T(t)=(y^T(t),y^T(t-1),\ldots)^T \tag{2.2}$$

where $u(t)$ is the input and $y(t)$ is the output of an unknown system with state space structure of the form

$$x(t+1)=\Phi x(t)+Gu(t)+w(t) \tag{2.3}$$

$$y(t)=Hx(t)+Au(t)+Bw(t)+v(t) \tag{2.4}$$

with $v(t)$ a measurement noise and $w(t)$ a process noise with respective cross spectral density matrices R and Q. From the theory of Markov processes and in particular the theory of stochastic realization, the minimal state vector defines the information from the past relevant to the future of the process and is called the predictor space (Akaike, 1974a).

The approach of canonical variables to system identification is to determine the optimal set of linear combinations $m(t)$ of the past $p(t)$ that best predict the future $f(t)$ in terms of minimizing the prediction error

$$E\,||\,f-\hat{f}\,||=E[(f-\hat{f})^T\,\mathrm{Cov}^{-1}\,(f,f)\,(f-\hat{f})] \tag{2.5}$$

where $\mathrm{Cov}(f,f)$ is the covariance matrix of the future f and $\hat{f}$ is the best prediction of f based upon the memory $m(t)$. This optimization problem involves the optimal selection of the dimension of $m(t)$ as well as the optimal selection of the linear combinations of the past.

The solution to this problem is derived in Larimore (1985a), included in appendix A, in terms of a generalized singular value decomposition (SVD).

This solution is precisely a generalization of the classical canonical correlation analysis problem of mathematical statistics (Hotelling, 1936). Modern computational procedures use a singular value decompositions (Golub, 1969) involving the covariance matricies of the past and future. The generalized SVD determines transformations J and L and a diagonal matrix D such that

$$JCov(p,f)L = Diag(\gamma_1 > \ldots > \gamma_1 > 0,\ldots,0) = D \tag{2.6}$$

$$JCov(p,p)J = I;\ LCov(f,f)L = I \tag{2.7}$$

The transformations can be interpreted as defining a new set of coordinates for the past and future in which the covariance are D, I and I as given in the last equation. If in (2.5) and (2.7), the covariance matrix Cov(f,f) is replaced by an arbitrary positive semidefinite weighting matrix A, then the above generalized SVD still gives the solution to minimizing the weighted prediction error (2.5) even though the covariance relationships no longer hold (Larimore, 1985a).

For a full order state model, the optimal memory or state x(t) is related to the past p(t) in terms of the first k canonical variables as m(t) = (Isub k, 0)Jp(t), i.e. the first k components of the canonical predictor variables Jp(t). A minimal order realization is obtained with this choice of state. The computation of the state space matricies is given in Larimore (1983) in Appendix B. The state space matricies and noise covariance matricies are given by a linear regression as specified by the state space equations (2.3) and (2.4).

In system identification, the covariance matrics are not known but are estimated from the observations. The statistical determination of rank in the canonical variate analysis is given approximatley using standard canonical correlation analysis methods (Akaike, 1976). A more refined compareson between the different order models is given by use of the Akaike information criterion (AIC) which is asymptotically optimal in minimizing entropy (Shibata, 1981). The use of entropy measures such as the AIC has a fundamental justification in terms of the basic statistical principles of sufficiency and repeated sampling (Larimore, 1983a).

The minimal order realization is unique independent of the weighting matrix Λ, but when a reduced memory is selected, the approximate system does not in general minimize the prediction error for that order. This is because the reduced rank canonical variables are not in general recursively computable. However in the case of the statistical rank determination problem, there is an insigificant difference between the state of the realized system corresponding to the statistically optimum choice of order and the full rank canonical variables.

2.3 Relationship with the Method of Instrumental Variables

The instrumental variables method gas a natural interpretation in terms of the generalized canonical variate problem. In the instrumental variables approach the state equations (2.3) are considered as unobserved structural relationships that are indirectly observed through the noisy measurement equations (2.4). A vector $m(t)$ of instrumental variables is constructed which is hopefully close to the true state $x(t)$. This is used in place of the true state in solving the problem. This apparently works well for an appropriate choice of the instrumental variables when the true order of the system is known or well chosen. In other cases, this approach may lead to inaccurate models.

A more general problem is the optimal choice of instrumental variables for a specified order k of the model as posed by Rao(1973, 1979) (see also Larimore, 1985a, in Appendix A). This is formulated as finding the optimal choice of k linear combinations of the past $p(t)$ that predict the future $f(t)$ as measure in terms of the squared error $(f-\hat{f})^T(f-\hat{f})$. This is precisely the generalized canonical variate problem with weighting matrix $\Lambda = I$. If k is chosen as full rank, then the memory and the state space realization are independent of the weighting. However, for lower rank k, there can be a considerable difference between the state space and reduced order system (Larimore, 1983). The squared error of instrumental variables relates to energy while the canonical correlation analysis relates to the statistical significance of the problem. Thus the canonical correlation analysis can be viewed as an optimal choice of the instrumental variables

using the appropriate weighting of the prediction errors for the determination of the statistically significant number of states.

Time recursive methods using instrumental variables and approximate maximum likelihood (IV-AML) are claimed to be an approximately efficient parameter identification method for large samples as shown in simulation examples (Young and Jakeman, 1979). This is shown by Monte Carlo simulation and by estimating the parameter estimation error covariance matrix. Below it is shown by Monte Carlo simulation that the canonical correlation method also gives efficient identification of the system dynamics. This is done by evaluating the spectral estimation error.

2.4 Maximum Likelihood Efficiency of CVA

The canonical variate system identification procedure has been found in moderate sample sized to be close to the lower bound of maximum likelihood estimation. There is no proof available for this, however simulations have shown this to be the case. There is some theory to suggest why canonical variate analysis is an efficient estimation procedure.

Conditional upon the choice of the state vector by the canonical variate analysis, the computation of the state space matricies by regression is a maximum likelihood procedure. The difficulty in proving the asymptotic efficiency of CVA is that for correlated time series there is no proof that CVA gives the choice of state that will result in maximum likelihood estimates unconditionally.

The lower bound for estimating the power spectrum and transfer function is given in Larimore (1985a, in Appendix A) as a function of frequency. From extensive simulations, the canonical variate analysis gives an identified system within the lower bound error of the maximum likelihood procedure at each frequency as shown in Larimore, Mahmood, and Mehra (1984, in Appendix D).

2.5 Input Design and Sampling

The accuracy of the identified plant model and subsequent control tracking error depends upon the sampling rate, sample size, the presence of implicit

or explicit extra input signals, and the presence of disturbance or output measurement noise. In fact the presence of a linear feedback control provides no information for identification of the plant (Ljung, Gustafson, and Soderstrom, 1974), and some additional input signal is required for plant identifiability. Recently, Anderson(1985) has shown that available adaptive control methods that do not have persistent excitation of the system necessarily exhibit burst phenomena of short periods with large tracking errors when the system parameters drift far from the true.

The requirement for additional information is easily seen since the presence of a linear feedback could be present in the plant internally and the actual input could be unconnected to the system and still give exactly the same response. On the presumption of a strictly linear plant, a nonlinear feedback can be used to provide identifiability. Also a switching between different linear feedback systems can provide identifiability. A better approach, however is to use an explicit additional input excitation. Such an excitation is best chosen to be a broad band noise type of spectrum which guarantees that it is persistently exciting.

In some applications, there are implicit excitations such as wind gust turbulence on an aircraft which provide some information about the plant. If the power spectrum of the turbulence is exactly known along with the input coupling to the plant state, then this can provide amplitude information about the transfer function from the gust input to the output. In particular, the relationship between the observed output spectrum $S_y(z)$ and the assumed input noise spectrum $S_n(z)$ and transfer function $H(z)$ is

$$S_y(z)=H(z)S_n(z)H^*(z) \tag{2.8}$$

Unfortunately, in most cases this is not very helpful since the gust spectrum is not accurately known and is highly variable with time. Also the gust input coupling to the state will generally be different than the control input. Furthermore, this provides only amplitude information, and for control the transfer function phase can be crucial.

The best input excitation is one that is incorrelated with the system state. The spectrum of the input excitation can be chosen on the basis of the plant transfer function, and the disturbance and output measurement noise spectra. The resulting plant identification error expected at each frequency is a complicated function of the above power spectrum and transfer functions as well as the parameterization of the model. A detailed derivation and description of the transfer function and noise spectrum estimation error variance at each frequency is given in Larimore (1985b, in Appendix E). These expressions are complicated but can be used to calculate the estimation error and produce simultaneous confidence bounds on the estimated transfer and spectral functions.

An additional consideration in identification accuracy is the sample rate and rate of reidentification of the system or equivalently the sample size. The issue of sample rate for representing a continuous time system is covered in Section 5. The primary consideration in choosing the sample rate is to insure that the important frequency information is preserved and that the higher frequencies of no interest do not degrade the estimation by aliasing. For large sample, the sample size has a simple relationship to the accuracy of the identified system which increases proportional to the inverse square root of the sample size. For moderate sample sizes of several hundred which is of primary interest, this relationship can be expected to hold approximately.

As an example of the accuracy bounds that are obtainable from the methods in Larimore (1985b, in Appendix E), consider the case identifying the transfer function of an ARMA(4,3) model discussed in Larimore et al (1984, in Appendix D) with a sample of 800 which is observed in closed loop with a white noise input excitation and a white output measurement noise with the signal to noise power ratio of the input to output equal to 0.10. Then the transfer function of the true, identified, and simultaneous confidence bands about the estimated are shown in figure 2.1. The confidence bands contain the true transfer function entirely within the bands across the entire frequency range with probability 0.95. Note that the confidence bands are quite tight in both phase and amplitude. For a lower sample size, the confidence bands are wider by a factor of the square root of the sample size.

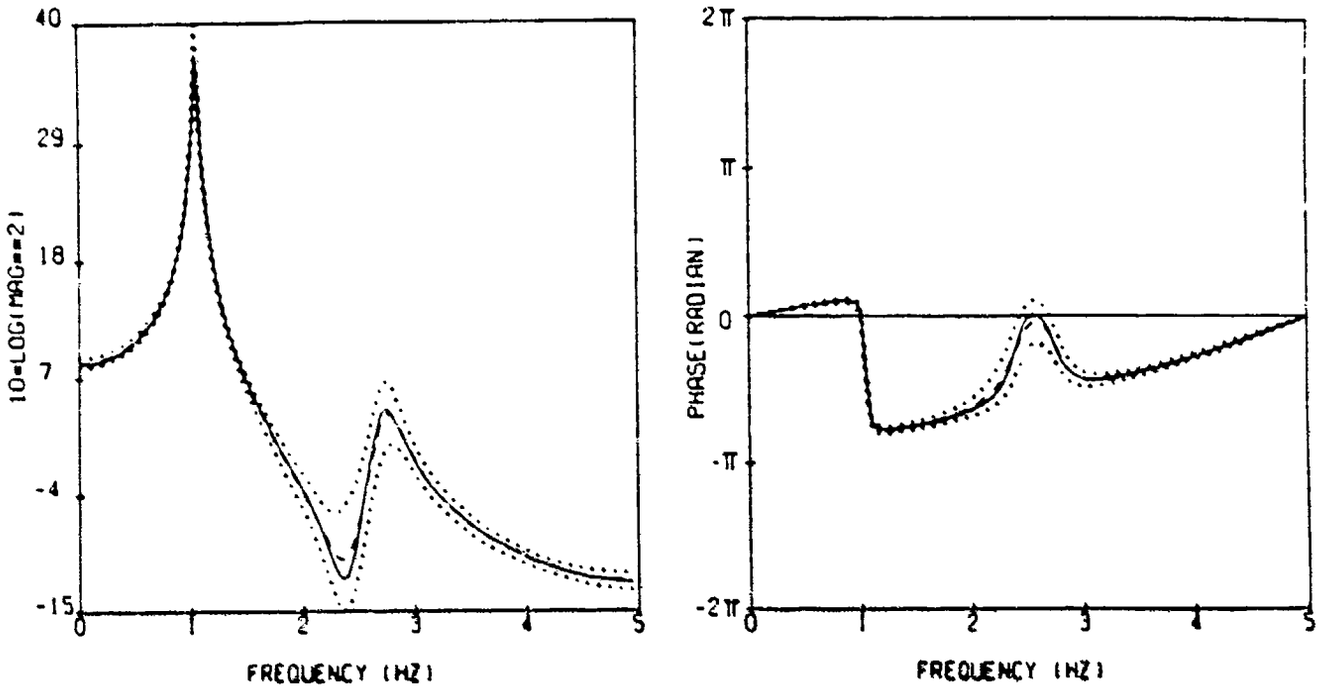


Figure 2.1 Power Spectral Density of ARMA(4,3) Process, True (solid), Estimated (dashed), and Simultaneous Confidence Band (dotted).

2.6 Stochastic and Dual Control

In stochastic and dual control, the effect of the stochastic input on both plant identification and control tracking error is taken into account. This is also possible in the adaptive MAC framework. In this section, we derive the tracking error as a function of the stochastic input excitation, plant disturbance and measurement noise, and the MAC controller plant mismodelling error.

The closed-loop transfer function from the plant input $u(z)$ and the composite plant disturbance and measurement noise $n(z)$ as seen at the plant output to the observed output $y(z)$ is given in Section 4.2 and can be expressed as

$$y(z) = \frac{(z-1)H(z)u(z)+(z-1)n(z)}{(z-1-\alpha)I+\alpha R(z)} \tag{2.9}$$

where the relative error $R(z)$ in estimating the plant transfer function is defined as

$$R(z) = \bar{H}^{-1}(z)[H(z)-\bar{H}(z)] \tag{2.10}$$

Here $H(z)$ is the true and $\bar{H}(z)$ is the identified plant open loop transfer function. Now for a complex differentiable function $w = f(x)$ of a complex random variable x with mean μ, the variance of the function is derived from

$$f(\alpha) = f(\mu) + f'(\mu)(x - \mu) \tag{2.11}$$

which holds to first order so

$$E\left| f(x) - f(\mu)\right|^2 = E\left| [f(x) - f(\mu)][f(x) - f(\mu)]^*\right.$$

$$= \left| f'(\mu)\right|^2 E\left| x-\mu\right|^2 \qquad (2.12)$$

In the context of the identification and control involving different segments of data, we have approximate independence between the processes $u(z)$, $n(z)$ and the transfer function relative estimation error $R(z)$. Thus the tracking error due to the input and disturbance excitation as well as the plant modelling error is

$$E\left| y(z)\right|^2 = [\left| G(z)\right|^{-2} S_u(z) + \left| J(z)\right|^2 S_n(z)][\ 1 + \frac{\alpha^2}{\left|(z-1)+\alpha\right|^2} \mathrm{Var}\ [R(z)]] \qquad (2.13)$$

where $G(z)$ and $J(z)$ are the closed loop transfer functions from the input excitation and disturbance noise excitations respectively to the plant output, and where $S_\eta(z)$ is the spectrum of the plant disturbance and measurement noise as seen at the plant output in open loop operation.

It is seen that as the input excitation is increased, the control tracking error increases for a fixed relative modeling error $R(z)$, but the increased excitation decreases the relative error in identification. The quantity $\mathrm{Var}\ [\ R(z)\ ]$, the relative squared error of identifying the transfer function is derived in Larimore (1985b, Appendix E). This is a function of the characteristics of the plant transfer function as well as those of the process and disturbance noise spectrum characteristics. The expressions for computing these quantities are straight forward but not easily expressed analytically. Thus as in the stochastic dual control literature, the optimal design is analytically intractable and requires a numerical approach.

2.7 Computational Considerations

In this section the major computational steps in the algorithm are described. The detailed computational equations are contained in the appendices.

The computational steps in the identification algorithm are shown in Figure 2.2. In the identification of the plant, first the covariance among the past and future are computed. Second, a canonical correlation analysis between the past and future is performed. From this, a comparison of the various state space model orders is computed using the AIC criterion. On the basis of this, the best state order is selected and the state space matrices computed by regression. This state space model is then used in the MAC controller. The detailed computations of these blocks are contained in Larimore (1983, in Appendix B) except for the AIC computation. An approximate AIC computation is given in Akaike (1976) as

$$AIC(k) = \sum_{j=1}^{k} \log(1-\gamma^2) + 2p_k \tag{2.14}$$

where p_k is the number of parameters fitted in the model.

To evaluate the AIC, the number of free parameters adjusted in the canonical variate procedure is required. For a state space model of state order k of the form of Equations (2.3) and (2.4), there are a number of implied constraints so that it is not correct to simply count the number of elements of the various matricies. The number of functionally independent free parameters p_k including the process and measurement noise covariance is (Candy, Bullock, and Warren, 1979)

$$p_k = 2kn + n(n+1)/2 + km + nm \tag{2.15}$$

where n and m are the vector dimensions of the number of outputs and inputs respectively at a given time. If there is no instantaneous feedfor-

ward, then the term nm is deleted, while if there is no input the terms km + nm are deleted.

The AIC expression (2.14) is only approximate, and the precise evaluation is given by computing the state space model $\hat{\theta}_k$ for competing order models and doing an exact evaluation of the AIC by

$$AIC(k) = -2 \log p(Y,\hat{\theta}_k)+2p_k \tag{2.16}$$

The state order is chosen which minimized the AIC(k).

The major computations are the covariance and the singular value decomposition Once the plant state order is determined, the computation of the state space matricies requires relatively little computation. For slow identification rates, the computation becomes proportional to the sample size times the the dimension of the past and future, while for fast identification rates, the computation is proportional to the cube of this dimension.

PART 2

CHAPTER 3

MULTIVARIABLE MAC IN A CLASSICAL CONTROL FRAMEWORK

3.1 Introduction

The theoretical properties of MAC have been studied in details in the previous report (AFWAL-TR-80-3125) using the impulse response (IR) model of the plant. The reason for using the IR description of the plant is that the MAC software (known as IDCOM) uses this description of the internal model in the computation of the control sequence. The IR description of the plant is the basis of the MAC technique where a quadratic optimization problem is formulated explicitly in terms of the future control sequence. The IR description of the plant is superb from the computational point of view, but it has a disadvantage that this description is not parsimonious i.e. it contains too many parameters and is therefore not suitable for analytical studies. Since one of the objective of this project is to investigate analytically various aspects of MAC, the MAC technique is described in this chapter in terms of a difference equation (DE) model of the plant. The DE description usually contains far fewer number of parameters than an IR description and is therefore suitable for analytical studies if a low order plant is selected in the analysis.

There is no mathematical model for a generalized MAC with multistep ahead optimization horizon, input blocking, input constraints etc. Therefore it is not possible to investigate analytically the properties of a generalized MAC control law. The MAC strategy generates an optimal control sequence by on-line optimization of a cost functional and the first element of this sequence is applied to the actual system. It has been shown in an earlier report that if the plant is minimum phase and the cost functional is optimized over one step ahead, then the MAC control law can be interpreted in a classical control framework. In this chapter we extend this interpretation to

multivariable systems and indicated how the robustness of MAC can be assessed in this framework.

Section 3.2 extend the earlier descriptions of MAC to multi-input multi-output (MIMO) systems which shows that MIMO MAC can also be interpreted in a standard unity feedback configuration. With a slight modification of this configuration it is shown that MAC can be explained in a multivariable root-locus framework. The root-locus technique gives the locations of the closed-loop poles as the output-feedback gain is changed from zero to infinity. Usually a rational transfer function or difference-equation (DE) model of the plant is used in this technique. Therefore in order to cast MAC technique in a root-locus framework, MAC has been described in section 3.3 using the DE model of the plant. Using this analysis, the root-locus interpretation of MAC is presented in section 3.4. Finally the MAC for a lightly damped system is discussed in section 3.5 where it has been shown qualitatively that one should not try to use a high gain output feedback to introduce sufficient damping in a lightly damped system, otherwise a high sampling rate may have to be selected. Conclusions are discussed in section 3.6.

3.2 What is MAC? - An Overview

MAC control strategy has been described and analyzed in earlier reports and publications [1,4,5,6]. We include here a simple description of MAC for the sake of completeness of this report. The following is an extended version of the earlier descriptions for MIMO plants.

The MAC methodology generates a control sequence by on-line optimization of a cost functional, and the algorithm is suitable for implementation on microprocessors. One of the attractive features of MAC is the clear and transparent relationship between system performance and various design parameters embedded in the design procedure. There are five basic elements in MAC (we assume in the following that the input sequence u(n) is m-dimensional and output sequence y(n) is p-dimensional):

(i) An actual stable plant, possibly not known exactly, with a pulse response sequence $\{H_n\}$, $n=1,2,\ldots N$ where each H_n is pxm dimensional matrix (we assume for simplicity that the plant has no time delay and is purely dynamic i.e. it has no feedthrough term). Then the input sequence u(n) and the output sequence y(n) are related by

$$y(n) = H_1\, u(n-1) + H_2\, u(n-2) + \ldots + H_N\, u(n-N) \qquad (3.1a)$$

or, $$Y(z) = H(z)U(z) \qquad (3.1b)$$

where U(z), Y(z) and H(z) are z-transforms of u(n), y(n) and $\{H_n\}$ respectively.

Here

$$H(z) = H_1 z^{-1} + H_2 z^{-2} + \ldots + H_N z^{-N} = H_p(z) z^{-N}$$

where $H_p(z)$ is a pxm dimensional polynomial matrix in z and is given by

$$H_p(z) = H_1 z^{N-1} + H_2 z^{N-2} + \ldots + H_N \qquad (3.1c)$$

This model is known as an "all-zero" model and $H_p(z)$ determines zeros of the plant. The locations of non-minimum phase zeros impose restrictions on achievable performance of MAC. We must remind the reader that the physical interpretation of zero in the impulse response model of the plant is different from that of a transmission zero in a rational transfer function model or equivalently difference equation (DE) model) of the plant. In the same way the physical interpretation of poles as natural modes of a plant are lost in this description. However this point will be elaborated further in the next section.

(ii) An internal model of the plant having the same input-output dimension pxm as that of the actual plant and the pulse response sequence $\{\tilde{H}_n\}$, $n = 1,2,\ldots\tilde{N}$. The input u(n) is the same as that to the actual plant and therefore the output $\tilde{y}(n)$ of the model is given by

$$\tilde{y}(n) = \tilde{H}_1\, u(n-1) + \tilde{H}_2\, u(n-2) + \ldots + \tilde{H}_{\tilde{N}}\, u(n-\tilde{N}) \qquad (3.2a)$$

or $$\tilde{Y}(z) = \tilde{H}(z)\, U(z) \qquad (3.2b)$$

where, as before

$$\tilde{H}(z) = \tilde{H}_p(z)\, z^{-\tilde{N}} \tag{3.2c}$$

and $\tilde{H}_p(z)$ is a pxm dimensional polynomial matrix. $\{\tilde{H}_n\}$ is generally different from $\{H_n\}$.

(iii) A p-dimensional reference trajectory $y_r(n)$, preferably smooth, initialized on the current output of the actual plant $y(n)$ that leads $y(n)$ to a possibly time varying p-dimensional set point c. If each of the reference trajectories $y_{ri}(n)$ has a first order dynamics with time constant α_i leading to set point c_i, $i=1,2,\ldots p$ and if the trajectories do not interact with each other then $y_r(n)$ evolves as

$$y_r(n+1) = \Lambda_\alpha\, y_r(n) + (I-\Lambda_\alpha)c, \quad y_r(n) = y(n) \tag{3.3a}$$

$$\text{or,} \quad zY_r(z) = \Lambda_\alpha\, Y_r(z) + (I-\Lambda_\alpha)\, C(z) \tag{3.3b}$$

where $\Lambda_\alpha = \text{diag}\,(\alpha_i)$

(iv) A closed loop prediction scheme for predicting the future output of the plant according to the scheme

$$y_p(n+1) = \tilde{y}(n+1) + y_p(n) - \tilde{y}(n) \tag{3.4a}$$

$$\text{or,} \quad Y_p(z) = \tilde{Y}(z) + z^{-1}\,[Y(z) - \tilde{Y}(z)] \tag{3.4b}$$

Here $y_p(n)$ is p-dimensional.

(v) A quadratic cost functional J based on the error between $y_p(n)$ and $y_r(n)$ over a finite horizon T_n (here T_n is an integer):

$$J = \sum_{k=1}^{T_n} [e^T(n+k)\, W(n+k)\, e(n+k) + u^T(n+k-1)\, R(n+k-1)\, u(n+1-1)] \tag{3.5a}$$

$$= T_r \sum_{k=1}^{T_n} [W(n+k)\, e(n+k)\, e^T(n+k) + R(n+k-1)\, u(n+k-1)\, u^T(n+k-1)] \tag{3.5b}$$

where W(.) and R(.) are positive semi definite time varying weights and $e(n+k) = y_p(n+k) - y_r(n+k)$. In most of MAC applications R(.) is set to be zero.

Given (i)-(v), MAC finds as optimal control sequence $\{u^*(n+i-1), i=1,\ldots T_n\}$ by minimizing J over the admissible input sequence $\{u(n+i-1)\varepsilon\Omega(i), i=1\ldots T_n\}$. Once the optimal control sequence is computed, the first element of the sequence is applied to the actual plant and the process repeats all over again.

In general, there is no analytic solution for the control sequence $\{u^*(n)\}$ - it is computed at each step using an algorithm known as IDCOM. In its greatest generality, MAC cannot be put into a classical control framework. However under the following simplifying assumptions MAC is equivalent to an inverse-control law and can be modelled as a feedback configuration.

(i) The actual plant $H(z)$ is minimum phase;

(ii) The plant model $\hat{H}(z)$ is minimum phase;

(iii) There are no input constraints, i.e. $\Omega(i) = R^m$ for all i;

(iv) $T_n=1$ i.e. the optimization is carried over one future step ahead: under this condition MAC is a one-step ahead predictive controller.

Under these simplifying assumptions, it is sufficient to select $u^*(n)$ to satisfy

$$y_p(n+1) = y_r(n+1) \text{ for all } n \geq 0 \tag{3.6}$$

for a minimum of the cost function J. The assumptions (i)-(iii) ensure the existence of an optimum control $u^*(n)$ that satisifies (3.6) - the resulting optimal cost J^* is zero in this case. However $U^*(z)$ is then implicitly generated by $Y_p(z) = Y_r(z)$ so that

$$U^*(z) = [(z-1)\hat{H}(z) + (I-\Lambda_\alpha)H(z)]^{-1}\,[I-\Lambda_\alpha]C(z) \tag{3.7a}$$

$$Y(z) = H(z)\,[(z-1)\hat{H}(z) + (I-\Lambda_\alpha)H(z)]^{-1}\,[I-\Lambda_\alpha]C(z) \tag{3.7b}$$

Equations (3.7a) and (3.7b) relate the setpoint $C(z)$ with the optimal input sequence $U^*(z)$ and output sequence $Y(z)$. It is easy to see that this simplified form of MAC is equivalent to the following MIMO unity feedback configuration (we have henceforth dropped the * superscript).

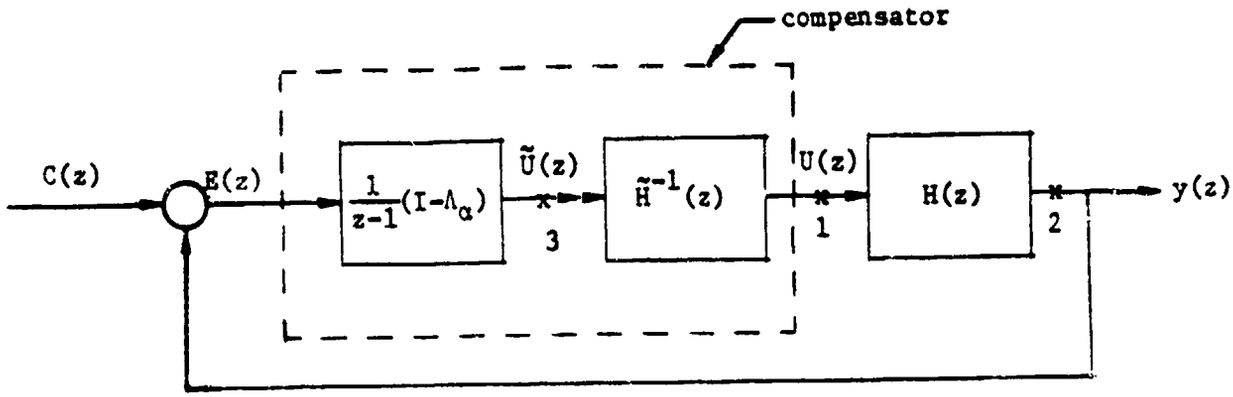


Figure 3.1 MIMO MAC in a classical framework

To see that the setup in Figure 3.1 indeed represents equation (3.7), note that at point 1 we have,

$$U(z) = \frac{1}{z-1}\tilde{H}^{-1}(z)(I-\Lambda_\alpha)E(z)$$

$$= \frac{1}{z-1}\tilde{H}^{-1}(z)(I-\Lambda_\alpha)\,[C(z) - H(z)U(z)]$$

Multiplying both sides of this equation by $(z-1)\tilde{H}(z)$ and rearranging we have,

$$[(z-1)\tilde{H}(z) + (I-\Lambda_\alpha)H(z)]\;U(z) = (I-\Lambda_\alpha)C(z)$$

from which (3.7a) and (3.7b) follow. The block within the dashed line can be thought of as a dynamic controller of the classical type. The loop transfer function when the loop is broken at the plant input (point 1) is given by

$$L(z) = \frac{1}{z-1}\tilde{H}^{-1}(z)(I-\Lambda_\alpha)H(z) \tag{3.8}$$

and determines the robustness of the feedback configuration at this

point. When we have perfect identification i.e. $H(z) = \hat{H}(z)$, then points 2 and 3 are the same in Figure 3.1 and

$$\tilde{U}(z) = Y(z) = \frac{1}{z-1}(I-\Lambda_\alpha)E(z)$$

$$\text{or,} \quad \tilde{U}(z) = \frac{1}{z-1}(I-\Lambda_\alpha)\,[C(z)-\tilde{U}(z)]$$

$$\text{or,} \quad z\tilde{U}(z) = \Lambda_\alpha\,\tilde{U}(z) + (I-\Lambda\alpha)C(z) \tag{3.9}$$

Equation (3.9) is equivalent to

$$\tilde{u}(n+1) = \Lambda_\alpha\,\tilde{u}(n) + (I-\Lambda_\alpha)c(n), \quad \tilde{u}(n) = y(n)$$

which shows that $\tilde{u}(n)$ is the reference trajectory sequence $y_r(n)$ as shown in equation (3.3a). This means that when the plant model is known exactly, the control sequence $U(z)$ is generated as

$$U(z) = H^{-1}(z)\tilde{U}(z) = H^{-1}(z)Y_r(z) \tag{3.10a}$$

Therefore the output of the actual plant is

$$Y(z) = H(z)U(z) = Y_r(z) \tag{3.10b}$$

which shows that, in steady state, the plant output $y(n)$ is identical to the reference trajectory $y_r(n)$ - perfect tracking has been achieved. Equation (3.10a) clearly shows the need for minimum phaseness of $H(z)$. This analysis has revealed another interesting property of MAC. Exact tracking could as well be achieved by inverting the plant to generate the sequence $\tilde{u}(n)$ in an open-loop configuration, but in MAC it does so in a closed-loop configuration and therefore the additional benefits of a feed-back configuration such as disturbance rejection, sensitivity reduction, etc are also obtained at the same time while achieving exact tracking.

Further insight is available if we interpret the above equations for SISO plants. The loop variables for SISO plants are denoted by corresponding small letters, e.g. $h(z)$ is a transfer function for a SISO plant and $H(z)$ is that for a MIMO plant. Also for a SISO loop $\Lambda_\alpha = \alpha$, and the Figure 3.1 takes the following simple form:

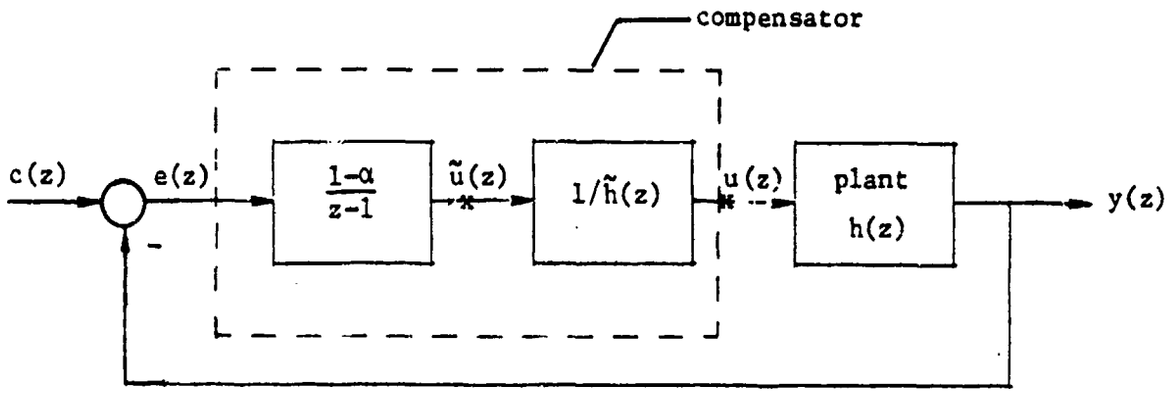


Figure 3.2 SISO MAC as a classical controller

Note that in this figure $(1-\alpha)$ can be treated as a gain and the usual classical root-locus technique can be applied to analyze the behavior of the closed loop poles as α changes from 0 to 1. But since the impulse response description of a plant has too many poles and zeros, the root-locus technique will not be useful and this is why we intend to describe MAC in terms of a difference-equation (DE) model of the plant in the next section.

3.3 Lightly damped system in terms of difference equation (DE) and impulse response (IR) model

Consider a generic lumped parameter linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{3.11a}$$

$$y(t) = Cx(t) \tag{3.11b}$$

where $x(t)$, $u(t)$ and $y(t)$ are n-, m- and p-dimensional vectors representing the states, inputs and outputs respectively and A, B, C have appropriate dimensions. The corresponding frequency domain description is

$$X(s) = \Phi(s)BU(s) \text{ and } Y(s) = C\Phi(s)BU(s) = H_c(s)U(s) \tag{3.12}$$

where $\Phi(s) = (sI-A)^{-1}$ and $H_c(s)$ is the impulse response of the system. If $\lambda_i = \alpha_i \pm j\omega_i$ is the i-th eigenvalue of A, then the system is

asymptotically stable if $\alpha_i < 0$ for each i and in this case each element of $H_c(s)$ is analytic in the closed right half plane. On the otherhand, the system is unstable if $\alpha_i > 0$. If $H_c(t)$ is the inverse Laplace Transform of $H_c(s)$, then for asymptotically stable systems each element of $H_c(t)$ approaches zero as $t \to \infty$, whereas for an unstable system some element diverges. If the impulse response $H_c(t)$ of a system takes a long time to settle down to zero, the system is generally known as a lightly damped system. The damping ratio associated with the i-th complex pole-pair $\lambda_i = \alpha_i \pm j\omega_i$ is defined as

$$\xi_i = \frac{|\alpha_i|}{\sqrt{\alpha_i^2 + \omega_i^2}} \tag{3.13}$$

so that $0 < \xi_i < 1$. The system is lightly damped if ξ_i is small which results when $|\alpha_i|$ is small, i.e. the system is lightly damped when at least one of the poles lies near $j\omega$-axis. These systems show undesirable behavior of "ringing" and excessive "overshoot" in open-loop transient response. The impulse response of these systems decays to zero very slowly, and therefore a large amount of data must be stored in the computer for representing the impulse response sequence model of the plant which directly affects MAC computation.

Since MAC is a digitally implemented control algorithm, we must find a sampled-data version of (3.11). There are several ways of implementing digital control schemes - one of these is the sample and zero-order hold mechanism which is equivalent to discretizing (3.11) by using an exponential transform. In this method the input is sampled every T seconds and held constant, i.e. $u(t) = u(n)$, $nT < At < (n+1)T$ between the two sampling instant. In this case the z-domain and s-domain descriptions are related through

$$z = e^{sT} \tag{3.14}$$

and the corresponding discrete-time system in state-space description is

$$x(n+1) = Fx(n) + Gu(n), \; x(0) = x_0 \tag{3.15a}$$

$$y(n) = Cx(n) \tag{3.15b}$$

where $F = \exp(AT)$, $G = (F-I)A^{-1}B$, provided that A^{-1} exists, otherwise

$$G = \int_0^T \exp(Aw)dwB \tag{3.15c}$$

If the system (3.15) is asymptotically stable, the zero-state solution of (3.15) is given by

$$y(n) = \sum_{i=0}^{n-1} H_{n-i}\, U(i),\ H_n = CF^{n-1}\, G,\ n \geq 1 \tag{3.16a}$$

which is the familiar discrete-time convolution. Notice that if T is very small, to the extent that $\max_{ij} |A_{ij}T| \ll 1$, where A_{ij} is the (i,j)-th element of A, then

$$H_n \approx C \exp(A(n-1)T)BT \tag{3.16b}$$

which also results if the integral in (3.15c) is approximated by the lower Riemann sum. Taking the z-transform of (3.15a) - (3.15b) we get the frequency domain description,

$$Y(z) = H_d(z)U(z), \tag{3.17a}$$

where

$$H_d(z) = C(zI-F)^{-1}G \tag{3.17b}$$

The power series expansion of H(z) gives

$$H_d(z) = C(I/z + F/z^2 + \ldots)\, G = \sum_n H_n z^{-n} \tag{3.18a}$$

with the region of convergence (ROC) $|z| > \max_i |\lambda_i(F)|$. We can recover $\{H_n\}$ from $H_d(z)$ using a Cauchy Integral as follows

$$H_n = \frac{1}{2\pi j} \quad H_d(z)z^{n-1}\, dz = CF^{n-1}\, G \tag{3.18b}$$

which is the same in (3.16a).

Ideally an IR sequence $\{H_n\}$ computed in the above manner has an infinite number of terms. Since MAC uses in its internal algorithm a

finite impulse response sequence $\{H_n\}$, the matrix valued sequence $\{H_n\}$ must be a fast converging one. The poles in the continuous time system λ_i and those of the sampled-data system z_i are related by $z_i = \exp(\lambda_i)$. Therefore the discrete time system is unstable if $|z_i| > 1$ for any i and is a lightly damped system if $|z_i|<1$ but close to unit circle i.e. $|z_i|\approx 1$. In the earlier case $\{H_n\}$ diverges and in the later case $\{H_n\}$ has a very large number of terms before it converges to zero. If the system is asymptotically stable $\{H_n\}$ converges, and given $\varepsilon > 0$ we can always find an integer $N(\varepsilon)$ such that $\| H_n \| < \varepsilon$ for all $n > N$ and we can truncate the impulse response sequence to any desired degree of accuracy. The finite impulse response description is also known to practicing engineers as a moving average (MA) or all zero model of the plant.

Now suppose that an impulse response has been truncated to obtain a finite sequence $\{H_n\} = \{H_1, H_2 \ldots H_N\}$. MAC uses this description of the plant model as shown in section 3.2 for a lightly damped system. This sequence is relatively long. The z-transform H(z) is given by

$$H(z) = \sum_{n=1}^{N} H_n z^{-n}. \tag{3.19}$$

Comparing with (3.18a) we find that

$$H_d(z) = H(z), \quad |z| \gg 1.0. \tag{3.20}$$

Here $H_d(z)$ will be called a difference equation (DE) description and H(z) an impulse response description. Although $H_d(z)$ and H(z) are approximately equal for all z within the region of convergence, the physical interpretation associated with the two description are different. To see the difference clearly, consider a SISO plant in which case $H_d(z)$ and H(z) are complex scalars and represented respectively by $h_d(z)$ and h(z). Then

$$h_d(z) = \frac{b(z)}{a(z)}$$

where a(z) and b(z) are polynomials in z, b(z) having a lower degree than a(z) for a causal system. The zeros of the denominator a(z) are

the 'poles' of the system $h_d(z)$ and are associated with the natural modes of the system. The impulse response (IR) of the system is composed of these modes. The zeros of $b(z)$ are transmission zeros of the plant which have the physical interpretation that if z_i is a zero of the plant and if z_i is also a mode of the input to the plant, then this mode of the input is blocked by the plant and does not appear at the output. On the otherhand the IR description $h(z)$ can also be written as

$$h(z) = \frac{n(z)}{d(z)}$$

where $n(z)$ and $d(z)$ are polynomials in z. Here $d(z) = z^N$, and $n(z)$ is a polynomial of degree N. This shows that $h(z)$ has N poles at the origin and N zeros - but these poles and zeros do not have any physical significance as in the rational transfer function model $h_d(z)$.

Since we want to explain the behavior of MAC in terms of standard pole-zero configuration, our immediate objective is to describe MAC using a difference equation model.

3.4 MAC with Difference-Equation Model: a Root Locus Approach

Consider again a pxm dimensional MIMO plant $H_d(z)$ with input $U(z)$ and output $Y(z)$. Then parallel to the description of MAC in section 3.2, we can describe the various elements of MAC as follows:

(i) The actual plant described by

$$Y(z) = H_d(z)\ U(z) \tag{3.21}$$

(ii) The internal model of the plant, also described by a rational transfer function description and given by

$$\hat{Y}(z) = \hat{H}_d(z)\ U(z) \tag{3.22}$$

(iii) A p-dimensional reference trajectory $y_r(n)$ which evolves as

$$y_r(n+1) = \Lambda_a\ y_r(n) + (I-\Lambda_a)c,\ y_r(n) = y(n)$$

or, $zY_r(z) = \Lambda_\alpha Y(z) + (I-\Lambda_\alpha)C(z)$ (3.23)

(iv) a closed-loop prediction scheme $y_p(n)$ for predicting the future output of the plant, according to the scheme

$$y_p(n+1) = \tilde{y}(n+1) + y_p(n) - \tilde{y}(n)$$

or, $Y_p(z) = \tilde{Y}(z) + z^{-1}\,[Y(z) - \tilde{Y}(z)]$ (3.24)

(v) and a cost functional as in (3.5)

If we compare the expressions in (3.21)-(3.24) with those in (32.1b)-(3.4b), we see that these expressions are the same mathematically although in (3.1b)-(3.4b) we have used the IR description of the plant whereas in (3.21)-(3.24) we have used the DE (rational transfer function) model of the plant. This comparison reveals the important fact that the basic principle of MAC does not depend upon the model description of the plant i.e., whether the model is described using a difference equation or impulse response. Therefore, for a one-step ahead prediction horizon, the interpretation of MAC as a feedback configuration (as shown in Figure 3.1) is also applicable in this case. The important difference in this case is that if we use the DE model of the plant, we can associate the traditional pole-zero interpretation to MAC. Indeed if we choose $\alpha_i = \alpha$ making the dynamics of all the reference trajectories the same, then we have $\Lambda_\alpha = \alpha I$ and the Figure 3.1 then is a familiar unit feedback MIMO configuration with $(1-\alpha)$ playing the role of a varying gain. There are two advantages of this configuration:

(i) the closed loop pole position can be ascertained apriori using the multivariable root-locus approach;

(ii) robustness of the closed loop can be examined in terms of the recently developed criteria employing the loop transfer function and return difference function at appropriate points in the loop.

3.4.1 Root Locus Analysis of MAC

Consider, for simplicity, a SISO plant with an actual transfer function $h(z)$ and suppose that its model is given by $\bar{h}(z)$. We assume that both the plant and this internal model is described by difference equation. Note that we have dropped the subscript d here from the previous section for notational convenience.

It is not obvious how $y(n)$ will be affected as α changes, but the effect can be analyzed as if we are finding the root locus of the closed-loop configuration in Figure 3.2. We can consider both α and $\bar{h}(z)$ as parameters that can be varied to regulate the closed-loop behavior of the system. Indeed if,

$$\frac{n(z)}{d(z)} = \frac{h(z)}{(z-1)\bar{h}(z)} \tag{3.25}$$

wher $n(z)$, $d(z)$ are polynomials in z,

$h(z)$ = plant transfer function in DE description

$\bar{h}(z)$ = model of the plant in DE description

the closed loop poles will trace a continuous path from the open-loop poles (i.e. poles of the plant, the zeros of the model and the zero at $z=1$) to the open-loop zeros (i.e. poles of $\bar{h}(z)$ and zeros of $h(z)$) as the gain varies from 0 to infinity. But here the gain $(1-\alpha)$ varies from 0 to 1 as α varies form 0 to 1. So the closed loop poles trace a path from the open loop poles to somewhere towards the open loop zeros. To put the problem into a standard framework of root locus, we introduce a one-to-one invertible mapping:

$$\beta = \frac{\alpha}{1-\alpha} \tag{3.26}$$

so that as α changes from 0 to 1, β changes from 0 to infinity.

Let $h(z) = \frac{n_h(z)}{d_h(z)}$ and $\bar{h}(z) = \frac{\bar{n}_h(z)}{\bar{d}_h(z)}$

From Figure 3.2 it can be shown that the input-output of the closed-loop is given by

$$h_{cl}(z) = \frac{(1-\alpha)n}{d+(1-\alpha)n} \qquad y(z) = h_{cl}(z)\ c(z)$$

where for simplicity we have written

$$n = n_h(z)\ \tilde{d}_h(z), \quad \text{and } d = (z-1)\ \tilde{n}_h(z)\ d_h(z). \tag{3.27}$$

For convenience henceforth we shall suppress the argument z. Using the transformation $\alpha = \frac{\beta}{1+\beta}$ gives

$$h_{cl}(z) = \frac{n}{d_{eq} + \beta n_{eq}} \tag{3.28}$$

where $\quad d_{eq} = n_h\tilde{d}_h + (z-1)\tilde{n}_h d_h$

$$n_{eq} = d = (z-1)\tilde{n}_h d_h$$

The closed loop characteristic polynomial is

$$\phi_{cl}(z) = d_{eq}(z) + \beta n_{eq}(z) \tag{3.28a}$$

It is obvious that

(i) as $\beta \to 0$ i.e. $\alpha \to 0$ (fast reference trajectory), the closed loop poles approach the zeros of $d_{eq}(z) = n_h\tilde{d}_h + (z-1)\tilde{n}_h d_h$. Depending on the characteristics of this polynomial the closed loop response may be oscillatory, damped and/or unstable.

(ii) as $\beta \to$ infinity i.e. $\alpha \to 1$ (slow reference trajectory), the closed loop poles approach the zeros of $n_{eq}(z)$, i.e. one pole approaches +1 and the remaining poles approach the poles of the plant and the transmission zeros of the model. The pole at $z=1$ will contribute to the sluggish response of the closed loop system.

So the problem of obtaining a specific response from MAC can be translated into the design of the polynomials $n_{eq}(z)$ and $d_{eq}(z)$. If the open loop poles are not located in the appropriate region of the z-plane, we can choose the model of the plant, i.e. $\tilde{n}_h$ and $\tilde{d}_h$ such

that the zeros of the polynomial $d_{eq}(z)$ are placed accordingly and the specific response can be obtained asymptotically as $\beta \to 0$. Note that the stability of the plant or the model is not required when analyzing MAC in a root locus framework. The problem is algebraic in nature, i.e., is a problem of synthesizing a specific polynomial $d_{eq}(z)$.

3.4.2 Examples

In this section we will demonstrate the above analysis through a simple example.

Example 3.1

Consider a scalar dynamic system

$$\dot{x}(t) = ax(t) + bu(t), \quad x(0) = x_0 \tag{3.29a}$$

$$y(t) = cx(t). \tag{3.29b}$$

Suppose the input and ouput are sampled every T seconds. Then the corresponding discrete-time (scalar dynamic system, as obtained by using the exponential transform (3.14), is

$$x(n+1) = fx(n) + gu(n) \tag{3.30a}$$

$$y(n) = cx(n), \tag{3.30b}$$

where

$f = e^{aT}$ and $g = \frac{e^{aT}-1}{a}$. Now suppose that the model of the plant is

$$x(n+1) = \tilde{f}\tilde{x}(n) + gu(n) \tag{3.31a}$$

$$\tilde{y}(n) = c\tilde{x}(n) \tag{3.31b}$$

For simplicity, let us chose $c=1/g$. Then using the notation of the previous section we have

$$h(z) = \frac{1}{z-f} = \frac{n_h(z)}{d_h(z)}, \quad \tilde{h}(z) = \frac{1}{z-\tilde{f}} = \frac{\tilde{n}_h(z)}{\tilde{d}_h(z)}.$$

Therefore, using the notation in (3.28),

$$d_{eq} = n_h\tilde{d}_h + (z-1)\tilde{n}_h d_h = z-\tilde{f} + (z-1)(z-f),$$

$$n_{eq} = (z-1)d_h\tilde{n}_h = (z-1)(z-f),$$

and the closed loop characteristics polynomial is

$$\phi_{cl}(z) = d_{eq}(z) + \beta n_{eq}(z)$$
$$= (z-\tilde{f}) + (z-1)(z-f) + \beta(z-1)(z-f).$$

As $\beta \to 0$ (i.e. $\alpha \to 0$: a fast trajectory), the closed loop characteristic polynominal asymptotically approaches

$$\phi_{cl}(z) \to z^2-fz+(f-\tilde{f})$$

and the closed loop poles approach

$$z_{1,2} \to \frac{f \pm \sqrt{f^2-4(f-\tilde{f})}}{2}. \tag{3.32}$$

Suppose $f=0{\cdot}9$ and the model is perfect, i.e, $\tilde{f}=0{\cdot}9$ too. Then since the plant is minimum phase, the closed-loop transfer function for all values of β is, from (3.27),

$$h_{cl}(z) = \frac{1-\alpha}{z-\alpha} \tag{3.33}$$

i.e., the perfect tracking has been achieved. This is shown in Figure 3.3, for $\alpha=0{\cdot}01$.

When $\tilde{f} = 0{\cdot}1$, and equation (3.32) indicates that the closed loop poles approach $0{\cdot}45 \pm j0{\cdot}77$. The closed loop response therefore is oscillatory which is demonstrated in Figure 3.4 for $\alpha=0{\cdot}01$. If $\tilde{f}=-0{\cdot}1$, the closed loop poles approach $0{\cdot}45 \pm j0{\cdot}90$ - the closed loop response becomes further oscillatory, which is shown in Figure 3.5 for the same value of α. Similary a choice of $\tilde{f}=-0.4$ places the closed loop poles at $0{\cdot}45 \pm j1{\cdot}047$ and the simulation has indeed shown the instability.

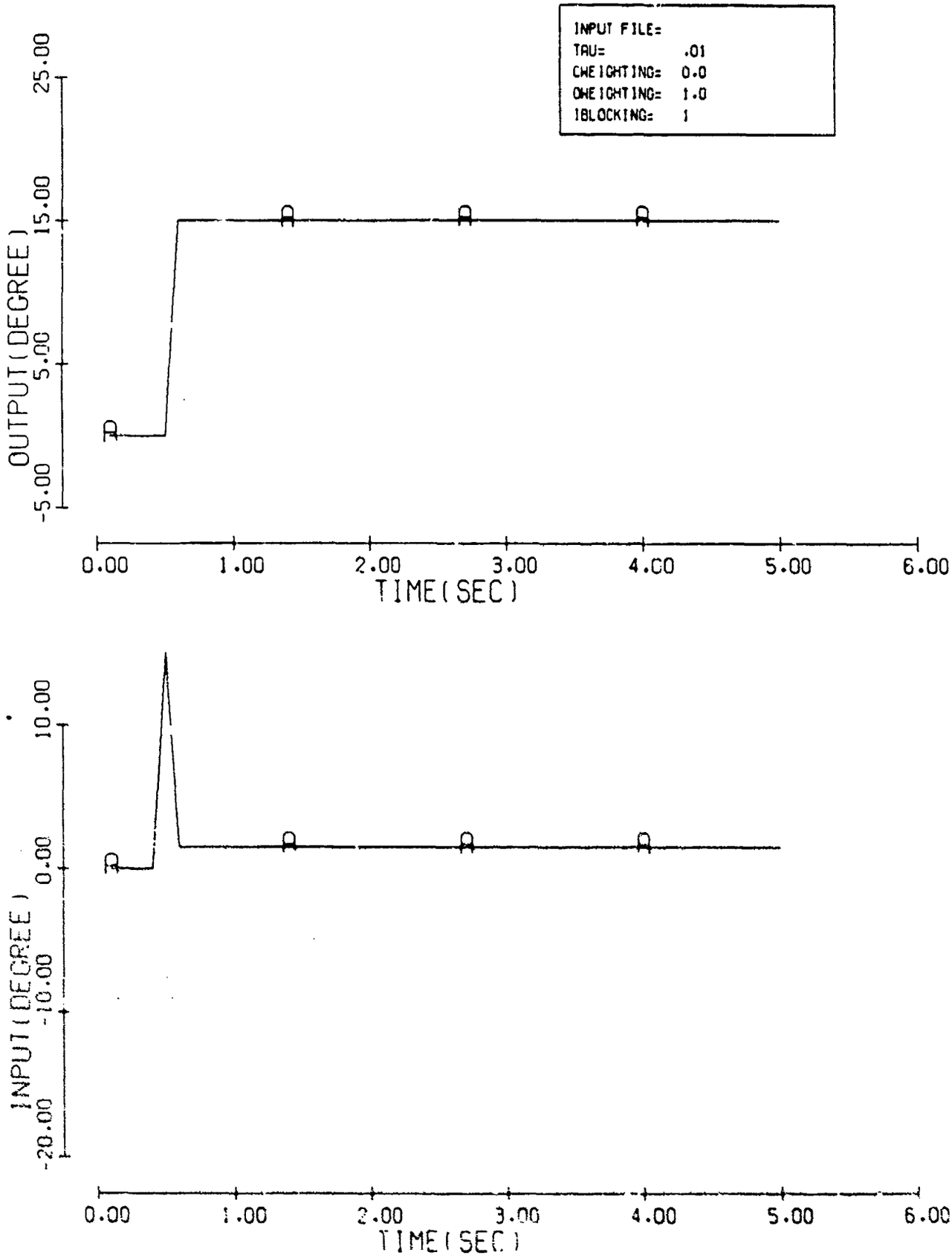


Figure 3.3 Perfect tracking when the model is exact.

It may also be noted that as $\beta \to \infty$, one of the closed loop poles approaches +1.0 which implies the loss of asymptotic stability or α very sluggish response therefore highly undesirable to operate MAC with a large β (or, α nearly equal to 1). But the problem with smaller β is that, along with a fast response, the bandwidth of the closed loop system is increased and the possibility of excitation of the unmodelled dynamics is also increased. A compromise, therefore, is needed while choosing the value of β.

To see how the root-locus interpretation helps in determining MAC behavior, let us consider the case of a perfect model and assume that the system is minimum phase. Then from equation (3.2),

$$\frac{n(z)}{d(z)} = \frac{1}{z-1}$$

and

$$n_h = \tilde{n}_h, \; d_h = \tilde{d}_h, \; s_{eq} = z n_h d_h, \; n_{eq} = (z-1) n_h d_h$$

$$\phi_u(z) = z n_h d_h + \beta(z-1)\, n_h d_h$$

Clearly then as $\beta \to 0$, (or $\alpha \to 0$), one closed loop pole approaches the origin $z=0$ and the others approach the zeros of $n_h d_h$. The later poles, however, get cancelled eventually (indicating that these mode become asymptotically either unobservable or uncontrollable) and the pole at $z=0$ becomes dominant, and a fast response is available from MAC. On the other hand as $\beta \to \infty$ (or, $\alpha \to 1$), the dominant pole is the one as $z=1$ and a sluggish response is obtained. All of these analyses agree with the observed behavior of MAC in everyday use.

3.5 Apriori fixed Gain Compensation of a Lightly Damped System or Unstable System.

A lightly damped system has a long impulse response (IR) sequence and therefore imposes burden on the computer storage. If the impulse response is sampled according to Nyquist sampling rate, an impulse response sequence of 60-150 elements are very common for a lightly damped system, particularly if the system has a frequency mode. It has been proposed that some additional damping may be introduced into

the system by applying output feedback and then MAC be applied to the overall system. It is the purpose of this section to investigate if apriori fixed gain output feedback can be useful for MAC application. Since there is no mathematical model available for a standard regular MAC with multistep prediction horizon, input blocking etc, we can not investigate analytically the effect of apriori output feedback on MAC. So the following analysis is based on the available properties of output feedback and our analysis is more qualitative then quantitative. We shall primarily emphasize on the issue that whether we can make the length of impulse response shorter using apriori fixed gain analog compensation of the plant.

3.5.1 Qualitative Analysis

In Section 3.4, we have characterised a lightly damped system by its pole positions. Roughly speaking, a continuous time dynamic system is lightly damped if any of its poles lies near the $j\omega$-axis in the s-plane. Similarly, in discrete time domain, a system is lightly damped if any pole lies near the unit circle on the z-plane. Physically it means tha the impulse response (IR) or the IR sequence is relatively longer. This fact plays an important role in the MAC technique, because the latter uses the IR descritpion of the plant. A lightly damped system has a relatively longer IR sequence and therefore uses more computer storage compared with a damped system. Since an unstable system has an infinitely long IR sequence, the current MAC implementation using the IR can not handle such systems.

If a system is open-loop unstable or lightly damped, it can be made stable or damping can be added apriori using constant or dynamic output feedback. The compensated plant with possibly a shorter IR sequence can be thought of as a new plant and MAC can then be applied to it for improved performance - the overall configuration is hybrid in nature. For simplicity, consider again a SISO plant

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0)=x_0 \tag{3.34a}$$

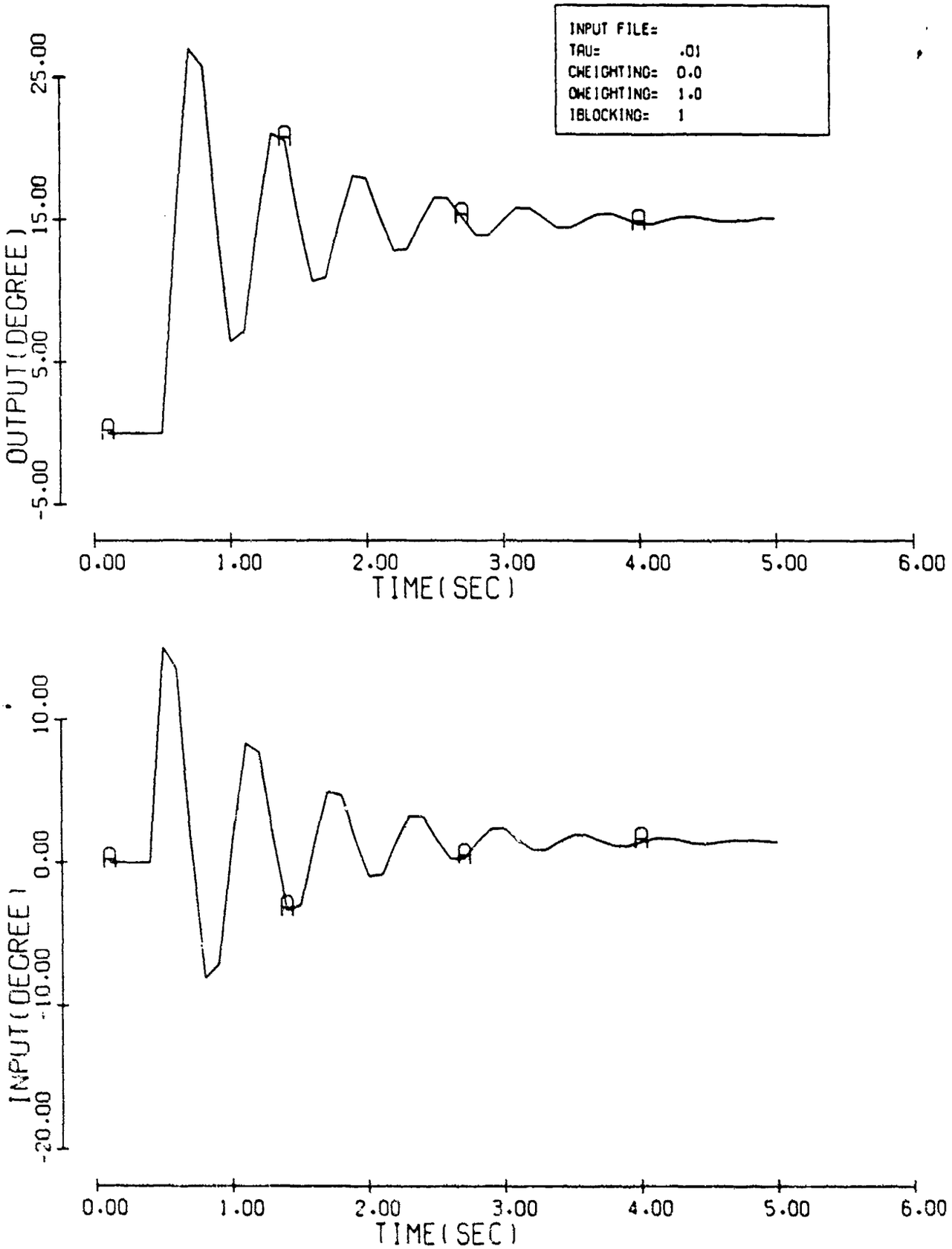


Figure 3.4 Oscillatory behavior for a fast trajectory.

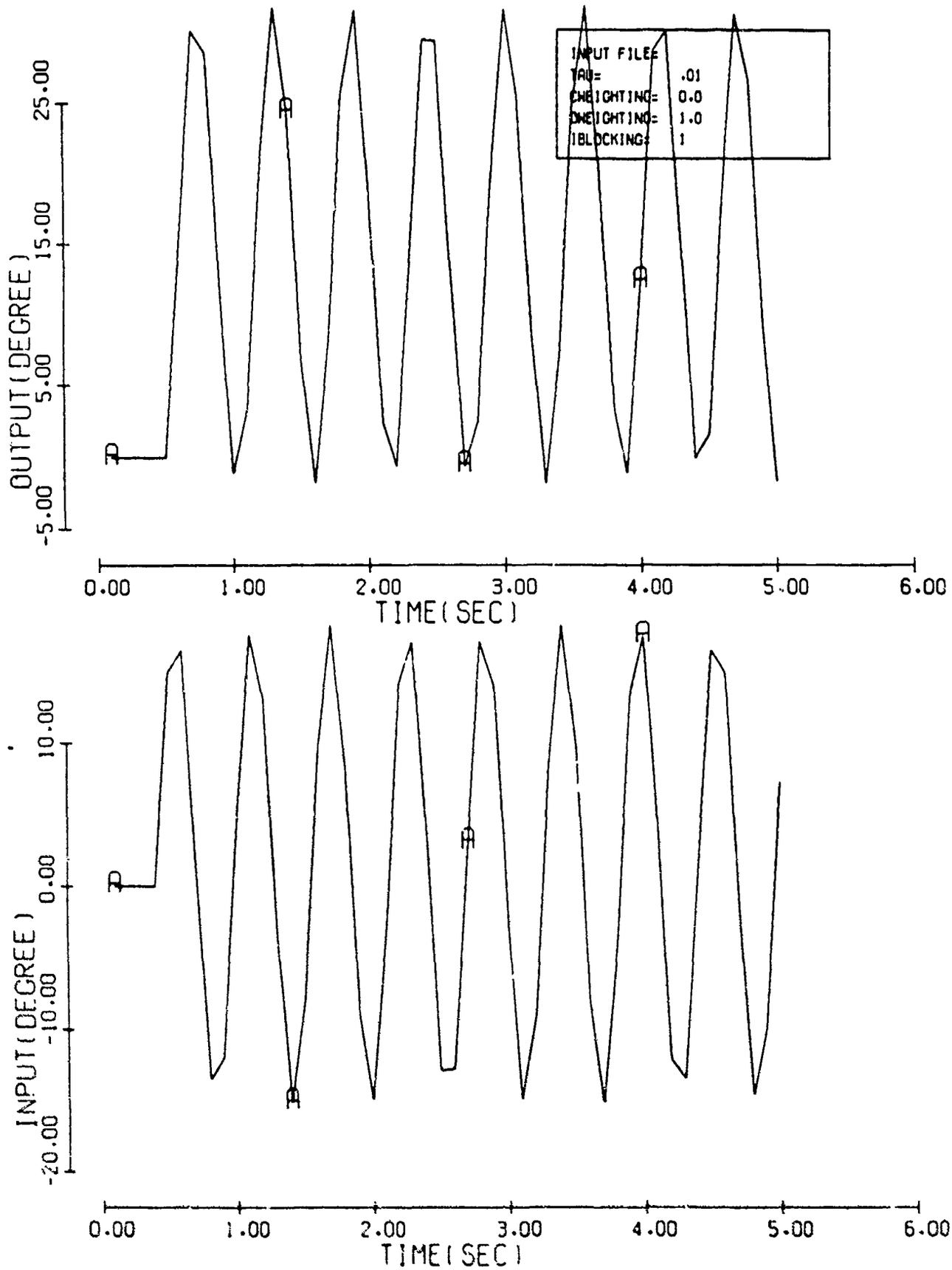


Figure 3.5 Oscillatory behavior of MAC as predicted by analysis.

$$y(t) = cx(t), \tag{3.34b}$$

where x(t) is n-dimensional and A,b,c have appropriate dimensions. If an output feedback control law is chosen of the form

$$u(t) = -ky(t) + v(t), \tag{3.35}$$

the closed loop system is given by

$$\dot{x}(t) = (A-bkc)x(t) + bv(t), \quad x(0)=x_0 \tag{3.36a}$$

$$y(t) = cx(t), \tag{3.36b}$$

and the closed-loop poles are given by the eignevalues of A-bkc. The hybrid system as a result of application of MAC is shown in Figure 3.6.

The speed of response and bandwidth of the system can be increased using output feedback. This makes it necessary to use a higher sampling rate for the compensated plant. This point needs some clarification. Although the Nyquist criteria holds for bandlimited signals, engineers have selected sampling rates according to this criteria, whether the signal is bandlimited or not, i.e., a rate of at least twice the highest frequency in the oscillatory modes in a plant. Similarly in a system without any oscillatory modes, the sampling rate is selected at a rate determined by the "Bandwidth (BW)" of the system. We may recall that the BW of such systems are defined as the frequency where the magnitude of the loop-transfer function drops off to half of its dc value. In this section we will see how apriori output feedback affects MAC performance via the sampling rate selection. The effect on robustness will be discussed in the next chapter.

Case 1. <u>When the Plant is Open-Loop Unstable</u>:

If the states are available for feedback, then it is well known that under the assumption of controllability, the closed loop poles can be placed arbitrarily in the complex plane using constant-gain state-feedback. But in the case of constant-gain output feedback,

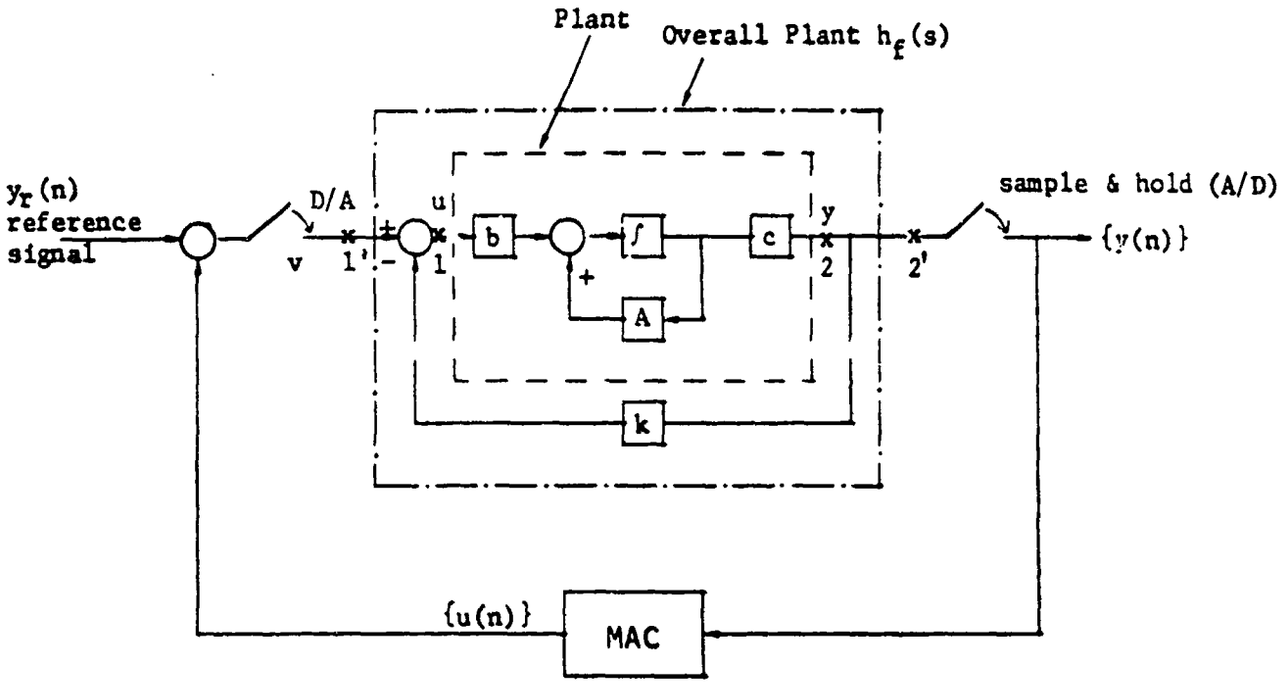


Figure 3.6 MAC applied to an apriori compensated plant

this freedom is lost and the poles are moved accordingly to the rules for root locus. But, as we know, it may not be possible to stabilize an unstable plant by constant-gain output-feedback - the interested reader may consult Youla's elegant work [Youla, et. al, 1974] for details. In such cases, the plant must be stabilized first by using dynamic output feedback before MAC can be used on the overall plant $h_f(s)$ in Figure 3.6. However once a stable $h_f(s)$ is obtained MAC treats it like any other stable plant.

Case 2. <u>When the Plant is Lightly Damped</u>:

If the open-loop plant has all the transmission zeros in the open left half of the s-plane (OLHP), the gain k can be made high and arbitrary fast response can be obtained without destabilizing the overall plant $h_f(s)$. As $k \to \infty$, some of the closed loop poles approach the finite transmission zeros of the plant and the remaining ones approach infinity. The limiting dynamical behavior of $h_f(s)$ is determined by the location of the transmission zeros. If the system has closed right half plane (CRHP) (in the s-plane) zeros, k can not be increased arbitrarily.

The BW of the overall system $h_f(s)$ in Figure 3.6 is determined by the fastest dynamics which in turn are determined by the poles that move toward infinity. Therefore as $k \to \infty$, the plant output must be sampled faster and faster to capture the dynamical characteristics of the overall plant $h_f(s)$. The situation is even worse if the transmission zeros are stable and lie near the $j\omega$-axis. In this case, as $k \to \infty$, some of the closed-loop poles arrive at these zeros and therefore $h_f(s)$ is lightly damped again. The IR of this system is composed of slow dynamics as well as of fast dynamics - the modes corresponding to slow dynamics make the impulse response of $h_f(s)$ long and the modes corresponding to fast dynamics dictate a fast sampling rate. The net result is that the IR sequence of the discretized system has possibly many more terms then the uncompensated plant. Therefore, there is a trade-off between the damping added to the system using output feedback and the resulting sampling rate.

Although the length of the impulse response gets smaller as a result of damping added, we may keep the sampling rate unchanged so that the number of terms in the IR sequence is smaller. This obviously deteriorates MAC performance. We illustrate these ideas with two simple examples.

3.5.2 Examples

Example 1.

Consider again the scalar system of the last section;

$$\dot{x}(t) = -ax(t) + bu(t), \quad x(0)=x_0 \tag{3.37a}$$

$$y(t) = cx(t), \tag{3.37b}$$

where a,b,c are scalars. Let us assume c=1, then the open-loop transfer function of the plant is $h_c(s)=b/(s+a)$. Although there is no oscillatory mode in this system, we will call it a lightly damped system if a=0. Using an output feedback control law u=-ky+v, the closed loop system is

$$\dot{x}(t) = -(a + bk)x(t) + bv(t), \quad x(0)=x_0 \tag{3.38a}$$

$$y(t) = x(t) \tag{3.38b}$$

and the closed-loop transfer function $h_f(s)=b/(s + a + bk)$. The power spectrum is

$$|h_f(j\omega)|^2 = \frac{b}{\omega^2 + (a+k)^2} \tag{3.39}$$

Clearly if the BW ω_0 of this system is defined as the frequency ω_0 at which $|h_f(j\omega_0)| = p\ |h_f(j0)|$, where p is a constant, then ω_0 is given by

$$\omega_0 = \sqrt{(1/p^2-1)}\ (a+k) = 2\pi f_0. \tag{3.40a}$$

The sampling time interval T is given by

$$T = \frac{1}{2f_0} = \frac{\pi}{\sqrt{(1/p^2-1)\ (a+k)}} \quad . \tag{3.40b}$$

The last equation shows that as k is increased to add more damping, (or, strictly speaking, to get a shorter duration IR sequence) the BW ω_0 is also increased and so does the sampling rate. The discretized system corresponding to (3.38) is obtained via an exponential transform as

$$x(n+1) = fx(n) + gu(n) \tag{3.41a}$$

$$y(n) = x(n) \tag{3.41b}$$

where

$$f = e^{-(a+k)T} \text{ and } g = \frac{f-1}{-(a+k)}\ b \tag{3.41c}$$

We shall examine how the MAC performance varies for a given T as k changes. Suppose a=1 and b=10 and consider the case for k=0; then for a choice of T=0.1 Sec, f=0.90484 and g=0.95163. MAC is applied to this system with a set point of 15.0, α=0·1. The result is shown in Figuree 3.7. Next k=10 is selected. For the same value of T, the discretized system parameters are f=0·332871 and g=0·60648. The result of applying MAC to this system is shown in Figure 3.8. Notice the difference between the control efforts in the two cases. In the later case, the same sampling interval of T=0·1 secs has captured less dynamical characteristics than the earlier case and the controller has spent more control effort in the stady-state tracking.

Example 2.

Next consider the decoupled longitudinal dynamics of an air-to-air missile (cf. AFWAL-TR-80-3125) [1].

$$\dot{x}(t) = \begin{pmatrix} -1.4868 & 1.00 \\ -149.93 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -281.11 \end{pmatrix} u(t)$$

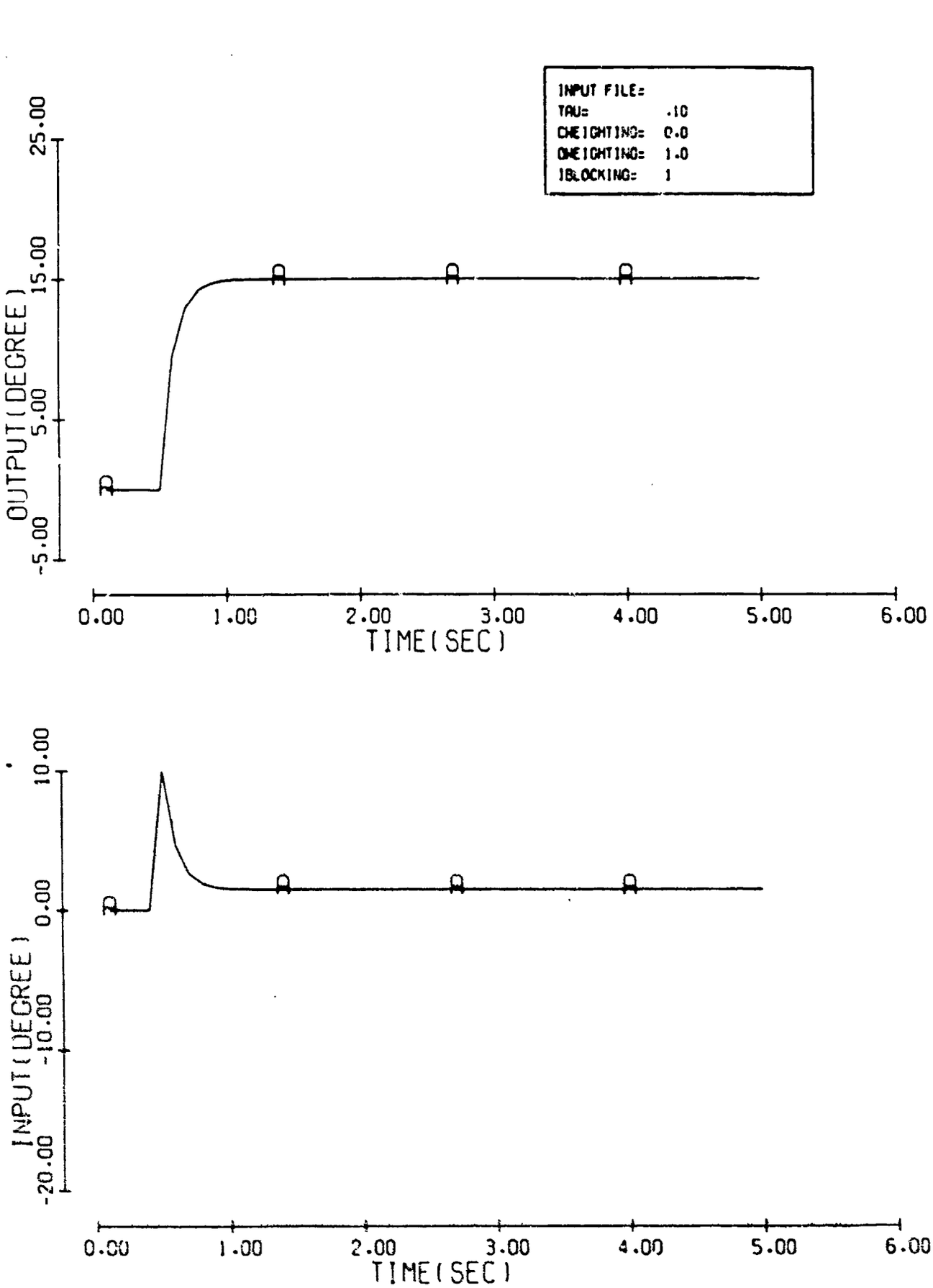


Figure 3.7 MAC applied to a lightly damped system, T= 0.1 secs.

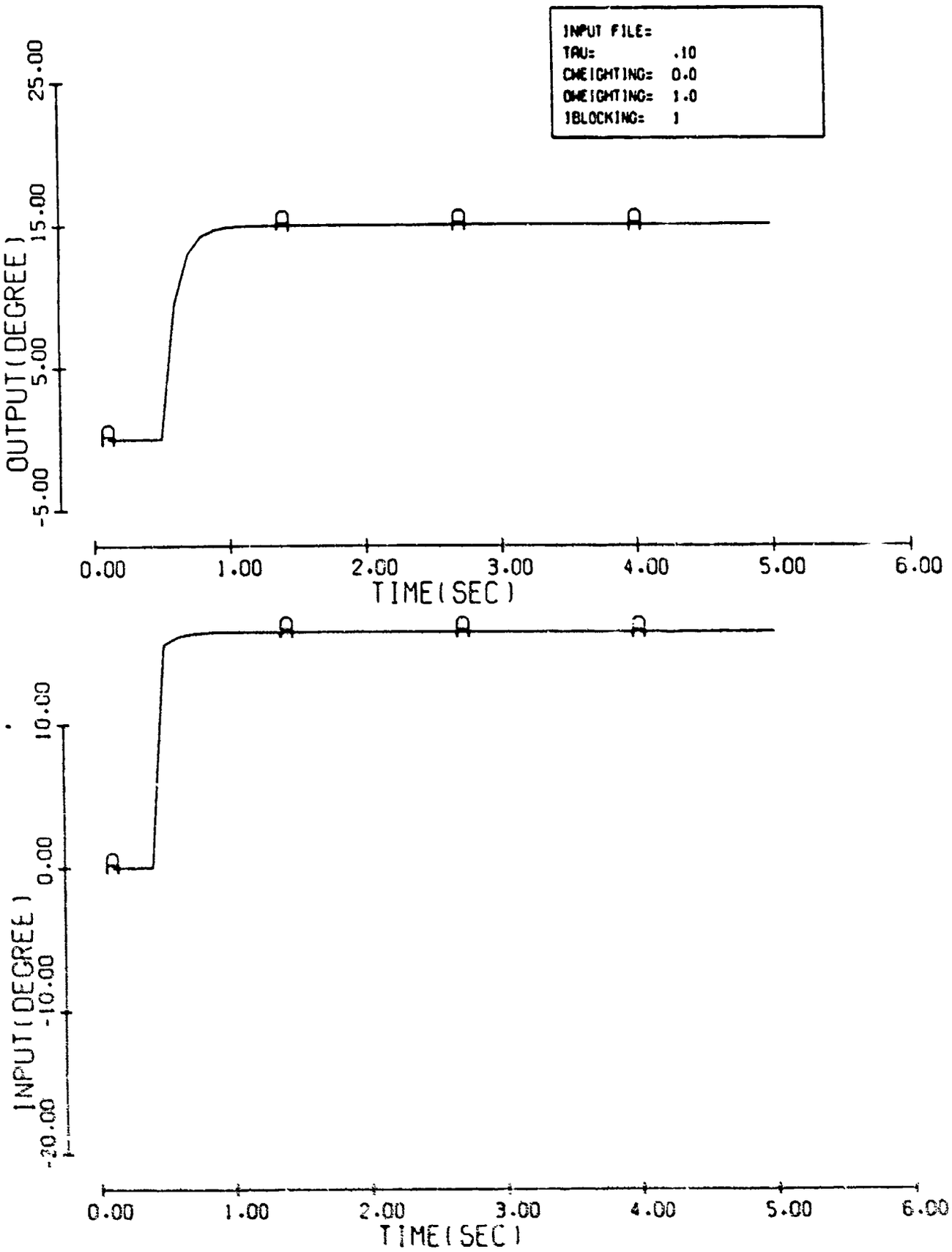


Figure 3.8 MAC applied to a system after damping is added to a lightly damped system, for the same sampling interval

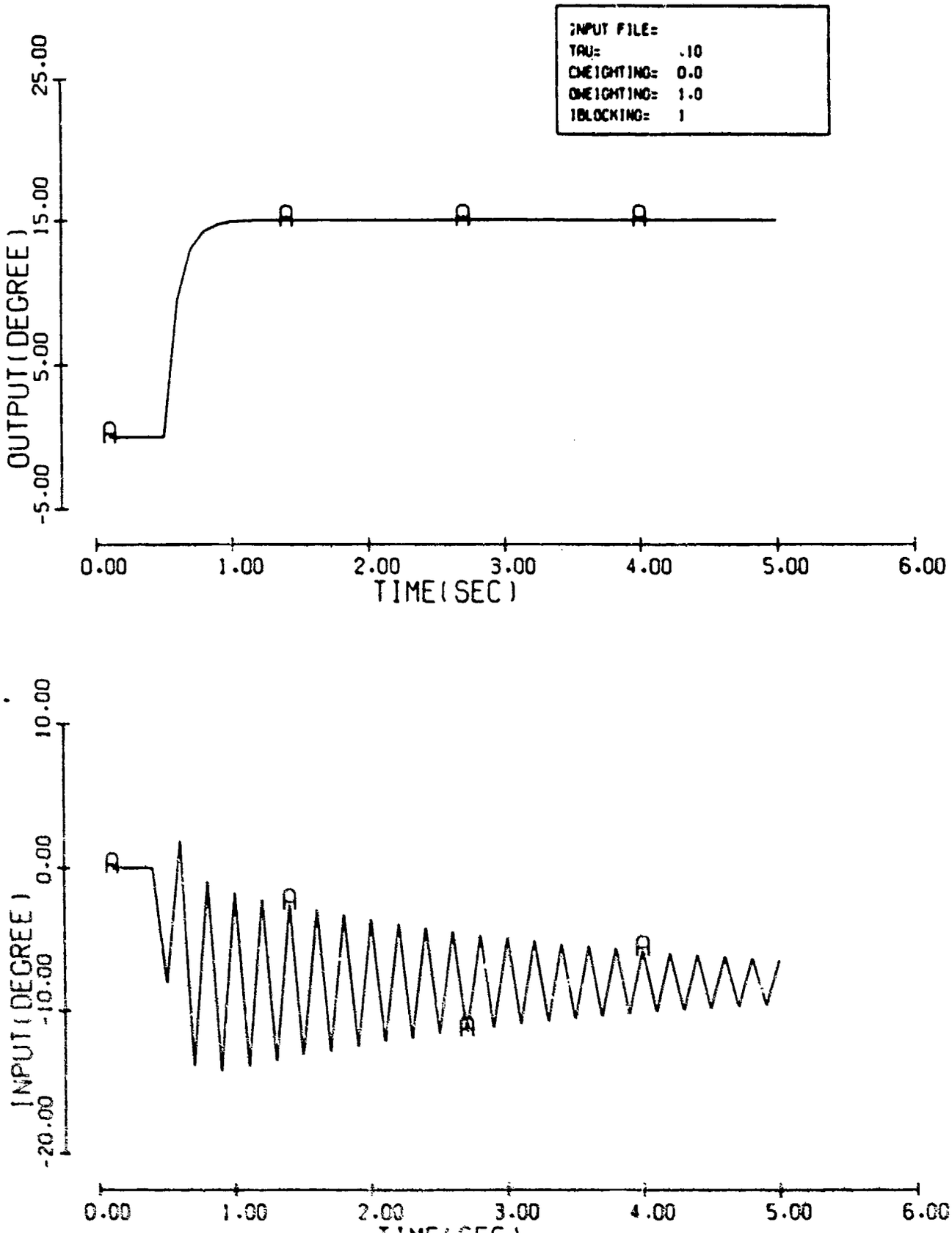


Figure 3.9 MAC applied to longitudinal dynamics of air-to-air missile, T = 0.1 secs.

$$y(t) = (1 \quad 0)x(t)$$

where $$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$x_1(t)$ = angle of attack in radian
$x_2(t)$ = perturbed pitch rate (rad/sec)
$u(t)$ = elevator angle (rad)

The eigenvalues are at $s_{1,2} = -0.7434 \pm j12.22$

The damping ratio $\xi = \dfrac{1.4868}{\sqrt{|1.4868^2 - 4\ 149.93|}} = 0.061$

The system is lightly damped with a natural frequency of 1.95 Hz. Therefore, the output must be sampled at least every 1/4 sec. Using negative feedback of the output, the closed loop system is given by,

$$\dot{x}(t) = \begin{pmatrix} -1.4868 & 1.00 \\ -149.93 + 281.11k & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -281.11 \end{pmatrix} v(t)$$

$$y(t) = (1 \quad 0)\ x\ (t)$$

The damping ratio of the closed loop system can be found as

$$\xi_o = \frac{1.4868}{\sqrt{|(1.4868)^2 - 4(149.93 + 281.11k)\ |}}$$

Clearly for $k > 0$, $\xi_o < \xi$. As k increases, the system approaches being undamped and accordingly the sampling rate decreases up to about k=0·531 when the system becomes critically damped. As k increases further both poles are real - one mode becomes fast and the other mode slow thus making the IR even longer until k=0·533 when the system is marginally stable.

As in the last example, T=0.1, set point = 15.0 and α=0.1 is selected. The result of application of MAC to the uncompensated plant, i.e. k=0, is shown in Figure 3.9. Now when k=0·53289, the compensated system is sampled in sampled at T=0.1 secs. and MAC is applied to the discretized system at this sample rate. The result is shown in Figure 3.10. As in the last example, the control effort

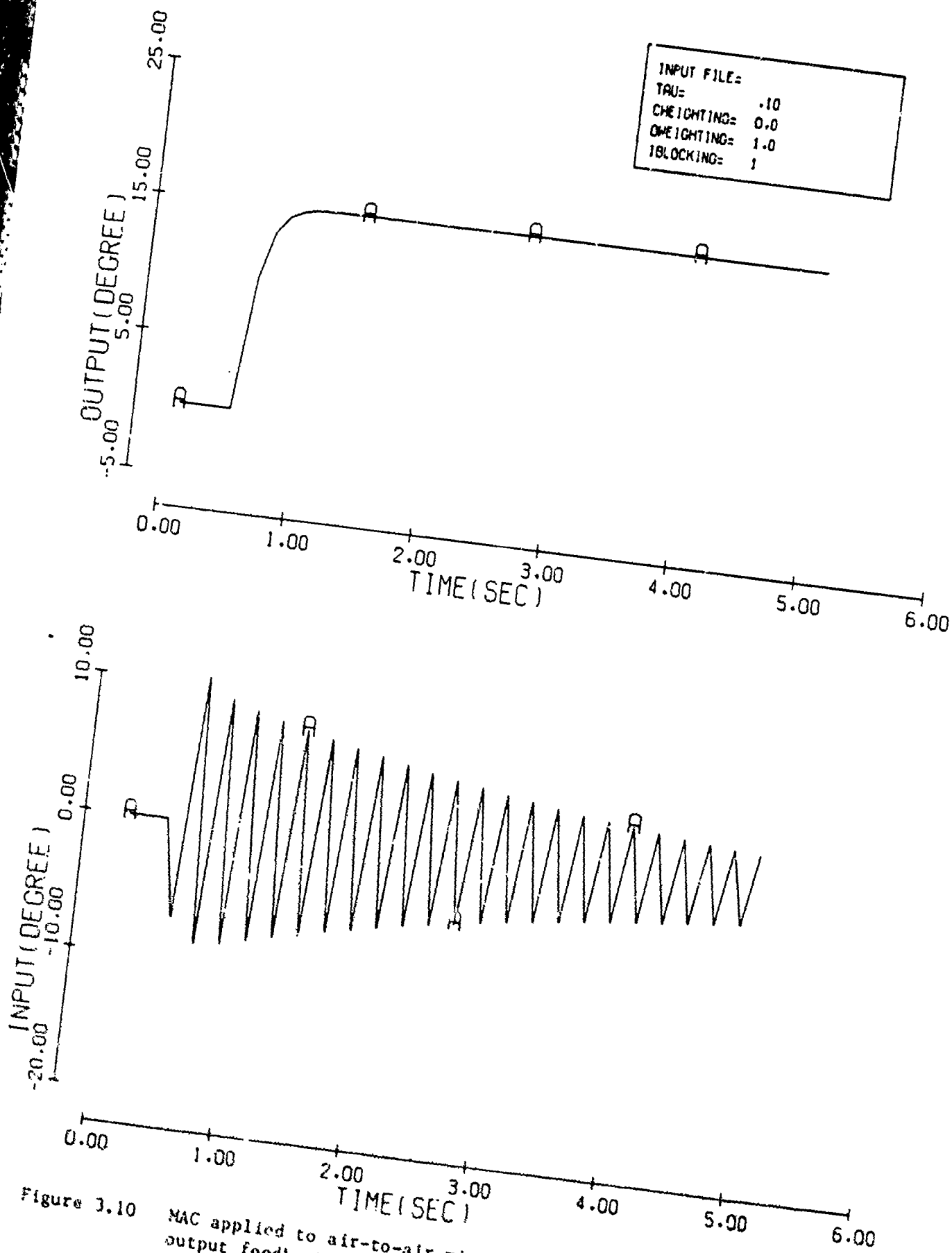


Figure 3.10 MAC applied to air-to-air missile: after using apriori output feedback, but at the same sampling interval of 0.1 secs.

needed to keep it in the right trajectory is larger than for uncompensated plant.

3.6 Conclusion

The main contribution of this chapter is the description of MAC for Multivariable system in section 3.2 where it has been shown that the classical-controller interpretation of MAC can be extended to MIMO systems. This interpretation of MAC will help the designer to apply the recently developed robustness analysis tool to MIMO MAC. Another important contribution of this chapter is the description of MAC using the rational transfer function (or difference equation) model of the plant - MAC can then be interpreted in a root-locus framework and explained using traditional pole and zeros of a rational transfer function. Finally in section 3.5, the effect of apriori analog compensation on the MAC performance has been investigated qualitatively. It has been found that if the addition of output feedback creates a faster mode than in the uncompensated plant, the sampling rate must be increased accordingly to capture the dynamical characteristics of the compensated plant. Otherwise MAC performance will deteriorate.

CHAPTER 4

ROBUSTNESS ANALYSIS OF MAC

4.1 Introduction

Any model of the plant is almost invariably different form the actual plant for many reasons. For the purpose of synthesizing a finite dimensional controller, the plant is modelled as finite dimensional even though the plant may be of a distributed nature or may have delays embedded in it. Usually the high frequency part of a plant is neglected and the model emphasizes the low frequency behavior of the plant. Even though a plant has been modelled accurately in the past, low frequency error is introduced eventually due to aging, deterioration etc. On the other hand a control law is designed on a nominal model and implemented on the actual plant. The nominal control law therefore must be robust enough to ensure the performance level for the actual plant. The purpose of robustness analysis is to examine the range of the nominal control law maintaining the closed-loop stability and performance level for all the plants around the nominal model. The classical designers measure the robustness (with respect to stability) of a nominal control law by its gain-margin (GM) and phase-margin (PM). In this chapter, the robustness of the MAC control law will be studied from the viewpoint of a classical controller and therefore MAC must be modelled as a classical controller. We have already developed a model of MAC of this type in the preceeding chapters which we summarize here again briefly. For simplicity of analysis, we shall consider SISO plants only. The MIMO plants are described in Larimore, Mahmood, and Mehra (1984).

This chapter is organized as follows. The MAC model developed in the previous chapter is briefly reviewed in Section 4.2 -- this model is the basis for all subsequent analysis of robustness. Classical gain margin (GM) and phase margin (PM) for MAC are analyzed in Section 4.2.1. The robustness in terms of GM and PM can handle a limited class of plant per-

turbations; therefore a more generalized class of perturbations are characterized and robustness evaluated in Section 4.2.2. Since a rational transfer function model usually has far fewer numbers of parameters than in the impluse response (IR) description of the plant, a robustness result is derived for such models in Section 4.2.3. A simple analytical example is presented in Section 4.3. Finally the chapter is concluded in Section 4.4.

4.2 Review of MAC Model for Robustness Analysis

Let us recall that under some simplifying assumptions, MAC can be modelled as in a classical control framework. The underlying assumptions are:

(i) the actual plant h(z) is minimum phase

(ii) there are no input constraints, i.e. $\Omega(i) = R$ for all i, where R is the real line

(iii) the optimization is carried over one future step ahead i.e., (T = 1); under this condition MAC is a one-step ahead predictive controller

The transfer functions under the MAC control law for MIMO plants have been developed in Equations (3.7a) and (3.7b). The corresponding quantities for SISO plants are:

$$\frac{u(z)}{c(z)} = \frac{1 - \alpha}{(z-1)h(z) + (1-\alpha)h(z)} \tag{4.1a}$$

$$\frac{y(z)}{c(z)} = \frac{h(z)(1-\alpha)}{(z-1)h(z) + (1-\alpha)h(z)} \tag{4.1b}$$

Equations (4.1a) and (4.1b) imply that MAC under assumptions (i)-(iii) is equivalent to the classical unit feedback configuration of Figure 3.2 in an input-output sense. The figure is again reproduced in the following for convenience:

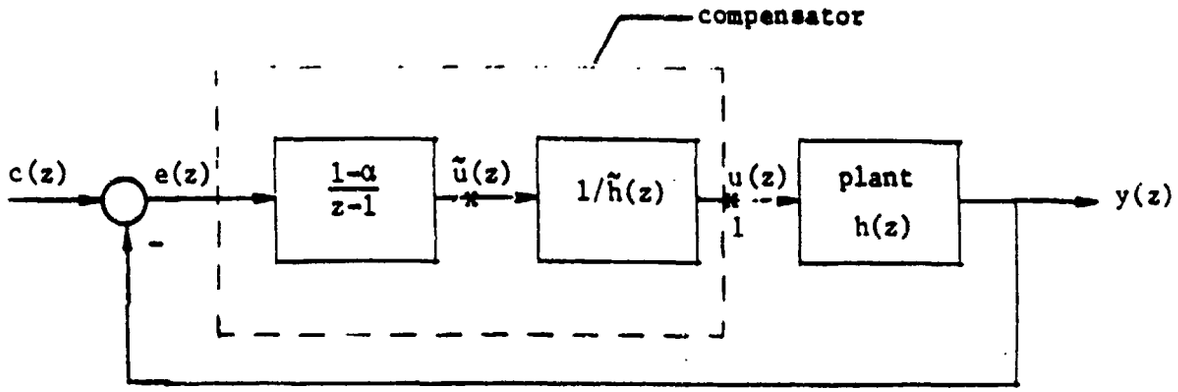


Figure 4.1. MAC as a Classical Controller

This interpretation of MAC is the basis of our analysis of MAC in the framework of classical control.

4.2.1. Phase and Gain Margins

The block within the dashed line can be considered as a dynamic controller of the classical type. The loop transfer function l(z) at point 1 is

$$l(z) = \frac{h(z)(1-\alpha)}{\tilde{h}(z)(z-1)} \tag{4.2a}$$

and the return difference function is

$$1 + l(z) = \frac{\tilde{h}(z)(z-1) + h(z)(1-\alpha)}{\tilde{h}(z)(z-1)} \tag{4.2b}$$

Note that since we are dealing with a SISO loop, the loop transfer function at any point of theloop is same. For MIMO loops, the loop transfer function depends on the point where the loop is broken because of the non-commutativity of matrices. However, in this case the error y(z) in tracking $e(z) = c(z) - y(z)$ is given by

$$e(z) = (1 + l(z))^{-1}\, c(z)$$

so that the steady state error due to a step input is

$$e_{ss}(t) = \lim_{z\to 1} (1+l(z))^{-1} = (1 + l(1)) = 0$$

whether the model is exact or not. This is a consequence of a built-in integrator in the compensator.

It may be noted from Figure 5 that at point 2, $\tilde{u}(z) = y_r(z)$ when $h(z) = \tilde{h}(z)$, where $y_r(z)$ is the reference signal. In this case the input $u(z)$ to the actual plant is generated as $u(z) = y_r(3)/h(z)$ and therefore $y(z) = h(z)u(z) = y_r(z)$. This shows why perfect tracking is possible under perfect identification. We will, however, now pursue this issue further.

It is obvious from Equations (4.1) and (4.2) that the closed-loop system is internally asymptotically stable if the roots of the rational function

$$\phi_{cl}(z) = (z-1)\tilde{h}(z) + (1-\alpha)h(z) \tag{4.3}$$

are within the open unit disk $z < 1$, and these roots are also the roots of the return difference function $1 + l(z)$. We can therefore find the stability margin in terms of the gain margin (GM) and phase margin (PM) from the Bode plot or Nyquist plot of the loop transfer function $l(z)$ evaluated on the contour $z = \exp(j\omega)$ appropriately indented around the poles on this contour. Recall that in continuous-time, the GM and PM are those values of k and ϕ respectively such that the perturbed loop $l(s) = k\exp(j\phi)l(s)$ is stable, where $l(s)$ is the nominal loop and s is the Laplace variable. A similar interpretation goes for the discrete-time systems (Kuo(1980)); but the PM, unless it is an integral value of the sampling interval, does not have any physical significance. Strictly speaking the complex constant $k\exp(j\phi)$ in continuous time should be replaced by kz^{-n}, n an integer, for measuring GM or PM of the discrete-time system.

Another way to compare with other continuous-time domain design techniques is that each element of the discrete-time loop should be transformed into an equivalent continuous-time element using the bilinear transformation, and PM of the fictitious continuous-time loop can be taken as the PM of the discrete-time loop. In this paper the word PM is used to mean the continuous-time equivalent phase margin. We can now state

Theorem 4.1

Under assumptions (i)-(iii), MAC has GM = $(0, 2/(1-\alpha))$, equivalent PM = $\mathrm{Cos}^{-1}\,(1-\alpha)/2$, and unity gain cross-over frequency ω_o = $2\sin^{-1}\,(1-\alpha)/2$.

Proof

The proof is trivial if we recall that PM and GM are measure on a nominal loop. Here we can assume that the nominal plant $h(z) = \tilde{h}(z)$, which implies $h_i = \tilde{h}_i$ and $N = \tilde{N}$ because both $h(z)$ and $\tilde{h}(z)$ are power series in z^{-1}. This nominal loop transfer function from (4.2a) is then

$$l(z) = \frac{1-\alpha}{z-1} \tag{4.4}$$

i.e. an integrator delayed by one-step. Evaluating on $z = \exp(j\omega)$, we get

$$l(\exp(j\omega)) = -\frac{1-\alpha}{2} - j\,\frac{1-\alpha}{2}\cot\frac{w}{2} \tag{4.5}$$

and $|l(\exp(j\omega 0)| = 1.0$ implies that unity gain cross-over frequency at

$$\omega 0 = 2\,\sin^{-1}\frac{1-\alpha}{2} \tag{4.6}$$

The Nyquist plot of the discrete-time loop in Equation (4.5) is quite simple and from the plot it is easy to see that the system is stable for all gains in the interval $(0,2/1-\alpha)$, and a pure delay $\phi = 90° - \mathrm{Sin}^{-1}(1-\alpha)/2$ will change the number of encirclements by the Nyquist contour, thus making the system unstable.

To get the equivalent PM we transform each element of the loop using the bilinear transformation $s = (z-1)/(z+1)^{-1}$ to get the equivalent continuous loop

$$l(s) = \frac{1-\alpha}{2}\left(\frac{1}{s} - 1\right) \tag{4.7}$$

From the Nyquist plot of $l(s)$ it is obvious that GM $(0,2/(1-\alpha))$ (same as found by analyzing the discrete-time Nyquist plot) and a PM = $\mathrm{Cos}^{-1}(1-\alpha)/2$.

Theorem 1, although very simple, reveals some intuitively appealing results about GM and PM of MAC. We can make the following remarks.

Remarks

1) Since $\alpha \varepsilon [0,1]$, the guaranteed upward GM is 2 and the PM is 60°.

2) We can always trade-off robustness against the speed of response. As response speed is increased by decreasing α, BW $\omega_0 = 2\sin^{-1}(1-\alpha)/2$ increases (which makes sense) with a consequent reduction of robustness in terms of GM and PM.

3) We get this remarkable PM even though MAC is an output-feedback controller possibly because the plant is inverted causally through the use of an optimization algorithm in the sense that at each time the algorithm provides the controller with the entire future input sequence. For the same reason, the discrete-time loop has a one pole roll-off for all frequencies - which is rather unusual.

4) Theorem 1 ensures that the controller can stabilize the loop for all the plants $\{h_i\}$ belonging to the set

$$\{h_i | h_i = k\bar{h}_i,\ i=1,\ldots,\bar{N},\ k\varepsilon(0,2/(1-\alpha))\}.$$

4.2.2. Plant Robustness Analysis for Generalized Perturbations

The nominal model $\bar{h}(z)$ is usually different from the actual plant $h(z)$ for various reasons. Sometimes $\bar{h}(z)$ is deliberately made simple to facilitate the control computation by retaining the modes in the active frequency range. On many occasion it is difficult to model high frequency modes, and these are simply neglected. Due to ageing, etc., the modes of the actual plant drifts slowly thus introducing low-frequency error. Thus the modeling error $e(z)$ has in almost every case, a dynamic structure; and the information about $e(z)$ must be incorporated in designing a nominal loop. As a minimum amount of

information $e(z)$ is expressed as an upperbound on $|e(\exp(j\omega))|$; and the purpose of robustness analysis is to find a requirement on the nominal loop interms of this upperbound so that the closed loop performance and stability is maintained in the face of modeling uncertainty.

Usually the admissible uncertainties are expressed in two ways: additively or multiplicatively. If we take $\bar{h}(z)$ as the nominal plant, then in an additively uncertain model, we express the actual plant $h(z)$ as

$$h(z) = \bar{h}(z) + \Delta h_a(z) \tag{4.8}$$

and in a multiplicatively uncertain model, the actual plant $h(z)$ is

$$h(z) = \bar{h}(z)(1 + \Delta\bar{h}_m(z)) \tag{4.9a}$$

or

$$h(z) = \bar{h}(z)\ \Delta h_m(z) \tag{4.9b}$$

For single-loop systems the order of multiplication in (4.9) is not relevant, but for MIMO cases the order is important because of the non-commutativity of matrices where input channel (left) uncertainty and output-channel (right) uncertainty must be distinguished. Both of the multiplicative forms in (4.9) are often used in analysis, but in this paper we shall be using (4.9b). Note that at nominal values of the plant, $\Delta h_a(z) = \Delta\bar{h}_m(z) = 0$ and $\Delta h_m(z) = 1$. Also note that the classical GM and PM ensures the stability of a perturbed plant of the form (4.9b). If the GM is k, then $\Delta h_m(z)=k$, and if the PM=n (in the sense of discrete-data system), $\Delta h_m(z)=z^{-n}$. These are undoubtedly a limited class of allowable perturbations and we must consider other possible error-structures in designing the nominal loop. The framework of (4.8) and (4.9) is more general in the sense that it can handle a constant, nonconstant and even dynamic model mismatch (say for example unmodelled poles, etc.). Let us rewrite $\bar{h}(z)$ and $h(z)$ as

$$\bar{h}(z) = \sum_{i=1}^{\bar{N}} \bar{h}_i z^{-i} = z^{-\bar{N}}\bar{h}_p(z) \tag{4.10a}$$

where $\bar{h}_p(z) = \sum_{i=1}^{\bar{N}} \bar{h}_i z^{\bar{N}-i}$ = a polynomial in z,

and $\qquad h(z) = z^{-N} h_p(z),$

$$h_p(z) = \sum_{i=1}^{N} h_i z^{N-i} \tag{4.10b}$$

then by straightforward manipulation, the closed loop characteristics polynomial is

$$\phi_{cl,p}(z) = z^N(z-1)\tilde{h}_p(z) + z^{\tilde{N}}(1-\alpha)h_p(z) \tag{4.11}$$

with p denoting that we are considering the polynomial part only. For closed-loop stability, $\phi_{cl,p}(z)$ must have all the roots strictly inside the unit disk $|z|=1$. For perfect identification $\tilde{N}=N$, $\tilde{h}_p(z)=h_p(z)$, and $\phi_{cl,p}(z)=z^N(z-\alpha)\tilde{h}_p(z)$. Of course the zeros of $\tilde{h}_p(z)$ will be cancelled eventually leaving the only closed loop pole at $z=\alpha$. However N, the order of the true plant, is usually unknown, and therefore in real-world situations (4.11) can not be evaluated. The actual plant h(z) must be considered as a perturbation of the nominal plant $\tilde{h}(z)$, and the stability conditions must be derived in terms of the nominal sequence $\{\tilde{h}_i\}$ and the perturbation $\Delta h_a(z)$ or $\Delta h_m(z)$. Let us assume that $\Delta h_a(z)$ and $\Delta h_m(z)$ can be expressed as in (4.10), i.e.,

$$\Delta h_a(z) = \sum_{i=1}^{N_a} h_{ai} z^{-i}$$

$$= z^{-N_a}\Delta h_{ap}(z),\ \Delta h_{ap}(z) = \text{a polynomial in } z \tag{4.12a}$$

$$\Delta h_m(z) = \sum_{i=1}^{N_m} \Delta h_{mi} z^{-i} = z^{-N_m} h_{mp}(z)\ . \tag{4.12b}$$

although the following theorem can be developed without such an explicit form. Note that the index in (4.12b) must start from 0 to accomodate constant multiplicative perturbation. We have the following theorem on robustness:

Theorem 4.2

(i) The system is closed-loop stable for all additive perturbations $\Delta h_a(z)$ satisfying

$$|\Delta h_{ap}(z)| < \frac{\sqrt{1 - 2\alpha \cos\omega + \alpha^2}}{1-\alpha} \; |\bar{h}_p(z)| \tag{4.13a}$$

and $z = \exp(j\omega)$

(ii) The system is closed-loop stable for all multiplicative perturbations $\Delta h_m(z)$ satisfying

$$|\Delta h_{mp}(z) - z^{N_m}| < \left|\frac{z - \alpha}{1 - \alpha}\right| \tag{4.13b}$$

on the unit circle where $\Delta h_{ap}(z)$ and $\Delta h_{mp}(z)$ are given by (4.8).

Proof: The proof is straightforward if we express $h(z)$ using the form (4.10)-(4.11), find the corresponding closed-loop characteristic polynomial, and finally use Rouch's theorem to prove (4.13) on the assumption that the nominal loop is internally stable and hence $(z-\alpha)\bar{h}_p(z)$ has all the roots strictly inside the unit disk $|z|=1$.

The tests of the type given in (4.13) are sufficient conditions and generally tend to be conservative. Nevertheless we can make the following remarks:

(i) Both tests (4.13a) and (4.13b) are useful. For example when an actual known model $\{h_i,\ i=1,\ldots,N\}$ is truncated to obtain $\{\bar{h}_i,\ i=1,\ldots,\bar{N},\ \bar{N} < N\}$, so that $\{\Delta h_{ai} = h_i,\ i=\bar{N}, \bar{N}+1,\ldots N$ and $\Delta h_{ai} = 0,\ i < \bar{N}\}$, stability around $\{h_i\}$ can be obtained from (4.13)

(ii) For constant multiplicative gain mismatch, i.e. $h_i = k\bar{h}_i$ for all i, $\{\Delta h_{mi} = k$ when $i=0$ and $\Delta h_{mi} = 0$ when $i > 0\}$, so that $\Delta h_{mp}(z) = kz^{N_m}$ and test (4.13b) yields that the system is stable for all k such that

$$|k - 1| < \frac{|z - \alpha|}{1 - \alpha}, \quad z = \exp(j\omega) \tag{4.14}$$

But it is easy to see that min $|\exp(j\omega)-\alpha| = 1-\alpha$ so that (4.10) becomes $|k-1|<1$ which implies $k\varepsilon(0,2)$. This clearly shows that these tests are conservative. (See remark (4) of the previous section).

It can be shown trivially that near $\omega=0$, the bound on the RHS of (4.13a) is meaningless; almost any reasonable perturbation will satisfy this sufficiency condition at low frequencies, but the above inequality must be obeyed for each $\omega\varepsilon[0,\Pi]$ particularly at high frequencies.

We note further that given any perturbation $\Delta h_p(j\omega)$, it is extremely difficult to come up with a stable design to accomodate it. On the other hand, given any stable design we can only make statements about the size of a perturbation the design can tolerate, and perhaps from our previous experience we can change the nominal design iteratively to accomodate the given perturbation.

4.2.3. Robustness Analysis When the Plant Model is Described by a Rational Transfer Function

In the previous section we analyzed the robustness of the MAC control law for systems represented by an impulse response sequence. In this section the analysis will be carried out for plants described by Difference Equations (DE) - this will yield more insight into the relation between the robustness of MAC and the design parameters embedded in it.

We analyze again under the usual assumptions, viz,

(i) the system is minimum phase

(ii) the optimizing horizon is one-step in the future

(iii) there are no constraints either on the input or any other loop variables

Under these assumptions, the MAC control law is given by equations (4.1a)-(4.1b) and the equivalent classical network is given in Figure 4.1. Obviously then the loop transfer function is

$$l(z) = \frac{(1-\alpha)h(z)}{(z-1)\bar{h}(z)} \qquad (4.15a)$$

so that the return difference function is

$$1 + l(z) = 1 + \frac{(1-\alpha)h(z)}{(z-1)\bar{h}(z)} = \frac{(z-1)\bar{h}(z) + (1-\alpha)h(z)}{(z-1)\bar{h}(z)} \qquad (4.15b)$$

Clearly the closed loop poles are given by the zeros of the numerator (4.15b). We have shown in the previous section that the MAC control law is nominally closed loop stable for all values of α, $0<\alpha<1$.

A typical Nyquist plot is shown in Figure 4.2. It is obvious from the figure that at any frequency ω_0, the loop transfer function $l(e^{j\omega_0})$ can tolerate a maximum perturbation of $|1+l(e^{j\omega_0})|$ and yet the Nyquist plot will not change the number of encirclements of the $-1+j0$ point. This observation leads to the following theorem on additive perturbations.

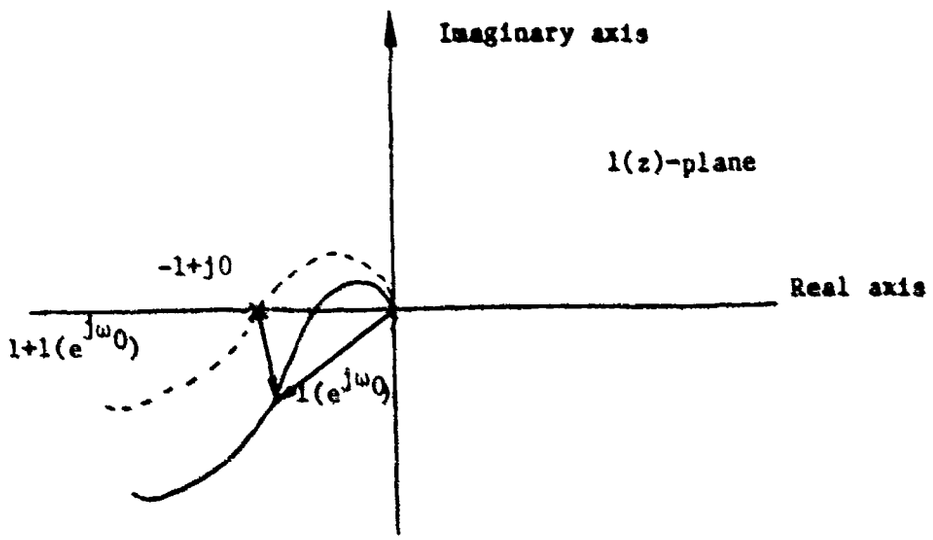


Figure 4.2. A Typical Nyquist Plot

Theorem 4.3

Suppose the loop is nominally stable. The the perturbed loop is stable for all additive perturbations $\Delta l(z)$ satisfying

$$|\Delta l(e^{j\omega})| < |1+l(e^{j\omega})| \qquad (4.16)$$

where ω varies over the unit circle if $l(z)$ is analytic on the contour $|z| = 1$ or over any suitable indented contour on the unit circle to bypass any singularity of $l(z)$ on the unit circle.

Proof: A heuristic proof should be obvious from Figure 4.2. A rigorous proof follows from a straightforward application of Rouches' Theorem as in the previous section.

A similar theorem can be developed for multiplicative perturbations. Theorem 4.3 gives the sufficiency condition for stability. Its usefulness lies in the fact that given an apriori knowledge of a perturbation that satisfies the inequality (4.16), the Theorem guarantees the stability of the closed loop system for such perturbation. For example if a high frequency mode is neglected or if the modes are not correctly modeled, the discrepancy is expressed in an additive form and a test of the type (4.16) must be carried out after a nominal control law has been found.

We can find a more specific form of Equation (4.16) as follows. Suppose the nomimal (or identified) plant is $\tilde{h}(z)$. The true plant $h(z)$ is assumed to lie in a neighborhood of $\tilde{h}(z)$, and suppose $h(z)$ is an additive perturbation of $\tilde{h}(z)$. In this case

$$h(z) = \tilde{h}(z) + \Delta h_a(z) \ . \tag{4.17}$$

The designer usually has a knowledge of an upperbound on $|\Delta h_a(e^{jw})|$. The nominal loop transfer function $l(z)$ and the nomimal return difference function $1 + h(z)$ can be found from Equation (4.2). These are

$$l(z) = \frac{1-\alpha}{z-1} \ , \quad 1 + l(z) = \frac{z-\alpha}{z-1} \tag{4.18}$$

Let $\Delta l(z)$ be an additive perturbation of the nominal loop $l(z)$ when the nominal plant $\tilde{h}(z)$ is perturbed to $h(z)$ as in Equation (4.17). Then the perturbed loop transfer function $l(z) + \Delta l(z)$ can also be evaluated using Equation (4.2) and we get

$$l(z) + \Delta l(z) = \frac{h(z)(1-\alpha)}{h(z)(z-1)}$$

from which we find

$$\Delta l(z) = \frac{\Delta h_a(z)}{h(z)} \; \frac{1-\alpha}{z-1} \quad . \tag{4.19}$$

Therefore using Theorem 4.3, we conclude that the closed-loop is stable for all perturbations $\Delta h_a(z)$ which satisfy

$$|\Delta h_a(z)| < \frac{|z - \alpha|}{1 - \alpha} \; |\tilde{h}(z)| \tag{4.20a}$$

on the unit circle $z=\exp(jw)$. This inequality can further be simplified to

$$|\Delta h_a(e^{j\omega})| < \frac{\sqrt{1 + \alpha^2 - 2\alpha \cos \omega}}{1-\alpha} \; |\tilde{h}(j\omega)| \; , \tag{4.20b}$$

which can be verified easily by plotting these functions.

It is very important to note that the conditions developed in Theorems 4.1, 4.2, and 4.3 are all sufficiency conditions and not necessary ones. If any perturbation $\Delta h_a(z)$ or $\Delta l(z)$ violates these conditions, the closed loop is not necessarily unstable; on the other hand, satisfaction of these conditions necessarily guarantees asymptotic stability of the perturbed closed-loop provided that the nominal closed-loop is stable.

4.3 Examples

In this section, the main features of the analyiss of the last section are demonstrated through a simple example. Since the IR description contains many more parameters than in the DE description, we use a rational transfer function model of the plant. The Theorem 4.3 will be used to evaluate the robustness against modelling mismatch of the true plant.

Example 4.1

Consider again the example of a scalar dynamic system of the last chapter. Suppose it has been modelled as

$$x(k+1) = \tilde{f}\, x(k) + gu(k) \tag{4.21a}$$

$$y(k) = cx(k). \tag{4.21b}$$

Assume for simplicity that cg=1 so that the rational transfer function of the model is

$$\tilde{h}(z) = \frac{1}{z - \tilde{f}} \quad . \tag{4.21c}$$

Then if the true plant $h(z) = \tilde{h}(z) + \Delta h_a(z)$, according to Equation (4.20b) the closed loop is stable for all $\Delta h_a(z)$ satisfying

$$|\Delta h_a(j\omega)| < \frac{\sqrt{1 + \alpha^2 - 2\alpha \operatorname{Cos} \omega}}{(1 - \alpha)\sqrt{1 + \tilde{f}^2 - 2\tilde{f} \operatorname{Cos} \omega}} \quad . \tag{4.22}$$

Now suppose that the actual plant is of the form

$$x(k+1) = f\, x(k) + gu(k) \tag{4.23a}$$

$$y(k) = cx(k) \tag{4.23b}$$

which is the same as the nominal model in (4.21) except that the true mode f is different from the nomimal mode $\tilde{f}$. Therefore

$$h(z) = \frac{1}{z - f} \quad , \tag{4.23c}$$

and $\Delta h_a(e^{j\omega})$ is of the form

$$\Delta h_a(j\omega) = \frac{f - \tilde{f}}{e^{j2\omega} - e^{j\omega}(f+\tilde{f}) + f\tilde{f}} \quad . \tag{4.24}$$

Given a nominal MAC loop for a specified $\tilde{f}$, the loop is stable for all f's if $|\Delta h_a(j\omega)|$ evaluated from (4.24) satisfies the inequality (4.22). We selected $\tilde{f}$= 0.3 and tested inequality (4.22) for three different f's: f = 0.8, f = -0.3, f = -0.8. In all cases, the set point = 15.0, and α=0.1 are selected. Since $\tilde{f}$ is the same for all three runs, the right-hand side of (4.22) is also the same and is shown as a thick line in all the plots. The left-hand side of (4.22) is shown in dotted lines.

Case 1: $\tilde{f} = 0.3$, $f = 0.8$

Here the true plant has the mode at 0.8. For perfect identification the MAC response is shown in Figure 4.3a. For an identified plant mode at $\tilde{f} = 0.3$, the sufficiency conditions are displayed in Figure 4.36 - which shows that this perturbation satisfies the inequality constraints in (4.22). The closed loop, therefore, is guaranteed to be stable as shown by the MAC performance for the perturbed loop in Figure 4.3c.

Case 2: $\tilde{f} = 0.3$, $f = -0.3$

MAC performance for the true plant $f=-0.3$ is shown in Figure 4.4a. The sufficiency conditions are displayed in Figure 4.4b which shows that the inequality has been violated. But because these conditions are only sufficient, we cannot say anything of the stability of the loop. In this particular situation, the perturbed closed loop has turned out to be stable as shown in Figure 4.4c.

Case 3: $\tilde{f} = 0.3$, $f = -0.8$

MAC performance for the true plant at $f = -0.8$ is shown in Figure 4.5a. The two sides of inequality (4.22) are drawn in Figure 4.5b, which shows that, as in Case 2, the sufficiency condition has been violated. But this time, the closed loop is unstable, as shown in Figure 4.5c.

4.4 Conclusion

An analytical model of MAC was developed in Chapter 3; we have used that model in this chapter to analyze the robustness of MAC. The robustness as been assessed ina classical control framework. The classical GM and PM of MAC are given in Theorem 4.1. The upward GM can be increased arbitrarily by slowing down reference the trajectory, PM can go up to 90°. GM and PM can guarantee stability against a limited class of plant perturbations, therefore a new framework for analyzing generalized perturbations has been developed in Section

4.2.2 and the main result in this direction is presented in Theorem 4.2. The corresponding analysis for models described by rational transfer functions is given in Section 4.2.3 and the main result is presented in Theorem 4.3. Theorem 4.1 and 4.2 can be readily verified by plotting transfer functions. These Theorems give the sets of plants in the neighborhood of the identified models which are guaranteed to be closed-loop stable whenever the nominal loop is stable.

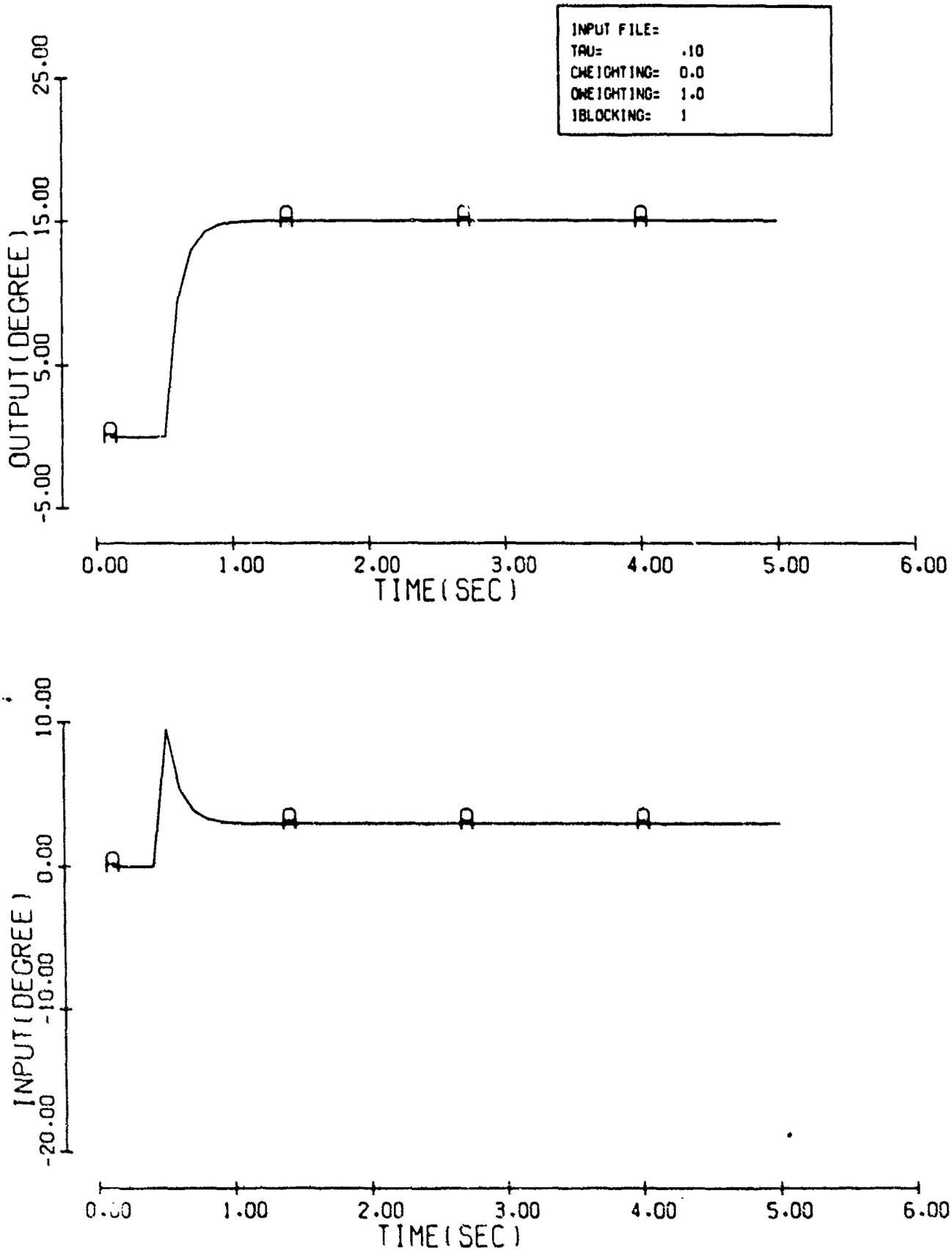
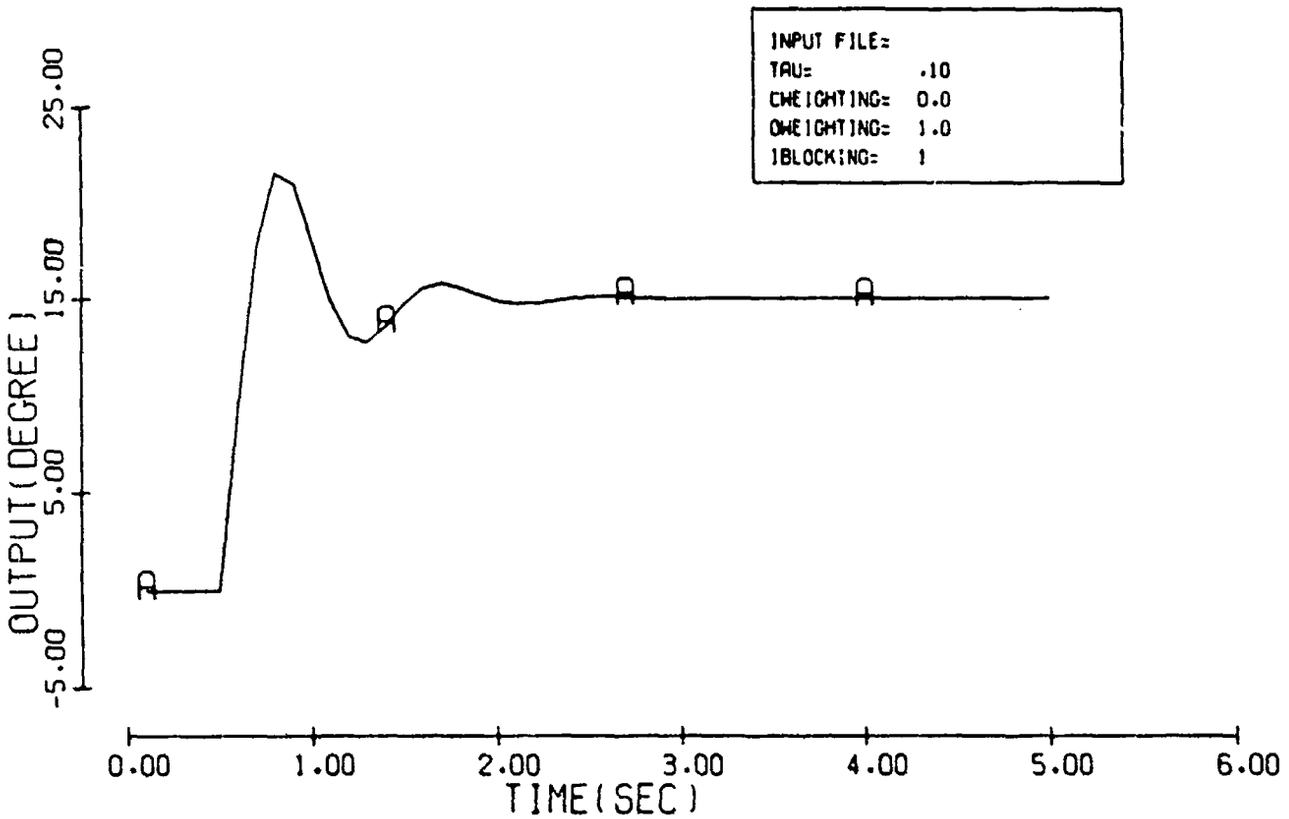


Figure 4.3a MAC Performance: True Plant Mode = 0.8, Identified Plant Mode = 0.8



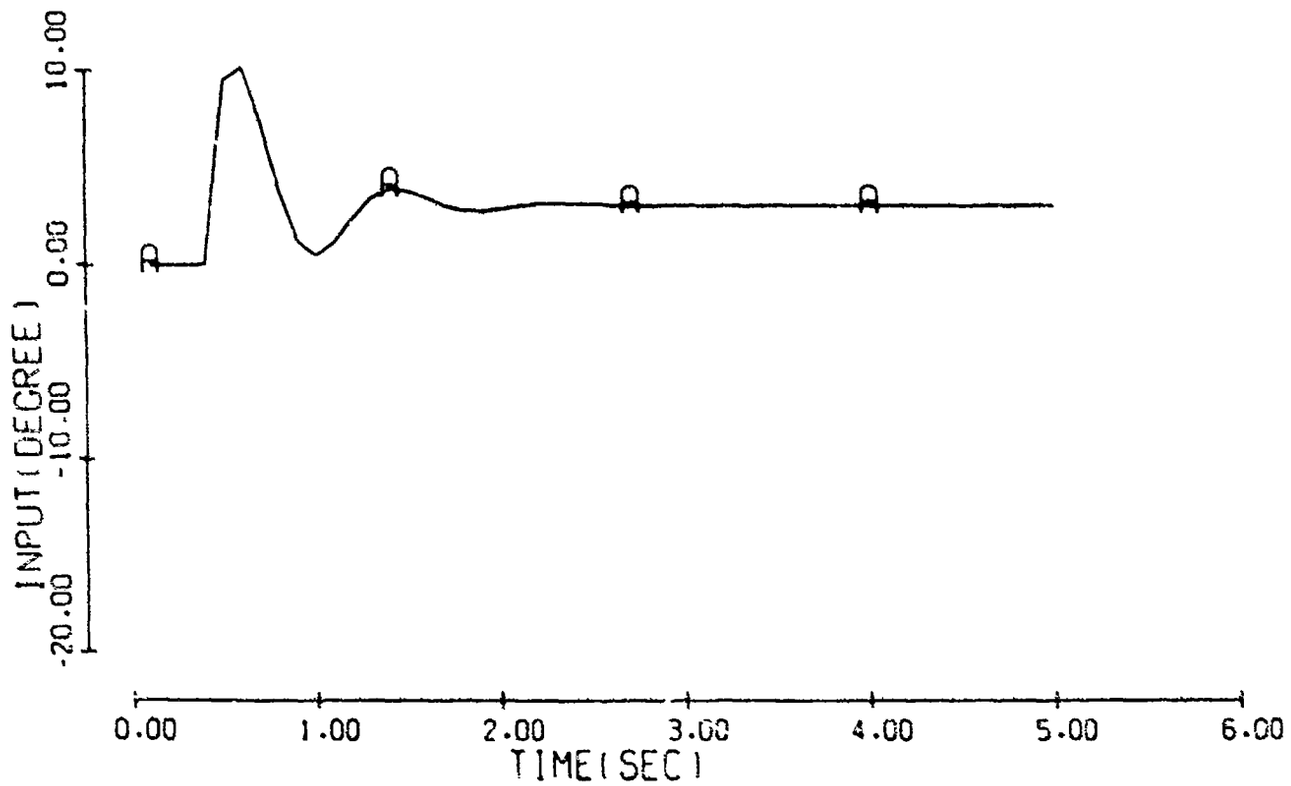


Figure 4.3c MAC Performance: True Plant Mode = 0.8, Identified Plant Mode = 0.3

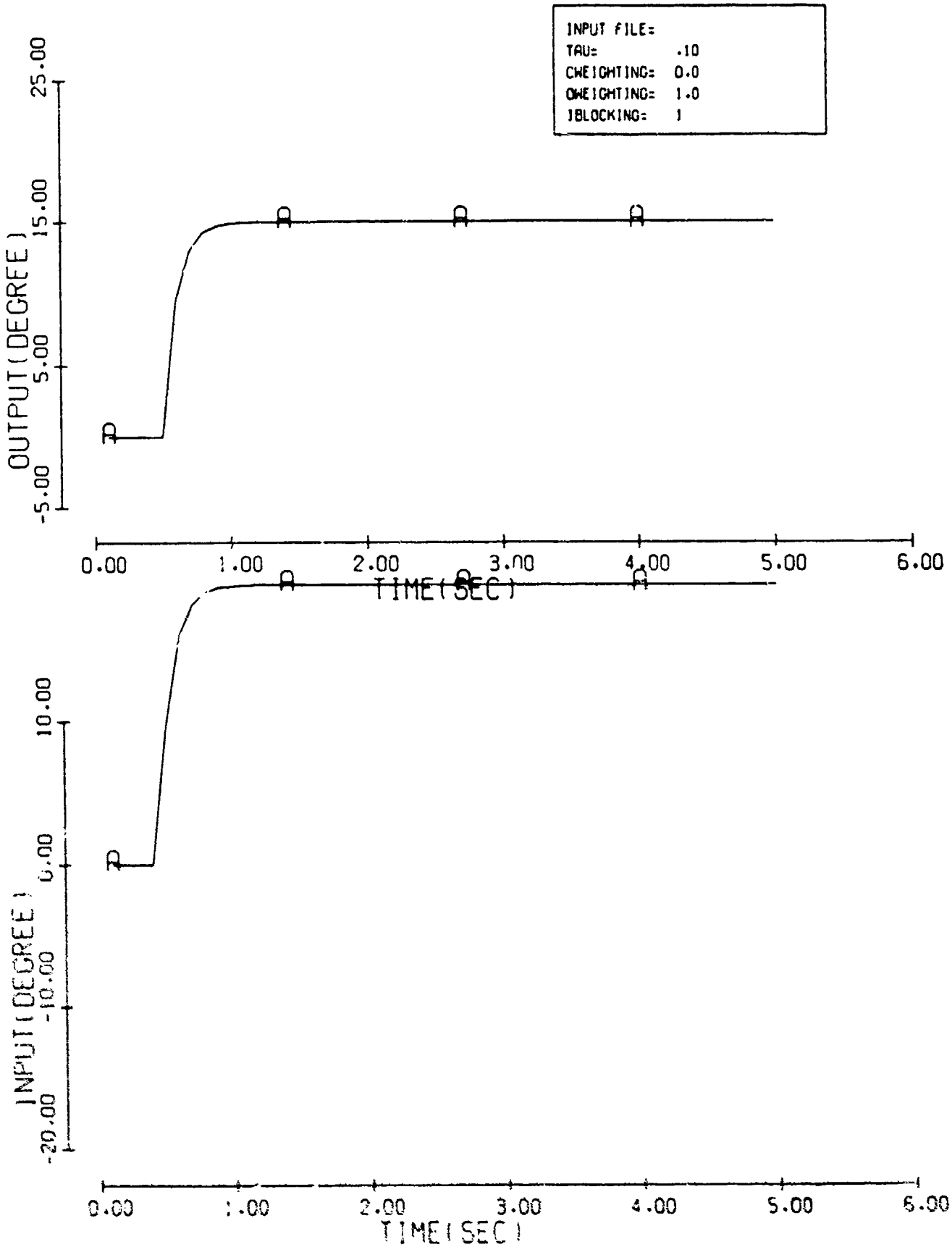


Figure 4.4a MAC Performance: True Plant Mode = -0.3, Identified Plant Mode = -0.3

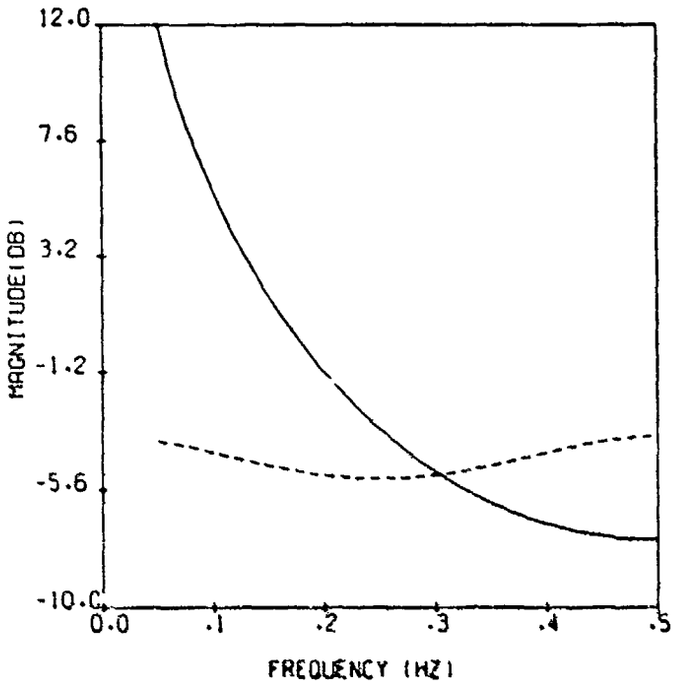


Figure 4.4b Plant Perturbation has Violated Sufficiency Conditions.

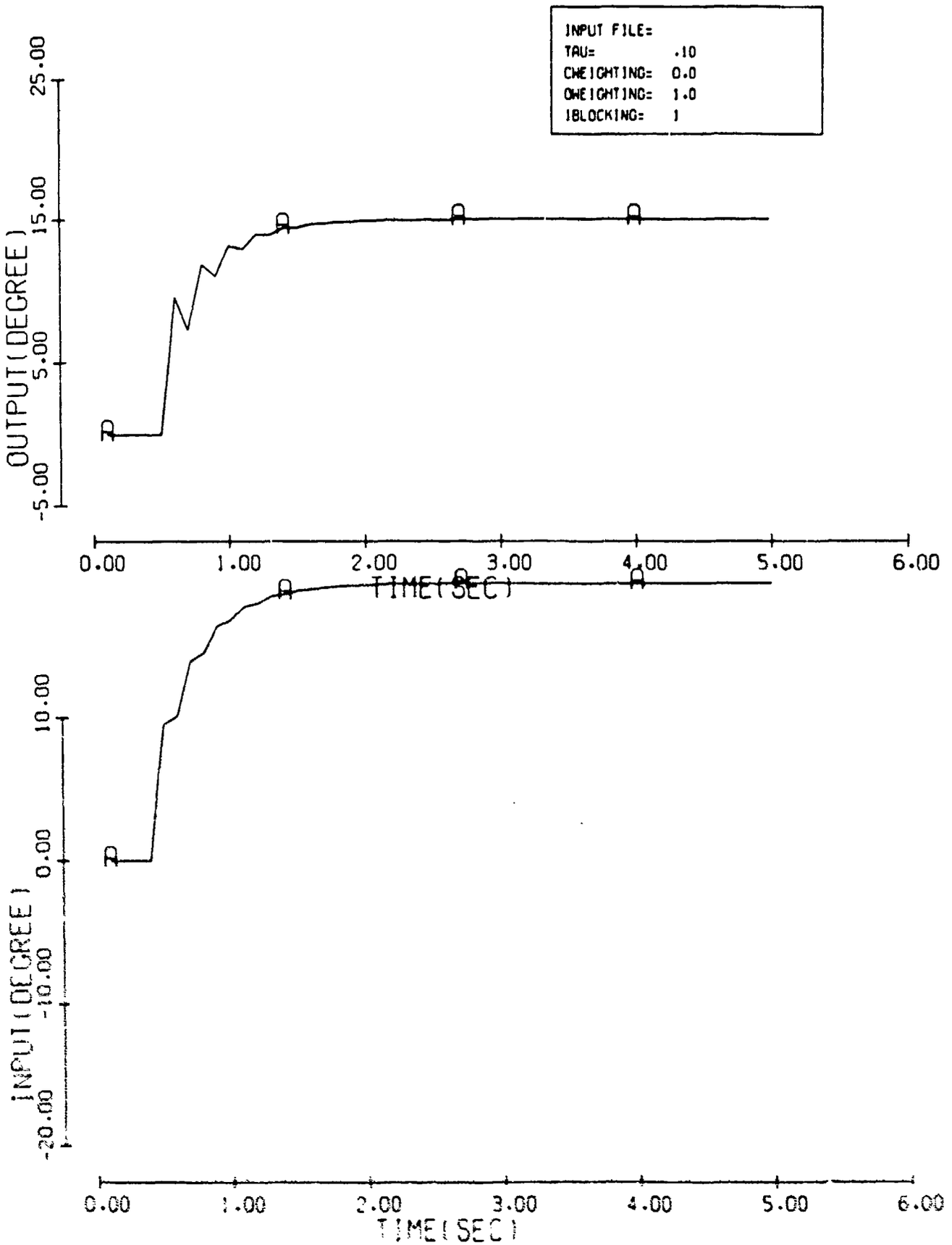


Figure 4.4c MAC Performance: True Plant Mode = -0.3, Identified Plant Mode = 0.3

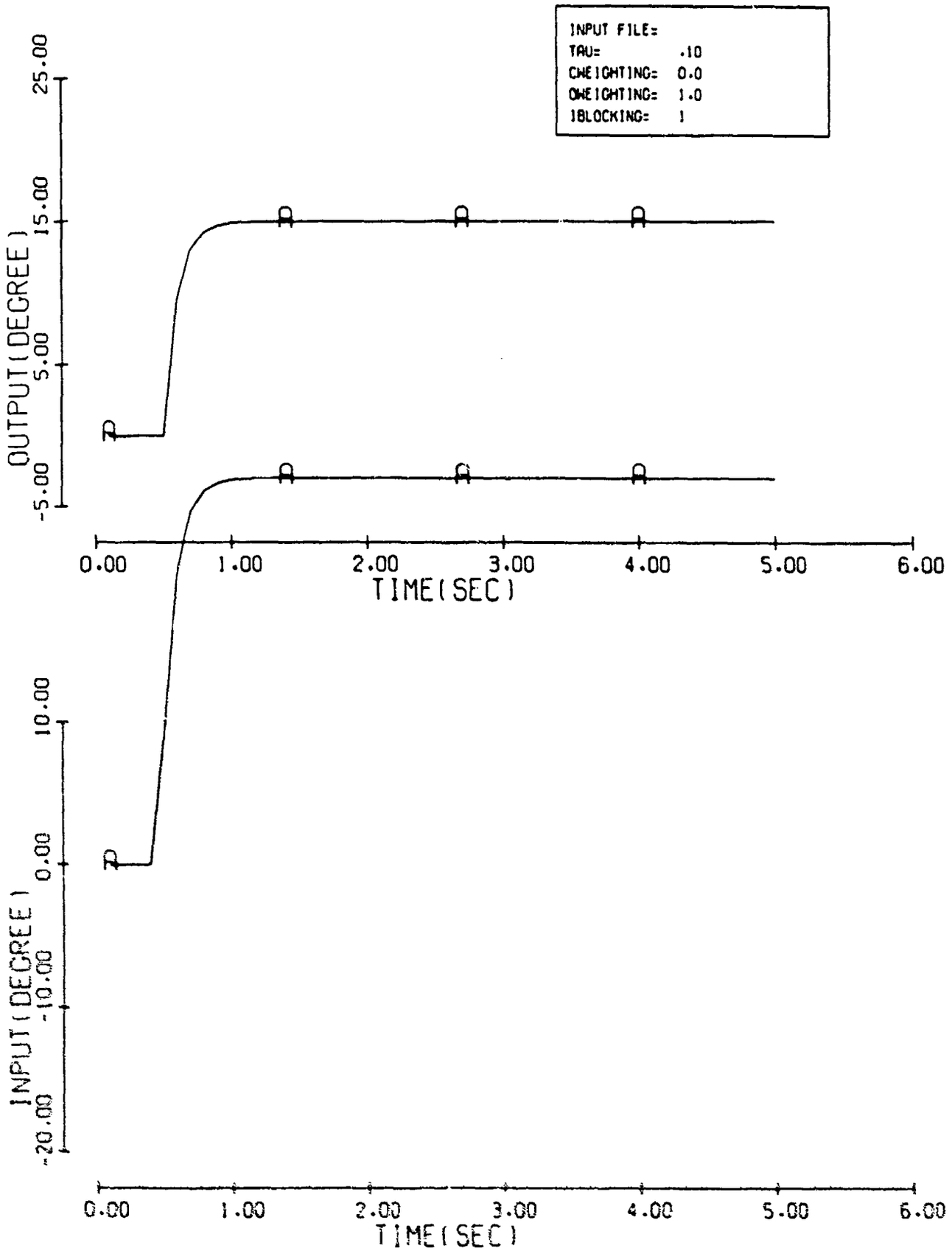


Figure 4.5a MAC Performance: True Plant Mode = -0.8, Identified Plant Mode = -0.8.

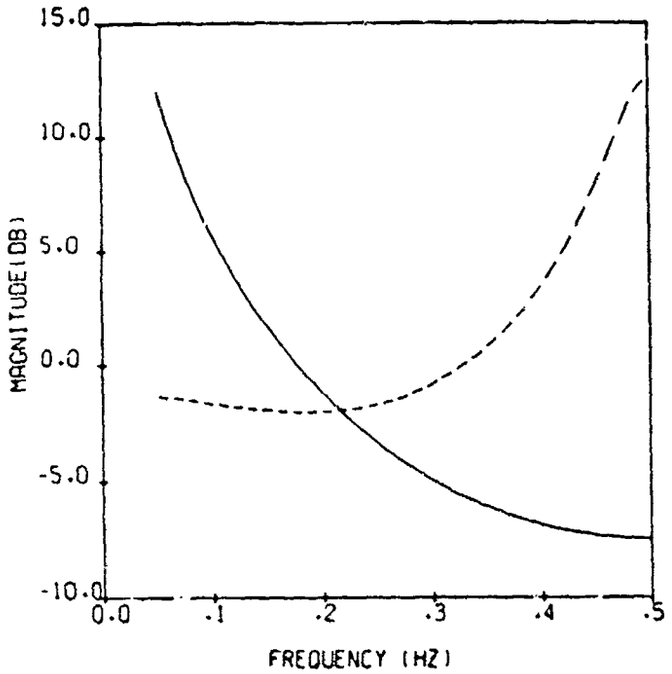


Figure 4.5b Plant Perturbation has Violated Sufficiency Condition.

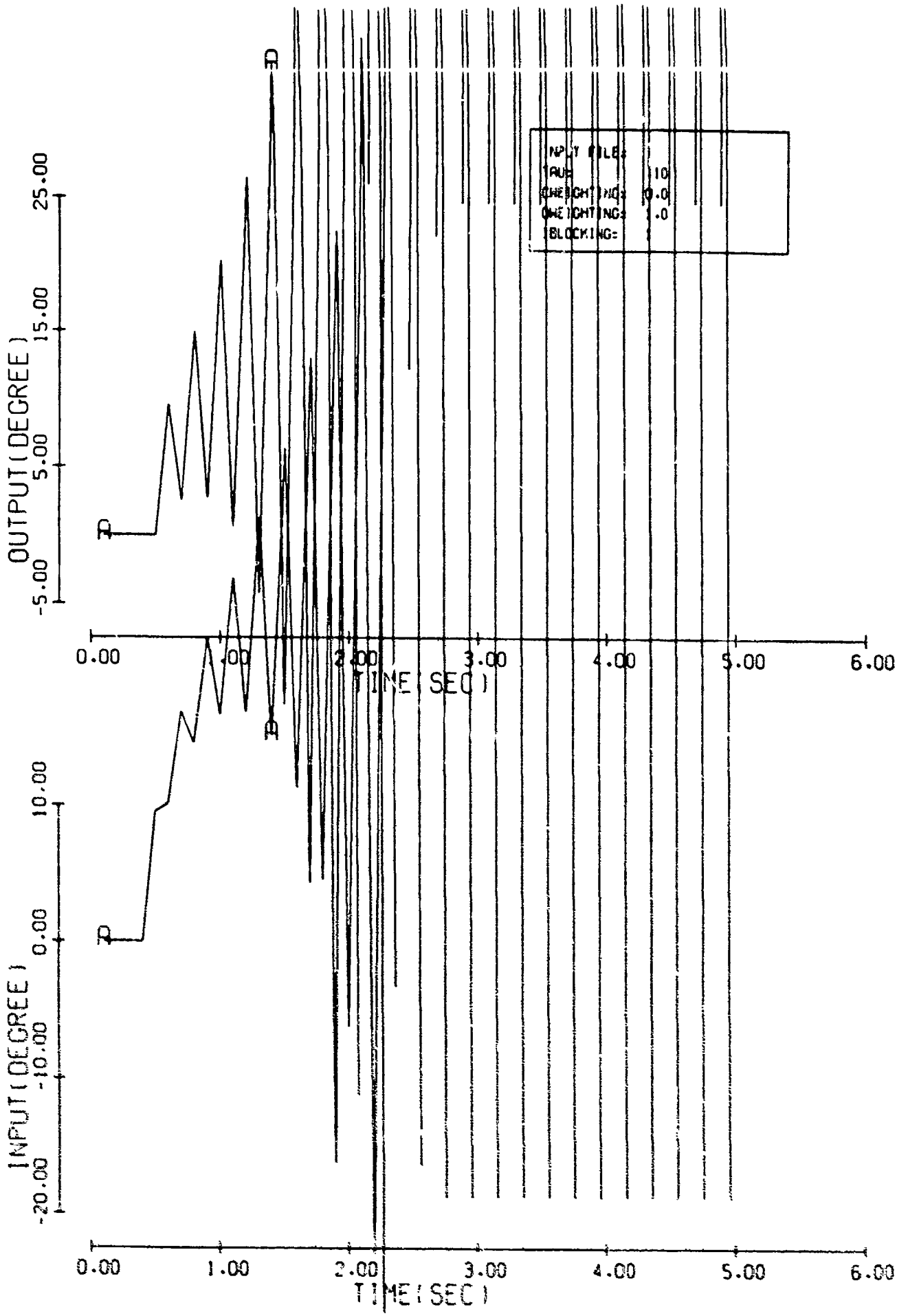


Figure 4.5c MAC Performance: True Plant Mode = -0.8, Identified Plant Mode = 0.3.

CHAPTER 5

SAMPLING INTERVAL & CONTROLLABILITY

5.1 Introduction

Sampled-data (SD) systems are becoming increasingly important with the advent of cheap computing power of microprocessors. Although these systems have been studied for a long time, very few researchers have explicitly dealt with the design of a suitable "sampling time interval 'T'. Almost all literature dictates a sampling rate satisfying the Nyquist rate--although the latter is applicable only for band-limited systems. For the systems with undamped modes, only certain discrete values of T are excluded (Chen, 1970) to guarantee the required rank of the "Controllability Matrix" of the SD systems. Nothing further is said as to what values of T should be chosen once the rank condition of this matrix is satisfied.

A recent study in this direction is by Reid et al.(1979), where T is uniquely chosen to maximize the robustness of a dead-beat control law. Although this is a significant step towards the characterization of a unique T, the procedure has limited application because not all of the SD systems will be used for the purpose of dead-beat control. Maximizing the determinant of the product of the controllability matrix and its transpose are much discussed in the literature, but without any rational justification.

In this study we have provided a logical and intuitively appealing framework for choosing an optimal, unique T. Our analysis is based on two intuitive ideas:

(1) that the amount of energy needed to drive a discrete system from an arbitrary initial state to the origin is a measure of the controllability of the system,

(2) that the amount of energy is also a measure of the degree of effectiveness of various control components.

A minimum energy terminal control problem is formulated which explains controllability in a quantitative framework and its relation with sampling time T. The solution is given in terms of the "Controllability Grammian," and a natural choice of T is made by maximizing the minimum singular value of the Grammian matrix over a compact interval of T. The excitation ability of various control components (or equivalently how effectively each control component influences the dynamics of the system) depends upon the relative orientation between the space spanned by left eigenvectors of the system matrix and the range space of the input distribution matrix. It is extremely difficult to visualize the interplay between a changing T and the relative orientation of the spaces. This has led us to solve the problem implicitly as a minimum energy problem where the relative orientation changes automatically as T varies to provide optimal effectiveness of the control components.

Sometimes control components may have different costs. We would prefer, then, that the two spaces adjust to reflect the relative costs so that the system uses more of the cheaper controls than others. We have implemented this idea by introducing an "input-weighted controllability Gramian" matrix.

The above ideas can be dualized to find an optimal T from the viewpoint of observability. Here T is chosen to minimize the maximum possible energy in the outputs for arbitrary initial states. Since the Hankel matrix is the product of the controllability and observability matrix, the corresponding values of T can be deduced from the singular values of the Hankel matrix, too.

In sections 5.2 and 5.3 we briefly discuss the relation between SD systems and the original continuous time systems and the previous results on the controllability of the SD systems. Section 5.4 also contains a brief discussion on modal controllability and observability. In section 5.5 we have formulated the minimum energy problem in the new perspective for finding an optimal T. Section 5.6 deals with the observability issues. Conclusions are given in section

5.7. The analysis has been kept limited to LTI systems for the sake of clarity although generalization to time varying systems are conceptually straightforward.

5.2 Problem Definition

Consider a linear time-invariant continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.1a}$$

$$y(t) = Cx(t) \tag{5.1b}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$ and A, B, C are matrices of compatible dimensions.

There are many sampling schemes to discretize the system (5.1). We shall be using here the "sample and zero-order hold" sampling mechanism, because it is easier to implement and probably the scheme most widely used in industries. Under this scheme the input is held constant during the sampling interval and the corresponding discrete system is given by

$$x(k+1) = Fx(k) + Gu(k) \tag{5.2a}$$

$$y(k) = Hx(k) \tag{5.2b}$$

$$F = \exp(AT) \tag{5.3a}$$

$$G = [\int_0^T \exp(Av)dv]B \tag{5.3b}$$

$$= (F-I)\,A^{-1}B, \text{ when A is nonsingular}$$

$$H = C \tag{5.3c}$$

and exp(AT) is the transition matrix associated with (5.1). The solution of equation (5.2) is given by

$$x(k) = F^k x(0) + \sum_{i=0}^{k-1} F^{k-1-i}Gu(i) \tag{5.4a}$$

$$y(k) = Cx(k) \tag{5.4b}$$

where F^k is the state-transition matrix of (5.2).

Roughly speaking the controllability of a system refers to its ability to steer any initial condition $x(0)$ at $k=0$ to the origin at $k>0$ for finite k_1 whereas the reachability refers to its ability to steer the system from the origin to any given state in finite time. Because of the non-singularity nature of $\exp(AT)$, the notions of controllability and reachability in continuous-time systems coincide. For discrete time systems, obviously a sufficient condition for the system to be controllable is that F^k be non-singular for each k, i.e., the system has the ability of backward transition whereas the reachability is the property of the range space $\{F^kG\}$, $k = 0,1,\ldots$. The controllability can be checked through the controllability Gramarian formed over a finite horizon; and for time-invariant systems, a horizon of n-steps is necessary and sufficient. The pair $\{F,G\}$ is controllable if the Controllability Grammian

$$W(0,k) = \sum_{i=0}^{k-1} F^{-i}G\, G'(F^{-i})' \tag{5.5}$$

is non-singular for any $k>n$. Equation (5.5) also shows why the non-singularity of F is necessary.

In a sample and zero-order hold mechanism, F is given by (5.3a) which means F is necessarily non-singular for any A. Thus the notion of controllability and reachability are the same, and we shall be using the word controllability hereafter to denote both concepts.

Sometimes the discretization mechanism (5.3a) goes by the name of "exponential transform." It is obvious from (5.3a) that under this mapping, both the continuous-time and the sampled-data system share the same eigenvectors and their poles are related through

$$z_i = \exp(s_iT)$$

where s_i and z_i are respectively the ith eigenvalue of F and A. Also note that

$\int_0^T \exp(Av)dv$ is always non-singular even if A is singular, because

$$\det\left[\int_0^T \exp(Av)dv\right] = \prod_{i=1}^{n} \rho_i \neq 0.0$$

where

$$\rho_i = \begin{cases} \dfrac{1 - \exp(s_i T)}{s_i} & \text{when } s_i \neq 0 \\ T & \text{when } s_i = 0 \end{cases} \tag{5.6}$$

Equality (5.6) is obvious from the Jordan form of A.

5.3 Controllability and Observability of SD system:

The controllability and observability of the time-invariant (TI) sampled-data system is a well-studied topic. Probably the mostly used criteria is the rank condition of the controllability matrix C and the observability matrix Θ, where

$$C = [G : FG : \ldots F^{n-1}G] \tag{5.7a}$$

and

$$\Theta = \begin{bmatrix} H \\ HF \\ \cdot \\ \cdot \\ HF^{n-1} \end{bmatrix} \tag{5.7b}$$

The system is controllable if $\rho(\mathscr{C})=n$ and the system is observable if $\rho(O)=n$, where $\rho(A)$ denotes the rank of A. The matrices F and G depend upon T whereas H does not. One way to determine the role of the sampling time interval T on the controllability and observability of the system is to check the rank condition of the matrices in (5.7) as T is continuously increased. The most significant results available in this direction are contained in the following theorem extracted from [2].

THEOREM 2.1

Assume that the continuous time system (5.1a) is controllable. A sufficient condition for the discrete time system (5.2a) with coefficients in (5.3) to be controllable is that $\mathrm{Im}[\lambda_i(A)-\lambda_j(A)] \neq 2\pi k/T$, $k=\pm 1, \pm 2, \ldots$ whenever $\mathrm{Re}[\lambda_i(A)-\lambda_j(A)]=0$. For the single-input case, the condition is necessary as well.

We can make the following remarks as a corollary of Theorem 2.1:

1. The conditions are also sufficient for maintaining the observability of the SD system, because the pair F, H is observable if and only if the pair $\{F', H'\}$ is controllable; and the Theorem gives the condition in terms of the eigenvalues of A, not in terms of H or G.
2. If $\lambda_i=\sigma_i \pm j\omega_i$ is any complex pole pair of A, T should not be chosen such that $T=k\pi/\omega_i$, $k=\pm 1, \pm 2 \ldots$. Therefore for SISO systems, as T_i is increased, SD system (5.2a) loses controllability for as many values of T and their integral multiples as there are complex pole pairs. Obviously by a continuity argument we can say that the controllability matrix will be ill-conditioned for T in the neighborhood of these T_i's.
3. Although not related to this theorem, another requirement on T to avoid aliasing effects is that we must sample the system at a Nyquist rate at the least. If $\omega_{max}=\max_i \omega_i$, then T should be selected such that

$$T < \frac{\pi}{\omega_{max}} \tag{5.8}$$

Note that if we choose $T=\pi/\omega_{max}$ exactly satisfying the Nyquist rate, we lose controllability for SISO systems.

5.4 Modal Controllability and Observability

Theorem (5.1) does not provide any "quantative" information on the "degree" of controllability which is best explained by modal controllability. The concepts of modal controllability and observability are old and can be found in any standard text on control theory. In this subsection we discuss briefly how the sampling time T is related to these ideas. Assume for simplicity that A is diagonalizable. The modal decomposition of A is

$$A = W\Lambda V' \tag{5.9}$$

where Λ is the diagonal matrix containing the eigenvalues λ_i of A, W and V' are respectively the matrices containing right- and left-eigenvectors of A.

If w_i and v_i are right- and left-eigenvectors respectively associated with ith eigenvalue λ_i, then

$$W = \text{col}\ (w_1,\ w_2, \ldots w_n) \tag{5.10a}$$

$$V' = \text{row}\ (v_1',\ v_2',\ \ldots\ v_n') \tag{5.10b}$$

and

$$WV' = V'W = I_n$$

Then the modal decomposition of F is

$$F = W\ \exp(\Lambda T)V' = W\ \Lambda_F\ V'$$

where $\{\Lambda_F\}$ $=\exp \lambda_i T \triangleq z_i$, the i-th mode of the SD system (5.2). By straightforward calculation, (5.4) simplifies to

$$x(k) = \sum_{i=1}^{n} (z_i)^k\ (v_i' x(0))\ w_i + \sum_{i=0}^{k-1} \sum_{j=1}^{n} (z_i)^{k-1\ i}\ (v_j' G) u(i) w_j \tag{5.11a}$$

$$y(k) = Hx(k) = \sum_{i=1}^{n} (z_i)^k (v_i' x(0))Hw_i + \sum_{i=0}^{k-1} \sum_{j=1}^{n} (z_i)^{k-1-i}(v_j' G)u(i)Hw(j) \tag{5.11b}$$

It should be apparent from (5.11) that it is the row vector $(v_j' G)$ that determines whether the control at the i-th instant u(i) will have any influence on the j-th mode of the system. If this row vector is identically zero for any j, i.e., if $v_j \varepsilon$ left ker (G) then the j-th mode is uncontrollable and the component of the state in the subspace spanned by j-th eigenvector cannot be controlled. Similarly if g_k is the k-th column of G, then $m_{jk} = \langle v_j, g_k \rangle$ determines the sensitivity of the k-th component of the control u_k on the j-th mode. In particular, if we form the n x m matrix $M = \{m_{jk}\}$, $j=1,\ldots n$, $k=1,\ldots m$, where

$$M = V'G \tag{5.12}$$

we can deduce the controllability as well as the "degree of controllability" of various imput-components from the entries of M. M is called the modal controllability matrix. To increase the sensitivity of the k-th control on the j-th mode, we should design g_k as much collinear with v_j as possible. It is easy to show that M is related with C in (5.7a) as

$$C = W[M : \Lambda_F M : \ldots : \Lambda_F^{n-1} M] \tag{5.13}$$

where W is the matrix of right eigenvectors as defined in (10), and if any row of M is identically zero then the controllability matrix C becomes rank deficient.

For the zeroeth order sample and hold (S&H) mechanism under consideration

$$M = V' \int_0^T \exp(Ax)dxB = \left(\int_0^T \exp(\Lambda x)dx\right)V'B \tag{5.14}$$

Now V'B and Λ are predetermined by the continuous-time system (5.1a). The only variable here is T which can be adjusted to regulate the elements of M.

Following the same argument as above we can deduce from equations (5.11) that the degree of the modal observability is given by the modal observability matrix N where

$$N = HW \tag{5.15}$$

and the observability matrix in terms of N is

$$\Theta = \begin{bmatrix} N \\ N\Lambda_F \\ \cdot \\ \cdot^{n-1} \\ N_F \end{bmatrix} V' \tag{5.16}$$

Since H=C and W is predetermined by the continuous-system, N is not affected by T , i.e., the modal observability matrix of a discretized system is the same as in the continuous-time system although the observability matrix Θ in (16) is dependent on T. This shows that the sampling time T will have more impact on the "degree of controllability" than on the "degree of observability" because T influences both the system matrix F and input matrix G forming the controllability matrix.

5.5 Sampling Time to Maximize the Degree of Controllability

In this section we formulate a minimum energy terminal control problem for the discretized system and explain why this scalar measure can be naturally taken as a "degree of controllability." Finally we choose T to optimize this scalar measure. Recall that a controllable discrete system can be driven to zero-state from any initial state in n-steps which motivates an optimization horizon of n-steps. Consider then the minimization of the cost functional

$$\min_{\{u(i),i=1,..n\}} J(x(0)) = \frac{1}{2} \sum_{i=1}^{n-1} u'(i)R(i)u(i), \quad R(i)=R'(i)>0 \tag{5.17a}$$

subject to

$$x(i+1)=Fx(i) + Gu(i) \tag{5.17b}$$
$$x(0) \text{ given, and } x(n)=0$$

Obviously, if the modes are sensitive to the control-components, the system can be driven to (n)=0 from (0) with lower expense of input energy than if the modes are insensitive to control components. This fact is reflected in the construction of J. The relative orientation between the left eigenspace of F and the range space of G (or equivalently the elements of M) and the elements of F are adjusted automatically while minimizing J. Note also that the relative cost of various input components can be reflected through the weighting matrix R, which possibly may be time varying.

The minimization in (5.17a) can be carried out using the ordinary-least-square technique or using Linear-Quadratic (LQ) theory from modern control, although we shall be using the latter to get a better perspective of the problem. The Hamiltonian sequence H(i)

$$H(i)=u'(i)R(i)u(i) + p'(i+1)[Fx(i) + Gu(i)] \tag{5.18}$$

where p(i) is the sequence of Lagrange multiplier.

The necessary condition of optimality gives [3],

$$x(i+1) = Fx(i) + Gu(i) \tag{5.19a}$$

$$p(i) = F'p(i+1), \quad i=0,1,\ldots n-1$$

subject to a given x(0) and x(n)=0, and the optimal control sequence is given by

$$u(i) = -R^{-1}(i)G'p(i+1) \tag{5.19b}$$

Solving in terms of p(0)(note that F is non-singular in our case) and matching the boundary values of x(i) at i=0 and n, we get, successively,

$$p(i) = (F')^{-i}p(0)$$

$$u(i) = -R^{-1}(i)G'(F^{1})^{-i-1}p(0)$$

$$p(0) = W^{-1}(0,n)x(0)$$

$$W(0,n) = \sum_{i=0}^{n-1} F^{-i-1}GR^{-1}(i)G'(F')^{-i-1} \tag{5.20}$$

Here $W'(0,n)=W(0,n)$ is the usual controllability Grammian except that it is weighted by a sequence $R(i)$, and consequently $W(0,n)$ in (5.20) may be called "Input-Weighted-Controllability Grammian." The optimal control sequence is

$$u^*(i)=-R^{-1}(i)G'(F)^{-i-1}W^{-1}(0,n)x(0), \quad i=0,\ldots n-1 \tag{5.21}$$

and the optimal cost J^* is

$$J^* = \sum_{i=0}^{n-1} u^{*\prime}(i)R(i)u^*(i) = x'(0)W^{-1}(0,n)x(0) = \sum_{i=1}^{n} (1/\sigma_i)\, c_i^2 \tag{5.22}$$

where σ_i=i-th eigenvalue of $W(0,n)$

$c_i = \langle x(0),u_i\rangle$ = projection of $x(0)$ on i-th orthonormal eigenvector of $W(0,n)$

<u>Remarks:</u>

1. $W'(0,n)=W(0,n)$ and is positive definite if the system is controllable. If the system is not controllable $\sigma_i=0$ for at least one i and therefore infinite energy is required to bring the initial state $x(0)$ to zero, which makes sense physically.

2. $W'(0,n)=W(0,n) > 0$ which implies the σ_i's are also the singular values of $W(0,n)$.

3. Since the matrices F, G depend on T (the sampling time), the σ_i's and consequently the minimum J are dependent on T. As we have seen from theorem 2.1, as T increases from zero to infinity, the discretized SISO system loses controllability around $T_i=\pi/\omega_i$, making some σ_i equal to zero and hence J^* in (22) goes unbounded. For other values of T, the σ_i's are non-zero and finite and J^* is also finite.

4. The use of the matrix $R(i)$ weights the share of various control components in the minimum energy. The cost of various control

components can be reflected through R(i). For the single-input case, the use of R(i) is superfluous and can be set equal to 1.

5. Note that an equivalent controllability Grammian $\tilde{W}(0,n)$ can be formed from the controllability matrix $\tilde{\mathscr{C}}$ weighted by the sequence R(i) as follows:

$$\tilde{W}(0,n) = \mathscr{C}'\mathrm{diag}\,(R(i))\mathscr{C}'$$

$$= \sum_{i=1}^{n} F^{n-i}G\, R^{-1}(i)\, G'(F')^{n-i} \tag{5.23}$$

Although rank $[\tilde{W}(0,n)]$=rank $[W(0,n)]$, the singular values are different. For this modified controllability Grammian $\tilde{W}(0,n)$, matrix inversion of F is not needed.

We are now in a position to find an optimal T on a rational basis The maximum possible normalized energy is

$$J_N^* = \max_{x(0)\in R^n} \frac{J^*}{x'(0)x(0)} = \max_{x(0)} \frac{x'(0)W^{-1}(0,n)x(0)}{x'(0)x(0)}$$

$$= ||W^{-1}(0,n)||_2 = \bar{\sigma}(W^{-1}(0,n)) = \frac{1}{\underline{\sigma}(W(0,n))} \tag{5.24}$$

where $||.||_2$ denotes the induced Euclidean norm and $\bar{\sigma}(.),\underline{\sigma}(.)$, is the maximum and minimum singular value respectively. From (5.22) it is obvious that J^* is bounded above and below as

$$\frac{1}{\sigma(W)}\,||x(0)||^2 < J^* < \frac{1}{\underline{\sigma}(W)}\,||x(0)||^2 \tag{5.25a}$$

$$0 < \frac{1}{\sigma(W)} < J_N^* < \frac{1}{\underline{\sigma}(W)} \tag{5.25b}$$

where W(0,n) has been denoted by W for the sake of brevity.

A rational choice of T is to minimize the upperbound of J_N^* as much as possible, i.e., the optimal T=T* should be such that

$$T^* = \operatorname*{Inf}_{T} J_N^* \tag{5.26a}$$

Since J_N^* is bounded below by zero and we shall be working with a compact interval $I=[0,t]$, (5.26a) is equivalent to

$$T^* = \min_{T\varepsilon I} J_N^* = \min_{T\varepsilon I} \frac{1}{\underline{\sigma}(W)} \tag{5.26b}$$

Therefore the complete procedure of obtaining T^* is

$$T^* = \min_{T\varepsilon I} \max_{\substack{x(0)\varepsilon R^n \\ ||x(0)||=1}} \min_{u(i)\varepsilon R^m} \sum_{i=0}^{n-1} u'(i)\, R(i) u(i) \tag{5.27}$$

subject to

$$x(i+1) = Fx(i) + Gu(i),\ x(0)=x(0)$$

$$F = \exp(AT),\quad G = \left(\int_0^T \exp(As)ds\right)B$$

Note that when the system loses controllability, then for some $T\varepsilon I$, $\underline{\sigma}(W)\to 0$, or, $1/\underline{\sigma}(W)$ blows up. So for computational and plotting purposes we may as well evaluate (5.26b) as

$$T^* = \max_{T\varepsilon I} \underline{\sigma}(W) \tag{5.28}$$

It is conjectured by many practitioners that T should be chosen to maximize the determinant of $\mathscr{C}\mathscr{C}'$ where $\mathscr{C}$ is the controllability matrix in (5.7a) without any rational justification. We explain here why this determinant of $\mathscr{C}\mathscr{C}'$ is not a good measure of the quantitative controllability ideas developed herein. Recall from remark (5.5) above that if $R(i)=I_m$ for all i,

$$\mathscr{C}\mathscr{C}' = \mathbb{W}(0,n)$$

and

$$W(0,n) = F^{-n}\, \mathbb{W}(0,n)(F')^{-n}, \qquad F = \exp(AT)$$

Therefore

$$\det(CC') = \det(W(0,n))$$

$$= \frac{\prod_{i=1}^{n} \sigma_i(W(0,n))}{\prod_{i=1}^{n} \exp(-2\mathrm{Re}[\lambda_i]nT)} \tag{5.29}$$

under the assumption that A is diagonalizable with eigenvalues λ_i. Expression (5.29) clearly shows the inadequacy of the determinant criteria, because for T, where the system almost loses controllability, the denominator of (5.29) is fixed and $\underline{\sigma}(W(0,n))$ is nearly zero. According to criteria developed herein, the system is nearly uncontrollable. Yet $\det(\mathscr{CC}')$ may be large if the remaining singular values are large; thus the "almost uncontrollability" situation of the discretized system remains undetected with the determinant criteria.

<u>Examples:</u>

Example 5.1. Consider a SISO continuous system

$$\dot{x} = \begin{pmatrix} -5 & 6 \\ -3 & 1 \end{pmatrix} \underline{x} + \begin{matrix} 0 \\ 1 \end{matrix} u$$

The poles are at $-2\pm j3$, with a Nyquist sampling rate $T_{Nyq}=1.04719$ sec. The $\underline{\sigma}(W(0,2^{\nu}))$ as a function of T is plotted in figure 5.1, which rightly shows that at T=1.047 sec, the system loses controllability. To avoid aliasing effects we must choose T smaller than the Nyquist sampling rate, and as seen from the plot the optimum T=0.65 second. Note also that near $T=T_{Nyq}$, the degree of controllability is poor.

Example 5.2. As another example consider the decoupled longitudinal dynamics of a missile in flight condition 1:

$$\dot{\underline{x}}(t) = \begin{pmatrix} -1.4868 & 1.00 \\ -149.93 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -281.11 \end{pmatrix} u(t)$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $x_1(t)$ = angle of attack in rad
$x_2(t)$ = perturbed pitch rate rad/sec.
$u(t)$ = elevator angle

The poles are at $-0.7434 \pm j112.22$ with a damping ratio $\xi=0.061$ and a Nyquist sampling interval rate $T_{Nyq}=0.257$sec. $\sigma(W(0,2))$ plot is given in figure 2 which shows that the system loses controllability at $T_i=kT_{Nyq}$, $k=1,2,\ldots$.

Although the optimal T^* is lower than T_{Nyq} by an infinitesimal amount, it is recommended that a sampling time between 0.1 and 0.2 sec. be chosen from practical considerations.

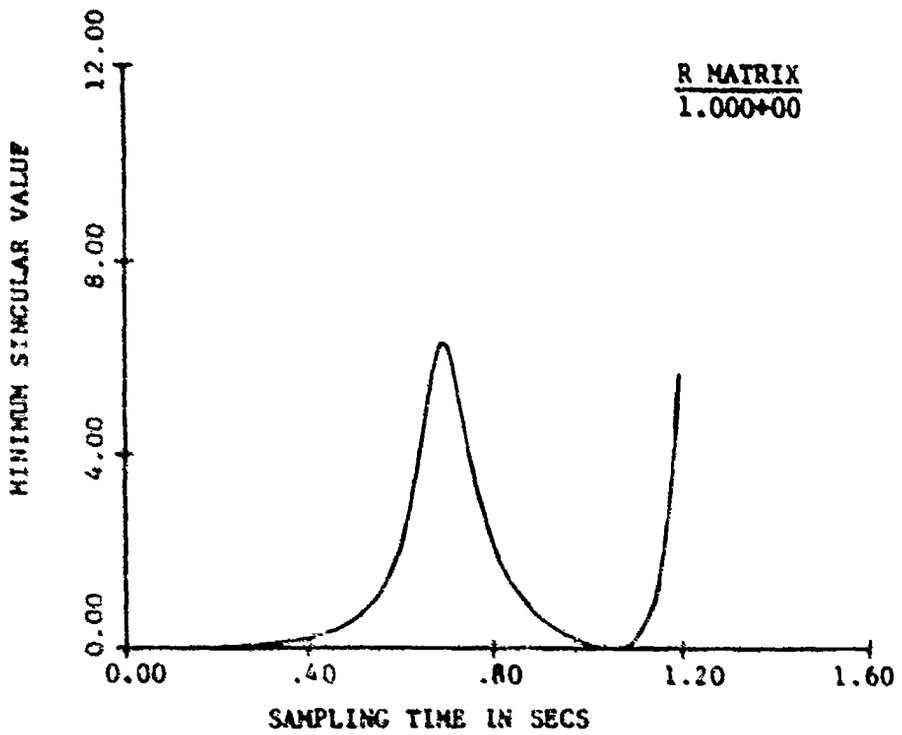


Figure 5.1. Sampling Time Interval and Degree of Controllability

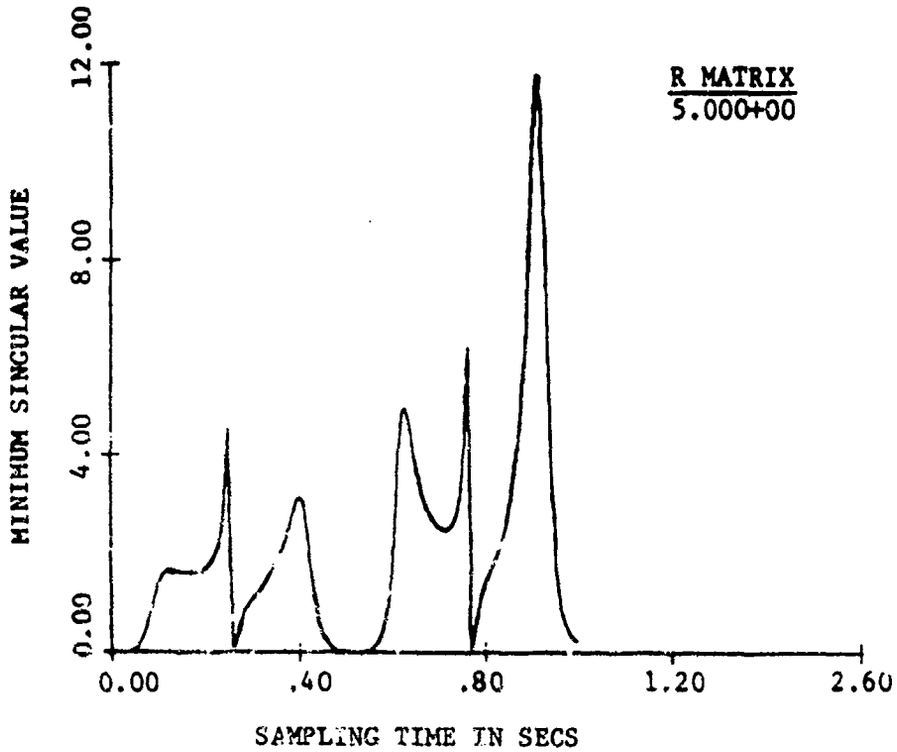


Figure 5.2. Sampling Time Interval and Degree of Controllability for an Air-to-Air Missile

5.6 Sampling Time Interval and the Observability of the Discretized Systems

In this section we formulate an optimization problem for finding an optimal sampling time interval T* from the observability viewpoint. The approach is analogous to that in the preceding section. The cost functional chosen for optimization is subjective and depends upon the application of the discretized system; but the point we want to emphasize is that this type of formulation yields an optimal unique T. It is shown here how to formulate the problem from the consideration of sensor sensitivity and optimal use of sensor measurements.

The observability of the SD system

$$x(k+1) = Fx(k), \quad x(0) \text{ unknown}$$
$$y(k) = Hx(k) \tag{5.30}$$

is concerned with the inference of the initial state $x(0)$ from n-observations, $y(k)$, $k=0,\ldots n-1$ and depends upon the observability matrix Θ in (7b).

Define

$$Y_n = [y'(0) : y'(1) : \ldots y'(n-1)]'$$

Then the estimate of $x(0)$ based on n-observations is

$$\hat{x}(0)|Y_n = \Theta^{\#}Y_n$$

where $\Theta^{\#}$ is the generalized inverse of the observability matrix Θ. If rank $(\Theta)=n$, Y_n lies in the range-space of Θ and $x(0)$ can be estimated exactly and

$$\hat{x}(0)|Y_n = (\Theta'\Theta)^{-1}\Theta'Y_n$$

When the system is unobservable, $\Theta'\Theta$ is rank deficient and the estimate is not perfect. The structure of the observability matrix Θ determines the "observability" of the system and the system continues to remain observable as long as rank $(\Theta)=n$. To embed the observability problem in a quantitative framework, note that the structure of this matrix also determines how a given initial condition $x(0)$ (or equivalently any given state $x(k)$) is distributed in the output sequence $\{y(k), k=0,\ldots n-1\}$. Maximizing observability by adjusting T implies in the sense of the L_2-norm that any initial condition $x(0)$ with energy $||x(0)||^2$ gives rise to maximum energy in the output sequence.

In the extreme case when the system is completely unobservable, the energy in the sequence $\{y(k), k=0,\ldots n-1\}$ is zero for any $x(0)$. There is another advantage of maximizing output energy. For a good performance from the sensors it is desirable to maximize the energy, because for a given $x(0)$ (or $\{x(k)\}$) and unmeasurable corrupting output noise, this is equivalent to maximizing signal to noise power ratio and consequently best sensor performance is obtained. There is another motivation that some sensors may be more efficient than others and less efficient sensors will need higher signal to noise ratio than

the more efficient ones. These observations suggest a weighted cost-functional (weighted energy in the output sequence), similar to (5.17a),

$$J = \sum_{i=0}^{n-1} y'(i)R(i)y(i), \quad R(i) = R'(i) > 0 \tag{5.32}$$

where R(i) determines the relative importance of various sensors. We should then maximize J. However (5.32) reduces to

$$J = x'(0)V(0,n)x(0) \tag{5.33a}$$

where

$$V(0,n) = \sum_{i=0}^{n-1} (F')^i C'R(i)C(F)^i \tag{5.33b}$$

may be called the "output-weighted observability Grammian."

The normalized energy is

$$J_N = \frac{J}{x'(0)x(0)}$$

and the minimum possible normalized energy is

$$J_N^* = \min_{x(0)\varepsilon R^n} J = \min_{x(0)\varepsilon R^n} \frac{J}{x'(0)x(0)} = \underline{\sigma}(V(0,n) \tag{5.34}$$

where $\underline{\sigma}(.)$, $\overline{\sigma}(.)$ denote as usual the minimum and maximum singular value respectively. Note that J_N is bounded below and above as

$$0 < \underline{\sigma}(V(0,n) < J_N < \overline{\sigma}(V(0,n)$$

and when the system is unobservable $\underline{\sigma}(V(0,n))=0$. The minimum singular value of V(0,n), $\underline{\sigma}(V(0,n))$ is a sensitive measure of unobservability, because the system need not be completely unobservable for $\underline{\sigma}((V)(0,n))$ to be zero. If any subspace of R^n is unobservable an arbitrary x(0) will have non-zero projection on this sub-space and $\underline{\sigma}(V(0,n))=0$. We

therefore should choose T to maximize J_N^* to take the system away from unobservability as much as possible. The optimal T=T* should then be chosen such that

$$T^* = \sup_T J_N^*$$

with the constraint that T* should be less than the Nyquist sampling rate.

Therefore, following the arguments of the previous section, we should find an optimal sampling time T* from the observability viewpoint by solving the following max-min problem:

$$T^* = \max_T \min_{\substack{x(0)\varepsilon R_n \\ ||x(0)||=1}} x'(0)\left(\sum_{i=0}^{n-1} (F')^i C'R(i)C(F)^i\right)x(0)$$

<u>Examples</u>

Example 5.3. We consider again the example 1 of the previous section with
the output matrix

$$H = (1 \quad 0)$$

The minimum singular value plot of $V(0,2)$ as a function of T is given in figure 5.3. Note the similarity with figure 1 and observe that the sampling time at which the system loses controllability is also the time at which the system loses observability. These happen at the Nyquist sampling interval of 1.04719 seconds.

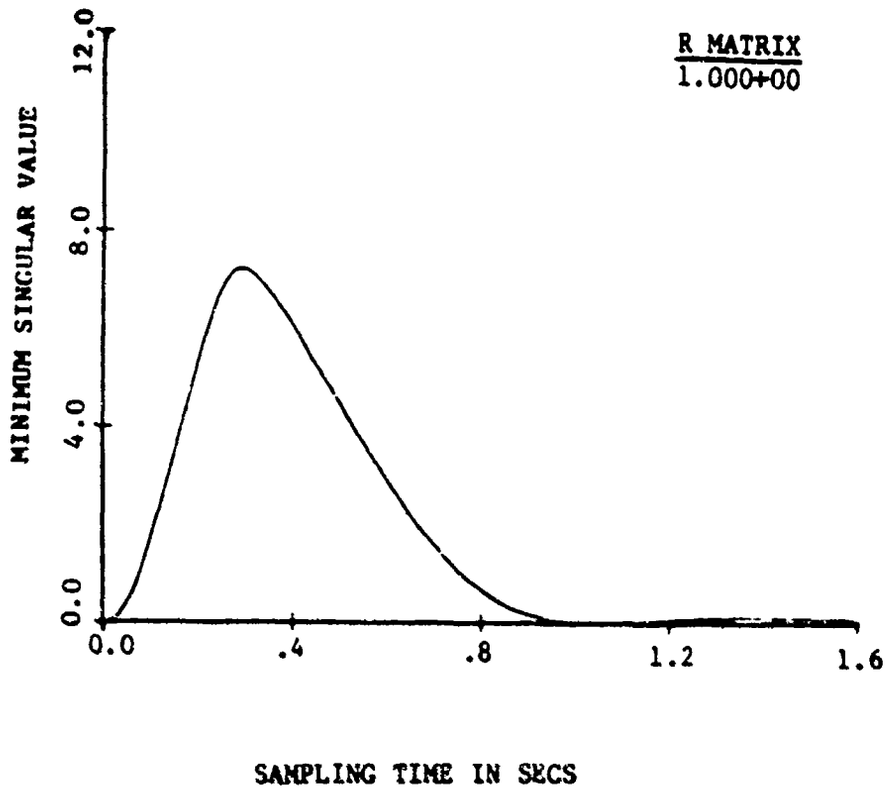


Figure 5.3: Sampling Time Interval and Degree of Observability

Example 5.4. Consider the example 5.2 with angle of attack as the output, i.e.

$$H = (1 \quad 0)$$

$\underline{\sigma}(V(0,2))$ plot is given in figure 4. Notice again the similarity with Figure 2. At T=0.257 sec. the observability is lost.

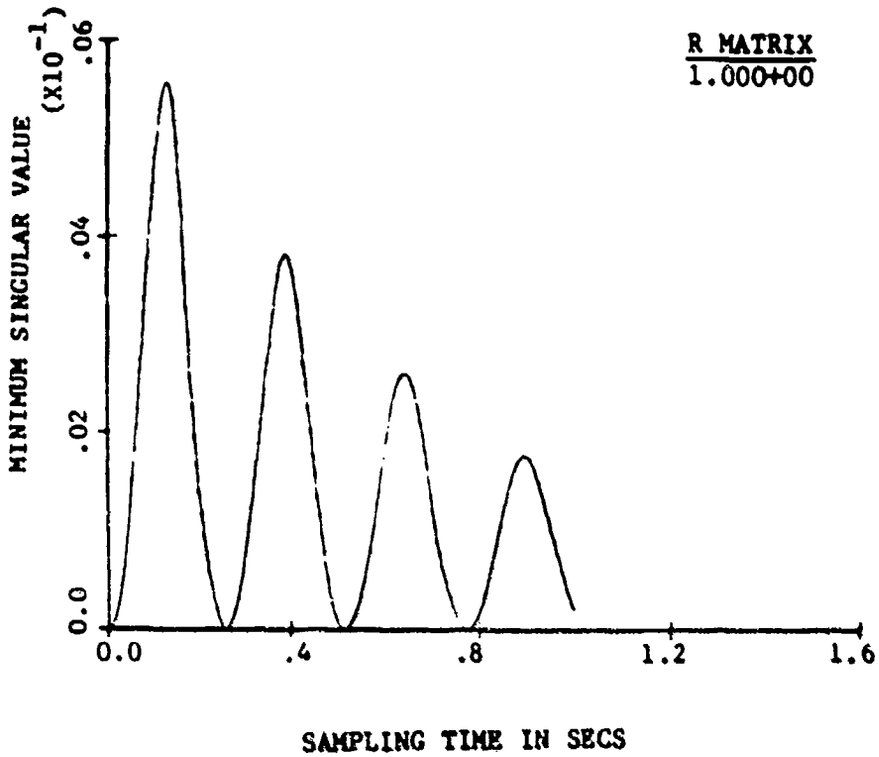


Figure 5.4: Sampling Time Interval and Degress of Observability of the Air-to-Air Missile

5.7 Conclusions

In this paper we have described a framework for determining a unique optimal sampling time T. The solution T is given by a mini-max problem when considered from the controllability viewpoint, and by maxi-min problem when considered from an observability viewpoint. The choice of cost-functionals as a basis of an optimization problem is very much a subjective matter and depends upon the application of the discretized system. But nevertheless, the framework developed in the

paper is based on practical considerations; the analysis is very simple, and the results are extremely useful to practicing control engineers.

CHAPTER 6

SIMULATION RESULTS

6.1 Introduction

The identification technique using CVA (Canonical Variate Analysis) has been described in Chapter 2 and the robustness analysis of the simplified MAC controller has been analyzed in Chapters 3 and Chapter 4. These results are combined in this chapter as an Adaptive MAC (AMAC) controller, and its performance will be demonstrated through realistic simulations in deterministic as well as in stochastic environments. The simulation runs have been designed to emphasize the effect of data length, dither strength (SNR), and closed loop identification capability of the CVA technique. It has also been shown how AMAC behaves for SISO and MIMO plants.

The primary purpose of this chapter is to exhibit the strength of the CVA technique as a closed-loop identifier and to demonstrate the reliable adaptive control scheme AMAC which utilizes the robust MAC technique. If the performance of the CVA technique degrades for some reason i.e. the identified plant is not 'close' to the actual plant, the robustness of MAC compensates for it in the sense that it enables the plant to maintain the closed-loop stability and follow the desired trajectory.

This chapter is orgnized as follows: The simulation models have been selected from the previous project report on MAC (AFWAL-TR-80-3125). For the sake of completeness of this report, the models and the various simulation parameters are described again in Section 6.2. Simulation results under various scenarios are presented in Section 6.3. Finally the summary and conclusions are given in Section 6.4.

6.2 Simulation Model and Simulation Parameters

The simulation models have been selected from the previous report on MAC [AFWAL-TR-80-3125]. The SISO and MIMO models are extracted from a single hypothetical air-to-air missile model with asymmetric aerodynamic properties. This model represents a simple, three-axis attitude control problem in flight condition 1 with independent pitch axis and coupled roll-yaw dynamics. In this flight condition (Mach 2 at 20,000 ft. and weighing 239.5 lb), this missile is flying at an equilibrium pitch angle of 9°, sideslip of 0° and roll angle of 0°.

6.2.1 SISO Model

The SISO Model consists of the decoupled pitch axis dynamics with 2 states. The model in the continuous time domain is

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1.4868 & 1 \\ -149.93 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -281.11 \end{pmatrix} u(t) \quad (6.1a)$$

$$y(t) = (1 \quad 0)x(t) \quad (6.1b)$$

The states are:

$x_1(t)$ = angle of attack,

$x_2(t)$ = perturbed pitch rate (rad/sec),

with input $u(t)$ = elevator angle (rad) and output $y(t)$ = angle of attack (rad). The open loop poles are at $-0.7434 \pm j12.222$ with a damping ratio of 0.061 which shows that the pitch axis dynamics are quite oscillatory.

The plant dynamics are discretized at a sampling rate of 10 Hz using the exponential transform (sample and zero order hold). The resulting poles of the discrete time system are

$$0.31711 \pm j0.87252 \quad (6.2)$$

with a modulus of 0.92836. The pulse response and step response when these are applied at t=0.4 seconds to this system are shown in Figure 6.1. The true poles in equation (6.2) will be subsequently compared with those of the identified systems.

6.2.2 MIMO Model

The coupled roll-yaw dynamics from the same air-to-air missile in section 6.2.1 are used for the MIMO Model. It has four states, two inputs and two outputs. The states are

$x_1(t)$ = sideship angle (rad)

$x_2(t)$ = perturbed roll rate (rad/sec)

$x_3(t)$ = perturbed yaw rate (rad/sec)

$x_4(t)$ = roll angle (rad)

with inputs

$u_1(t)$ = aileron angle (rad)

$u_2(t)$ = rudder angle (rad)

and outputs

$y_1(t)$ = sideship angle (rad)

$y_2(t)$ = roll angle (rad).

An early analysis of these dynamics indicated a very severe roll instability. Since MAC can work only for systems with a finite impulse response, roll angle and rate feedback were added to the aileron command to add damping to the system (see the previous report, page 125). With such compensation, the dynamics are

$$\dot{x}(t) = \begin{pmatrix} -0.91237 & 0.15708 & -1.0 & 0.015431 \\ -1559.2 & -4385.3 & 0 & -4385.3 \\ 290.48 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 8770.6 & 0 \\ 0 & 281.11 \\ 0 & 0 \end{pmatrix} u(t) \tag{6.3a}$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x(t) \tag{6.3b}$$

The open-loop poles are at

$$\begin{aligned} &-4384.24, \\ &-1.00040, \quad -0.484 \pm j17.035. \end{aligned} \tag{6.3c}$$

As in the SISO case, the plant dynamics are discretized using an exponential transform for a sampling interval of 0.1 seconds. The open-loop poles of the discretized system are:

$$0.00000654,\ 0.9047,\ -0.12609 \pm j0.9444 \tag{6.4}$$

The response of this system to a pulse and a step in aileron input is shown in Figure 6.2. The corresponding responses to similar excitations in rudder input are shown in Figure 6.3. As in SISO case, these inputs are applied at t=0.4 seconds. In all the figures involving MIMO plant simulations, the following notations have been used:

on output plots:

A = sideship angle,

B = roll angle,

on input plots:

A = aileron angle,

B = rudder angle.

It is obvious from Figures 6.2 and 6.3 that the first output is insensitive to changes in the first input and the second output is similarly related to the second input.

6.2.3 Simulation Parameters

In order to facilitate comparison between related plots, the scales have been kept constant, if possible, within each series of runs. Unless otherwise noted, the following conditions existed in the simulations:

- The sample time was 0.1 seconds.
- The controls were computed for the three blocks ending at one, three and five steps in the future (for details of the input blocking techniques see the previous report on MAC).
- The reference trajectory time constant was 0.1 seconds for all outputs.
- No input constraints were imposed.
- It was assumed that the plant model was completely unknown at the beginning.

Therefore the missile was allowed to run open-loop for a while under the effect of dither excitation and measurement noise. The plant was identified at the end of this period which was then used by MAC as an internal model of the plant. The set points were then changed at the end of this interval as follows:

For the SISO plant, angle of attack was set from 0° to 15°,

For the MIMO plant, sideslip was set from 0° to 10° and the roll set point remained at 0°.

- The output weights were all equal to 1 and no input weights were used.
- The input excitation noise (dither) and measurement noise were white Gaussian noise processes generated by the subroutine GGNML from IMSL library.

6.3 Simulation Under Various Scenarios

Under each condition, AMAC was applied to the SISO plant of Section 6.2.1 and the MIMO plant of Section 6.2.2. These results are exhibited separately.

6.3.1 MAC Applied to Perfectly Known Plants

Extensive simulation results under this condition, i.e. when the plant model is perfectly known, have been reported in the previous report on MAC [AFWAL-TR-80-3125]. Two of these results are reproduced here for later comparison with AMAC performances. The control and the output of the SISO plant under the same simulation parameters of Section 6.2.3 when the set point is changed from 0° to 15° at 0.4 seconds is shown in Figure 6.4. Similar response for the MIMO plant for a set point change at 7.0 seconds is shown in Figure 6.5.

6.3.2 AMAC Applied to Unknown Plants

The adaptive MAC was applied to the plants of Sections 6.2.1 and 6.2.2 and the results are shown in the subsequent figures. The variance of the excitation signal (dither) was 0.1 and that of the measurement noise was 0.05 so that the signal-to-noise ratio (SNR) was 6db. This ratio is considered to be realistic by many practicing engineers. The dither was superimposed on the normal input obtained from MAC algorithm and the measurement noise was added to the actual output of the plant.

The SISO plant was identified at the end of every 7-second interval and the optimal state order was selected using the AIC criteria (see Chapter 2 for details). As mentioned earlier, the plant was running open loop during the first interval and closed loop in the subsequent intervals. The control and the output sequences are plotted in Figure 6.6 - the vertical dotted lines in this and the subsequent figures indicate the length of the intervals. The plant is identified at the instants indicated by these dotted lines. This figure clearly shows that under AMAC, the plant can track the

reference input albeit at the expense of ride comfort (or oscillations in the output). To see how CVA performs when combined with MAC, we have compared the transfer function of the identified plant in the first interval (i.e. open-loop identification) with the actual one in Figure 6.7(a) and that from the 3rd interval (closed-loop identification) in Figure 6.7(b). The optimal state order and the identified poles during various intervals (see Figure 6.6) are found as follows:

	State Order	Poles
Section I	3	0.588, 0.3437±j0.8509
Section II	3	0.988, 0.3042±j0.8455
Section III	3	0.966, 0.2664±j0.8308

These poles of the identified system can be compared with those of actual plant which are at 0.31711±j0.87252.

The MIMO plant was identified every 20 seconds under similar conditions, the plant being run open loop in the first interval. The servo performance of AMAC under this run is shown in Figure 6.8. The set point was changed at the 20th second. The optimal order and the identified poles are:

	State Order	Poles
Section I	3	0.8089,-0.12066±j0.9312
Section II	5	0.5888, 0.7479, 0.835, -0.1358±j0.9308

Again these identified poles may be compared with the actual ones in equation (6.4). Each element of the identified transfer function from Section I (i.e., open-loop identification) is compared with the corresponding element of the actual transfer function in Figure 6.9. The comparison of the closed-loop identified system (i.e. from Section II) is made in Figure 6.10. Note that the accuracy of the transfer function identification is essentially the same for both the open-loop

and closed-loop identification which is a theoretical property of the CVA identification method as discussed in Chapter 2.

6.3.3 Effect of Data Length and Dither Strength

The adaptation interval for the SISO plant was reduced from 7 seconds to 4 seconds and the AMAC was applied to the plant, keeping other simulation parameters unchanged. But this time the identified plant was too far away from the true plant and the inherent robustness of MAC was not adequate to enable the plant to track the reference input. The closed-loop was unstable as is shown in Figure 6.11. The dither strength was then raised to 1.0 thus making SNR 26 db. The adaptation interval was fixed at 4 seconds. This time the quality of the identified plant was better and the plant under MAC was able to track the reference input again albeit at a cost of much higher oscillation. The resulting tracking behavior is shown in Figure 6.12. The identified plant in the open-loop and closed-loop environments are compared in Figure 6.13. The optimal state orders for Sections I, II and III were respectively 4, 3 and 6.

For the MIMO plant the data length was reduced from 200 to 100 and similar effect was observed - the closed loop was unstable as shown in Figure 6.14. As in the SISO case above SNR was raised to 26 db by increasing the dither strength to 1.0. As shown in Figure 6.15, the tracking capability of AMAC was revived again. The identified system from the closed loop operation is compared in Figure 6.16. The optimal state order was 6 in both sections I and II.

The simulations in this section clearly indicate that the servo quality of AMAC can be improved either by increasing data length or dither strength.

6.3.4 No Measurement Noise

In this set of runs, it was assumed that there was no measurement noise and the dimension of the state-space was known apriori. The intensity of the dither signal was taken to be 0.1.

The SISO plant is identified every 2.5 seconds, i.e. only 25 data points were used in the identification algorithm. The result of applying AMAC is shown in Figure 6.17 and the transfer function of the identified plant is compared in Figure 6.18. The identified poles are follows:

	State Order	Poles
Section I	2	0.2981±j0.8751
Section II	2	0.3161±j0.8629
Section III	2	0.3094±j0.8729

Under similar conditions, AMAC was applied to MIMO plant for a data length of 50, i.e. the identification scheme was invoked every 5 seconds. The result is shown in Figure 6.19. The transfer function of the identified plant in closed loop operation (i.e., from segment III) is compared in Figure 6.20. The identified poles from different segments of the run are as follows:

	State Order	Poles
Section I	4	0.907, -0.029, -0.145±j0.885
Section II	4	0.676, 0.912, -0.115±j0.967
Section III	4	0.888±j0.037, -0.11±j0.948

These plots show that when there is no observation noise, the CVA technique can reliably identify the plant from a relatively small data length.

6.3.5 Gust Noise Excitation

To demonstrate the effect of colored noise excitation on the accuracy of the identified trasnfer function, a wind gust excitation of the form described in MIL-F-8785 (Hoh et al, 1982) is used. This is in contrast to the white noise input excitation used in the other simulations of this chapter. The wind gust excitation was simulated using a white noise excitation of unit variance into a transfer function shown in Figure 6.21 along with the plant transfer function. The gust excitation level was chosen so that the total variance of the input excitation was the same as the white noise excitation used in Figure 6.6 and 6.7.

The control and output sequences are shown in Figure 6.22. The identified transfer functions corresponding to the time intervals I and III are shown in Figures 6.23 with the use of open and closed loop data respectively. In theory, the accuracy of the identified transfer function at different frequencies is proportional to the ratio of the input excitation power to the measurement noise power at the frequency. Thus one would expect to see a slightly greater accuracy of the transfer function near the peak of the gust spectrum and slightly lower accuracy at the frequencies with low power when compared with Figure 6.7. This is consistent with the simulation run, however the statistical variability is high in comparing identification accuracy on only two data sets.

An implicit input excitation where the excitation is not observed was also considered. The result is of little use in transfer function identification since only the magnitude of the trasnfer function is obtainable and not the phase. In addition the accuracy of the magnitude function is considerably worse than in the case of an explicit input excitation. Thus the presence of wind gusts are of very limited value in plant transfer identification unless the gust excitations are accurately measured.

6.4 Conclusion

The simulations of this chapter have demonstrated the fact that the combination of CVA and MAC results in a reliable adaptive control scheme. This scheme can be used in an environment where the plant model is completely unknown and/or slowly time varying. The satisfactory performance of AMAC demonstrated that:

(i) CVA can identify a plant satisfactorily in an open loop as well as in closed loop operation of the plant.

(ii) The optimal state-order selection criteria (using AIC) is extremely helpful when the state-space dimension of the true plant is not known apriori. The comparison between the identified and the true transfer function shows that this order selection technique works very well in a low SNR environment.

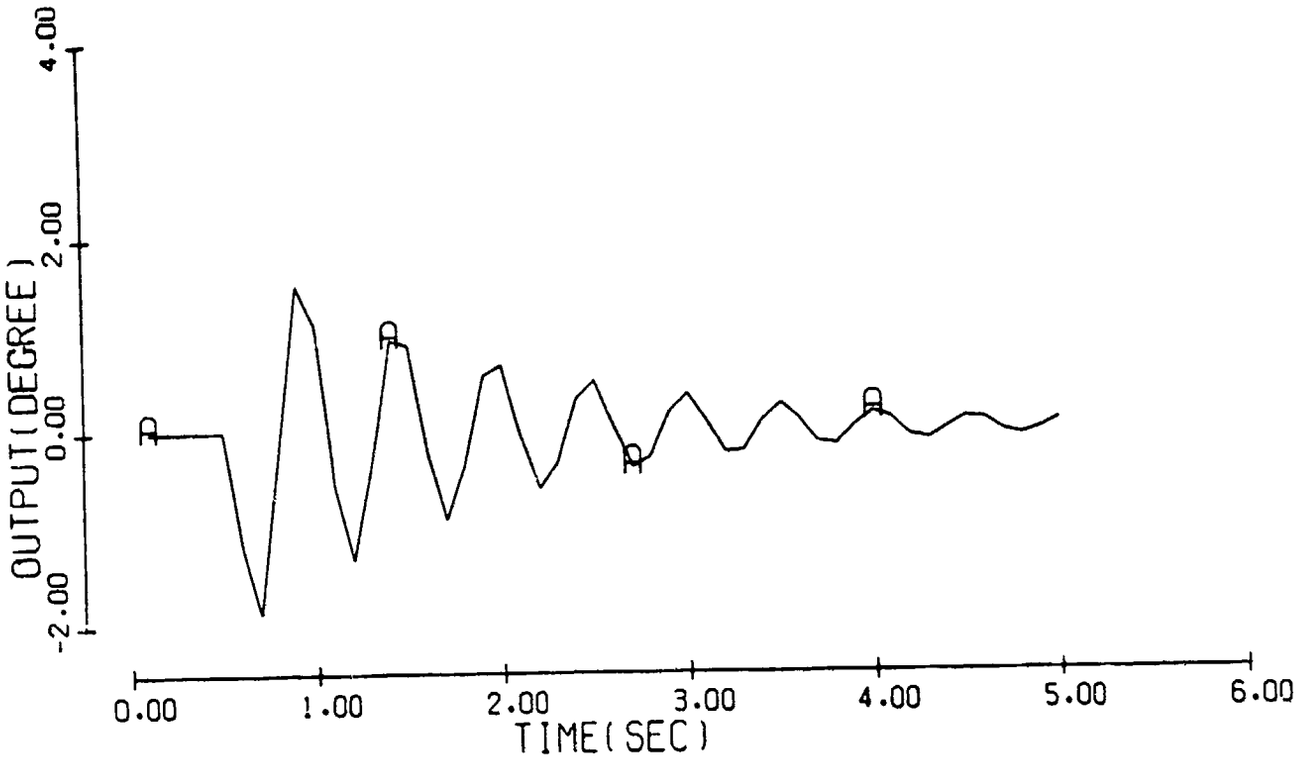
(iii) The accuracy of the identified plant (and hence the performance of AMAC) depends upon data length and SNR. However these factors can be traded between one another - CVA performance can be maintained by using shorter data length and larger SNR and vice versa.

(iv) MAC has excellent robustness properties. As a result the closed loop performances can be maintained in many instances, even when the quality of identification has been degraded.

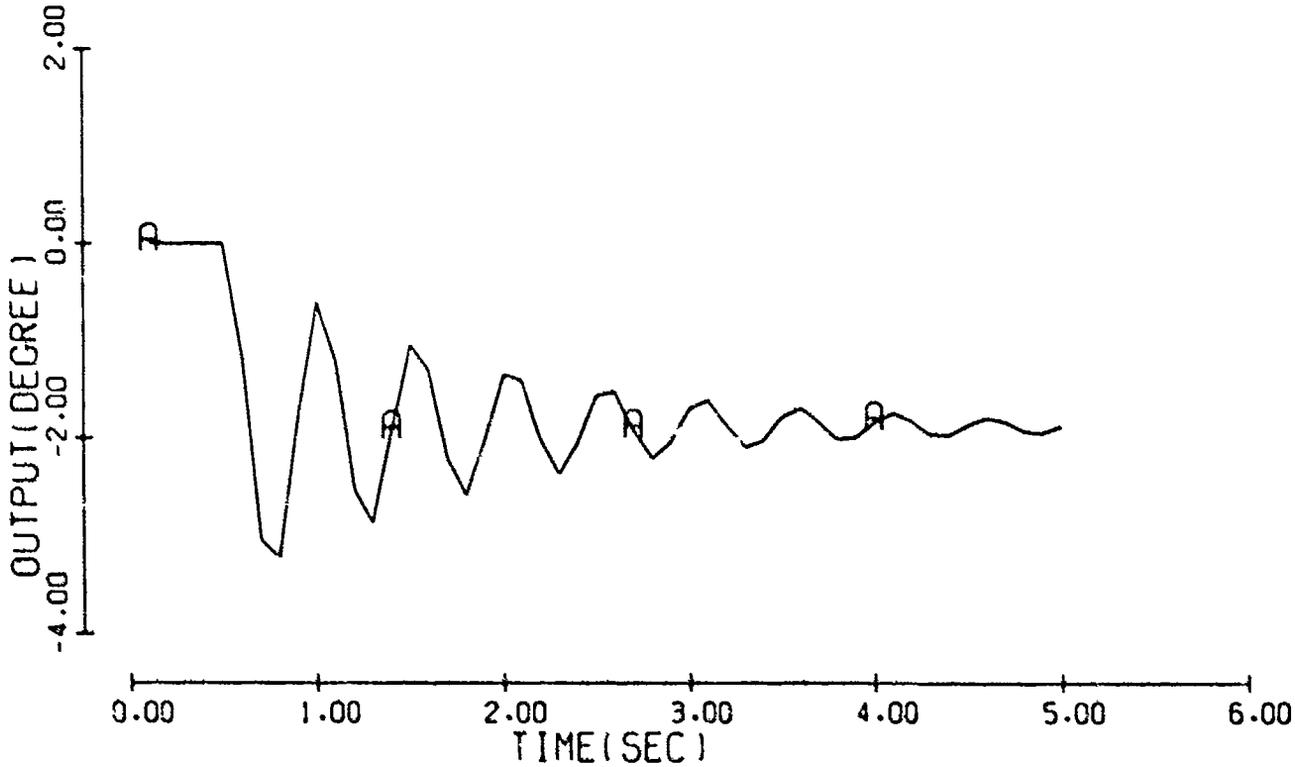
(v) If there is no measurement noise, the plant can be identified from a much smaller sample size compared to the situations having measurement noise.

It is worth noting that the MAC control technique is based upon the impulse response model of the plant and therefore MAC can be used only for controlling stable plants. This causes no problem in a deterministic environment if the plant is a stable one. But in an adaptive control scheme where the plant is reidentified frequently, the identified plant may turn out to be unstable if the data length is too short or the signal-to-noise ratio too low even if the true plant

is an asymptotically stable one. We indeed faced this problem in some of the simulations of this chapter, but the effect was not dramatically visible because the intervals of simulations were too short. However this problem can be remedied by using Model Predictive Control (MPC) technique - a newer version of MAC which can handle stable and unstable systems with equal ease in the same framework.



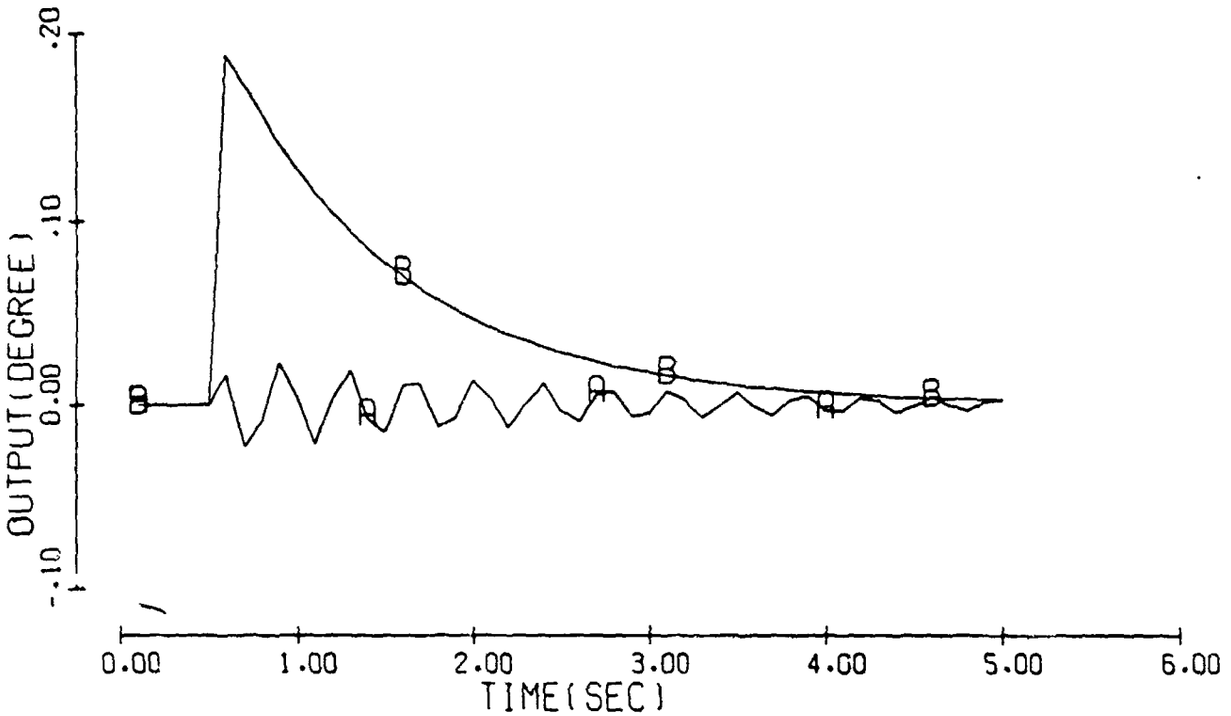
(a) Pulse response



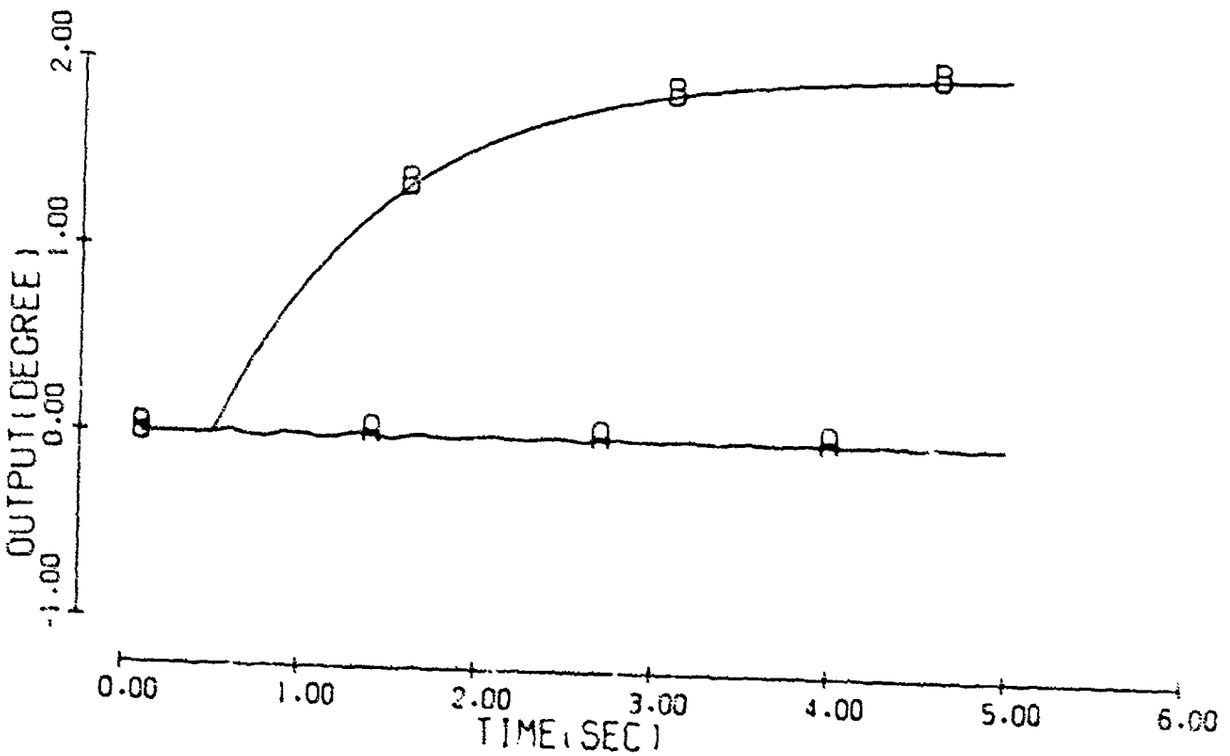
(b) Step response

Figure 6.1 Pitch axis dynamics for simulation

A = Sideship angle
B = Roll angle

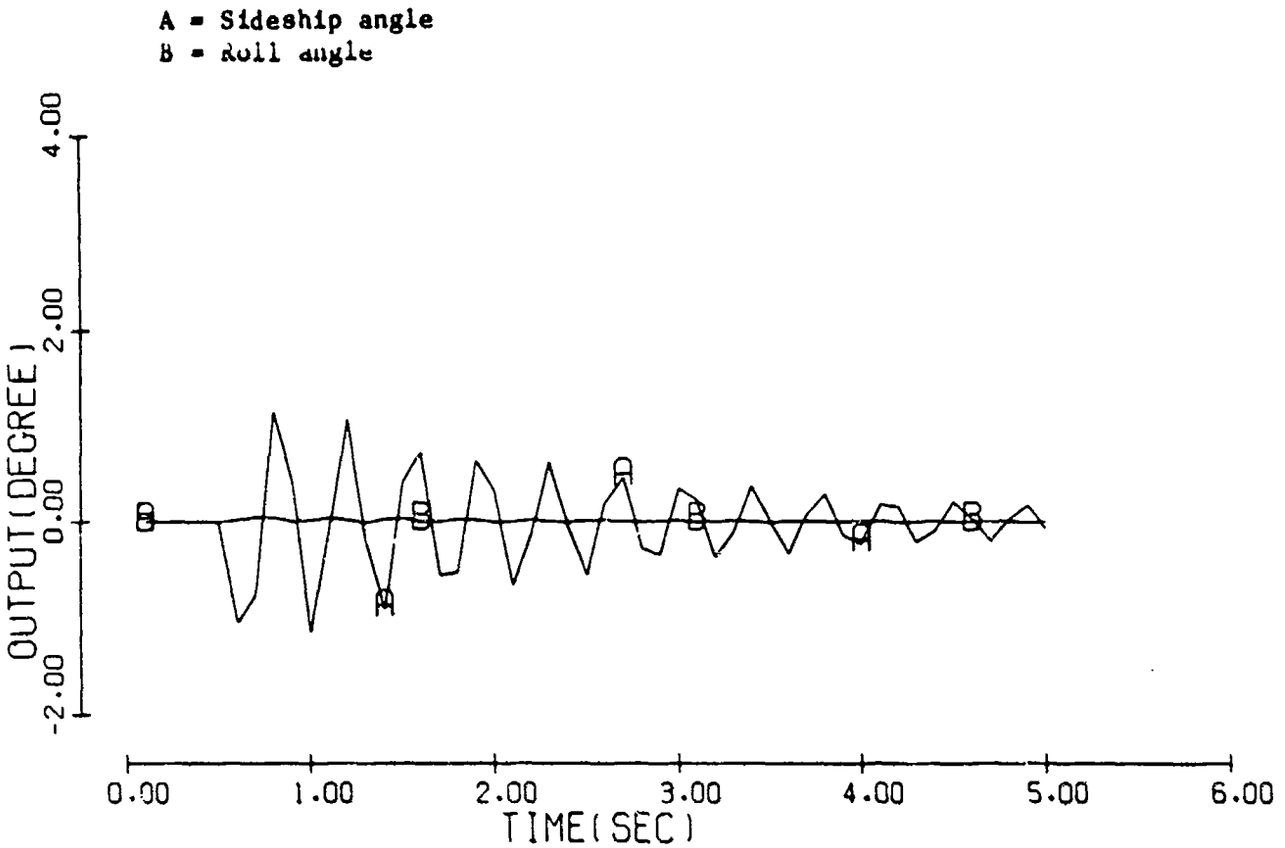


(a) Pulse response

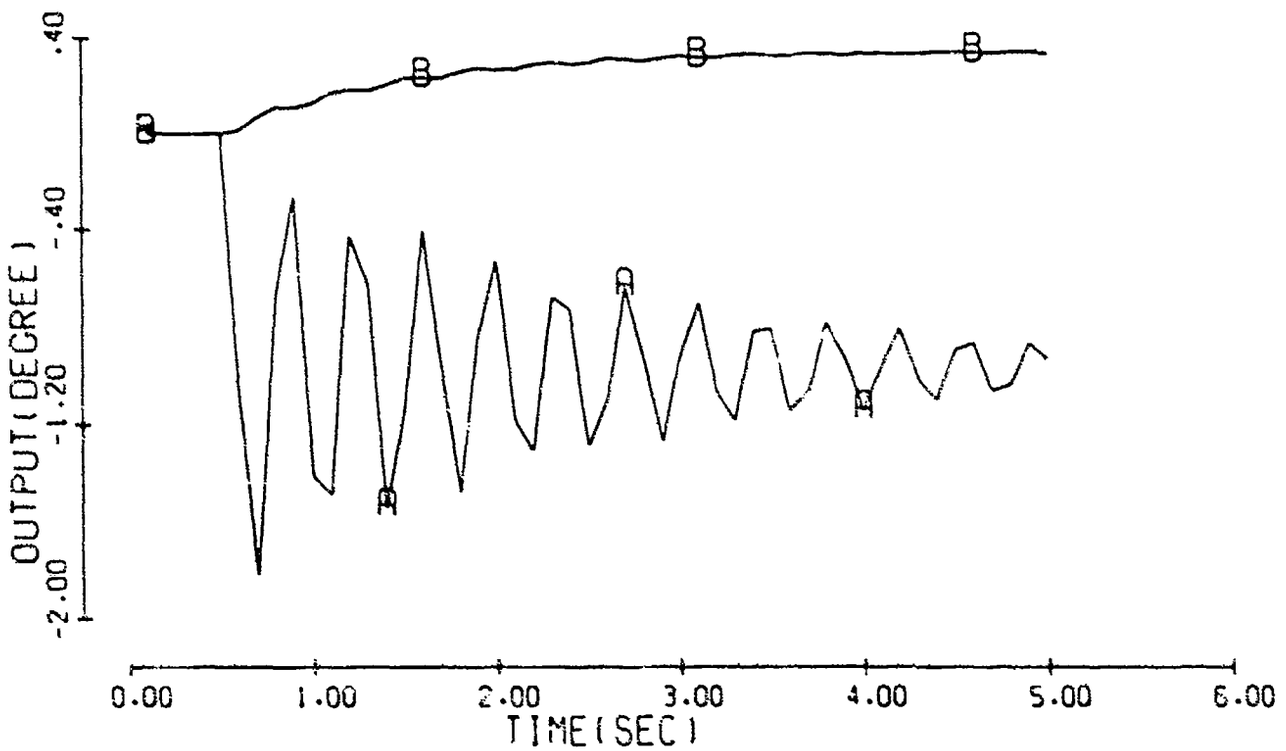


(b) Step response

Figure 6.2 Roll-yaw responses due to aileron movement

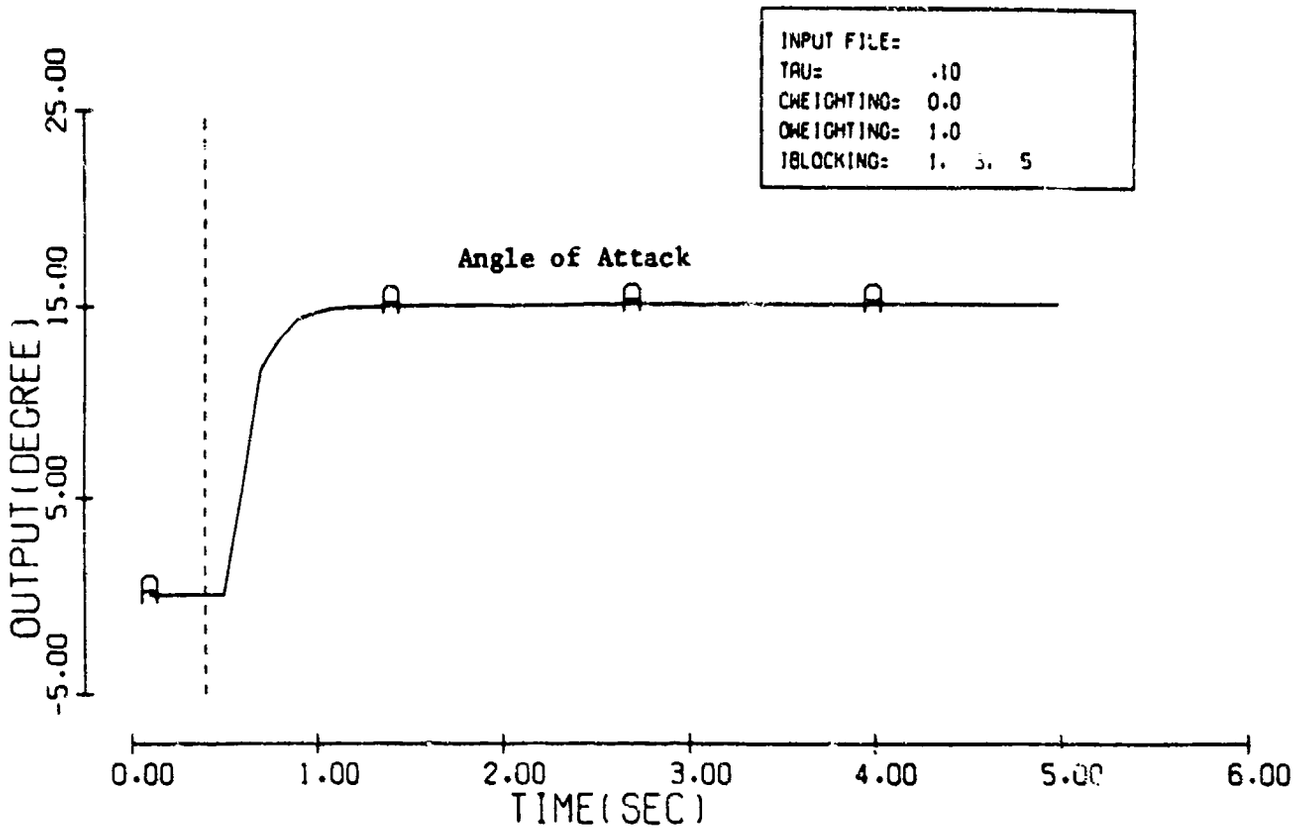


(a) Pulse response



(b) Step response

Figure 6.3 Roll-yaw responses due to rudder movement



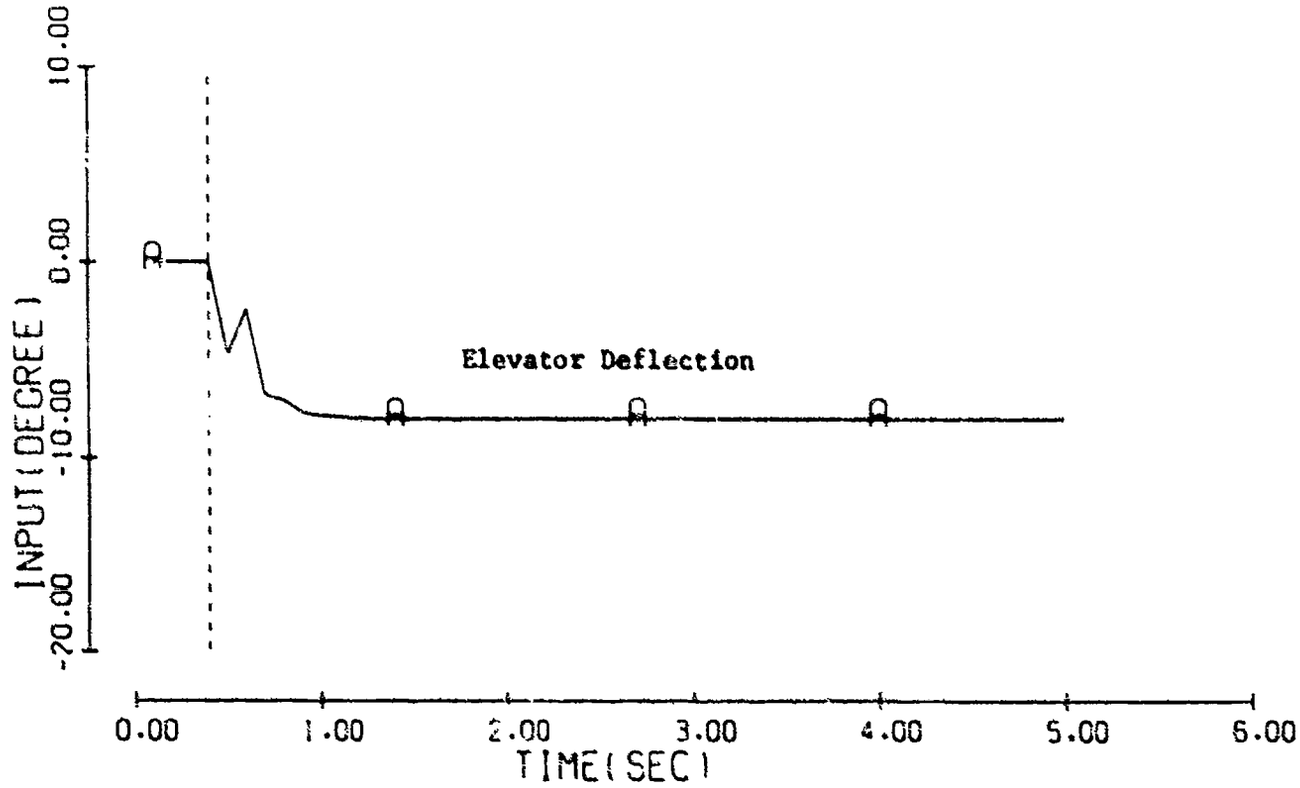


Figure 6.4 MAC applied to perfectly known SISO plant

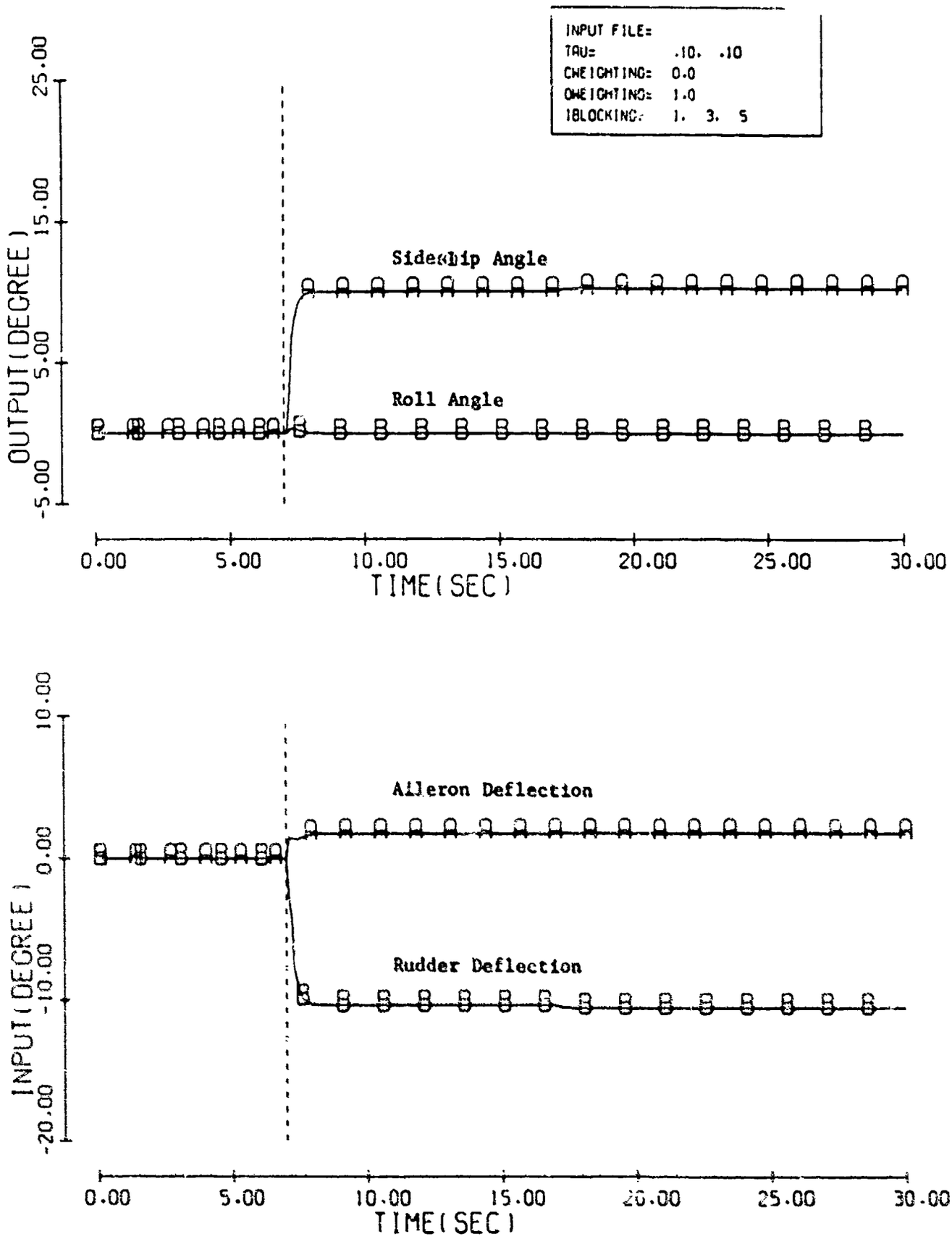


Figure 6.5 MAC applied to perfectly known MIMO plant

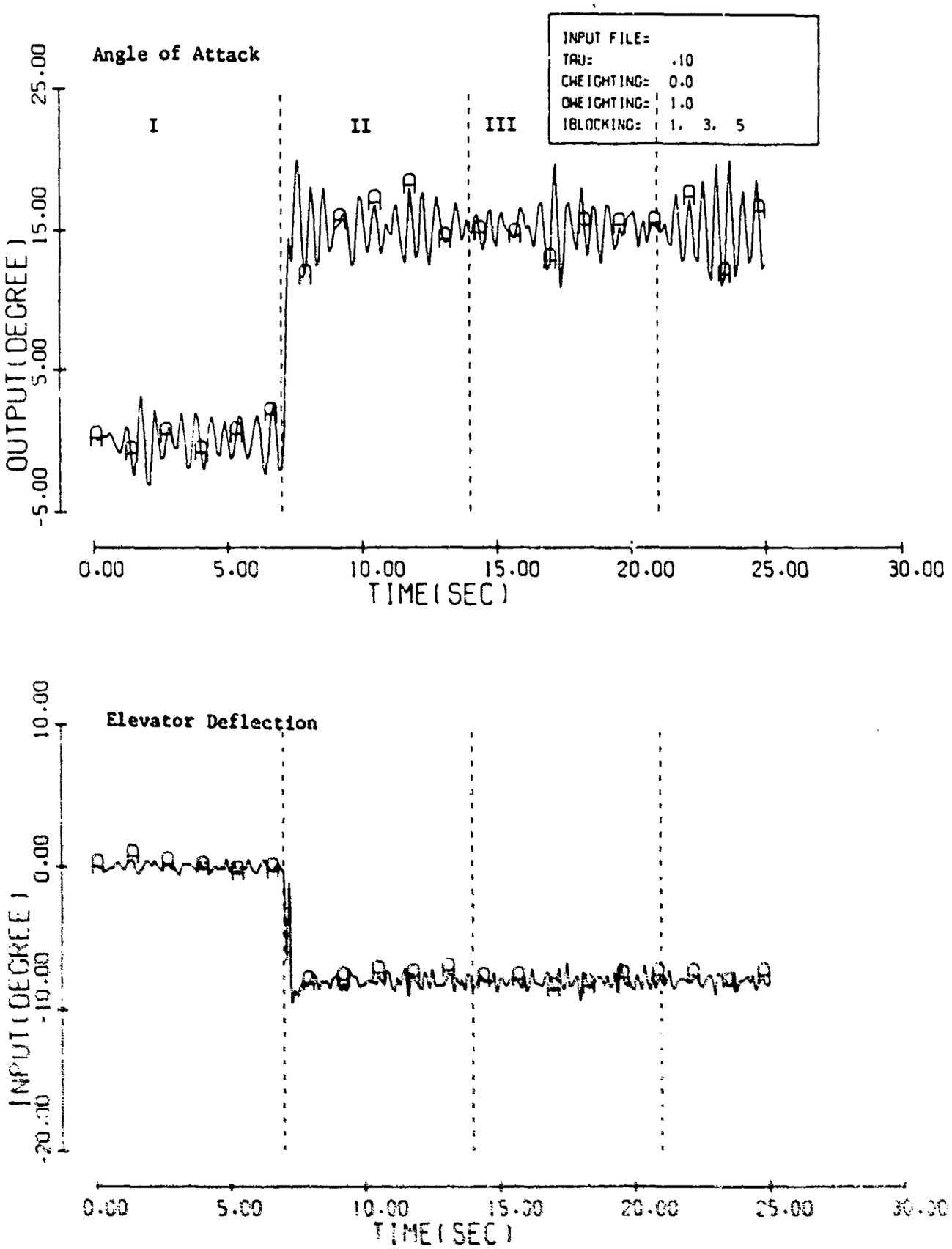
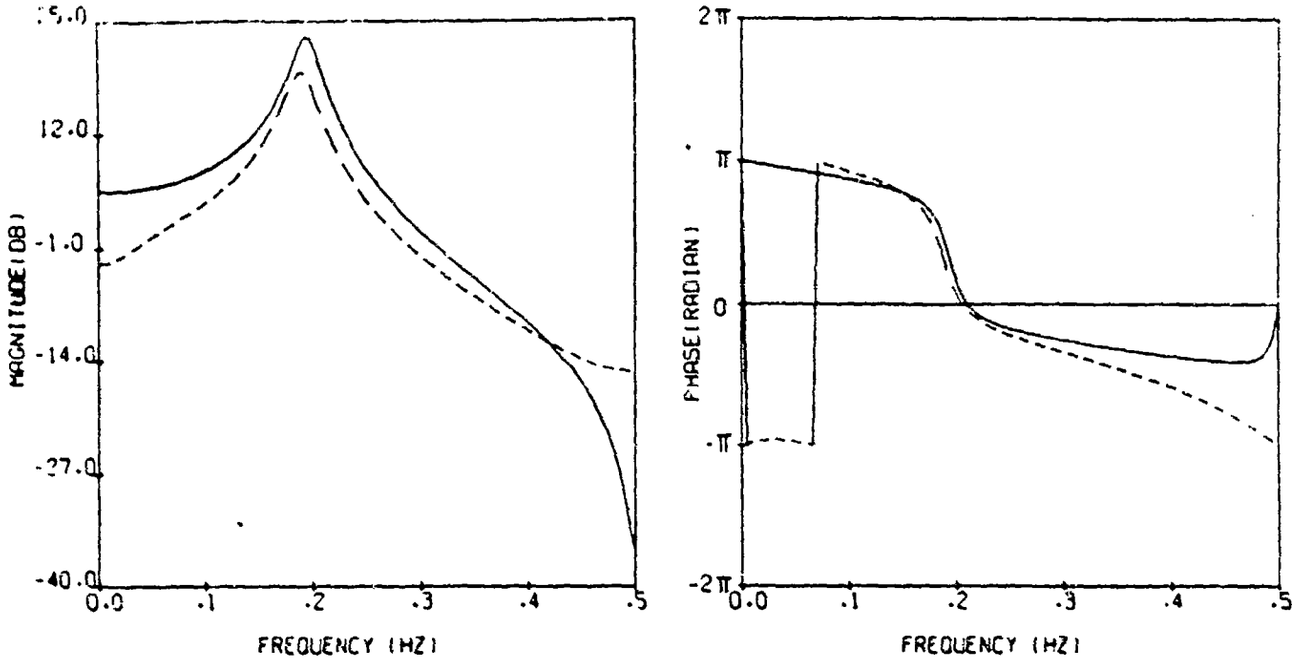
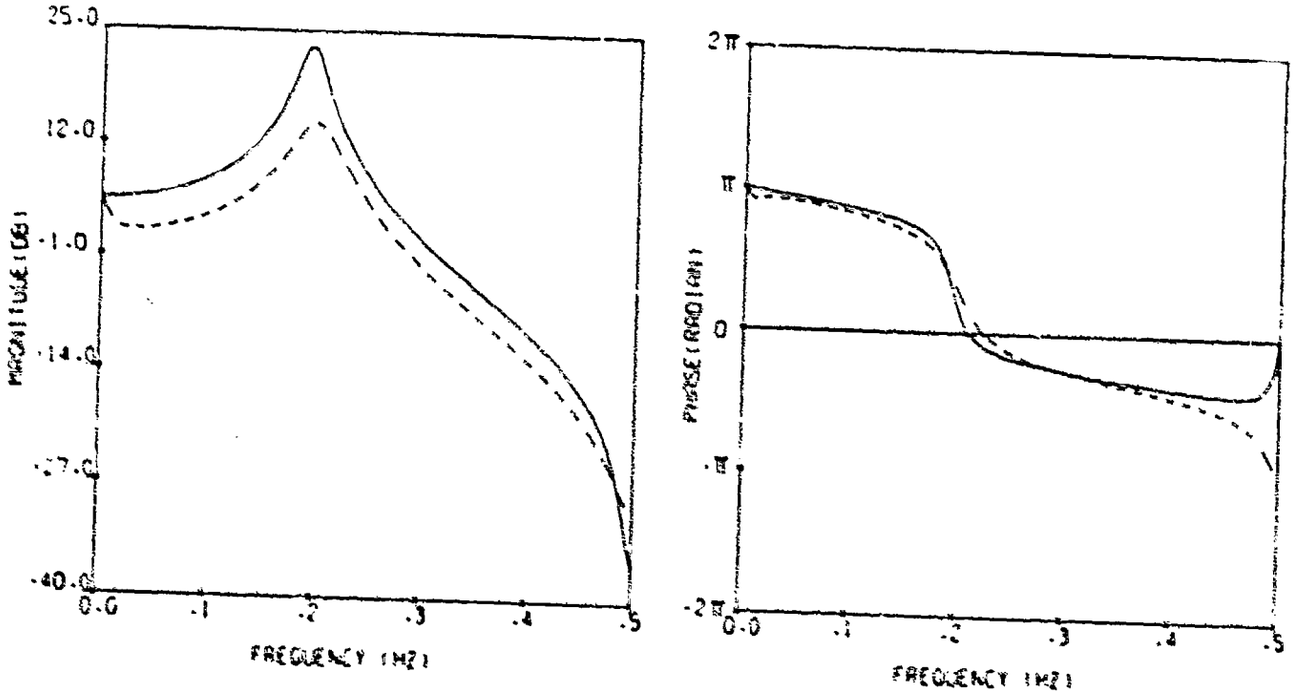


Figure 6.6 AMAC applied to SISO plant



(a) Open loop identification form segment I



(b) Closed loop identification from segment III

Figure 6.7 Actual Plant vs Identified Plant from the run of Figure 6.6

——— true plant

- - - - identified plant

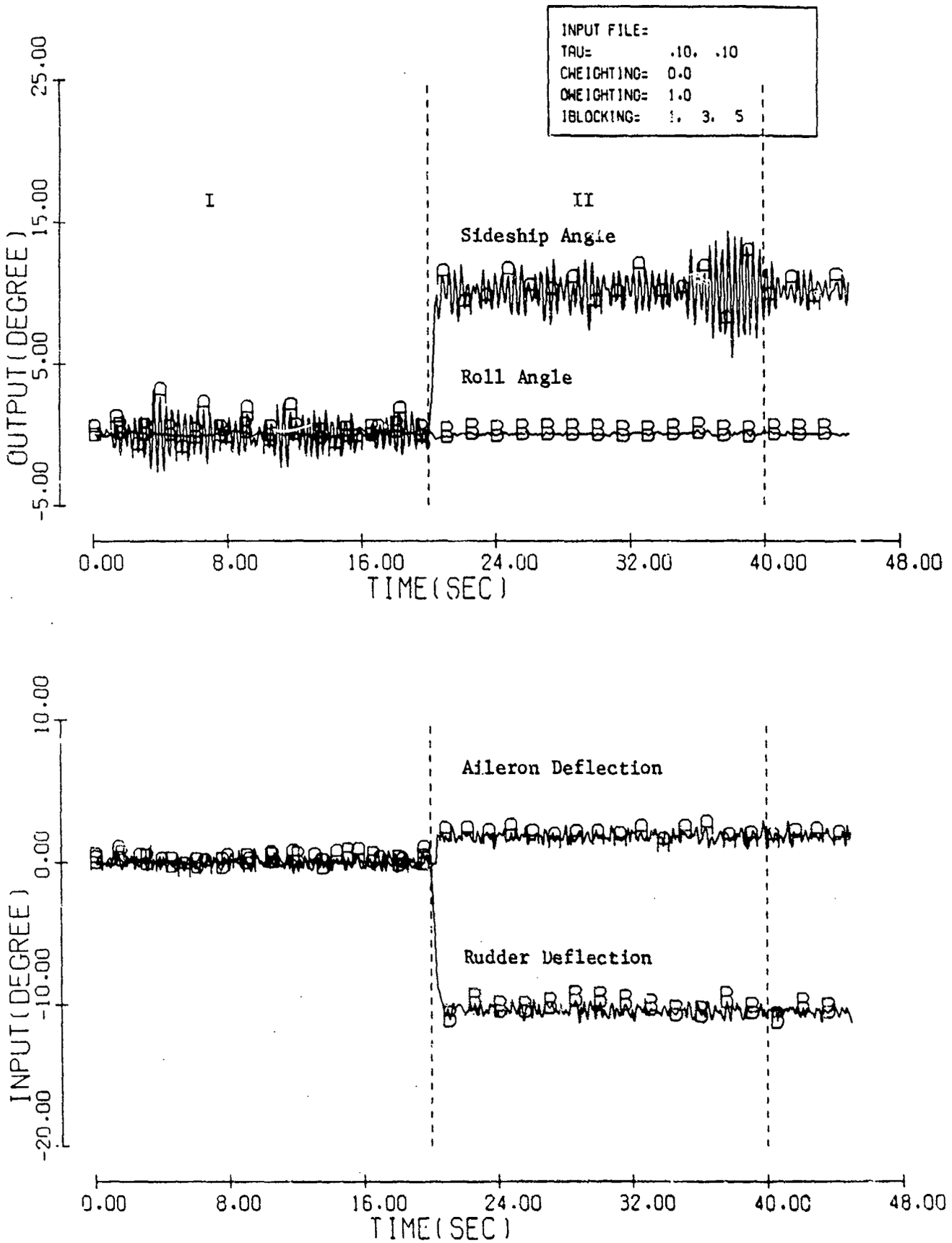
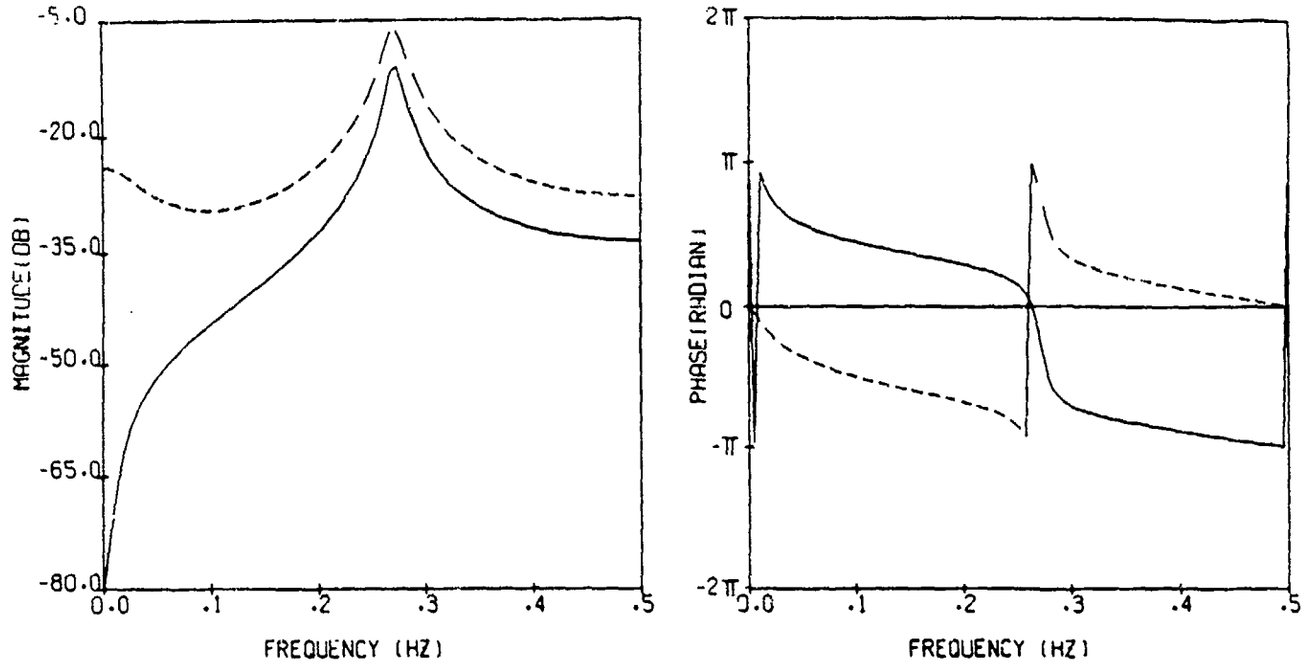
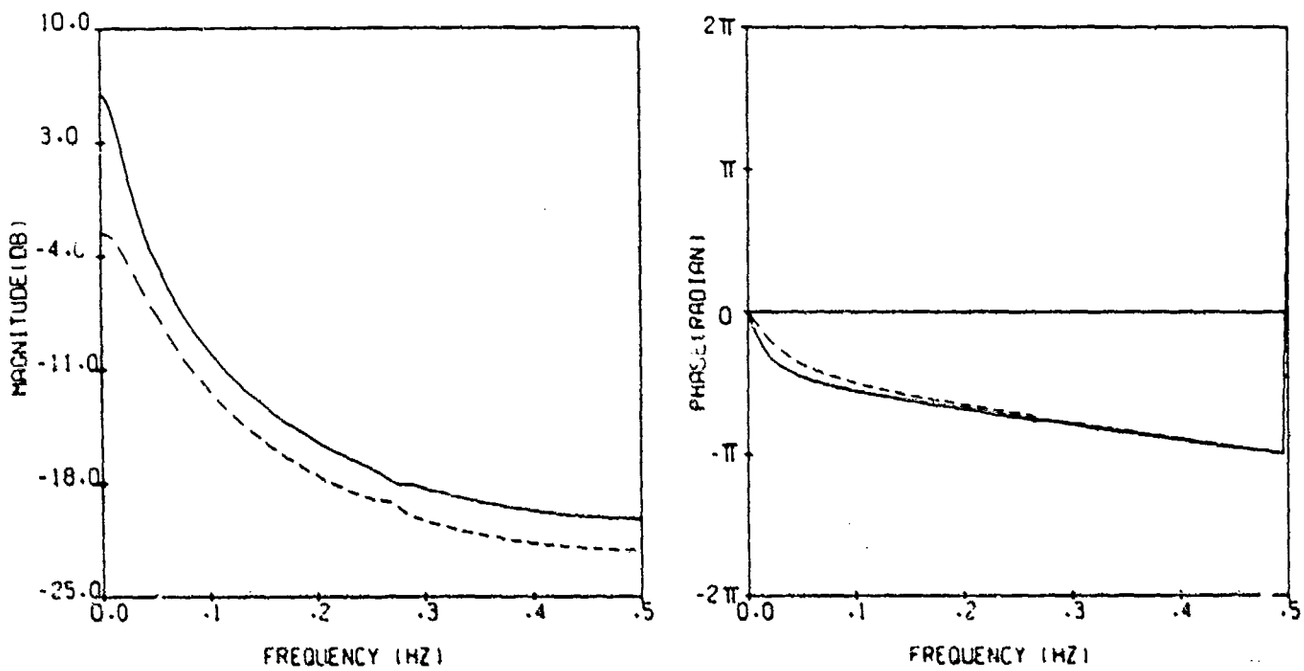


Figure 6.8 AMAC applied to the MIMO plant

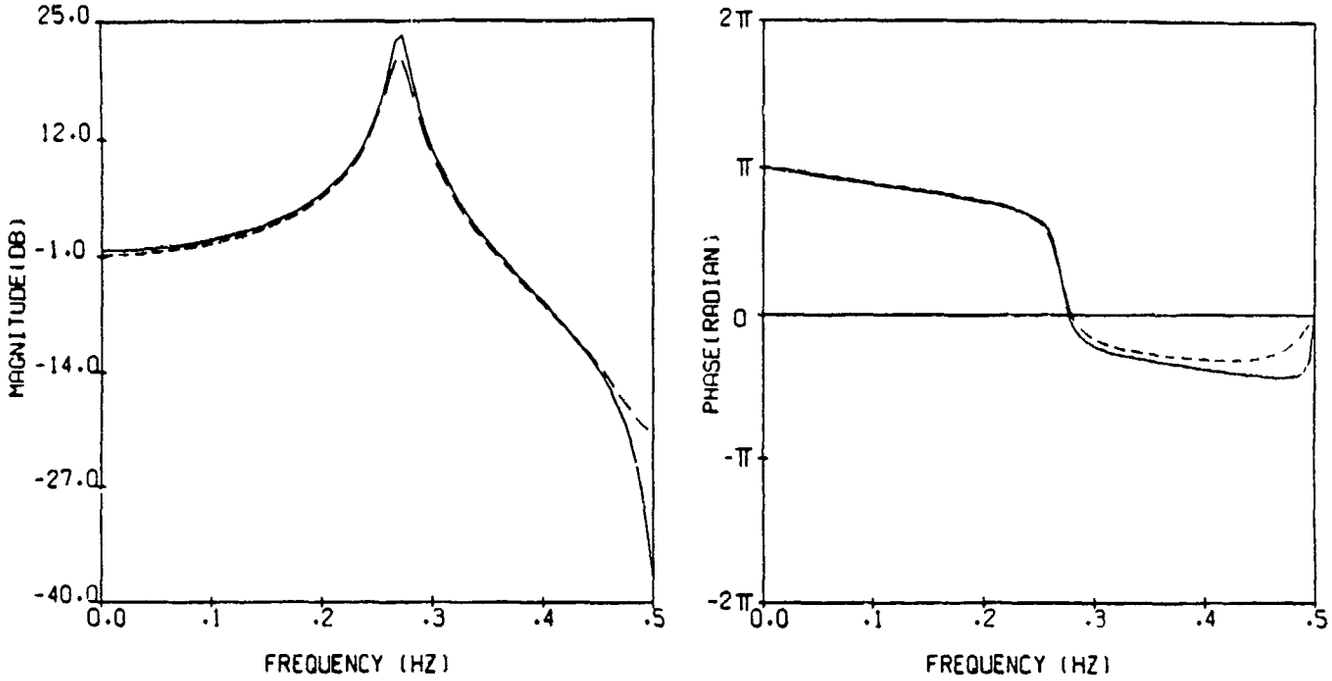


(a) (1,1) element

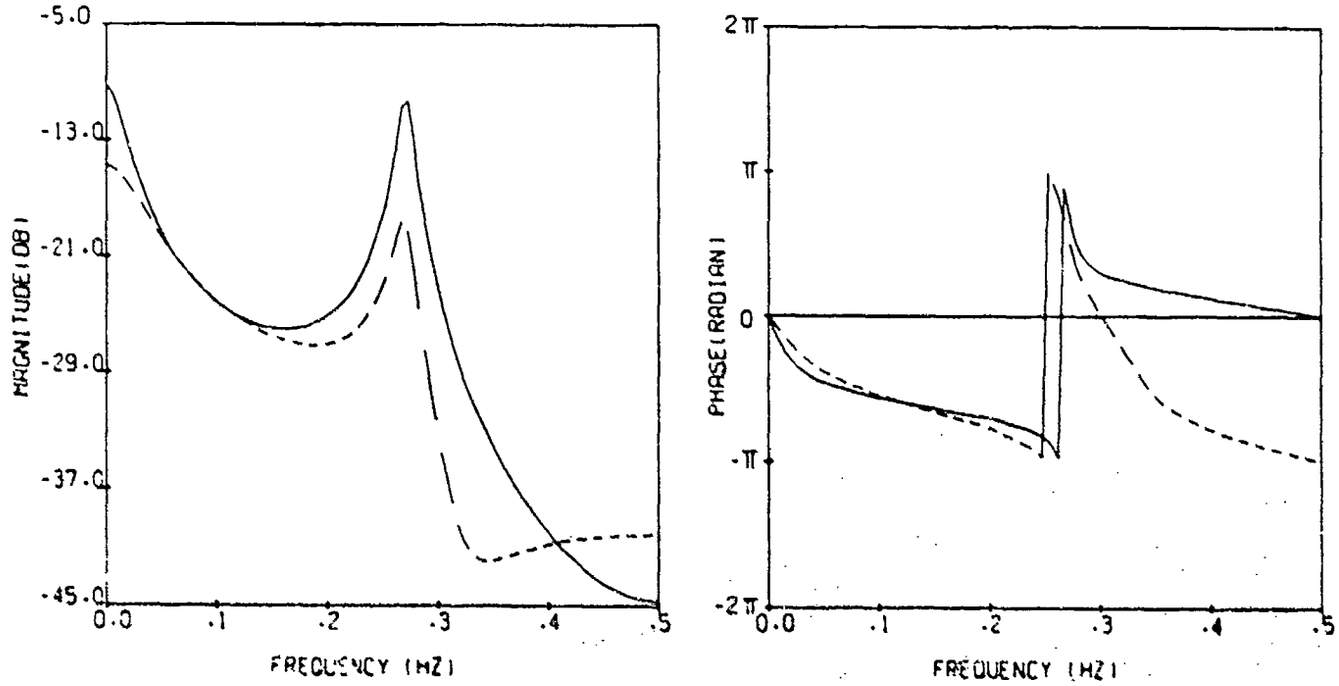


(b) (1,2) element

Figure 6.9 Open loop identification of the MIMO plants (Section I, Figure 6.8)



(c) (2,1) element

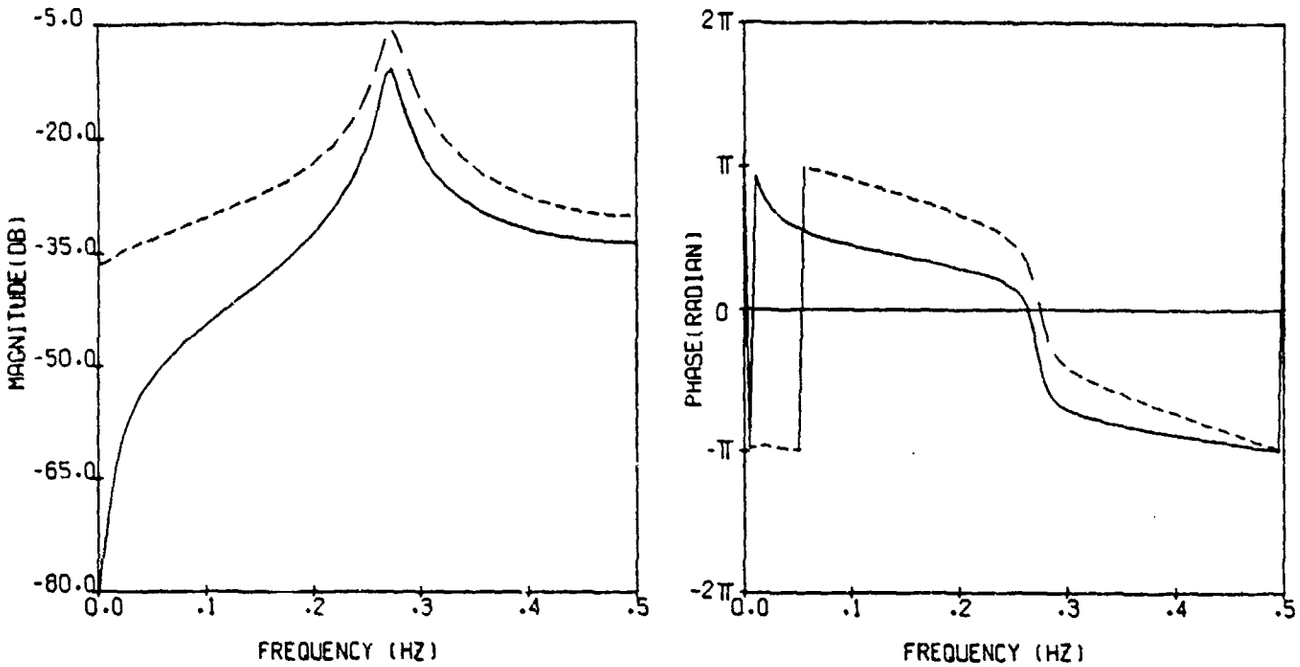


(d) (2,2) element

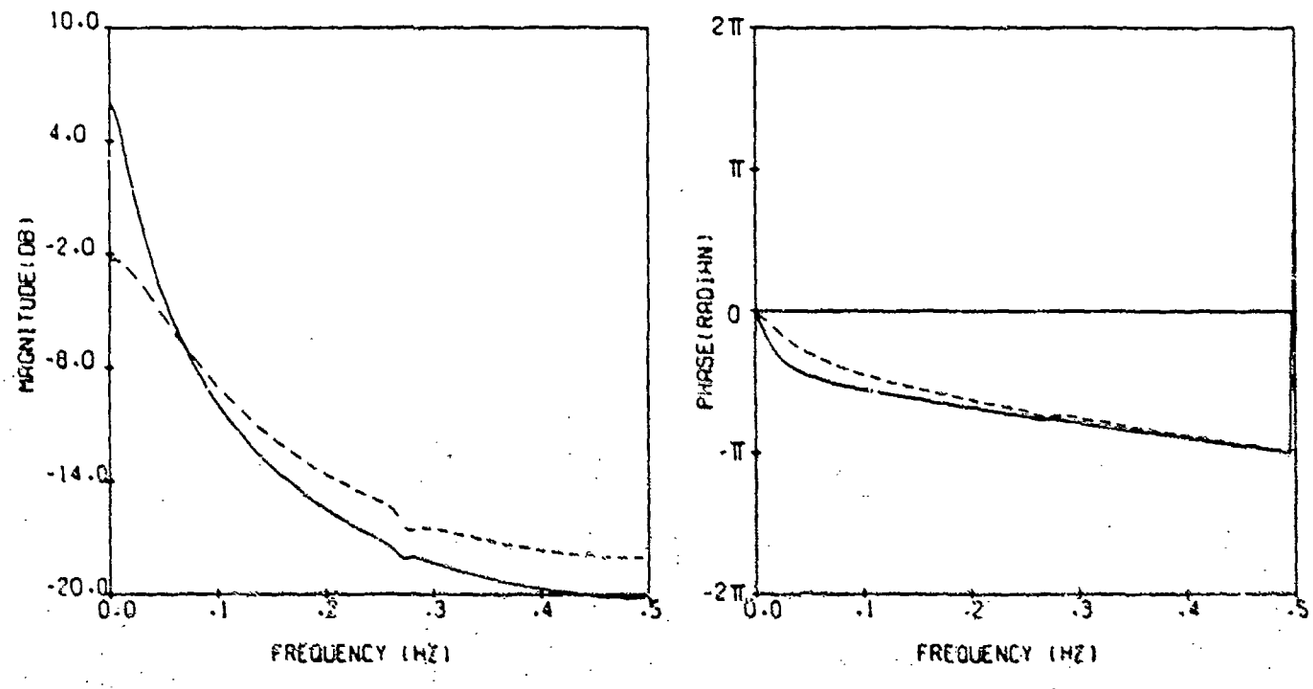
Figure 6.9 (Continued)

——— true plant

----- identified plant

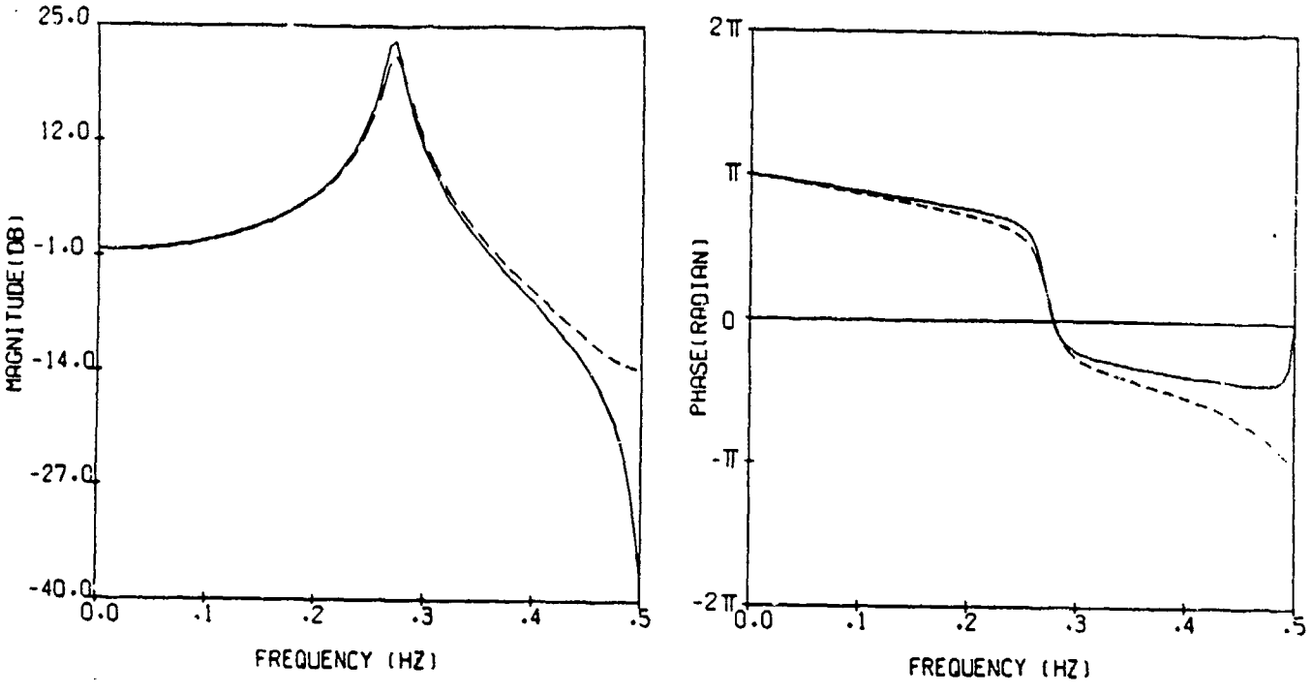


(a) (1,1) element

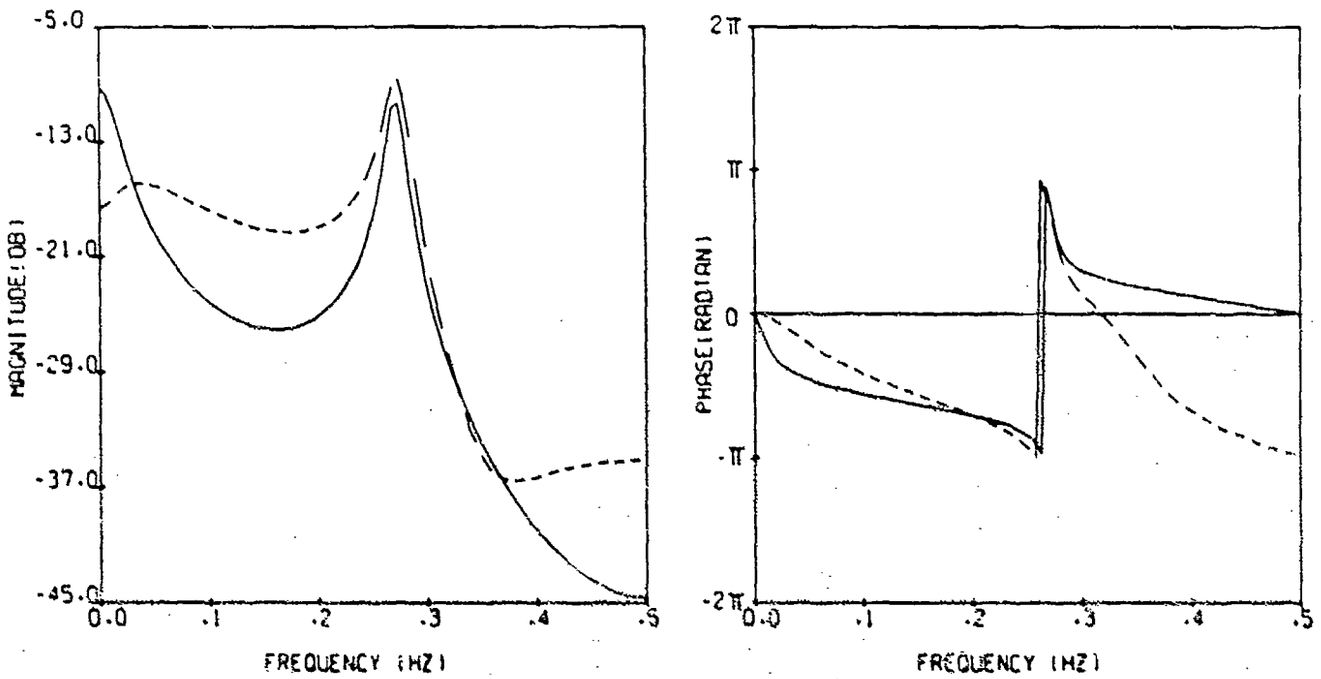


(b) (1,2) element

Figure 6.10 Closed loop identification of the MIMO plant (Section II, Figure 6.8)



(c) (2,1) element



(d) (2,2) element

Figure 6.10 (Continued)

——— Actual plant
- - - - - identified plant

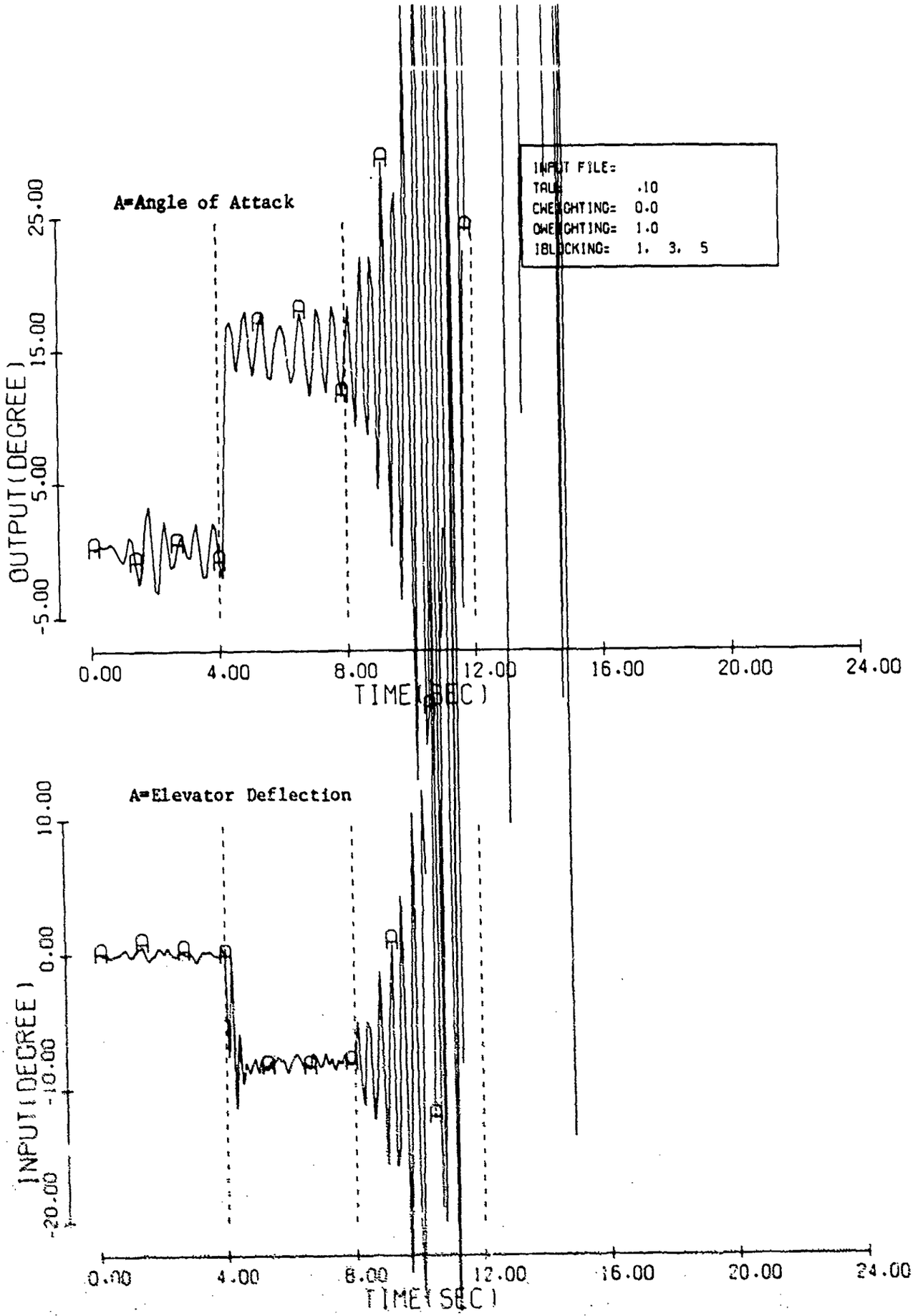


Figure 6.11 Instability of the SISO loop due to short data length and/or poor SNR

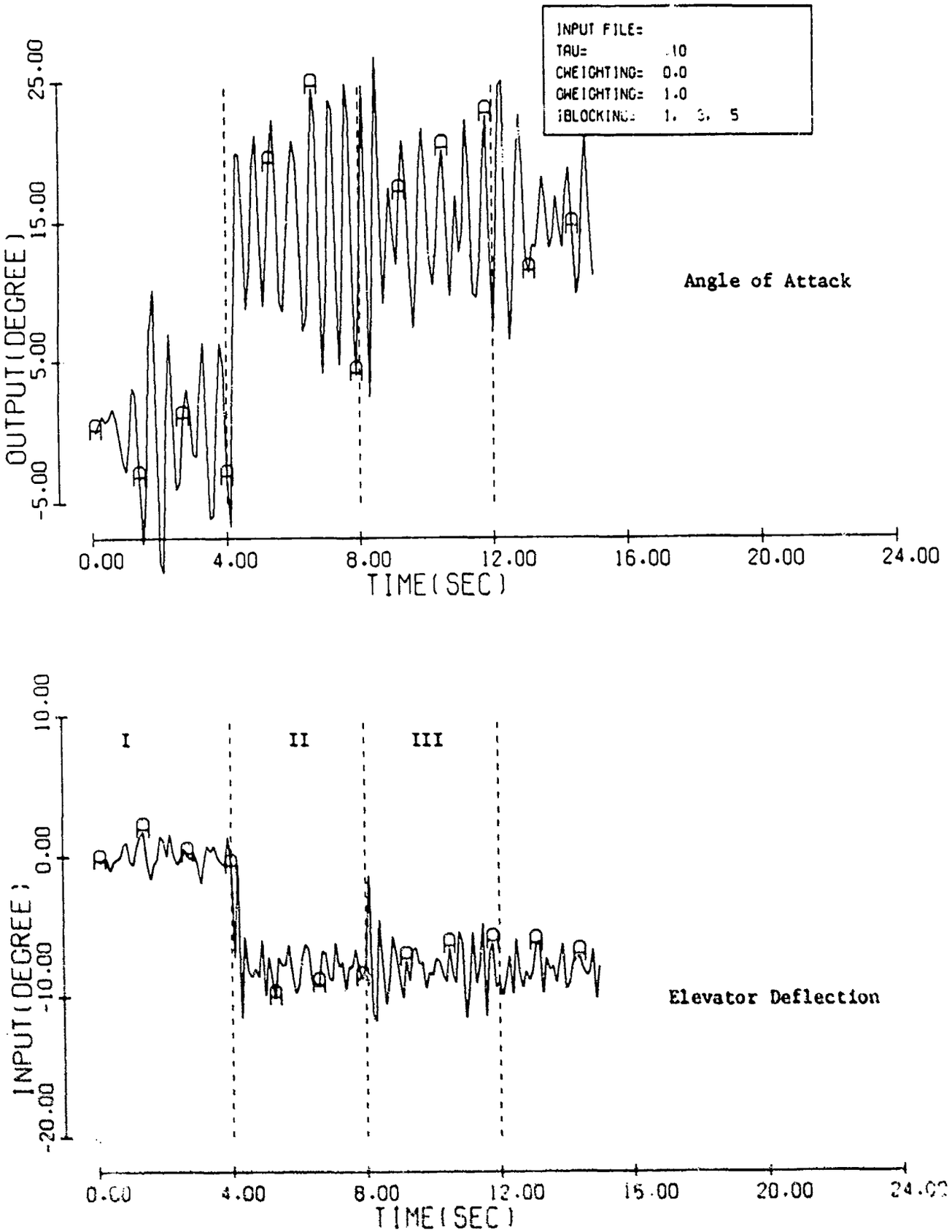
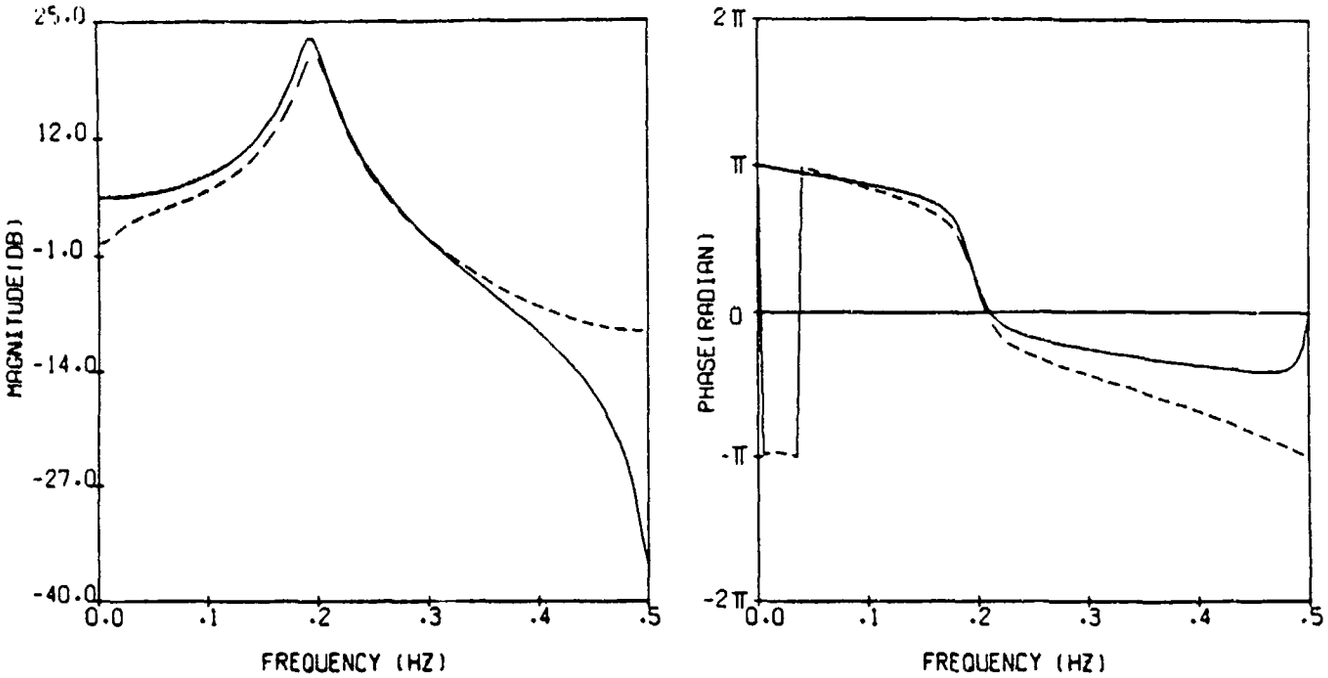
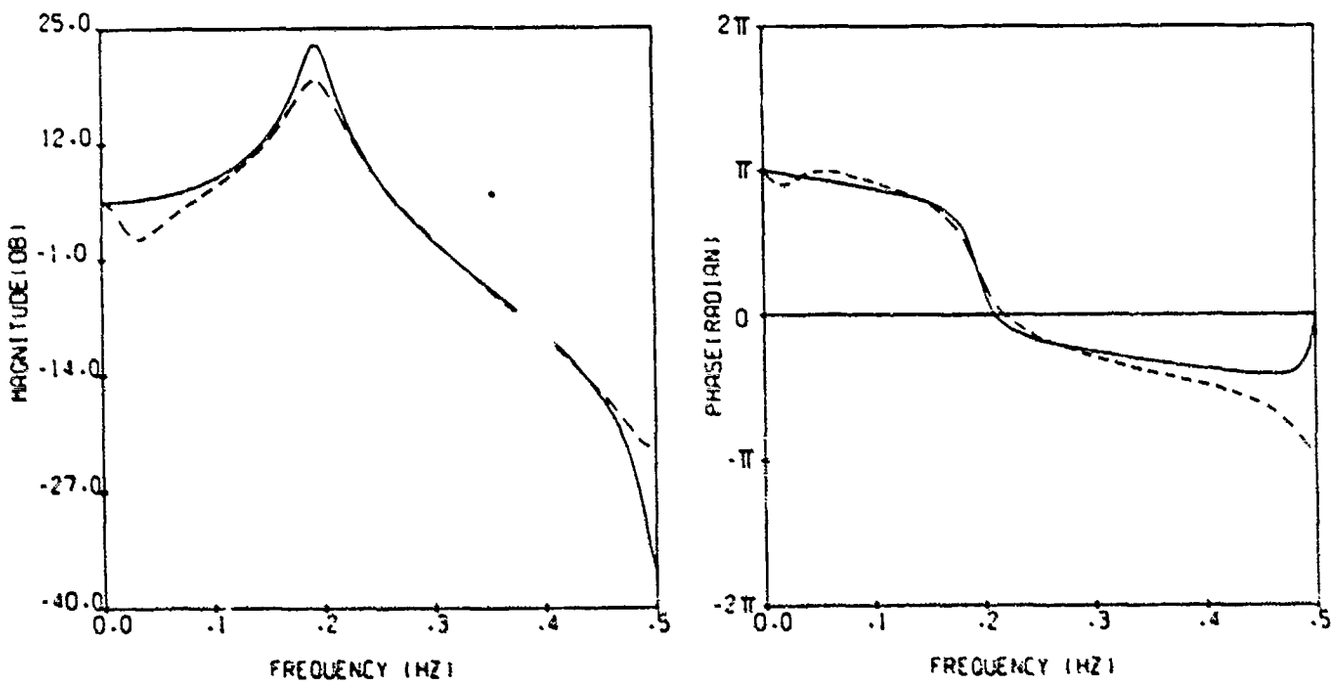


Figure 6.12 Effect of higher SNR on AMAC



(a) Open-loop identification, (Section I, Figure 6.12)



(b) Closed-loop identification, (Section III, Figure 6.12)

Figure 6.13 Comparison of Identified Plants (See Figure 6.12)

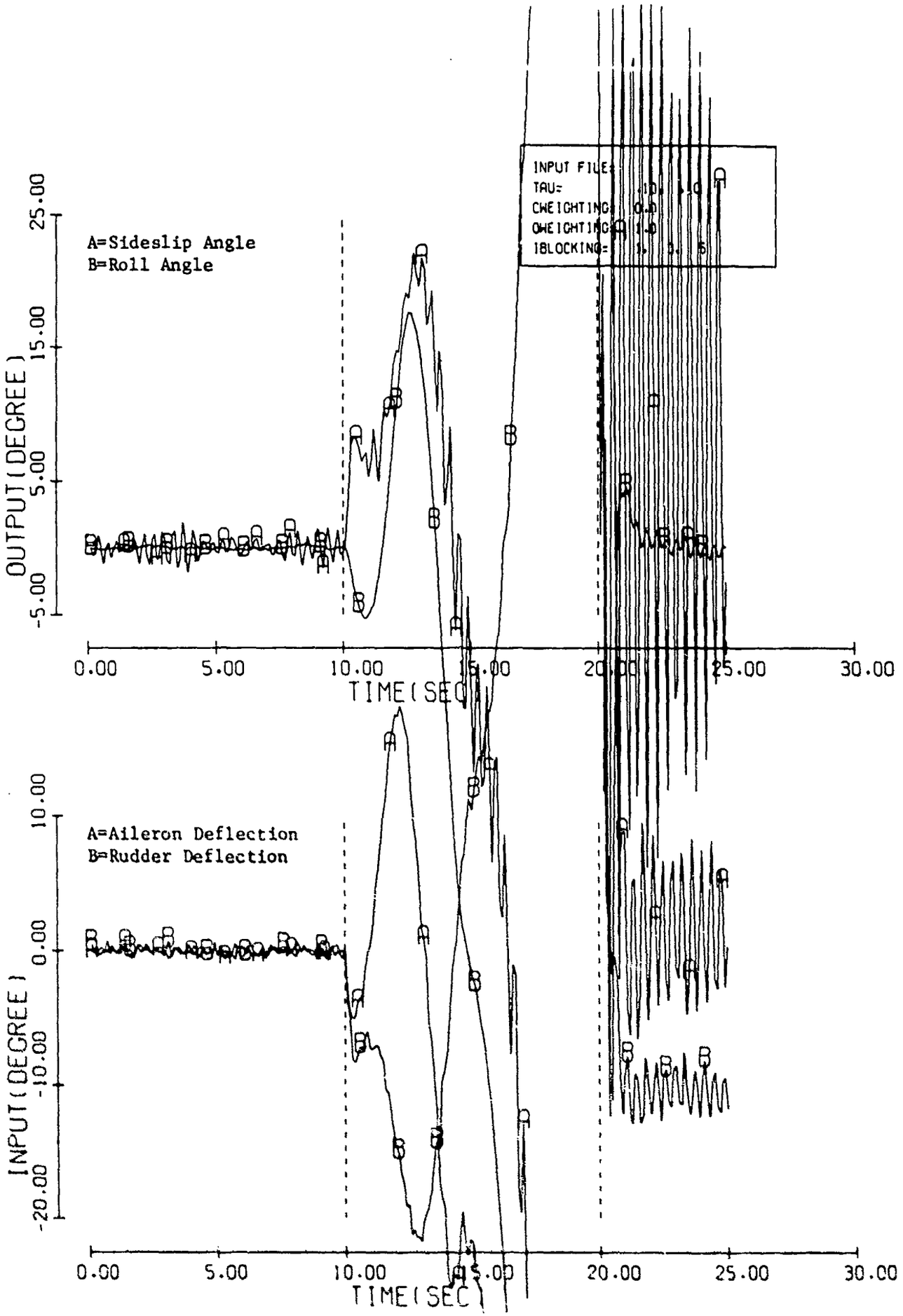


Figure 6.14 Closed loop instability of the MIMO loop due to short data length

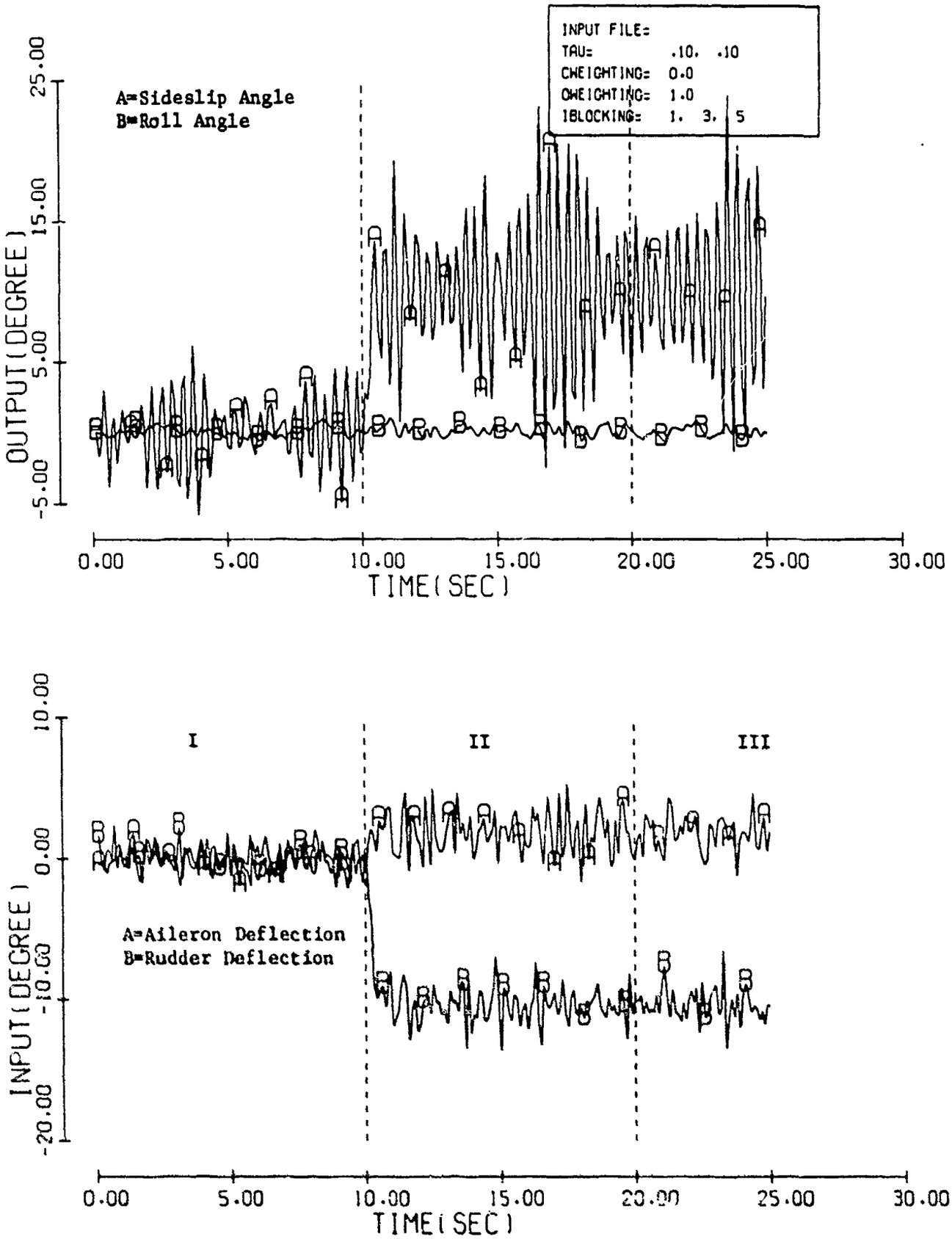
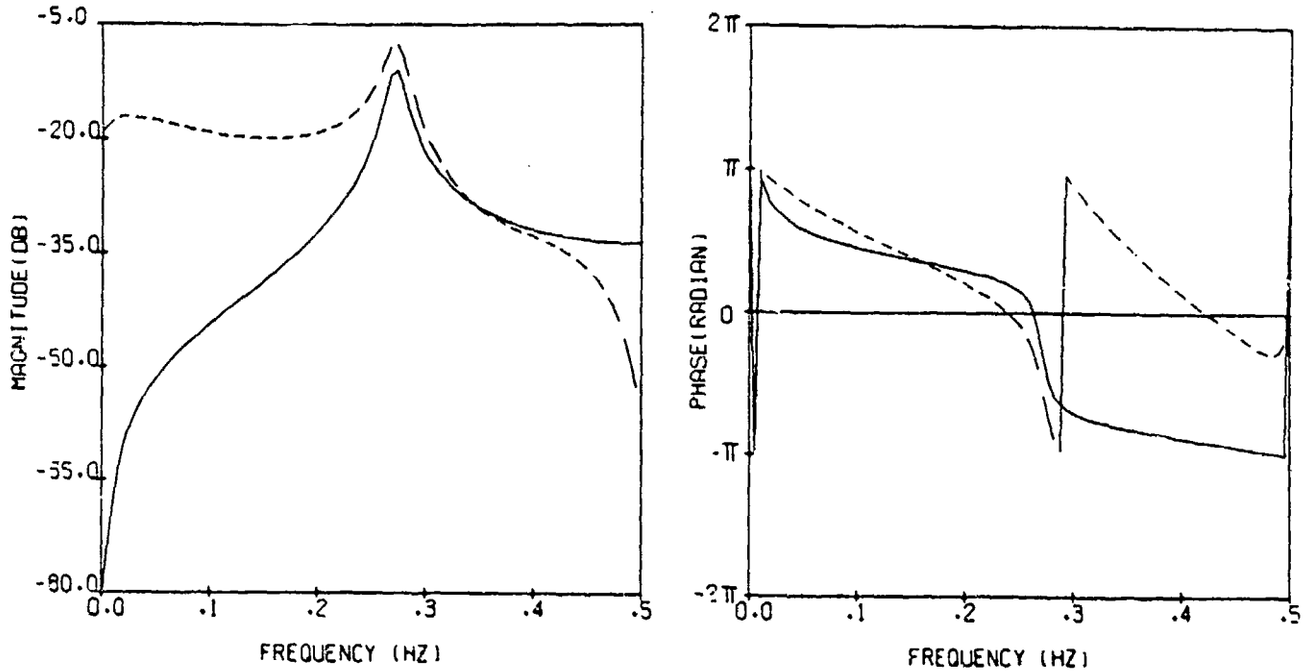
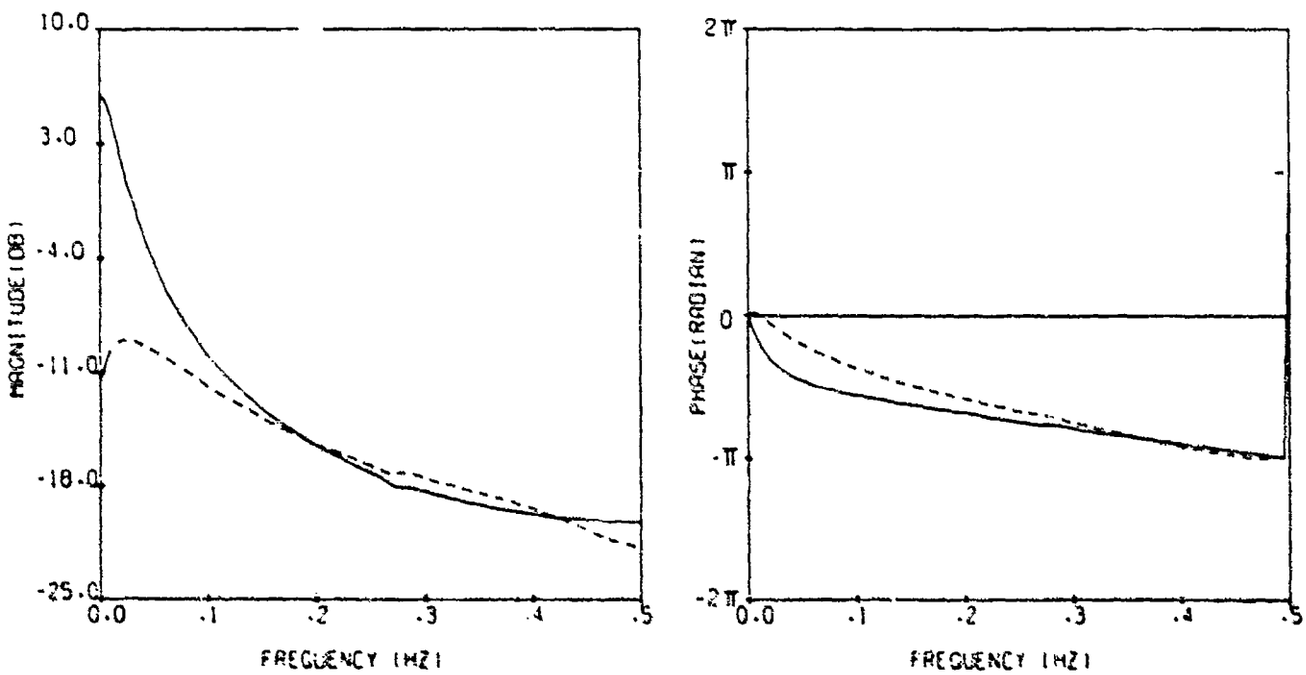


Figure 6.15 Effect of increasing dither strength on MIMO AMAC

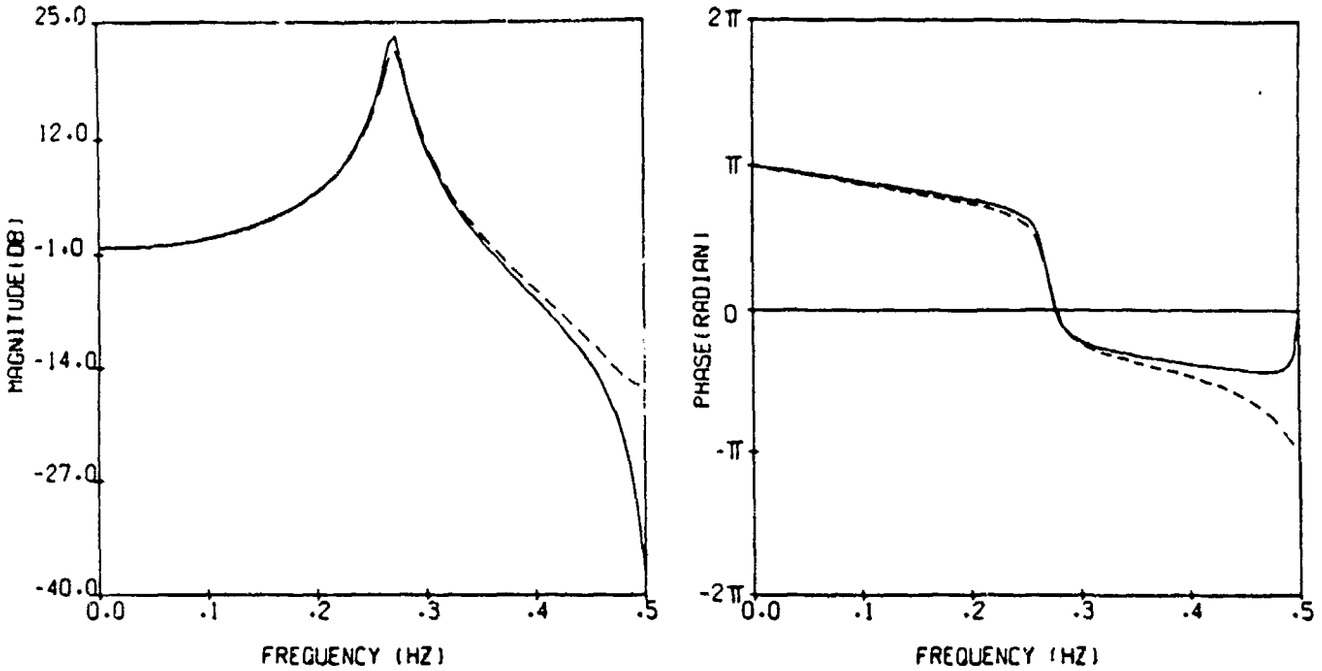


(a) (1,1) element

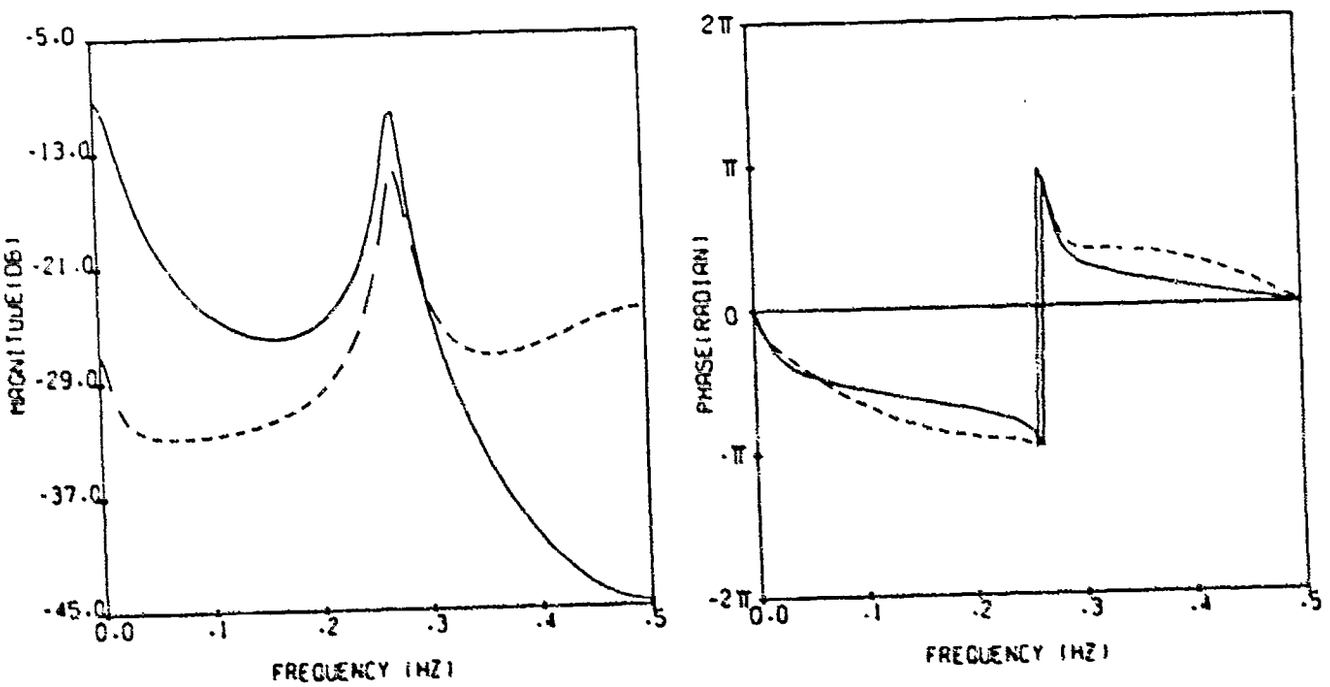


(b) (1,2) element

Figure 6.16 Identified plant by increasing dither strength (Section II, Figure 6.15)



(c) (2,1) element



(d) (2,2) element

Figure 6.16 (Continued)

——— Actual plant

– – – identified plant

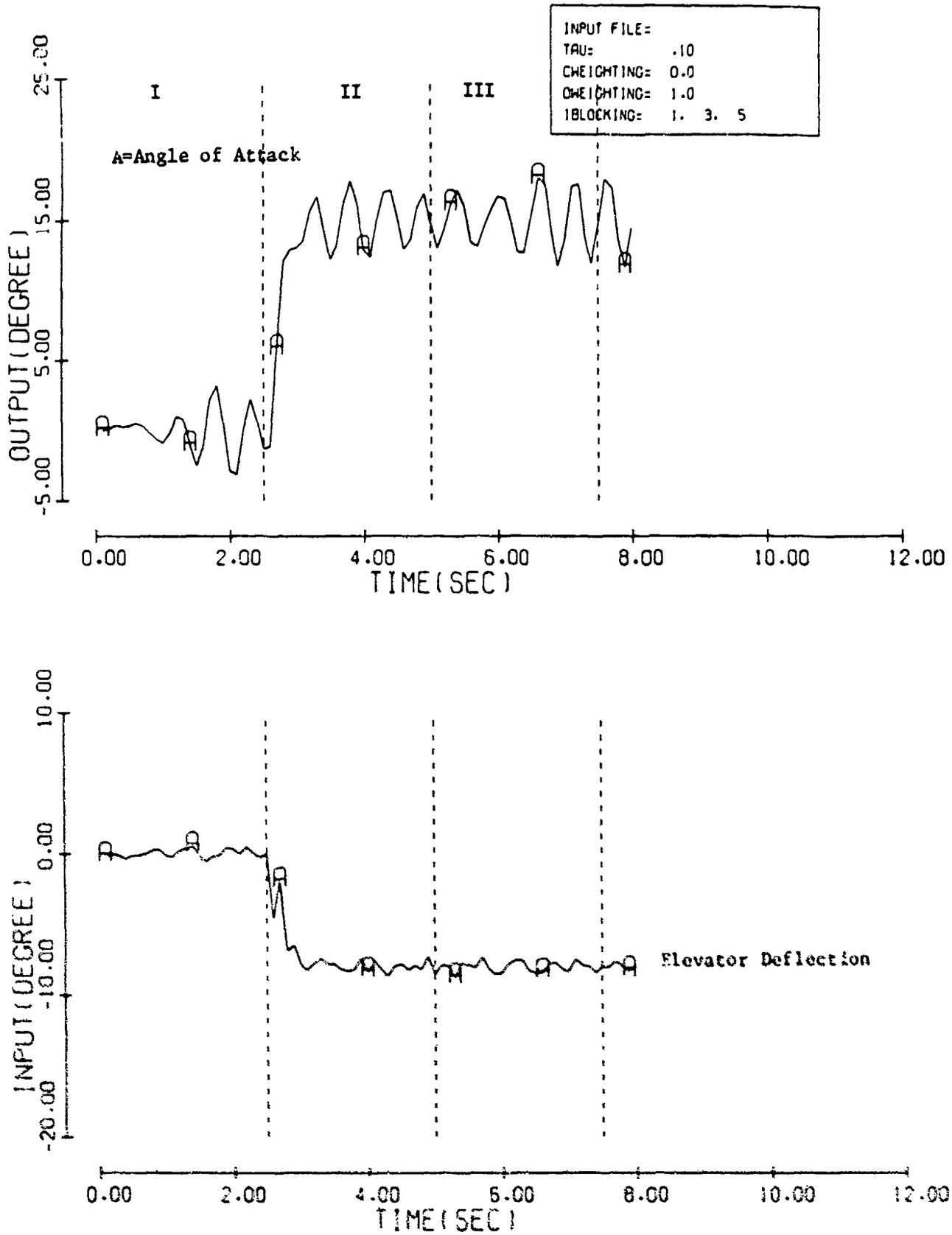
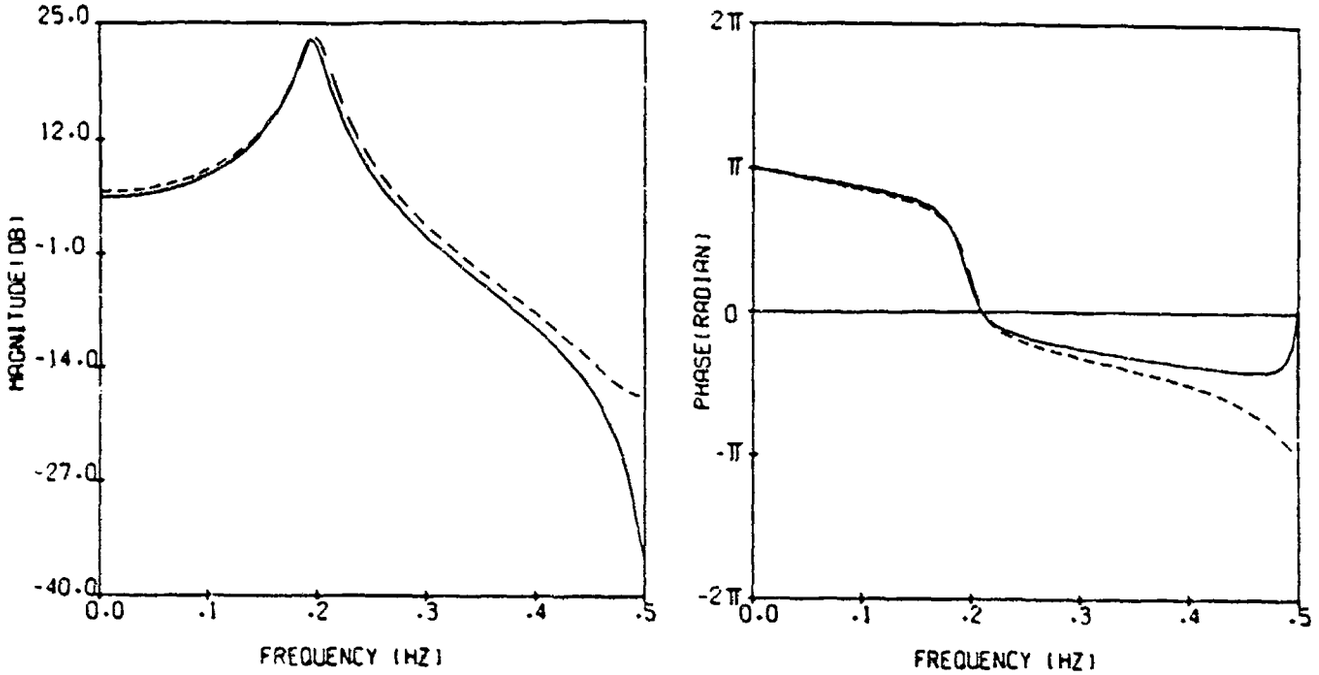
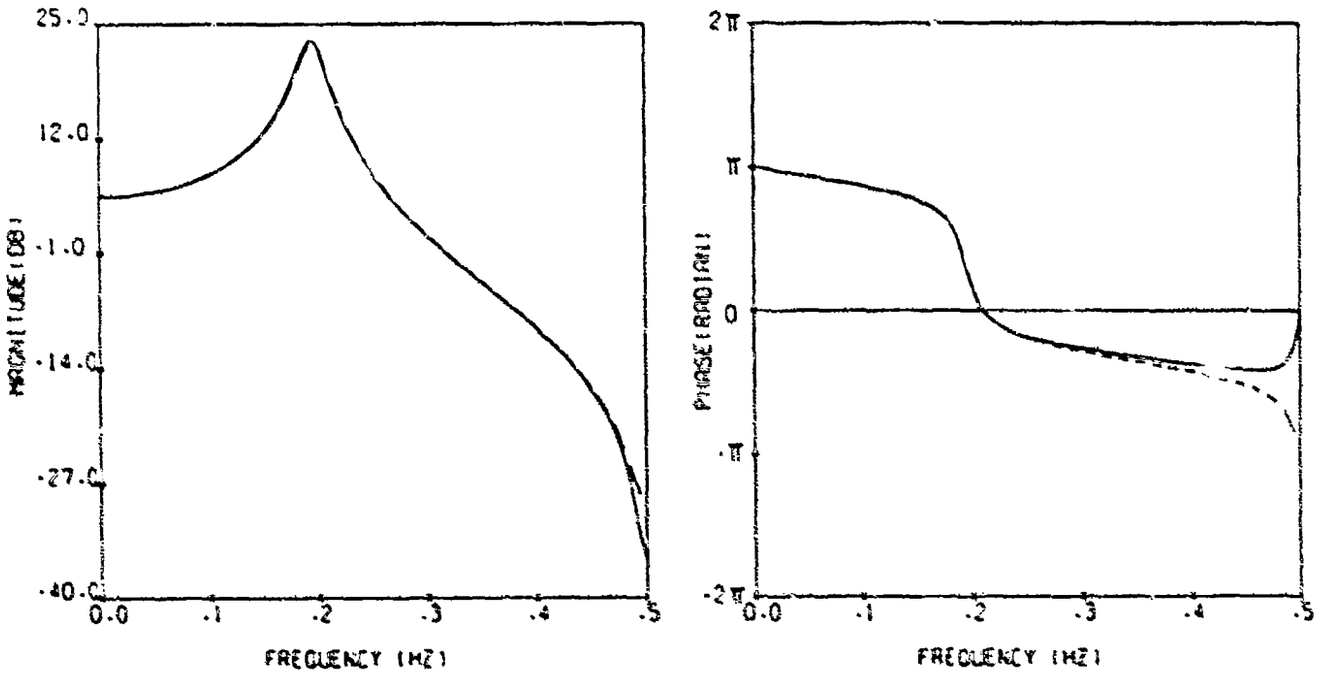


Figure 6.17 ANAC with no observation noise and short data length



(a) Open-loop identification, (Section I, Figure 6.17)



(b) Closed-loop identification, (Section III, Figure 6.17)

Figure 6.18 SISO plant identified from 25 data points

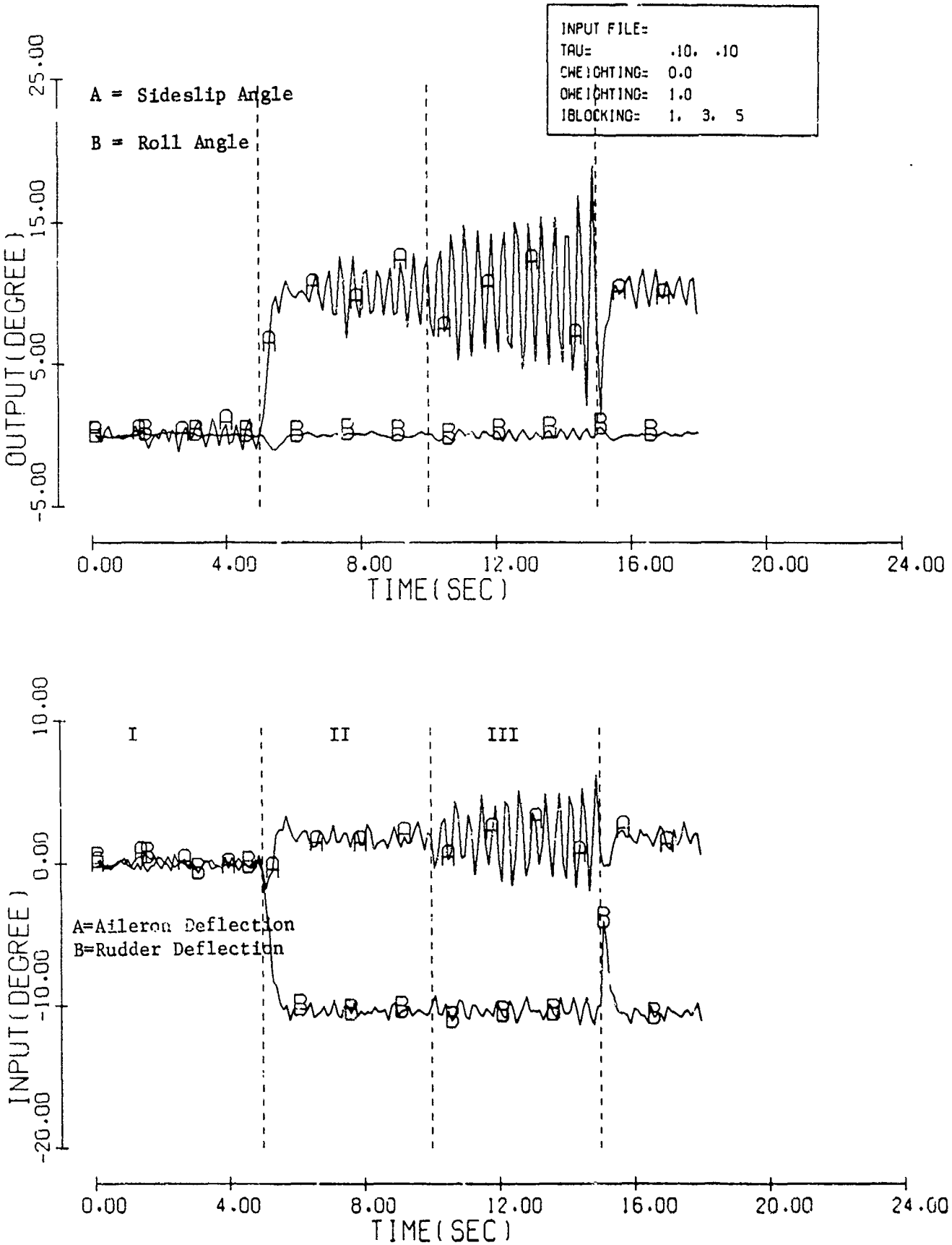
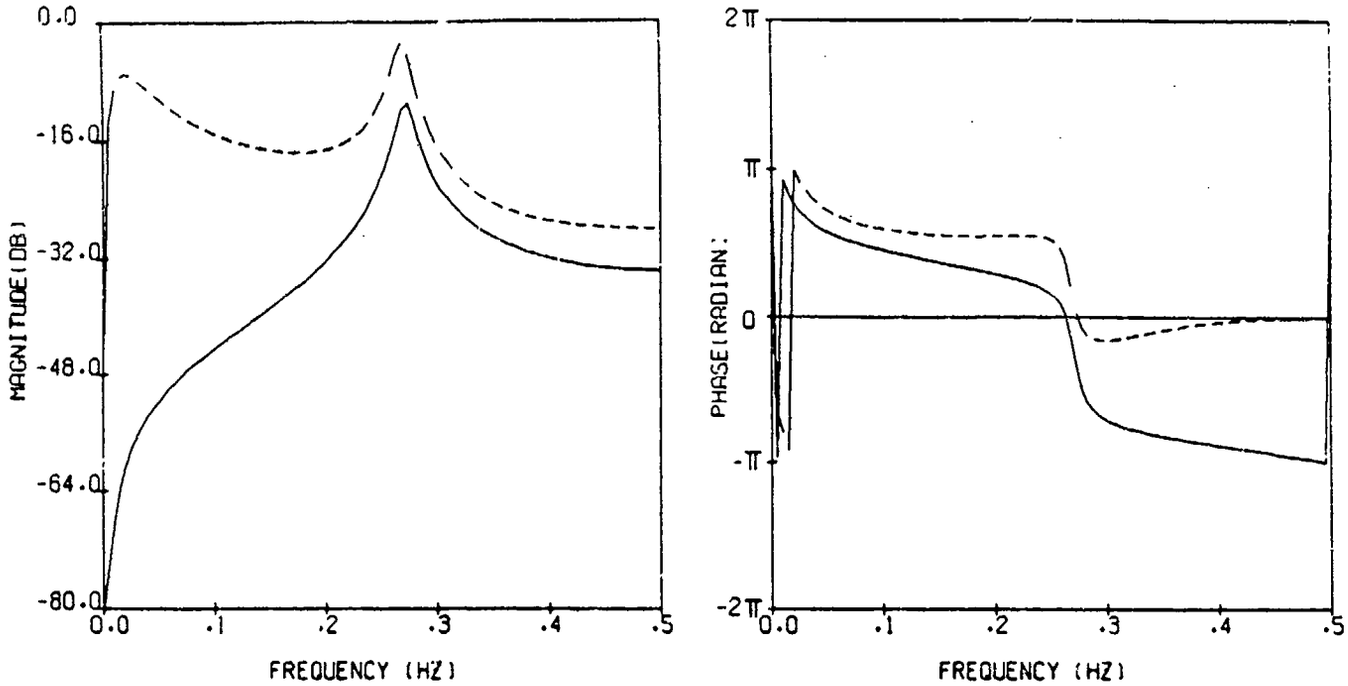
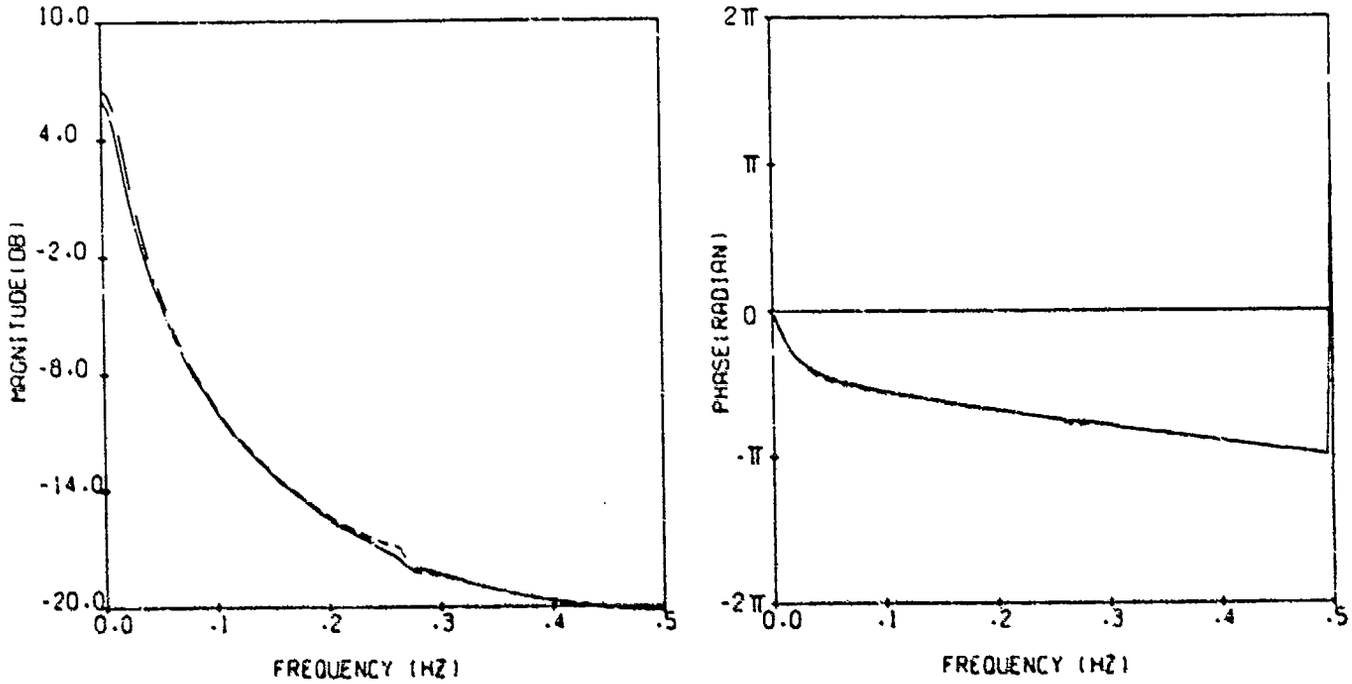


Figure 6.19 AMAC applied to MIMO plant (no measurement noise) and shorter data length

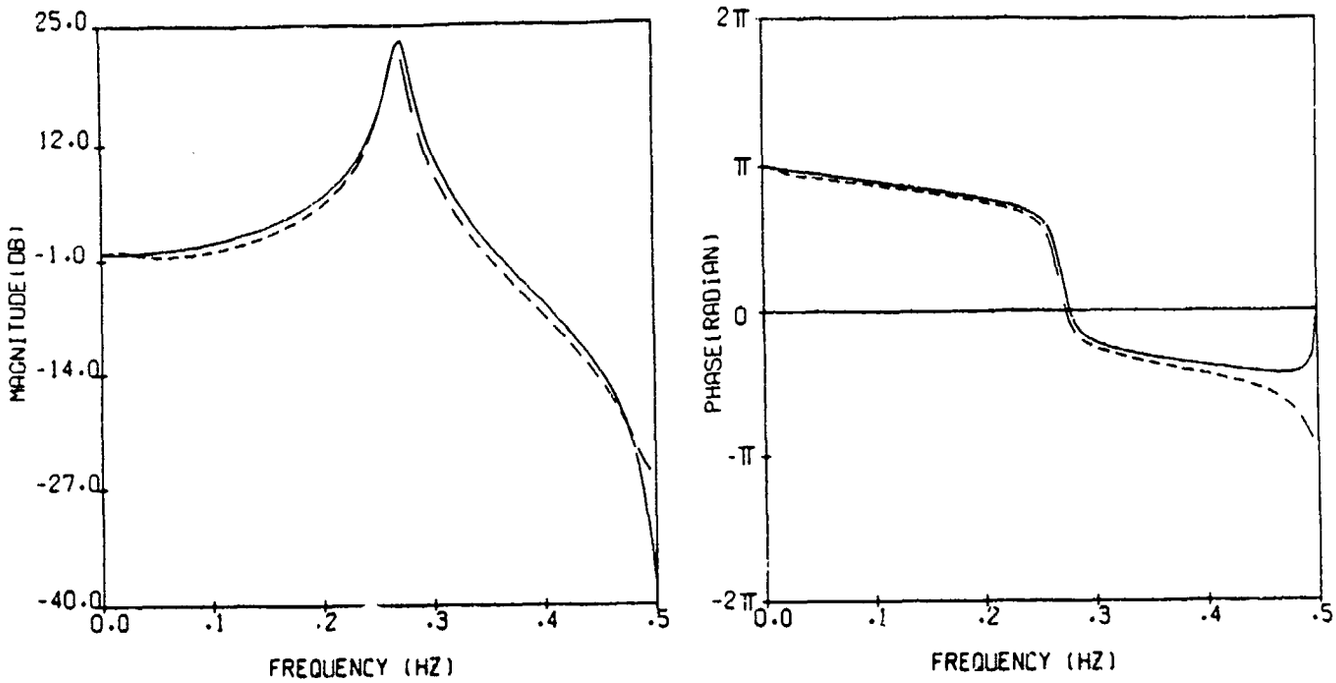


(a) (1,1) element

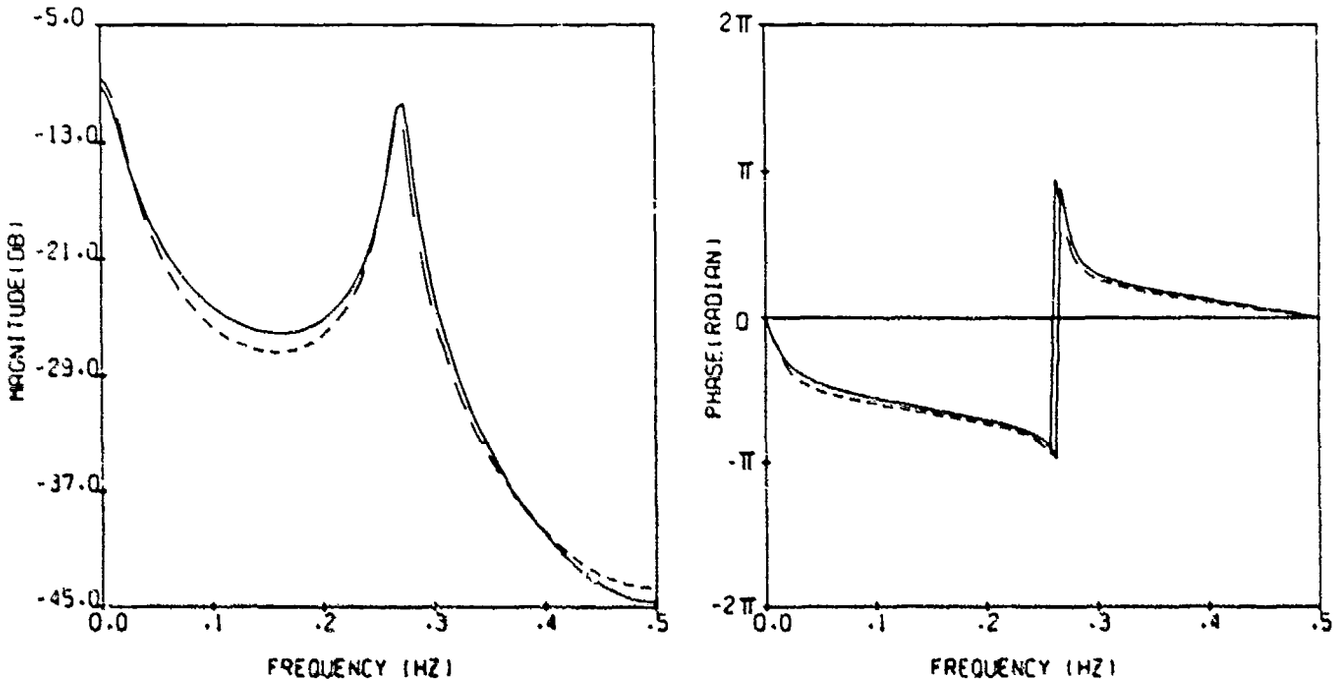


(b) (1,2) element

Figure 6.20 Closed loop identification of MIMO plant, 50 data points and no measurement noise (Section III, Figure 6.19)



(c) (2,1) element



(d) (2,2) element

Figure 6.20 (Continued)

——— Actual plant

- - - - - identified plant

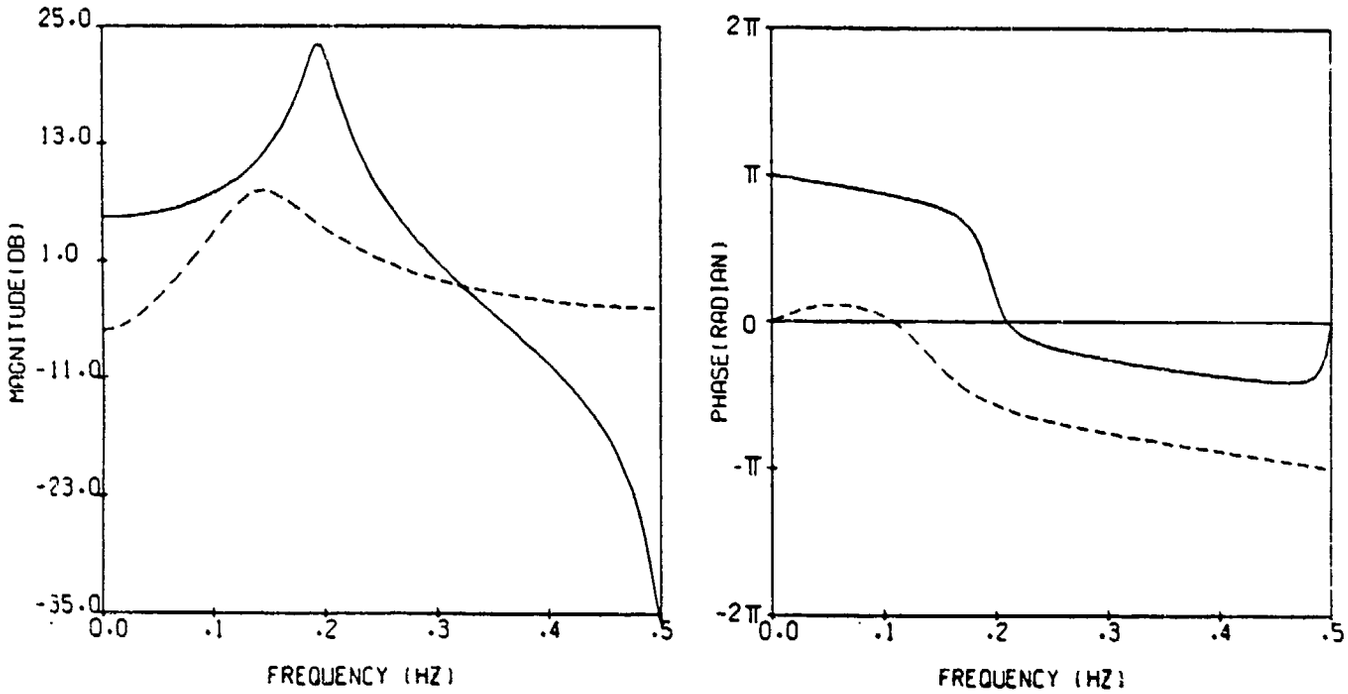


Figure 6.21 Transfer Function of Wind Gust Model (---) and plant model (——).

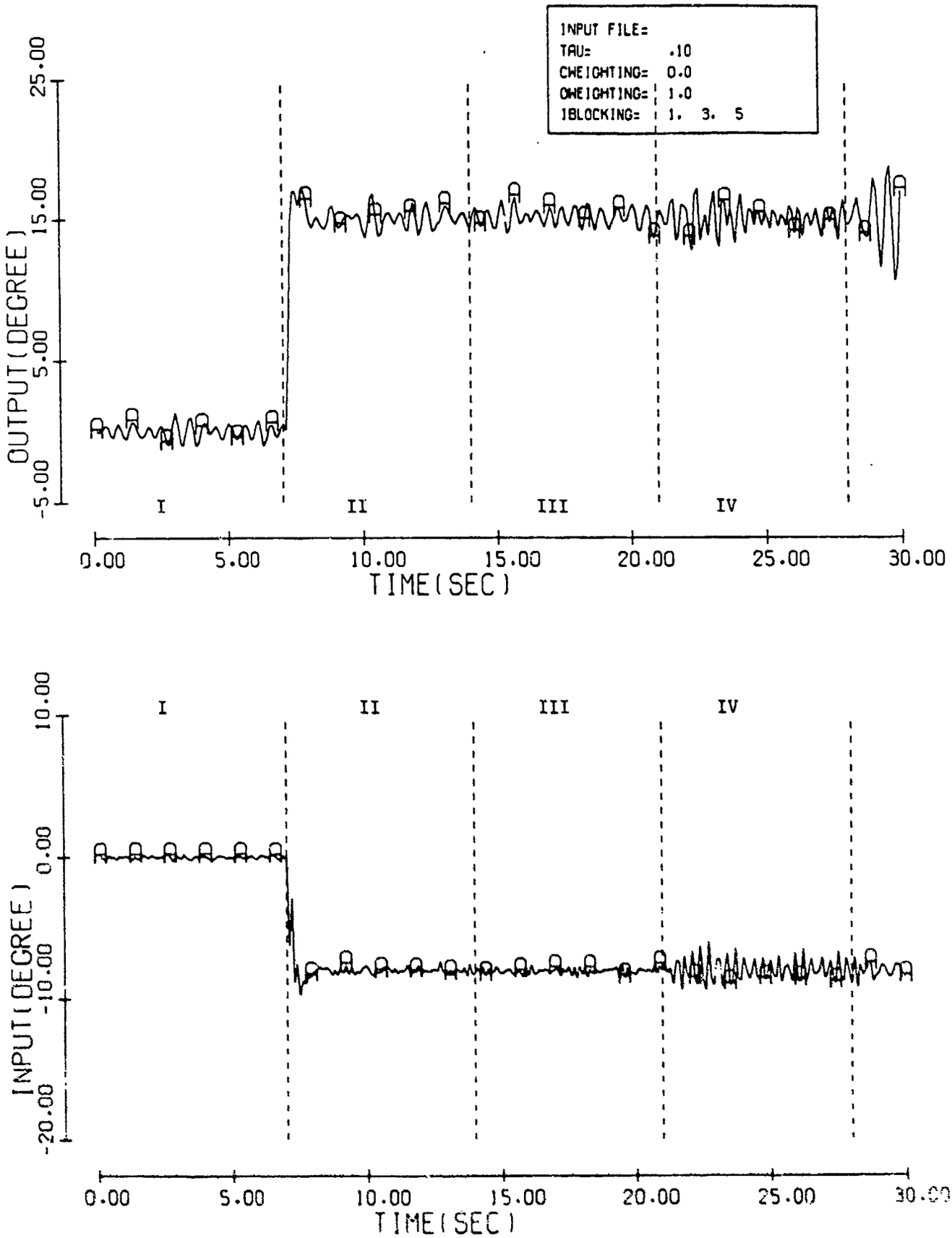
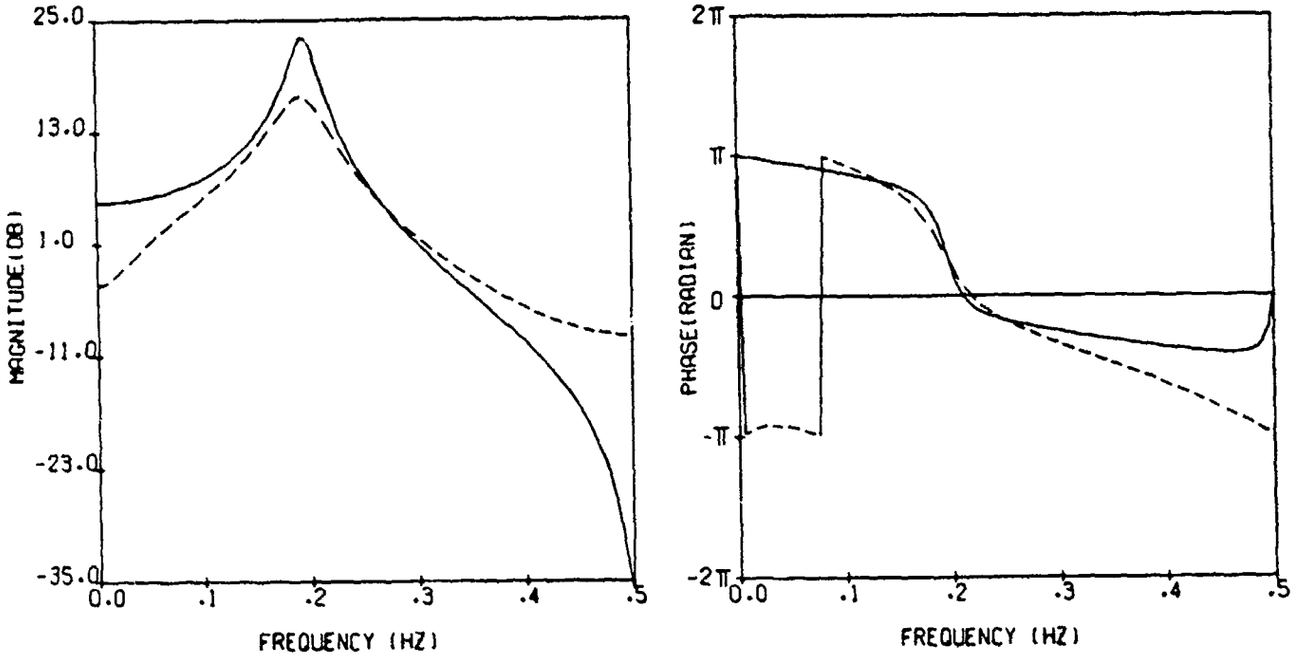
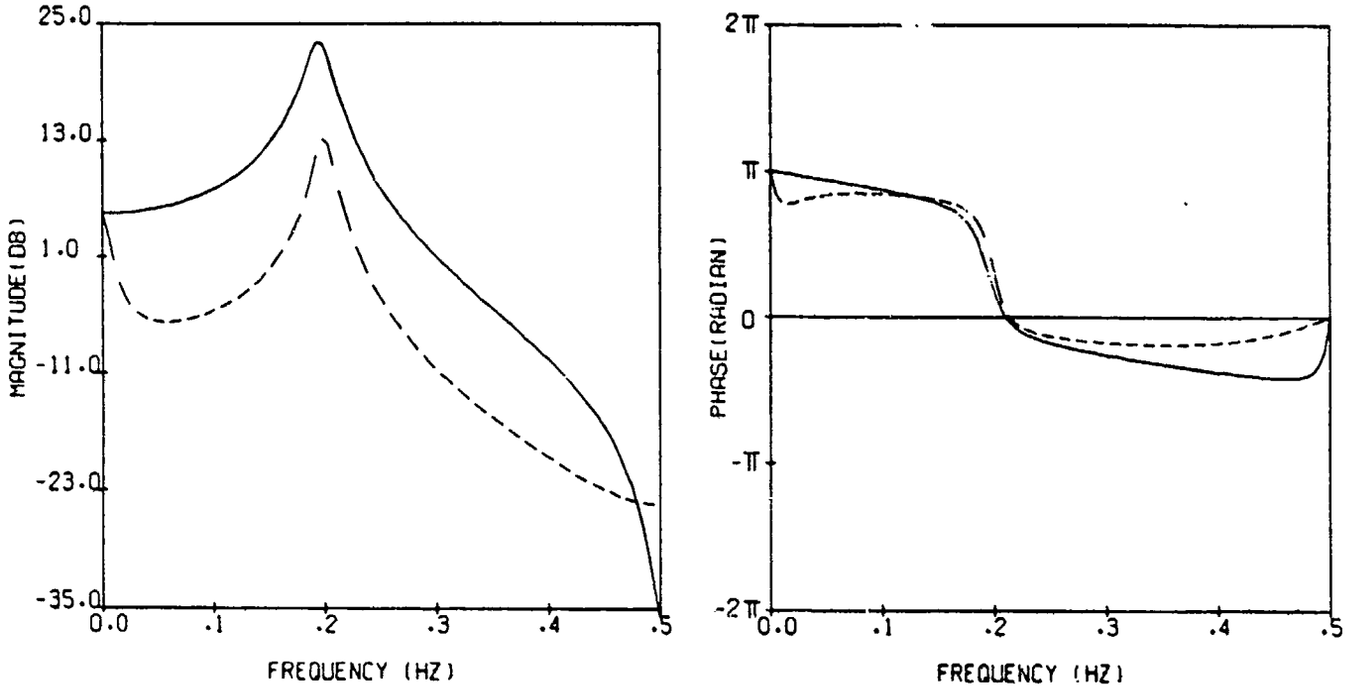


Figure 6.22 AMAC Applied With Gust Input Excitation.



(a) Open Loop Identification from Segment I.

Figure 6.23 Actual Plant vs. Identified Plant for Run of Figure 6.22.



(b) Closed Loop Identification from Segment III.

Figure 6.23 (Continued)

CHAPTER 7

CONCLUSIONS

The overall conclusion of this study is that MAC control design technique can be used in situations where the plant model is not know exactly and/or slowly time varying by incorporating a suitable on-line parameter estimation technique in the existing MAC software. Many of the available techniques for system identification suffer from the fact that these can not identify the system in a closed-loop configuration. But the one developed in Part 1 of this report is based on canonical variate analysis and has the same performances in both open-loop and closed-loop configurations. The robustness analysis in Part 2 gives the neighborhood of stability around the identified model provided that the nominal MAC loop is stable for the identified plant. Thus combining the results of Parts 1 and 2, adaptive MAC provides an analytically sound and very useful control design technique in an uncertain environment such as in the missile attitude control problem in different flight conditions where the plant model drifts from one flight condition to another. The problem of under sampling and over sampling can be avoided by using the optimum selection technique developed in Part 2.

Specific conclusions of this study are:

(i) MAC software uses impulse response description of the plant and therefore cannot be used if the plant is unstable to start with. On the other hand if the plant is lightly damped, the impulse response sequence contains a large number of terms and computational requirements become large. For these systems, it is recommended that the plant be made stable and/or damping be added to the dynamics of the plant apriori by using constant gain output feedback and then MAC be applied to the overall compensated plant. However if the overall dynamics are made very fast using high gain, the sampling rate must be high too in order to satisfy Nyquist's sampling criteria. However if

the plant model is not known exactly, gain should not be made arbitrarily high because if the unmodelled dynamics have non-minimum phase zeros, the overall combination will again be unstable.

(ii) The standard frequency domain robustness analysis can also be applied to a one-step-ahead MAC control law and thus the MAC robustness can be compared to that of other conventional control design techniques under similiar situations. The robustness results obtained in this report for SISO plants can be extended to MIMO plants if the magnitude function is replaced by the operator norm of the transfer function. Every nominally stable design guarantees the stability of a class of plants in the neighborhood of the nominal one, and the boundary of this neighborhood has been identified in Part 2 of this report. It is recommended that, before applying MAC to any real world situation, the region of guaranteed stability be calculated and, if unsatisfactory, enlarge by slowing down the trajectory time constants and/or other parameters.

(iii) Any conventional on-line parameter identification technique can be embedded in the existing MAC software to generate the internal model of the plant and the resulting control technique in an "Adaptive MAC". It is recommended that the identification technique based on canonical variate analysis developed in Part 1 of this report be used for identifying and updating the system parameters. The advantage of this technique is that it can identify the plant equally well in open-loop and closed-loop configuration and it can give the simultaneous confidence band on the transfer function for all frequencies. In this technique the parameters are updated intermittantly whereas in other conventional techniques this is done in every step. Although the computational requirement is comparatively higher in this technique, the quality of the estimate and computational reliability of the solution justifies this additional burden.

(iv) Sampling interval is an important parameter in the MAC design process. Usually sampling rate is selected satisfying the constraints of Nyquist rate, but yet the designer is confronted with a

choice from infinitely many rates satisfying this constraint. The optimum (possibly unique) sampling rate selection technique developed in this report relieves the designer form this problem. The use of this technique is not limited to MAC control design only - it can be used in any situation where a sampling rate is to be selected.

(v) The simulation results in Chapter 6 demonstrate that the use of MAC and a suitable system identification method such as CVA or Maximum Likelihood provide a reliable adaptive control method if there is sufficient input excitation or data length. The AMAC procedure is demonstrated on multiinput multioutput systems in closed loop operation under MAC feedback control. The accuracy of the parameter identification is shown to be the same in either open or closed loop operation as is predicted by theory. The selection of state order using the AIC procedure in the CVA method is shown to give accurate model selection in the cases where state order is unknown. The accuracy of the identified plant can be increased by increasing the data length or the input excitation amplitude. The robustness of MAC can accomodate a moderate uncertainty in the identified plant, but for too large an error the closed loop system may become unstable.

(vi) The results of this study suggest a number of fruitful areas for future research. The MAC approach uses the impulse response representation of the plant dynamics which has the difficulty of being unbounded for unstable systems and very long for very lightly damped systems. Constant gain feedback is used in this study to obtain a closed loop system that is well damped. A more direct approach is that of Model Predictive Control (MPC) using a state space representation of the system. Such a representation is in fact the natural representation given in the CVA identification. The CVA procedure can be easily extended to nonlinear systems of polynomial form. This would greatly widen the areas of application of the AMAC. Another area for research is the use of the confidence intervals on the identified transfer function and the robustness bounds on the MAC controller to determine the required sample size or input excitation to maintain stable closed loop operation.

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APPENDIX A

A UNIFIED VIEW OF REDUCED RANK MULTIVARIATE PREDICTION USING A GENERALIZED SINGULAR VALUE DECOMPOSITION

By Wallace E. Larimore

A Unified View of Reduced Rank Multivariate Prediction Using a Generalized Singular Value Decomposition

By Wallace E. Larimore

Scientific Systems Inc., Cambridge, Massachusetts, U.S.A.

SUMMARY

A generalized reduced rank prediction problem, which is a generalization of a number of multivariate analyses including the classical canonical correlation analysis, is formulated as an explicit prediction problem: given two sets of random variables and an integer p, find p linear combinations of the first set which best predict the second set as measured in terms of a specified quadratic form in the prediction error. Use of a generalization of the singular value decomposition reduces this problem to a simple form with an explicit geometric interpretation, includes the case of singular covariance matrices, is the preferred numerical procedure for actual computation, and gives a complete characterization of nonuniqueness in the case of multiple solutions. The optimal solution is shown to be a formal application of classical canonical correlation analysis to a "pseudo" covariance matrix. Special cases include the classical canonical correlation analysis, the standard as well as a generalized principal component analysis, the optimal selection of instrumental variables, and reduced rank regression.

Partial support for the research was provided by Air Force Wright Aeronautical Laboratories under contract F33615-82-C-3600, Basic Research on Adaptive Model Algorithmic Control.

AMS 1980 subject classification. Primary 62H20, secondary 62H25.

Key words and phrases. Canonical variables, Canonical correlations, Instrumental variables, Linear predictors, Quadratic prediction error, Reduced rank regression, Singular covariance matrix, Singular value decomposition, Principal component analysis.

Submitted to *Annals of Statistics*

1. Introduction

In recent years there has been considerable interest in unifying concepts in multivariate analysis and research into a number of generalizations (Izenman, 1975; Muller, 1982; Rao, 1979). The approach taken in this paper is to formulate a single generalized prediction problem which includes a number of multivariate analysis procedures such as principal component, canonical correlation, reduced and full rank regression, instrumental variables, as well as some generalizations of these. Some of these multivariate analysis procedures are not traditionally formulated or considered as prediction problems, and this extends the range of useful applications for these methods (Yohai and Garcia Ben, 1980). The prediction problem is very naturally considered as a generalized canonical variate analysis.

A primary objective of this paper is to give a complete characterization of the solutions of the generalized prediction problem in the cases of multiple solutions and/or singular covariance matrices. Such multiple solutions may arise in the reduced rank case with repeated singular values or, in terms of the traditional formulation, with repeated generalized eigenvalues. Multiple solutions have received little attention and seem not to have been characterized from the geometric point of view in terms of subspaces as is given in this paper. The singular case has also received little attention. This is probably due to the rather considerable complexity in the derivation and description of procedures such as canonical correlation analysis.

The classical approach to reduced rank or rank constrained problems such as principal component and canonical variate analyses has been the use of canonical representations which are obtained by the solution of related generalized eigenvalue-vector problems (Hotelling, 1936). The canonical variables have a particularly simple covariance structure although the means of obtaining them are often quite complicated involving the solution

of a constrained maximization problem by differentiation leading to the generalized eigenproblem. Most treatments do not prove that the conditions sufficient for the existence of such a maximum are satisfied as noted by Stuart (1982). Rarely is there any discussion of the multiplicity of solutions, an exception being Yohai and Garcia Ben (1980). The case of singular covariance matrices is not included in these approaches and has received very little attention in the literature (see Khatri, 1976).

In recent years, the simple structure of the covariance matrix of the canonical variables has been expressed in terms of a singular value decomposition (SVD) of appropriate quantities depending upon the particular problem such as principal components or canonical variates. In a few discussions, the derivations were considerably simplified by the use of the singular value decomposition as compared with the classical eigenproblem (Good, 1969; Mandel, 1982; Rao, 1979; Stuart, 1982). While this greatly simplifies the derivation and interpretation, a unified treatment of the various reduced rank problems is not available.

The approach of this paper using a generalization of the singular value decomposition includes simply the cases of multiple solutions and singular covariance matrices. This approach involves concepts and methods from the singular value decomposition which in recent years has become a standard tool of linear algebra for the investigation of reduced rank and illconditioned problems from both an analytical as well as a computational point of view (Golub, 1969; Lawson and Hanson, 1974;). This approach focuses immediately upon the central algebraic and geometric properties of the problem and gives the generalized canonical variables directly. The generalized singular value decomposition reduces the optimal prediction problem to a simple form which is directly and easily solved using elementary properties of orthonormal matrices. This avoids the need to solve a constrained maximization problem by differentiation using Lagrange multipliers which is the

traditional approach to canonical variate analysis.

The generalized singular value decomposition provides a unification on several levels. A single mathematical framework using the generalized singular value decomposition solves a single generalized problem that can be specialized to the various reduced rank prediction problems. This unified treatment gives the complete multiplicity of solutions for cases with repeated singular values and simultaneously includes the case of a singular covariance matrix largely missing in the literature. Also there is unification using the generalized singular value decomposition in the derivation of the proof, the mathematical statement of the results, the geometric interpretation of the prediction problem and its solution, and the computation of the solution using modern numerical methods that are numerically accurate and stable. This gives a considerable unification of the teaching, understanding, interpretation, and application of these methods. The diversity and complexity of the present literature makes the learning and understanding of such methods as canonical correlation analysis difficult for many potential users, and is considered by some to be largely responsible for its relative neglect in applications.

2. A Generalized Prediction Problem

Consider two sets of zero mean random variables $X^T = (x_1, \ldots, x_m)^T$ and $Y^T = (y_1, \ldots, y_n)^T$ with a joint covariance matrix of $(X^T, Y^T)^T$ given by

$$\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \tag{1.2}$$

where Σ_{xx} and Σ_{yy} are possibly singular. In this paper, the following constrained prediction problem is considered: for given p, find a p-dimensional vector $Z = H_p X$ of linear combinations of X such that the optimal prediction $\hat{Y}_z$ of Y based upon Z minimizes the

general quadratic prediction error measure

$$\| Y-\hat{Y}_z \|^2_{\Lambda^\dagger} = E\{(Y-\hat{Y}_z)^T \Lambda^\dagger (Y-\hat{Y}_z)\} \tag{2.2}$$

where Λ is an arbitrary nonnegative definite symmetric matrix of rank $\bar{n}$, and $\dagger$ denotes the pseudoinverse, i.e. the inverse of the full rank part of Λ. Let L satisfy $L\Lambda L^T = I$ where L is full rank with dimension $\bar{n}\times n$, then it will be convenient to express $\Lambda^\dagger = L^T L$. From the eigenvector decomposition of a matrix, the rows of L span the same subspace as the eigenvectors of Λ with nonzero eigenvalues. Such an L will occur naturally in the generalized singular value decomposition. Although the use of the inverse or pseudoinverse in the definition of the prediction problem may appear awkward, it will lead to considerable simplicity in formulating the mathematical problem to be solved and in the geometrical and statistical interpretation of the resulting solution. The prediction problem (2.2) is considered in the case that Λ is full rank by Izenman (1975) and Rao (1979). Larimore (1983) extends the prediction problem to the case of time series analysis of Markov processes of constrained Markov (state) order p.

In the paper, the geometrical interpretation will play an important part. A linear vector space V of random variables generated by a set S of random variables is defined as the set V of all random variables that are linear combinations of S. In the sequel, several inner products $<u,v>_\Gamma = Eu^T\Gamma v$ for $u,v \in V$ will be defined for various positive semidefinite symmetric matrices Γ. Two random variables u and v are *orthogonal* with respect to the inner product $<\cdot,\cdot>_\Gamma$ if $<u,v>_\Gamma = 0$, and a set of random variables $u_1,\ldots,u_n$ are orthonormal if they are orthogonal and in addition $<u_i,u_i>_\Gamma = 1$. Then all of the usual properties of inner product spaces apply to such a space of random variables such as subspace, rank of a subspace, and linear independence of vectors.

Consider the case where Σ_{xx} is full rank. Then for an arbitrary p-vector $Z = H_p X$ of linearly independent combinations of X where $p \le m$ so H_p is rank p, the optimal estimate $\hat{Y}_z$ of Y given Z is

$$\hat{Y}_z = \Sigma_{yz} \Sigma_{zz}^{-1} Z = \Sigma_{yx} H_p^T (H_p \Sigma_{xx} H_p^T)^{-1} Z \tag{2.3}$$

and the prediction error is

$$\| Y - \hat{Y}_z \|_{\Lambda^\dagger}^2 = tr \Lambda^\dagger \Sigma_{yy} - tr \Lambda^\dagger \Sigma_{yx} H_p^T (H_p \Sigma_{xx} H_p^T)^{-1} H_p \Sigma_{xy} \tag{2.4}$$

Now H_p does not uniquely specify Z in terms of estimating Y since from inspection of (2.3) any nonsingular transformation of Z will leave $\hat{Y}_z$ invariant. An orthonormalization of Z will give an equivalent $Z = J_p X$ with

$$\Sigma_{zz} = J_p \Sigma_{xx} J_p^T = I_p \tag{2.5}$$

where I_p is the $p \times p$ identity and where the last equality is satisfied if $Rank(\Sigma_{xx}) \ge p$.

In the singular case where $Rank(\Sigma_{xx}) < p$, then by an orthonormalization of Σ_{zz} a new set of random variables $\bar{Z} = AZ = AJ_p X = \bar{J}_{\bar{p}} X$ of lower dimension $\bar{p}$ can be chosen with a full rank covariance matrix equal to the identity. For this new orthonormalized set of random variables, dropping the bar notation we have precisely (2.5). Note that by replacing H_p by J_p, the inverses in (2.3) and (2.4) are then also well defined. We may thus in any case introduce the constraint (2.5) on J_p without loss of generality. The optimum prediction problem (2.2) can thus be stated mathematically as choosing a J_p to minimize

$$\| Y - \hat{Y}_z \|_{\Lambda^\dagger}^2 = tr \Lambda^\dagger \Sigma_{yy} - tr \Lambda^\dagger \Sigma_{yx} J_p^T J_p \Sigma_{xy} \tag{2.6}$$

subject to the constraint

$$J_p \Sigma_{xx} J_p^T = I_p \tag{2.7}$$

This problem is of great practical interest. The classical canonical correlations and variates analysis will be shown to be equivalent to minimizing (2.2) with $\Lambda = \Sigma_{yy}$. The principal component analysis problem is equivalent to $Y = X$ so $\Sigma_{xx} = \Sigma_{yy} = \Sigma_{xy}$ and in addition setting $\Lambda = I$. More general weightings are afforded by other choices of Λ which can reflect a cost of prediction error of practical value such as dollars or a second order approximation to a nonlinear cost function. The particular weighting Λ used in a given problem can make a considerable difference in the solution, which suggests that the classical canonical correlation analysis in some cases does not give the most appropriate choice of Λ . The generalized canonical variate analysis provides a unified framework for canonical correlation analysis and principal component analysis as well as more general prediction problems.

3. A GENERALIZED SINGULAR VALUE DECOMPOSITION

A very intuitive approach to finding the canonical decomposition is through one particular generalization of the singular value decomposition. The usual singular value decomposition is given by the following (Lawson and Hanson, p. 20-1, 1974).

Theorem 1. If A is a real $m \times n$ matrix of rank r, then there exist orthonormal matrices $B(m \times m)$ and $C(n \times n)$ such that

$$B^T A C = Diag(d_1 \geq \cdots \geq d_r > 0,\ldots,0)\ ,\ B^T B = I_m\ ,\ C^T C = I_n \tag{3.1}$$

where Diag denotes a $m \times n$ diagonal matrix with nonnegative elements in decending order.

In generalizing this, let P be a nonnegative definite symmetric matrix of rank r. Then we define a matrix J to be *P–orthonormal* (rowwise) when $JPJ^T = I_r$. Note that this definition includes the requirement that the dimension of J is $RankP \times DimP$ with J full rank. This can all be conveniently stated simply as $JPJ^T = I_{RankP}$ so that the dimension and ranks of P and J do not have to be explicitly stated. Also throughout the paper, $D = Diag(d_{11}, \cdots, d_{ii}, \cdots)$ will denote a general rectangular matrix with all elements zero except for elements d_{ii} on the main diagonal. Then the *(R,S)–singular value decomposition* is given by the following theorem.

Theorem 2. Let R and S be nonnegative definite symmetric matrices of order m and n and ranks $\bar{m}$ and $\bar{n}$ respectively, and let A be a $m \times n$ matrix. Then there exist transformations J and L such that

$$JAL^T = D = Diag(\gamma_1 \geq \ldots \geq \gamma_r > 0, \ldots, 0) \ , \ JRJ^T = I_{RankR} \ , \ LSL^T = I_{RankS} \qquad (3.2)$$

Thus the transformations J and L are R– and S-orthonormal respectively and in addition satisfy the following:

(i) For distinct singular values γ_i's, the row vectors of J and L are unique except for a sign change.

(ii) For repeated singular values γ_i's, the rows of J corresponding to a given repeated singular value must span a fixed subspace, and similarly for L.

(iii) Any transformations J and L satisfying the decomposition (3.2) are related to a particular solution J_*, L_* in terms of a block diagonal orthonormal matrix of the form $J = Diag(P_1, \ldots, P_h, P_u)J_*$, $L = Diag(P_1, \ldots, P_h, P_v)L_*$ where the blocks P_j are arbitrary orthonormal matrices which for $j \leq h$ have dimension $k_j \times k_j$ corresponding to the j-th nonzero value of γ that repeats k_j times and where P_u and P_v are orthonormal matrices of dimension $Rank(R) - r$ and $Rank(S) - r$ respectively. Thus for any J and L, the rows

corresponding to the same singular value are orthonormal linear combinations of the corresponding rows of J_* and L_* .

Proof: Existence: Let B and C be any R– and S–orthonormal matrices respectively so that $BRB^T = I_{RankR}$ and $CSC^T = I_{RankS}$. Now consider the singular value decomposition of Theorem 1 applied to BAC^T , so $\bar{B}^T BAC^T \bar{C} = D$ with $\bar{B}^T \bar{B} = I = \bar{C}^T \bar{C}$. Then $J = \bar{B}^T B$ and $L = \bar{C}^T C$ satisfy (3.2).

Uniqueness: To determine all solutions, let $\bar{J}$, $\bar{L}$, and $\bar{D}$ be another solution satisfying (3.2). Then $\bar{J}^T R\bar{J} = I_{RankR} = J^T RJ^T$ implies that the row vectors of J and $\bar{J}$ span the full rank subspace of R . Thus there exists a nonsingular matrix F such that $\bar{J} = FJ$ and similarly there exists a nonsingular matrix G with $\bar{L} = GL$. From the decomposition (3.2), $I_{RankR} = \bar{J}R\bar{J}^T = FJRJ^T F^T = FF^T$ and similarly $GG^T = I_{RankS}$ so that F and G are orthonormal matrices. Also $\bar{D}\bar{D}^T = FDD^T F^T$, and from the uniqueness of the eigenvalues and eigenvectors of a symmetric matrix it follows that $DD^T = \bar{D}\bar{D}^T$ so $D = \bar{D}$ and that F is block diagonal with blocks corresponding to the repeated singular values. A similar result holds for G by considering $\bar{D}^T \bar{D}$. Now $D = \bar{D} = FDG^T$, so using the block diagonal forms of D, F, and G with diagonal blocks D_i, F_i, and G_i respectively, we have for every block i with $\gamma_i = 0$ that $\gamma_i I = F_i \gamma_i I G_i^T$ so $F_i G_i^T = I = F_i F_i^T$ which implies $F_i = G_i$ since they are both square matrices which proves the Theorem.

One generalization of the singular value decomposition proposed by Van Loan (1976) is somewhat different defining P-orthonormality column wise and using the inverse of the transformation J so that the decomposition satisfies the following: $J^T RJ = I_{RankR}$, $L^T SL = I_{RankS}$, $J^{-T} AL = Diag(\gamma_1 > ... > \gamma_r, 0, \ldots, 0)$. If we make the identification $\bar{J} = J^{-T}$, $\bar{L} = L^T$, and $\bar{R} = R^{-1}$, then $\bar{J}, \bar{L}, \bar{R}, S$ and D satisfy the generalized singular value decomposition (3.2). From a statistical point of view, this decomposition is much

more intuitive as is seen in the next section, and treatment of the case of singular covariance matrices would be considerably more involved using the Van Loan decomposition.

4. GENERALIZED CANONICAL VARIATE ANALYSIS

Now consider the (Σ_{xx},Λ)-singular value decomposition of Σ_{xy} given as

$$J\Sigma_{xy}L^T = D = Diag(\gamma_1,\ldots,\gamma_r,0,\ldots,0) \ , \ J\Sigma_{xx}J^T = I_{\bar{m}} \ , \ L\Lambda L^T = I_{\bar{n}} \tag{4.1}$$

where $\bar{m} = Rank(\Sigma_{xx})$ and $\bar{n} = Rank(\Lambda)$. This decomposition has the very intuitive interpretation of a new basis defined by the *generalized canonical variables* or *variates*

$$U = JX \ , \ V = LY \tag{4.2}$$

of dimensions $\bar{m}$ and $\bar{n}$ respectively for which:

(i) $\Sigma_{uu} = I$, so that the components of U are uncorrelated with variance unity.

(ii) $\Sigma_{uv} = Diag(\gamma_1,\ldots,\gamma_r,0,\ldots,0)$, so that the components of U and V are uncorrelated except for the i-th pairs with $cov(u_i v_i) = \gamma_i$. The γ_i's will be called *canonical covariances*.

(iii) the norm of the prediction error

$$\| Y-\hat{Y}_z \|^2_{\Lambda'} = E\{(Y-\hat{Y}_z)^T L^T L(Y-\hat{Y}_z)\} = E\{(V-\hat{V}_z)^T(V-\hat{V}_z)\} = \| V-\hat{V}_z \|^2_I \tag{4.3}$$

is a sum of squares in $V-\hat{V}_z$ where $\hat{V}_z = L\hat{Y}_z$, and the inner product induced by the transformation L is $<Y_1,Y_2>_\Lambda = E\{Y_1^T L^T L Y_2\} = <V_1,V_2>_I$, the inner product with respect to the identity.

(iv) the projection of the prediction error $Y-\hat{Y}_z$ on the full rank subspace of Λ, i.e. where the prediction error has nonzero weighting, is

$$\Lambda\Lambda^{\dagger}(Y-\hat{Y}_z) = \Lambda L^T L(Y-\hat{Y}_z) = \Lambda L^T(V-\hat{V}_z) \tag{4.4}$$

which gives the inverse transformation from V onto the full rank subspace of Λ.

As in the discussion following (2.5), U contains the part of X involving the full rank part of Σ_{xx}. Thus without loss of generality X may be expressed as $X = KU$. In terms of the canonical variables U , the p linear combinations Z are $Z = J_p X = J_p KU = M_p U$ where we define the $p \times \bar{m}$ matrix $M_p = J_p K$. Using the constraint (2.5) gives the equivalent constraint

$$M_p M_p^T = M_p \Sigma_{uu} M_p^T = \Sigma_{zz} = I_p \tag{4.5}$$

so that M_p has orthonormal rows. Furthermore since $Z = J_p X = M_p U = M_p JX$, we can substitute into (2.6) the relationships $J_p = M_p J$ and $\Lambda^{\dagger} = L^T L$. Use of the generalized singular value decomposition (4.1) then gives the simple expression

$$\| Y-\hat{Y}_z \|^2_{\Lambda^{\dagger}} = \| V-\hat{V}_z \|^2_I = tr\,\Lambda^{\dagger}\Sigma_{yy} - trM_p DD^T M_p^T \tag{4.6}$$

Thus the generalized singular value decomposition (2.3) reduces the original problem of minimizing (2.6) subject to the constraint (2.7) to the problem of finding a $p \times \bar{m}$ matrix M_p with orthonormal rows maximizing $trM_p DD^T M_p^T$ with $D = Diag(\gamma_1 \geq \cdots \geq \gamma_r > 0, \cdots, 0)$. To solve this maximization problem requires only the elementary properties of orthonormal matrices as stated in the following Lemma.

Lemma 1: Let the integer $p \leq \bar{m}$ be fixed, let m_i be the columns of the $p \times \bar{m}$ matrix M_p with orthonormal rows, and suppose the $\bar{m} \times \bar{m}$ diagonal matrix D is $D = Diag(\gamma_1, \ldots, \gamma_q > \gamma_{q+1} = \cdots = \gamma_p = \cdots = \gamma_{q+k} > \gamma_{q+k+1}, \cdots)$ for k repeated values equal to γ_p, so for γ_p unique we have $q+1 = p = q+k+1$. Let $M^q = [m_1, \ldots, m_q]$, $M^k = [m_{q+1}, \ldots, m_{q+k}]$, $M^{\bar{m}} = [m_{q+k+1}, \ldots, m_{\bar{m}}]$. Then $TrM_p DD^T M_p^T$ is a maximum if and

only if all of the following hold:

$$M^{qT}M^{q} = I \quad , \quad M^{qT}M^{k} = 0 \quad , \quad M^{\bar{m}} = 0 \tag{4.7}$$

Proof: By Gram-Schmidt orthonormalization, the $\bar{m}\times p$ matrix M_p^T may be extended to a square $\bar{m}\times\bar{m}$ matrix $[M_p^T \; N^T]$ with orthonormal columns. Thus

$$\begin{bmatrix} M_p \\ N \end{bmatrix}\begin{bmatrix} M_p^T & N^T \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_{\bar{m}-p} \end{bmatrix} = \begin{bmatrix} M_p^T & N^T \end{bmatrix}\begin{bmatrix} M_p \\ N \end{bmatrix} \tag{4.8}$$

where the second equality follows since a right inverse is also a left inverse. In particular, denoting the i-th column of M_p and N by m_i and n_i respectively, we have $m_i^T m_i + n_i^T n_i = 1$ so that $m_i^T m_i \le 1$. Furthermore

$$p = trI_p = tr(M_p M_p^T) = tr(M_p^T M_p) = \sum_{i=1}^{\bar{m}} m_i^T m_i$$

$$= trM^{qT}M^{q} + trM^{kT}M^{k} + trM^{\bar{m}T}M^{\bar{m}} \tag{4.9}$$

Using $m_i^T m_i \le 1$ implies the inequality

$$tr(M_p^T M_p)DD^T = \sum_{i=1}^{\bar{m}} \gamma_i^2 m_i^T m_i \le \sum_{i=1}^{p} \gamma_i^2 \tag{4.10}$$

By considering $m_i^T m_i = \alpha_i$ as arbitrary positive numbers whose sum is p, it is easily shown that the equality is achieved if and only if $m_i^T m_i = 1$ and $n_i^T n_i = 0$ and $m_i^T m_i = 0$ for $q+k < i \le \bar{m}$.

(Only if). Now if we partition N similar to that of M_p so $N = [0 \;\; N^k \;\; N^{\bar{m}}]$, then when the maximum is achieved we must have from (4.8)

$$\begin{bmatrix} M^{qT} & 0 \\ M^{kT} & N^{kT} \\ 0 & N^{\bar{m}T} \end{bmatrix}\begin{bmatrix} M^{q} & M^{k} & 0 \\ 0 & N^{k} & N^{\bar{m}} \end{bmatrix} = \begin{bmatrix} M^{qT}M^{q} & M^{kT}M^{q} & 0 \\ M^{kT}M^{q} & M^{kT}M^{k}+N^{kT}N^{k} & N^{kT}N^{\bar{m}} \\ 0 & N^{\bar{m}T}N^{k} & N^{\bar{m}T}N^{\bar{m}} \end{bmatrix} = I_{\bar{m}} \tag{4.11}$$

which implies that $M^{qT}M^q = I_q$ and $M^{qT}M^k = 0$.

(If). Suppose that (4.7) is true. Then from $M^{qT}M^q = I_q$, $M^{\bar{m}} = 0$ and (4.9),

$$trM^{kT}M^k = \sum_{i=q+1}^{q+k} m_i^T m_i = p-q \tag{4.12}$$

Then using (4.10),

$$Tr(M_p^T M_p)DD^T = \sum_{i=1}^{\bar{m}} \gamma_i^2 m_i^T m_i = \sum_{i=1}^{q} \gamma_i^2 + \gamma_{q+1} trM^{kT}M^k = \sum_{i=1}^{p} \gamma_i^2 \tag{4.13}$$

so that the maximum is achieved which proves the Lemma.

5. Optimal Prediction via Generalized Canonical Variables

Using the above reduction to canonical variables and previous Lemma, solutions to minimizing the prediction error (2.2) are characterized simply in terms of the generalized canonical variables from the generalized singular value decomposition (4.1). The solution is given by essentially choosing Z as the first p canonical variables, although for repeated singular values it is somewhat more involved in that any $p-q$ dimensional subspace corresponding to the repeated singular value may be chosen. The uniqueness of the generalized singular value decomposition exactly characterizes the nonuniqueness of the canonical variables and the solution to the optimal prediction problem (2.2). This is precisely stated in the following theorem.

Theorem 3: Consider the problem of choosing p linear combinations $Z = H_p X$ of X for predicting Y such that

$$\| Y - \hat{Y}_z \|^2_{\Lambda^1} = \mathcal{E}\{(Y-\hat{Y}_z)^T \Lambda^1 (Y-\hat{Y}_z)\} \tag{5.1}$$

is minimized where Σ_{xx} and Λ are possibly singular positive semidefinite symmetric matrices with ranks $\bar{m}$ and $\bar{n}$ respectively. Then existence and uniqueness are given as follows:

(i) Existence: $Z = H_p X$ is a solution minimizing (5.1) if and only if there exist transformations J and L satisfying the (Σ_{xx}, Λ)-generalized singular value decomposition

$$J\Sigma_{xx}J^T = I_{\bar{m}}\ ,\ L\Lambda L^T = I_{\bar{n}}\ ,\ J\Sigma_{xy}L^T = Diag(\gamma_1 \geq \ldots \geq \gamma_r, 0, \ldots, 0) \tag{5.2}$$

such that

(a) $Z = H_p X$ spans the first p of the canonical predictors $U = JX$, i.e. we have $Z = QU_p = Q[I_p \ \ 0]U = Q[I_p \ \ 0]JX$ for some nonsingular Q . Thus $H_p = Q[I_p \ \ 0]J$, so the rows of H_p are linearly independent linear combinations of the first p rows of J. In addition we have:

(b) the prediction error is reduced in the span of the corresponding first p canonical variables V, and the corresponding subspace of Y is the span of the random variables consisting of linear combinations of Y given by the first p rows of L.

(c) there is no reduction in the prediction error in the span of the last $\bar{n}-p$ variables of V, and the corresponding subspace of Y is the span of the random variables consisting of linear combinations of Y given by the last $\bar{n}-p$ rows of L.

(ii) Uniqueness:

(a) If $\gamma_{p+1} > \gamma_p$ then the solution in (i) is essentially unique, i.e. the subspaces in (a), (b), and (c) are unique and given by any particular representation (5.2).

(b) If $\gamma_p = \gamma_{p+1}$ with k equal singular values $\gamma_{q+1} = \cdots = \gamma_p = \cdots = \gamma_{q+k}$, then the subspaces in (i) are not unique. The subspace span by Z contains the first q canonical predictors $U_q = [I_q \ \ 0]JX$ and in addition contains an arbitrary selection of $p-q$ linear combinations of the canonical variables $u_{q+1}, \cdots, u_{q+k}$. In particular, H_p has the form

$$H_p = Q\begin{bmatrix} I_q & 0 & 0 \\ 0 & C_{p-q} & 0 \end{bmatrix} J \tag{5.3}$$

where C_{p-q} is $(p-q)\times k$ with orthonormal rows so $C_{p-q}C_{p-q}^T = I_{p-q}$ and Q is an arbitrary nonsingular matrix.

(iii) The minimum value is

$$\min_{H_p} \| Y - \hat{Y}_z \|^2_{\Lambda^\dagger} = tr\,\Sigma_{yy}\Lambda^\dagger - \gamma_1^2 - \cdots - \gamma_p^2 \tag{5.4}$$

Proof: (i) Existence - (only if). Suppose $Z = H_p X$ is given which minimizes (5.1), then we seek a generalized singular decomposition satisfying (5.2) and (i)(a). To simplify the derivation we work with the equivalent $Z = J_p X$ as in (2.6) subject to the constraint (2.7). Now consider a fixed decomposition (5.2) with J and L given with corresponding canonical variables U and V. From the discussion following (4.4), there exists a $M_p = J_p K$ satisfying (4.5) and minimizing (4.6). We use M_p to construct $\bar{J}$ and $\bar{L}$ satisfying (5.2) and (i)(a).

As in Lemma 1, suppose that $M_p = [M^q \;\; M^k \;\; 0]$ minimizes the prediction error. From the nonuniqueness of the generalized singular value decomposition, the problem is to select a new basis from among the columns of M^k which is full rank and use this in the construction. This is simply accomplished by considering the generalized singular value decomposition of the matrix M_p with respect to the identity matrices given by $FM_pG^T = D$, $FF^T = I_p$, $GG^T = I_{\bar{m}}$. From the orthogonality of M_p, we have $I_p = FF^T = FM_pM_p^TF^T = DD^T$ so that $D = [I_p \;\; 0]$. Partitioning the various matrices in this singular value decomposition corresponding to the partitioning of $M_p = [M^q \;\; M^k \;\; 0]$ gives $[FM^qG^q \;\; FM^kG^k \;\; 0] = [I_p \;\; 0]$. A reordering of columns of F gives of the generalized singular value decomposition of the $p\times k$ matrix M^k with respect to the identity matrices

as

$$BM^kC^T = \begin{bmatrix} I_{p-q} & 0 \\ 0 & 0 \end{bmatrix} \ , \ BB^T = I_p \ , \ CC^T = I_k \tag{5.5}$$

where $k \geq p-q$. The rank of M^k is thus obviously $p-q$. Let $B_{p-q} = [I_{p-q}\ 0]B$ and $C_{p-q} = [I_{p-q}\ 0]C$ be the first $p-q$ rows of B and C respectively.

Now consider the transformation on the variables Z to $\bar{Z}$

$$\bar{Z} = \begin{bmatrix} M^{qT} \\ B_{p-q} \end{bmatrix} Z \tag{5.6}$$

In the sequel, we will need the property that $B_{p-q}M^q = 0$ which is the case if and only if $M^{qT}B_{p-q}^TC_{p-q} = 0$ since C_{p-q} is full rank. This indeed follows from Lemma 1 since

$$M^{qT}B_{p-q}^TC_{p-q} = M^{qT}B^T\begin{bmatrix} I_{p-q} \\ 0 \end{bmatrix}[I_{p-q}\ 0]C = M^{qT}B^TBM^kC^TC = M^{qT}M^k = 0 \tag{5.7}$$

Thus the matrix of (5.6) is orthonormal, and using $Z = M_pU = M_pJX$ it follows that

$$\bar{Z} = \begin{bmatrix} M^{qT} \\ B \end{bmatrix}[M^q\ \ M^k\ \ 0]U = \begin{bmatrix} I_q & 0 & 0 \\ 0 & [I_{p-q}\ \ 0]BM^kC^TC & 0 \end{bmatrix}U = \begin{bmatrix} I_q & 0 & 0 \\ 0 & C_{p-q} & 0 \end{bmatrix}JX$$

$$= [I_p\ \ 0]\bar{J}X = [I_p\ \ 0]\bar{U} = \bar{U}_p \tag{5.8}$$

where $\bar{J} = Diag[I_q, C\,J_{\bar{m}-k-q}]J$. From Theorem 2 (iii), the transformations $\bar{J}$ and $\bar{L} = Diag[I_q, C\,J_{\bar{m}-k-q}]L$ are just an alternate set of matrices satisfying the generalized singular value decomposition (5.2). Thus (i)(a) is satisfied since Z in (5.6) is given by a nonsingular linear transformation of $\bar{Z}$ which is by construction $\bar{U}_p$.

(i)(if). Suppose $H_p = Q[I_p\ 0]J$, with J and L satisfying (5.2). Then by Lemma 1, with $M_p = [I_p\ 0]$, (4.6) is minimized so that $Z = H_pX$ is a solution minimizing (5.1).

To show (i)(b) and (c), consider the prediction error $\bar{V}-\hat{\bar{V}}$ in $\bar{V}$ where $\hat{\bar{V}} = \Sigma_{vu}[I_p\ \ 0]^T I_p[I_p\ \ 0]\bar{U}$ as in (2.3). The reduction in prediction error is

$$\Sigma_{vv} - \Sigma_{(v-\hat{v})(v-\hat{v})} = I - I + D^T[I_p\ \ 0]^T[I_p\ \ 0]D = Diag[\gamma_1^2, \ldots, \gamma_p^2, 0, \ldots, 0] \qquad (5.9)$$

which proves (b) and (c).

(ii) Uniqueness: Suppose that there are two solutions satisfying (5.2) which minimize the prediction error (5.1). Then by the uniqueness of the generalized singular value decomposition from Theorem 2, the respective J and L matrices are related by a block diagonal orthonormal matrix. If γ_p is unique, then so is the subspace span by the rows of J_p which proves (ii)(a). If γ_p is not unique, then a choice of a different generalized singular value decomposition relates to a different choice of basis for the k-dimensional basis corresponding to the singular value γ_p. Thus there is an arbitrary choice of a $p-q$ dimensional subspace from the rows $q+1, \ldots, q+k$ of J giving the canonical variables $u_{q+1}, \ldots, u_p$. The matrix $C_{p-q} = [I_{p-q}\ \ 0]C$ is constructed in (5.5) which proves (ii)(b). The minimum value is given by setting $M_p = (I_p, 0)$ in (4.6) so $trM_pD^2M_p^T = \gamma_1^2 + \cdots + \gamma_p^2$. This proves the theorem.

6. Multivariate Reduced Rank Prediction

The Theorem 3 includes a number of special cases that arise in the analysis of multivariate data. A particular solution to the general prediction problem (2.2) in the case of Λ nonsingular is given by Izenman (1975) and Rao (1979), although the solution is not unique if the generalized singular values are not distinct. The classical canonical

correlation analysis problem is obtained if we set $\Lambda = \Sigma_{yy}$, since then the normalization $L\Sigma_{yy}L^T = I_{Rank\,\Sigma_{xx}}$ implies that the variables $V = LY$ are orthonormal and hence Σ_{uv} is a correlation matrix. Infact the solution (5.2) then reduces to the canonical relationships $\Sigma_{uu} = I_m$, $\Sigma_{vv} = I_n$, $\Sigma_{uv} = Diag(\gamma_1, \ldots, \gamma_r, 0, \ldots, 0)$ which are a central aspect of canonical correlation analysis (see e.g. Rao, 1973, Sec. 8f.2(iv)). If $\Lambda = I$, the solution is different from canonical correlation analysis unless $\Sigma_{yy} = I$.

The principal component analysis problem is given by $X = Y$ and $\Lambda = I$, so that the norm is $\| X-\hat{X}_z \|_I^2 = E\{(X-\hat{X}_z)^T(X-\hat{X}_z)\}$. A generalization of principal component analysis is obtained by setting $Y = X$ so $\Sigma_{xx} = \Sigma_{xy} = \Sigma_{yy}$ and using an arbitrary positive definite symmetric weighting Λ so the norm to be minimized is $\| X-\hat{X}_z \|_{\Lambda^{-1}}^2 = E\{(X-\hat{X}_z)^T\Lambda^{-1}(X-\hat{X}_z)\}$. A different generalization, principal component analysis of instrumental variables, is discussed in Rao (1965). The problem is equivalent to setting $\Lambda = I$ so the prediction error norm is $\| Y-\hat{Y}_z \|_I^2 = E\{(Y-\hat{Y}_z)^T(Y-\hat{Y}_z)\}$. The canonical variables U are called the principal components of the instrumental variables X

.

In the above particular cases, the derivation and proofs in the cited references all assume that the matrices Σ_{xx} and Λ (i.e. Σ_{yy} or I) are nonsingular. The only discussion of the singular case seems to be Khatri (1976) for the canonical correlation analysis which is much more complicated than the present approach.

The definition and properties of the generalized singular value decomposition clearly express the fundamental properties of these multivariate prediction problems. Mathematically, geometrically and statistically the fundamental relationship is the selection of the canonical variables U and V by selecting the transformations J and L of the random variables X and Y. The fundamental geometrical properties of these transformations are that

J and L are orthonormal with respect to the matrices Σ_{xx} and Λ while simultaneously they are orthonormal with respect to Σ_{xy} except in corresponding pairs. This is concisely stated mathematically by the generalized singular value decomposition which includes the general case of singular matrices. These mathematical orthonormality relationships have immediate and direct statistical interpretation in terms of the identity covariances of U and V , the mutual zero correlation between U and V except in pairs, and the sum of squares property of the prediction error $V-\hat{V}$ with the addition of more predictor variables from U. The different multivariate prediction problems correspond only to a different selection of the random variables X and Y and the matrix Λ involved in the weighting of the prediction error.

7. Computational Aspects

Modern computer algorithms for canonical correlation analysis use a standard singular value decomposition to compute the generalized singular value decomposition (2.3) with $\Lambda = \Sigma_{yy}$ by first finding square root factors of Σ_{xx} and Λ , and then doing a standard singular value decomposition on $A = \Sigma_{xx}^{-1/2}\Sigma_{xy}(\Lambda^{-1/2}) = QSR^T$ where $QQ^T = I = RR^T$ and S is diagonal. Then the generalized singular value decomposition (2.3) is given by $J = Q^T\Sigma_{xx}^{-1/2}$, $L = R^T\Lambda^{-1/2}$ and $D = S$. Thus the joint orthonormalization of X and Y in the norms Σ_{xx} and Λ to give the canonical covariance structure D is very naturally viewed as a generalized singular value decomposition both in terms of the simple reduction discussed in Section 2 as well as the actual computational algorithms. This can be determined computationally using a standard singular value decomposition which is numerically very accurate and stable as compared with the earlier eigenvalue computational methods (Bjorck and Golub, 1973). An open topic is the investigation of numerical methods that directly compute the generalized SVD rather than transforming the problem to the

standard SVD. Such a direct approach may have better overall numerical accuracy.

A second problem is specified in terms of the observed data given as N repeated observations $(X^1, \ldots, X^N) = C$ and $(Y^1, \ldots, Y^N) = D$ on X and Y respectively. The usual sample covariances are computed as $\Sigma_{xx} = CC^T$, $\Sigma_{xy} = CD^T$ and $\Sigma_{yy} = DD^T$ which mathematically are used in the generalized singular value decomposition. Numerically, however, the formation of these products defining the sample covariances results in a halving of the numerical precision of the computation. In the case of given data, Bjorck and Golub (1973) give computational procedures that avoid these squaring operations and operate directly on the observed data.

Another computational aspect that may have a considerable effect upon statistical computing in the future is parallel computers. A very efficient algorithm for computing the singular value decomposition has been recently devised for highly parallel systolic arrays by Brent and Luk (1985). Such an $n \times n$ square array of processors requires communication between only the nearest neighbor processors in synchrony with the processor computational cycle. The computation of a singular value decomposition of a $n \times n$ matrix using a $n \times n$ array of processors requires only order n processor cycles as compared to order n cubed for a serial computer with a single processor. Such parallel processors and algorithms could make routine the analysis of very large sets of variables such as arise naturally in multivariate time series (Larimore, 1983).

From remarks above, it is obvious that the optimal solution to minimizing the quadratic prediction error measure (2.2) has exactly the same structure as solving the "pseudo" canonical correlation analysis problem using singular value decomposition methods with Σ_{yy} in (1.1) replaced by Λ . Although the matrix (1.1) is no longer a covariance matrix, a formal application of canonical correlation analysis indeed gives the optimal solution to minimizing (2.2). Thus a sufficiently general computational algorithm can be devised

which will solve all of the particular multivariate problems described above. Available algorithms for canonical correlation analysis may not be sufficiently general if for example they assume that the matrix (2.1) with Σ_{yy} replaced by Λ is a covariance matrix or that the canonical covariances are correlation coefficients ($(\gamma_i)^2 < 1$).

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APPENDIX B

SYSTEM IDENTIFICATION, REDUCED-ORDER FILTERING AND MODELING VIA CANONICAL VARIATE ANALYSIS

By Wallace E. Larimore

SYSTEM IDENTIFICATION, REDUCED-ORDER FILTERING AND MODELING VIA CANONICAL VARIATE ANALYSIS[2]

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ABSTRACT

Very general reduced order filtering and modeling problems are phased in terms of choosing a state based upon past information to optimally predict the future as measured by a quadratic prediction error criterion. The canonical variate method is extended to approximately solve this problem and give a nearoptimal reduced-order state space model. The approach is related to the Hankel norm approximation method. The central step in the computation involves a singular value decomposition which is numerically very accurate and stable. An application to reduced-order modeling of transfer functions for stream flow dynamics is given.

1. Introduction

Many complex random phenomena are modeled as high order of infinite order Markov processes. Often, however, most of the behavior of interest can be adequately approximated by a Markov process model of much lower order. Many of the modeling, control, and filtering methods depend upon a Markov or state-space structure. Even implementation of the general Wiener filter theory often requires use of finite-order state devices. Thus, it is frequently necessary to reduce a complex process to a limited number of states at some point in the analysis or implementation. In this paper, the problem of modeling or filtering with a restricted order state-space is addressed with emphasis on how best to determine approximate models or filters when the state order is restricted.

There have been a number of papers dealing with reduced-order modeling, filtering and system identification. Here we review only those related to the canonical variate approach, with more technical details contained in the appropriate sections. The theory of canonical correlations and variables was developed independently by Hotelling (1936) and Obukhov (see Gelfand and Yaglom (1959)). The solution of the canonical variate problem was first reduced to finding the eigenvectors of several symmetric matrices (Hotelling (1936), also see Anderson (1958)) A more computationally efficient, numerically accurate and stable method was developed by Golub (1969) based upon the singular value decomposition of a matrix.

Gelfand and Yaglom (1959) generalized the canonical variate method to describe the correlation structure between two discrete- or continuous-time random processes on possibly different time intervals. They expressed the mutual information between two such random processes simply in terms of the canonical correlations (see Section 5). Yaglom (1970) considered the relationship between the past outputs of a process and the future outputs (or any two disjoint intervals) and has shown there are a finite number of nonzero canonical correlations if and only if the process has a rational power spectrum, i.e., is a finite order Markov process.

Using a canonical variate analysis between the past and future of a discrete time stochastic process, Akaike (1975) constructed a minimal realization procedure for Markov processes. This resulted in a stochastic minimal realization algorithm similar to the algorithm of Ho and Kalman (1965). Later, Akaike (1974a) gave an abstract (coordinate free) description of the projection of the future of a process on the past and called it the predictor space. The canonical variate realization provides a particular basis for the predictor space. He used the concept of the predictor space to characterize any minimal realization for a discrete time Markov process as a particular choice of basis for the predictor space. The predictor space concept has been widely used in stochastic realization theory (Clary (1977), Fujishige et al. (1975), Picci (1976)).

Fujishige et al. (1975) addresses the reduced order modeling problem using the predictor space, but they do not use the canonical variate structure for model reduction. The criterion they define is the sum square prediction error of all output components for all the future which is a special case of the prediction error criterion discussed in Section 3. Their procedure requires an initial state-space model; however, it results in needing only to solve for eigenvectors of a symmetric matrix whose dimension is the original system state order - a very small amount of computation compared with most reduced order modeling schemes. A very interesting but brief discussion of canonical variate and predictor space methods and their relation to filtering problems is given by Kailath (1974). He talks about an approximation problem and the possible usefulness of canonical variates, but he does not explicitly discuss a reduced-order filtering problem.

The minimal splitting field of past and future is the continuous-time analog to the predictor space and predates Akaike's work although he was the first to propose a realization algorithm. The methods involve abstract Hilbert spaces to accommodate continuous time processes (Levinson and McKean (1964), McKean (1963), Pitt (1972), Rozanov (1976), (1977)).

The optimal Hankel norm approach of Adamjan et al. (1978) has received much recent attention in reduced order modeling (see Kung and Lin (1981) and cited references). Canuto and Menga (1982) discuss relationships between the canonical variate approach of Akaike and the optimal Hankel norm approach.

2. Approach

A major departure of this paper from previous work is the use of canonical variate analysis to optimally choose k linear combinations of the past for prediction of the future. The very natural measure of quadratically weighted prediction errors at possibly all future time steps is used. In Section 3 we formulate the problem and show how a generalized canonical variate analysis problem solves it explicitly. The interpretation of canonical variates as optimal predictors is central in motivating interest in such a problem formulation and is scarcely found in the statistical literature. The optimal k-order predictors are not in general recursively computable, but the optimal state-space structure for approximating them is expressed simply in terms of the canonical variate analysis. The problem of finding an optimal Hankel norm reduced order model is related to the canonical variate approach

[1]Part of this work was performed while the author was at The Analytic Sciences Corporation, Reading, MA.

[2]Reprinted from Proceedings of the 1983 American Control Conference, June 22-24, San Francisco, CA

3. Statement of the Problem

Consider the problem of choosing an optimal system or model of specified order for use in predicting the future evolution of the process. We will distinguish between the past $p(t)$ of one vector process $r(t)$ at time t or before and the future $f(t)$ of another vector process $s(t)$ at time later than t so

$$p^T(t) = (r^T(t), r^T(t-1), \ldots) \tag{3-1}$$

$$f^T(t) = (s^T(t+1), s^T(t+2), \ldots) \tag{3-2}$$

We assume that the processes $r(t)$ and $s(t)$ are jointly stationary.

The major interest is in determining a specified number k of linear combinations of the past $p(t)$ which allow optimal estimation of the future $f(t)$. Any set of k linear combinations of the past $p(t)$ are denoted as a $k \times 1$ vector $m(t)$, memory of the past of order k. The optimal linear prediction $\hat{f}(t)$ of the future $f(t)$ which is a function of a reduced order memory $m(t)$ is measured in terms of the prediction error

$$\varepsilon = \| f - \hat{f} \|^2_{\vartheta^{-1}} = E[(f - \hat{f})^T \vartheta^{-1} (f - \hat{f})] \tag{3-3}$$

where ϑ^{-1} is an arbitrary quadratic weighting and E is the expectation operation. The reduction problem is to determine an optimal k-order memory

$$m(t) = J_k p(t) \tag{3-4}$$

for which the optimal linear predictor $\hat{f}(t, x(t))$ minimizes the prediction error.

In various particular problems, the process $r(t)$ of the past will include outputs of a system and/or inputs of a system. The process $s(t)$ of the future may be the same as $r(t)$ or different. The general case of interest is the reduced order filtering and modeling problem: given the past of the related random processes $u(t)$ and $y(t)$, we wish to model and predict the future of $y(t)$ by a k-order state-space structure of the form

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{3-5}$$

$$y_t = Hx_t + Du_t + Bw_t + v_t \tag{3-6}$$

where w and v are white noise processes that are independent with covariance matrices Q and R respectively. These white noise processes model the covariance structure of the error in predicting y from u. A special case of the reduced-order filtering problem is the transfer function approximation problem where u and y are the input and output processes and an approximate state-space model is desired.

Once the optimal k-order memory $m(t)$ is determined, we will develop state-space equations for approximately computing the memory or recursively describing its evolution. A major part of the problem is, however, the choice of the optimal k-order memory.

4. Canonical Variates as Optimal Predictors

In this section the solution to choosing the optimal k-order memory is described in terms of the canonical variate analysis method of mathematical statistics. Here the solution is explicitely described in terms of a singular value decomposition. To treat the prediction problem of section 3 involving an arbitrary ϑ in the prediction error (3-3) requires extension of the classical canonical variate analysis method. The derivation of this extension is rather lengthy and will be described elsewhere.

In the statistical literature, the canonical variate problem is delt with as one of maximizing correlation between two sets of variables (i.e., $p(t)$ and $f(t)$); whereas our interpretation will be choosing variables from $p(t)$ that optimally predict $f(t)$, which is rarely the conceptual framework used in statistics.

We treat here explicitly the case of finite past and future, i.e., $p(t)$ and $f(t)$ of finite dimension, to avoid the technicalities of the infinite dimensional case which is discussed in detail in Gelfand and Yaglom (1959).

4.1 Canonical Variate Solution

The solution to the canonical variate problem is expressed quite simply by putting the covariance structure of past $p(t)$ and future $f(t)$ in a canonical form. We seek nonsingular transformations of p and f

$$c = Jp, \quad d = Lf \tag{4-1}$$

such that in this new basis the norm (3-3) for weighting prediction errors of the future is a sum of squares

$$\|\varepsilon\|^2_{\vartheta^{-1}} = d^T(L\vartheta L^T)^{-1}d = d^Td \tag{4-2}$$

In addition the covariances among the past and between the past c and future d have a canonical structure

$$\mathrm{cov}\,(c,c) = I, \tag{4-3}$$

$$\mathrm{cov}\,(c,d) = \mathrm{Diag}(\gamma_1, \ldots, \gamma_k, 0, \ldots 0) = D \tag{4-4}$$

with the canonical covariances $\gamma_1 \geq \ldots \geq \gamma_k > 0$ in descending order. Thus, the components of the past c are mutually uncorrelated. Of all linear combinations of p and f, the first component of c has maximum covariance with the first component of d.

It can be shown that for any order k, that the first k components of c, i.e., corresponding linear combinations of the past p, lead to the best prediction $\hat{f}$ of the future f. The optimal choice of a k-order memory is then

$$m_k = J_k p = (I_k, 0)J_p \tag{4-5}$$

The minimized prediction error for order k is simply expressed in terms of the canonical covariances as

$$\min_{J_k} \|f - \hat{f}\|^2_{\vartheta^{-1}} = \mathrm{tr}\,\vartheta^{-1}\Sigma_{ff} - (\gamma_1^2 + \ldots + \gamma_k^2) \tag{4-6}$$

The sum of squared canonical covariances $\gamma_{k+1}^2 + \ldots + \gamma_r^2$ corresponding to the neglected variables gives the increased prediction error from using memory order k rather than r.

4.2 Calculations Using Singular Value Decomposition

The requirements of (4-3) through (4-4) are equivalent to finding J and L such that

$$J\Sigma_{pf}L^T = \mathrm{Diag}(\gamma_1, \ldots, \gamma_k, 0, \ldots 0) = D \tag{4-7}$$

$$J\Sigma_{pp}J^T = I \quad (4-8), \qquad L\Sigma_{ff}L^T = I \quad (4-9)$$

This is easily accomplished using a singular value decomposition (Golub (1969)) which is computationally very efficient and numerically very accurate and stable. Dimensions of p(t) and f(t) as high as several hundred can be handled efficiently and accurately using these computational techniques.

To find this decomposition, first the square roots $\Sigma_{pp}^{-1/2}$ and $\vartheta^{-1/2}$ are computed by either a Cholesky procedure or an eigenvector procedure. Since a singular value decomposition procedure is used latter and is numerically much more accurate and stable, it can be used to find the eigenvalues and eigenvectors

$$\Sigma_{pp} = U_1 S_1 V_1^T, \qquad \vartheta = U_2 S_2 V_2^T \tag{4-10}$$

where $U_1 = V_1$ and $U_2 = V_2$ are matrices of the eigenvectors, S_1 and S_2 are diagonal and contain the eigenvalues, and the equality of the U's and V's follows since Σ_{pp} and ϑ are positive definite and symmetric. The square roots are

$$\Sigma_{pp}^{-1/2} = U_1 S_1^{-1/2} V_1^T, \quad \vartheta^{-1/2} = U_2 S_2^{-1/2} V_2^T \tag{4-11}$$

Now form the matrix

$$A = \Sigma_{pp}^{-1/2} \Sigma_{pf} \vartheta^{-1/2} \tag{4-12}$$

and do a singular value decomposition

$$A = USV^T, \qquad U^T U = V^T V = I \tag{4-13}$$

where S is diagonal with nonnegative elements in descending order

$$S = \mathrm{dia}\,(s_1 \geq s_2 \geq \ldots \ldots s_n) \tag{4-14}$$

The canonical variate decomposition is obtained by setting

$$J = U^T \Sigma_{pp}^{-1/2}, \qquad L = V^T \vartheta^{-1/2}, \qquad D = S \tag{4-15}$$

4.3 Relationship to the Hankel Norm

Consider the deterministic input-output case which can be cast in the canonical variate framework by choosing a white noise stochastic input so $\Sigma_{pp} = I$ and letting $\vartheta = I$. The covariance matrix Σ_{pp} is

$$\Sigma_{fp} = E\begin{pmatrix} H_1 & H_2 & H_3 \cdots \\ H_2 & H_3 & \cdots \\ H_3 & \vdots & \\ \vdots & & \end{pmatrix}\begin{pmatrix} u_t \\ u_{t-1} \\ \vdots \end{pmatrix}(u_t, u_{t-1} \ldots) = H\Sigma_{pp} = H \tag{4-16}$$

where the Hankel matrix H involves the impulse response matrices H_i. The Hankel norm between H and an approximation A is defined as

$$\|H - A\|_s \tag{4-17}$$

where $\|\ \|_s$ is the spectral norm

$$\|B\|_s = \max_{\|x\|_2 = 1} \|Bx\|_2, \quad \|x\|_2^2 = \sum x_i^2 \tag{4-18}$$

Now consider a singular value decomposition of H

$$H = USV^T, \quad UU^T = I, \quad VV^T = I \tag{4-19}$$

where S is diagonal with nonnegative singular values in nonincreasing order. A property of the singular value decompositions is

$$\inf_{A\,:\,\mathrm{rank}(A) \leq k} \|H - A\|_S = \sigma_{k+1} \tag{4-20}$$

As shown in Adamjan et al (1978), this bound is achieved for A restricted to a Hankel matrix for single input-output systems, and is at least a lower bound for multivariable systems. This solution is a minimax solution - for a given approximation A the norm (4-17) measures the largest possible error in output future sequences f over all possible past sequences p with $\|p\|=1$. For finite order systems, the exponential decay of the impulse response will cause the worst sequence to be concentrated near the origin. This is a very atypical input sequence to use as a basis for measuring closeness

By contrast, in the canonical variate formulation, the norm is

$$E\|f_H - \hat{f}_A\|^2 = \mathrm{tr}(H-A)(H-A)^T = \|H-A\|_F^2 \tag{4-21}$$

the Frobenius norm. From (4 - 6) this norm has the lower bound

$$\inf_{A\,:\,\mathrm{rank}(A) = k} \|H-A\|_F = (\sigma_{k+1}^2 + \ldots + \sigma_l^2)^{1/2} \tag{4-22}$$

A fundamental difference here is that the norm (4-21) measures the average overall output sequences resulting from random input sequences with unit average power, i.e., $E(u(t))^2 = 1$ for $t \leq 0$; whereas for (4-20), $\sum_{t=-\infty}^{0} (u(t))^2 = 1$.

Camuto and Menga (1982) discuss some relationships between the canonical correlation and Hankel norm approaches. There is no interpretation of the canonical correlations as minimizing a norm as in Equation (4-6), and further they note that the singular values in the two approaches do not coincide. They conclude that "because they (the canonical correlations) do not have any practical significance about the energetic structure of the dynamics of the process, the properties of the resulting approximated models are not clear". The present canonical variate approach makes clear that the energetic structure of the dynamics is better accounted for in the prediction error measure than it is by the Hankel norm.

4.4 Related Literature

In the classical canonical correlation analysis (Hotelling (1936), Anderson (1958)), $\vartheta = \Sigma_{ff}$ so that the prediction errors of the future are weighted by their inverse covariance matrix, and consequently the future f is normalized to have identity covariance matrix. Also the canonical covariances are then correlation coefficients. The traditional criterion, to the extent that there has been such discussions, has concerned the mutual information (Shannon and Weaver (1962)) in one random vector p about another random vector f defined by

$$I(p;f) = \int p_{pf}(p,f) \log \frac{p_{pf}(p,f)}{p_p(p)\, p_f(f)}\, dp\, df \tag{4-23}$$

The base of the logarithm is arbitrary and determines the particular units of information, and p_{pf} is the joint and p_p and p_f the marginal probability densities.

Gelfand and Yaglon (1959) showed that the mutual information is simply expressed in terms of the canonical correlations $\gamma_1,\ldots,\gamma_n$ between the two vectors by

$$J(p;f) = -\tfrac{1}{2}\sum_{j=1}^{n} \log(1-\gamma_j^2) = -\tfrac{1}{2}\log w \qquad (4\text{-}24)$$

where Hotelling (1936) defines the vector alineation coefficient

$$w = (1-\gamma_1^2)\ \ldots\ (1-\gamma_n^2) \qquad (4\text{-}25)$$

as a measure of independence of p and f. Gelfand and Yaglom (1959) extend the definition of mutual information to vectors of infinitely many random variables, e.g., random processes in both continuous time and discrete time. This development also provides the basis for extending canonical variates to random processes (Yaglom (1970)).

Now, if a restricted number k of linear combinations $(c_1,\ldots,c_k)$ of the past of one random process r(t) are used to predict the future of another random process s(t), then the choice maximizing the mutual information is the first k canonical variates and the mutual information is expressed by the first k canonical correlations

$$\max_{c_1,\ldots,c_k} J(c_1,\ldots,c_k;f) = -\tfrac{1}{2}\log\sum_{j=1}^{k}(1-\gamma_j^2) \qquad (4\text{-}26)$$

Thus, the canonical correlation method provides an optimal procedure in terms of mutual information for choosing a finite number of linear combinations of one random process for prediction of another.

Recently, in the statistical literature Yohai and Garcia Ben (1980) point out the use of canonical variates as optimal predictors. They show that the canonical variates (with $A=\Sigma_{ff}$ in our scheme) minimize the prediction error

$$E(f-\hat{f})(f-\hat{f})^T \qquad (4\text{-}27)$$

where denotes determinant, and the minimum value is

$$(1-\gamma_1^2)\ldots(1-\gamma_k^2). \qquad (4\text{-}28)$$

The logarithm of this expression is, within a fixed additive constant, proportional to the minimized mutual information (4-26). Rao (1973) gives a problem in which the canonical variates for the measure

$$E\,\|f-\hat{f}\|^2 \qquad (4\text{-}29)$$

are to be determined ($A=I$ in our scheme) which gives a different solution from the classical canonical correlation problem ($A=\Sigma_{ff}$). The general canonical variate problem using a general prediction error measure (1-3) was formulated and solved in Porter and Larimore (1976a) for a nondynamical problem. This was first applied to reduced order modeling for the canonical correlation case ($A=\Sigma_{ff}$ and no input u(t)) in Larimore et al. (1977). The general prediction error measure (arbitrary A) was considered for deterministic impulse response modeling in Goldstein and Larimore (1980).

5. State Space Realizations

As discussed in Section 3, there are a variety of problems of interest including reduced-order stochastic modeling and filtering. The most general form is the state space model

$$x_{t+1} = Ax_t + Gu_t + w_t \qquad (5\text{-}1)$$

$$y_t = Hx_t + Au_y + Bu_t + v_t \qquad (5\text{-}2)$$

where u_t is an input process, x_t is the state vector, w_t is white process noise with covariance matrix Q, and v_t is white measurement noise uncorrelated with w_t with covariance matrix R. It has been shown (Lindquist and Pavon (1981)), that for no input u_t, the form (5-1) and (5-2) is the most general state-space realization of a Markov process, and that the state dimension is equal to the Markov order. Other Markov realizations as in Akaike (1975) and Baram (1981) which have A=B=G=R=0 are not the most general and may require much higher state order for a suitable approximation. These latter forms are particularly inefficient in the presence of moving average terms or additive white measurement noise. As will be seen below, a regression interpretation of the state-space matrices makes it clear that the error in regression is ignored. A further point of Lindquist and Pavon (1981) is that for a parsimonious state defined by the predictor space, the past and future must be nonoverlapping as in (3-1) and (3-2).

For the purely stochastic case with u(t)=0, let

$$r(t) = y(t); \qquad s(t) = y(t) \qquad (5\text{-}3)$$

in setting up the past p and future f as in (4-1). For the case of a deterministic input u(t), the input must also be included in the past so that

$$r^T(t) = (u^T(t),\ \ y^T(t)); \quad s(t)=y(t) \qquad (5\text{-}4)$$

Another case is the deterministic input-output system with no process and measurement noises w_t and v_t so

$$r(t) = u(t), \quad s(t) = y(t) \qquad (5\text{-}5)$$

Note that in the case of a known input u(t) present, the covariance function of u(t) is required and used in specifying the most important components of the past of u(t) to include in the state for prediction of the future of y(t).

Now for a given order k for a model, we wish to find a best k-element state. This is equivalent to finding the k linear combinations of the past p which have the best ability to predict the future f and which are also computable recursively in time. The optimal k linear combinations x(t) given by the canonical variate analysis (4-9) through (4-9) is not generally recursively computable except when k is the full order n. To approximately find the best k-order state x(t), the following procedure is used. First find the optimal k-order memory from the canonical variate analysis for any k<n with the minimal-order realization given for k=n. Considering k fixed below, we have

$$x(t) = L_k p(t) \text{ where } L_k = (I_k,\ 0) \qquad (5\text{-}6)$$

with I_k the kxk identity.

To determine a state $x(t)$ satisfying the recursive relations (5-1) and (5-2) which approximately gives the optimal k order memory $m(t)$, the optimal prediction of $m(t+1)$ and $y(t)$ using $m(t)$ and $u(t)$ is determined from simple regression relationships.

The dynamical equations (5-1) and (5-2) express (x_{t+1}, y_t) as a linear combination of (x_t, u_t) plus a white noise vector with correlated components. Thus, using simple multivariate regression procedures (Anderson (1958)), the matrix for optimal prediction of (m_{t+1}, y_t) from (m_t, u_t) is

$$\begin{pmatrix} \Phi & G \\ H & A \end{pmatrix} = \mathrm{cov}\left[\begin{pmatrix} m_{t+1} \\ y_t \end{pmatrix}, \begin{pmatrix} m_t \\ u_t \end{pmatrix}\right] \mathrm{cov}^{-1}\left[\begin{pmatrix} m_t \\ u_t \end{pmatrix}, \begin{pmatrix} m_t \\ u_t \end{pmatrix}\right] \quad (5\text{-}7)$$

and the error in prediction has covariance matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \mathrm{cov}\left[\begin{pmatrix} m_{t+1} \\ y_t \end{pmatrix}, \begin{pmatrix} m_{t+1} \\ y_t \end{pmatrix}\right]$$

$$-\mathrm{cov}\left[\begin{pmatrix} m_{t+1} \\ y_t \end{pmatrix}, \begin{pmatrix} m_t \\ u_t \end{pmatrix}\right] \mathrm{cov}^{-1}\left[\begin{pmatrix} m_t \\ u_t \end{pmatrix}, \begin{pmatrix} m_t \\ u_t \end{pmatrix}\right] \mathrm{cov}\left[\begin{pmatrix} m_t \\ u_t \end{pmatrix}, \begin{pmatrix} m_{t+1} \\ y_t \end{pmatrix}\right] \quad (5\text{-}8)$$

The matrices Q, R, and B are simply e[illegible] in terms of S by

$$\begin{aligned} Q &= S_{11} \\ B &= S_{21} S_{11}^{+} \\ R &= S_{22} - S_{21} S_{11}^{+} S_{12} \end{aligned} \quad (5\text{-}9)$$

where (+) denotes the pseudoinverse.

Explicit computation of the covariance matrices is obtained using the decomposition and the covariance of p, f, y and u as

$$\mathrm{cov}\left[\begin{pmatrix} m_{t+1} \\ y_t \end{pmatrix}, \begin{pmatrix} m_t \\ u_t \end{pmatrix}\right] = \mathrm{cov}\left[\begin{pmatrix} J_k p_{t+1} \\ y_t \end{pmatrix}, \begin{pmatrix} J_k p_t \\ u_t \end{pmatrix}\right]$$

$$\mathrm{cov}\left[\begin{pmatrix} m_t \\ u_t \end{pmatrix}, \begin{pmatrix} m_t \\ u_t \end{pmatrix}\right] = \mathrm{cov}\left[\begin{pmatrix} J_k p_t \\ u_t \end{pmatrix}, \begin{pmatrix} J_k p_t \\ u_t \end{pmatrix}\right]$$

This, then, gives the covariance matrices explicitly in terms of the covariance functions involving $u(t)$ and $y(t)$. In the purely stochastic case that there is no input $u(t)$ present, the components of u_t in the vector (m_t^T, u_t^T) are deleted and G and A are then not computed. For the deterministic ca[illegible] where w_t and v_t are zero for the full order realization, it may be of use in some reduced-order modeling problems to compute Q, B, and R to give a reduced-order model. The reduced-order model (5-7) of a stable system can be shown (Fujishige et. al. (1975)) always to be stable.

The existing literature on the use of the canonical variate method in deriving the above reduced-order state-space models is based upon Larimore et al (1977). Baram (1981) and Koehler (1981) describe a restricted stochastic modeling problem using the canonical correlation approach ($\theta = \Sigma_{ff}$, no input $u(t)$, no measurement noise $v(t)$, and no prediction error interpretation) essentially as it was presented in Larimore et al (1977). White (1983) gives a more recent development and full acknowledgement for the origin of these ideas.

6. Example in Impulse Response Modeling

In a recent study (Goldstein and Larimore (1980)) for the National Weather Service, the canonical variate procedure was used for deriving reduced-order state-space models of stream-flow dynamics. This was a necessary component in that study which investigated the application of Kalman filtering and maximum likelihood parameter identification to hydrologic forecasting.

The problem is formulated in terms of a given unit hydrograph $h(\tau)$ that specifies the response at lag τ to a unit pulse input at time zero. It is desired to find a state-space model, preferably of low order, which is a good approximation in some sense to the given unit hydrograph. This problem cannot be separated from the characteristics of the [illegible]ut process since the modes of $h(\tau)$ that are excited and, hence, the output depend strongly upon the input process. Nominally, it will be assumed that the input process is white noise which excites all frequencies proportionately. If the typical input signal power spectrum is known and different from white noise, this fact can be easily included and would lead to an alternative approximating state-space model. It will be shown that the white noise assumption leads to excellent approximations of the unit hydrograph with low-order state-space models. A schematic descri[illegible]tion of the problem is shown in Figure 1.

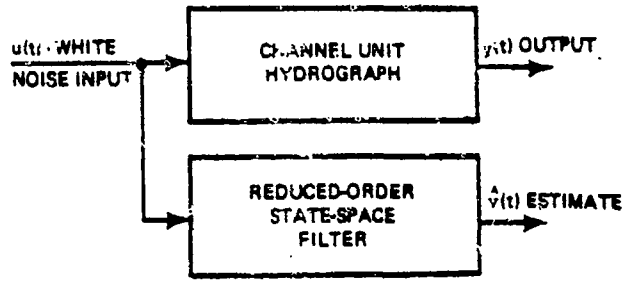
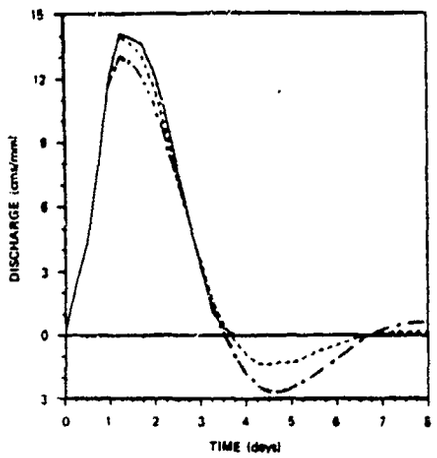


Figure 1 Approximation of Unit Hydrograph by a Reduced Order Filter

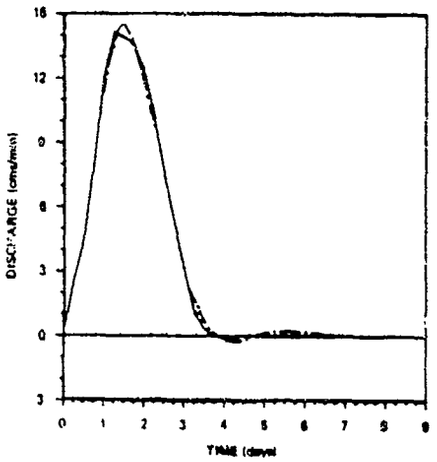
The reduced-order state-space modeling described above has been applied to unit hydrographs for a number of river basins supplied by NWS. The character of the reduced-order models is illustrated below and described in more detail in Goldstein and Larimore (1980).

Two different weightings θ of errors in predicting the future f were used, $\theta = I$ giving a sum squared error or energy measure and $\theta = \Sigma_{ff}$ giving a squared relative error measure.

The differenc[illegible]s in reduced-order models obtained from these two measures of prediction errors depend very strongly upon the spectral shape of the hydrograph transfer function. A striking comparison in fit using the two criteria was obtained for the Bird Creek basin which is order 14. The six-hour unit hydrographs based upon the input hydrographs for 4- and 8-state models are shown in Figure 2 respectively for the two cases $\theta = I$ and $\theta = \Sigma_{ff}$. The respective squared magnitude transfer functions are shown in Figure 3. [illegible]ote in Figure 2 that even the 8-state unit hydrograph from the case $\theta = \Sigma_{ff}$ has a significant nonzero tail whereas the 4-state unit hydrograph from the case $\theta = I$ produces an excellent fit. Figure 3a

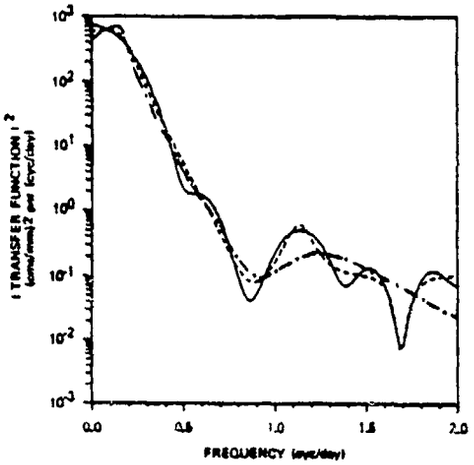


(2a) Squared Relative Error $\theta = \Sigma_{ff}$

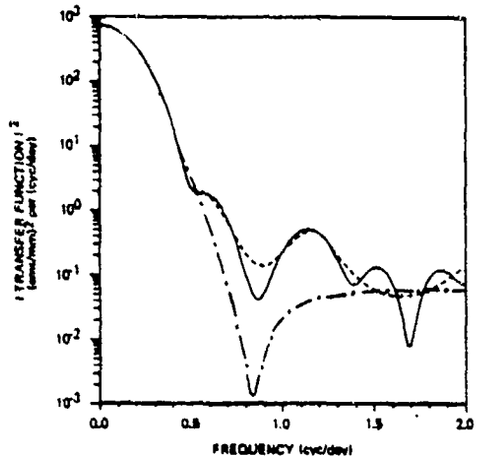


(2b) Sum Squared Error, $\theta = I$

Figure 2. Six-Hour Unit Hydrographs, Original (Solid), Eight-Order (Dashed), and Fourth-Order (Dashed and Dotted).



(3a) Squared Relative Error $\theta = \Sigma_{ff}$



(3b) Sum Squared Error, $\theta = I$

Figure 3. Squared Magnitude Transfer Function, Original (Solid), Eight-Order (Dashed), and Fourth-Order (Dashed and Dotted).

clearly illustrates the tendence of the case $\theta = I$ to fit all frequencies with nearly equal percent error, whereas from Figure 3b it is seen that in the case $\theta = I$ the frequency bands of highest energy are empnasized. Thus, for a hydrograph with a large spectral peak and complicated spectral shape, i.e., requiring a high order rational function for a good approximation, the case $\theta = I$ can be expected to excel in fitting the unit hydrograph.

10. Conclusion

The canonical variate approach provides a powerful and general procedure for reduced-order modeling, filtering and system identification. The procedure is computationally noniterative and incorporates use of a singular value decomposition which is numerically accurate and stable. This guarantees a computational solution in every case. All reduced-order models are easily computed from one singular value decomposition.

This paper extends the pioneering work of Akaike's in a number of directions. A generalized canonical variate procedure is explicitly described in terms of minimizing an arbitrary quadratic weighting of the error in prediction of the future from the past. This considerably extends the usefulness of the method. While Akaike considered only the case of process noise, we include any combination of inputs, process and measurement noise. This extends the approach of Akaike to the reduced-order filtering and transfer function modeling problems as well as modeling in the presence of an input function. The use of a non-overlapping past and future lead to lower-order state-space models. Using a finite past and future, simple and computationally efficient expressions are explicitly given for determining reduced-order system state-space matrices. A particular specialization of the canonical variate procedure is related to the Hankel norm method for deterministic input-output systems, however the former has an interpretation in terms of the prediction error of the future.

The example modeling river basin dynamics illustrates the flexibility of the general quadratic weighting of the error in predicting the future from the past. The classical canonical correlation procedure leads to uniform fitting in the frequency domain while the sum square error criterion leads to a uniform fitting of the unit pulse response.

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APPENDIX C

ADAPTIVE MODEL ALGORITHMIC CONTROL

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ADAPTIVE MODEL ALGORITHMIC CONTROL[1,2]

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Abstract. Model Algorithmic Control (MAC) is a relatively new design methodology successfully used by industries for the last several years. The objective of this paper is to investigate robustness properties of MAC, and evaluate the use of adaptive methods for real-time identification of the plant under closed-loop control. Some theoretical robustness properties of MAC are given in terms of classical qualities such as gain margin and phase margin for a wide class of systems. Although MAC is an output-feedback controller, it has a guaranteed continuous-time equivalent phase margin of 60°, and the upward gain margin can be made arbitrarily large by slowing down the reference trajectory. Some robustness properties of MAC are also given by a perturbation analysis of a miss-modeled plant impulse response. Preliminary results are discussed for on-line identification of the closed-loop plant using the canonical variate method. Performance of the identification of the plant in the presence of both input and measurement noise is given.

Keywords. Adaptive control; identification; robustness; canonical variate analysis; model algorithmic control.

INTRODUCTION

The MAC methodology generates a control sequence by on-line optimization of a cost-functional, and the algorithm is suitable for implementation on microprocessors. One of the attractive features of MAC is the clear and transparent relationship between system performance and various design parameters embedded in the design procedure. MAC has been described elaborately in the literature (Mehra et. al. (1977, 1979, 1980), Mereau et. al. (1978), Richalet et. al. (1978), and Rouhani and Mehra (1982)), and therefore only a brief description of MAC is given below. The z-transform or s-transform of a time function is denoted by replacing the time-argument by z or s respectively; for example $y(z)$ denotes the z-transform of $y(n)$. For the sake of simplicity a single-input single-output system is considered although the extension to multiinput multi-output plants is conceptually straightforward.

There are five basic elements in MAC;

(i) An actual plant with a casual pulse response function $h(t) = \{h_i, i = 1,\ldots N\}$, an input $u(t)$ and an output $y(t)$ to be controlled. The $y(t)$ are related through a convolution operator(*)

$$y(t) = h(t) * u(t) = \sum_{i=1}^{N} h_i u(t-i)$$

or,

$$y(z) = h(z)u(z), \quad h(z) = \sum_{i=1}^{N} h_i z^{-i} \qquad (1.1)$$

(ii) A model of the plant $\hat{h}(t) = \{\hat{h}_i, i = 1,\ldots N\}$ with output $\hat{y}(t)$ and input $u(t)$ so that

$$\hat{y}(t) = \sum_{i=1}^{N} \hat{h}_i u(t-i)$$

or,

$$\hat{y}(z) = \hat{h}(z)u(z), \quad \hat{h}(z) = \sum_{i=1}^{N} \hat{h}_i z^{-i} \qquad (1.2)$$

(iii) A smooth trajectory $y_r(t)$ initiated on the current output $y(t)$ that leads $y(t)$ to a possibly time varying set point c. The $y_r(t)$

1 This work was supported by the Air Force Wright Aeronautical Laboratory.

2 Reprinted from Proc. IFAC Workshop on Adaptive Systems in Control and Signal Processing, June 20-22, 1983, San Francisco.

evolves as follows:

$$y_r(t+1) = \alpha y_r(t) + (1-\alpha)c(t), \quad y_r(t) = y(t) \quad (1.3a)$$

$$y_r(z) = \alpha z^{-1}y(z) + (1-\alpha)z^{-1}c(z) \quad (1.3b)$$

where α is a constant determining the speed of response;

(iv) a closed loop prediction scheme for predicting the future output $y_p(t)$ of the plant according to the scheme

$$y_p(t+1) = \tilde{y}(t+1) + y(t) - \tilde{y}(t) \quad (1.4a)$$

$$y_p(z) = \tilde{y}(z) + z^{-1}(y(z) - \tilde{y}(z)) \quad (1.4b)$$

and finally

(v) a quadratic cost functional J based on the error between $y_p(t)$ and $y_r(t)$ over a finite horizon T:

$$J = \sum_{i=1}^{T} \{ (y_p(t+i)-y_r(t+i))^2 w(i) + u^2(t+i-1)r(i-1)) \} \quad (1.5)$$

where w(i) and r(i) are time varying weights. Usually r(i) is chosen to be zero.

Given (i) - (v), MAC finds an optimal control sequence $\{u^*(t+i-1), i=1,\ldots T-1\}$ by minimizing J over the admissible input sequence $\{u(t+i-1)\in U(i), i = 1,\ldots T-1\}$. Once the optimal control sequence is computed, the first element of the sequence is applied to the actual plant and the process repeats all over again.

To investigate the theoretical properties of MAC and to interpret MAC from the classical control viewpoint we make the following assumptions:

(i) the actual plant h(z) is minimum phase;

(ii) there are no input constraints, i.e. $U(i) = R$ for all i, where R is the real line;

(iii) the optimization is carried over one future step ahead i.e., (T=1); under this condition MAC is a one-step ahead predictive controller.

Under these simplifying assumptions, it is sufficient to select $u^*(t)$ satisfying

$$y_p(t+1) = y_r(t+1) \text{ for all } t \quad (1.6)$$

for a minimum of the cost function J. The assumptions (i) - (ii) ensure the existence of an optimum control $u^*(t)$ satisfying (1.6). $u^*(t)$ is then implicitly generated by $y_p(z) = y_r(z)$ so that

$$(z-1)\tilde{h}(z)u(z) + (1-\alpha)h(z)u(z) = (1-\alpha)c(z) \quad (1.7)$$

By further manipulation (1.7) can be expressed as

$$\frac{u(z)}{c(z)} = \frac{1-\alpha}{(z-1)\tilde{h}(z) + (1-\alpha)h(z)} \quad (1.8a)$$

$$\frac{y(z)}{c(z)} = \frac{h(z)(1-\alpha)}{(z-1)\tilde{h}(z) + (1-\alpha)h(z)} \quad (1.8b)$$

Equations (1.8) imply that MAC under assumptions (i)-(iii) above is equivalent to the following classical unity feedback configuration in an input-output sense.

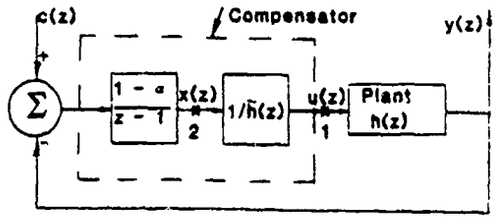

Fig. 1. MAC as a Classical Controller

This interpretation of MAC is the basis of our analysis of MAC in terms of classical control.

PHASE AND GAIN MARGINS

The block within the dashed line can be considered as a dynamic controller of the classical type. The loop transfer function at point 1 is

$$L(z) = \frac{h(z)(1-\alpha)}{\tilde{h}(z)(z-1)} \quad (2.1a)$$

and the return difference function is

$$1+L(z) = \frac{\tilde{h}(z)(z-1) + h(z)(1-\alpha)}{\tilde{h}(z)(z-1)} \quad (2.1b)$$

The error $e(z) = c(z) - y(z)$ in tracking is given by

$$e(z) = (1+L(z))^{-1} c(z)$$

so that the steady state error due to a step input is

$$e_{ss}(t) = \lim_{z\to 1} (1+L(z))^{-1} = (1 + L(1))^{-1} = 0$$

which is a consequence of a builtin integrator in the compensator. It may be noted that using the set-up of Fig. 1 and by treating $(1-\alpha)$ as a gain, the usual classical root-locus technique can be applied to analyze the behavior of the closed-loop poles as α changes from 0 to 1. To make the root-locus picture complete, the characteristic polynomial can be rearranged with a modified gain $\lambda = \alpha/(1-\alpha)$ so that as α changes from 0 to 1, λ changes from 0 to infinity.

It may be noted from Fig. 1 that at point 2, $x(z)=y_r(z)$ when $h(z) = \tilde{h}(z)$, where $y_r(z)$ is the reference signal. This shows why perfect tracking is possible under perfect identification. We will, however, not pursue this approach here.

It is obvious from (1.8) and (2.1) that the closed-loop system is internally asymptotically stable if the roots of the rational function

$$\Phi_{cl}(z) = (z-1)\tilde{h}(z) + (1-\alpha)h(z) \qquad (2.2)$$

are within the open unit disk $|z|<1$, and these roots are also the roots of the return difference function $1 + L(z)$. We can therefore find the stability margin in terms of the gain margin (GM) and phase margin (PM) from the Bode plot or Nyquist plot of the loop transfer function $L(z)$ evaluated on the Nyquist contour $z = \exp(j\omega)$ appropriately indented around the poles on this contour. Recall that in continuous-time, the GM and PM are those values of k and ϕ respectively such that the perturbed loop $\tilde{L}(s) = k\exp(j\phi)L(s)$ is stable, where $L(s)$ is the nominal loop and s is the Laplace variable. A similar interpretation goes for the discrete-time systems (Kuo (1980)); but the PM, unless it is an integral value of the sampling interval, does not have any physical significance. Strictly speaking the complex constant $k\exp(j\phi)$ in continuous time should be replaced by kz^{-n}, n an integer, for measuring GM or PM of the discrete-time system.

Another way to compare with other continuous-time domain design techniques is that each element of the discrete-time loop should be transformed into an equivalent continuous-time element using bilinear transformation, and PM of the fictitious continuous-time loop can be taken as the PM of the discrete-time loop. In this paper the word PM is used to mean the continuous-time equivalent phase margin. We can now state

Theorem 1:

Under assumptions (i)-(iii), MAC has GM = $(0, 2/(1-\alpha))$, equivalent PM = $\cos^{-1}(1-\alpha)/2$, and unity gain cross-over frequency $\omega_o = 2\sin^{-1}(1-\alpha)/2$.

Proof: The proof is trivial if we recall that PM and GM are measured on a nominal loop. Here we can assume that the nominal plant $h(z) = \tilde{h}(z)$, which implies $h_i = \tilde{h}_i$ and $N = \tilde{N}$ because both $h(z)$ and $\tilde{h}(z)$ are power series in z^{-1}. The nominal loop transfer function from (2.1a) is then

$$L(z) = \frac{1-\alpha}{z-1} \qquad (2.3)$$

i.e. an integrator delayed by one-step. Evaluating on $z = \exp(j\omega)$, we get

$$L(\exp(j\omega)) = -\frac{1-\alpha}{2} - j\frac{1-\alpha}{2}\cot\frac{\omega}{2} \qquad (2.4)$$

and $|L(\exp(j\omega_o))| = 1.0$ implies the unity gain cross-over frequency at

$$\omega_o = 2\sin^{-1}\frac{1-\alpha}{2} \qquad (2.5)$$

The Nyquist plot of the discrete-time loop (2.4) is quite simple and from the plot it is easy to see that the system is stable for all gain $\in(0, 2/1-\alpha)$, and a pure delay $\phi = 90^o - \sin^{-1}(1-\alpha)/2$ will change the number of encirclement by the Nyquist contour, thus making the system unstable.

To get the equivalent PM we transform each element of the loop using the bilinear transformation $s = (z-1)/(z+1)^{-1}$ to get the equivalent continuous loop

$$L(s) = \frac{1-\alpha}{2}\left(\frac{1}{s} - 1\right). \qquad (2.6)$$

From the Nyquist plot of $L(s)$ it is obvious that GM $\in(0, 2/(1-\alpha)$ (the same as found by analyzing the discrete-time Nyquist plot) and a PM = $\cos^{-1}(1-\alpha)/2$.

Theorem 1, although very simple, reveals some intuitively appealing results about GM and PM of MAC. We can make the following remarks.

Remarks:

(i) Since $\alpha\in[0,1]$, the guaranteed upward GM is 2 and the PM is 60° respectively.

(ii) We can always trade-off robustness against the speed of response. As response speed is increased by decreasing α, BW $\omega_o = 2\sin^{-1}(1-\alpha)/2$ increases (which makes sense) with a consequent reduction of robustness in terms of GM and PM.

(iii) We get this remarkable PM even though MAC is an output-feedback controller possibly because the plant is inverted causally through the use of an optimization algorithm in the sense that at each time the algorithm provides the controller with the entire future input sequence. For the same reason, the discrete-time loop has a one pole roll-off for all frequencies - which is rather unusual.

(iv) Theorem 1 ensures that the controller can stabilize the loop for all the plants $\{h_i\}$ belonging to the set $\{h_i \mid h_i = kh_i,\ i=1,\ldots,\tilde{N},\ k\in(0, 2/(1-\alpha))\}$.

PLANT ROBUSTNESS ANALYSIS

The nominal model $\tilde{h}(z)$ is usually different from the actual plant $h(z)$ for various reasons. Sometimes $\tilde{h}(z)$ is deliberately made simple to facilitate the control computation by retaining the modes in the active frequency range. On many occasions it is

difficult to model high frequency modes, and these are simply neglected. Due to ageing, etc., the modes of the actual plant drifts slowly thus introducing low-frequency error. Thus the modelling error $e(z)$ has in almost every case, a dynamic structure; and the information about $e(z)$ must be incorporated in designing a nominal loop. As a minimum amount of information $e(z)$ is expressed as an upperbound on $|e(\exp(j\omega))|$; and the purpose of robustness analysis is to find a requirement on the nominal loop in terms of this upperbound so that the closed loop performance and stability is maintained in the face of modelling uncertainty.

Usually the admissible uncertainties are expressed in two ways: additively or multiplicatively. If we take $\tilde{h}(z)$ as the nominal plant, then in an additively uncertain model, we express the actual plant $h(z)$ as

$$h(z) = \tilde{h}(z) + \Delta h_a(z) \tag{3.1}$$

and in a multiplicatively uncertain model, the actual plant $h(z)$ is

$$h(z) = \tilde{h}(z)(1 + \Delta\tilde{h}_m(z) \tag{3.2a}$$

$$\text{or} \quad h(z) = \tilde{h}(z)\,\Delta h_m(z) \tag{3.2b}$$

For single-loop system, the order of multiplication in (3.2) is irrelevant, but for MIMO cases the order is important because of the non-commutativity of matrices where input channel (left) uncertainty and output-channel (right) uncertainty must be distinguished. Both of the multiplicative forms in (3.2) are often used in analysis, but in this paper we shall be using (3.2b). Note that at nominal values of the plant, $\Delta h_a(z) = \Delta\tilde{h}_m(z) = 0$ and $\Delta h_m(z) = 1$. Also note that the classical GM and PM ensures the stability of a perturbed plant of the form (3.2b). If the GM is k, then $\Delta h_m(z) = k$, and if the PM = n (in the sense of discrete-data system), $\Delta h_m(z) = z^{-n}$. These are undoubtedly a limited class of allowable perturbations and we must consider other possible error-structures in designing the nominal loop. The framework of (3.1) and (3.2) is more general in the sense that it can handle a constant, non-constant and even dynamic model mismatch (say for example unmodelled poles, etc.). Let us rewrite $\tilde{h}(z)$ and $h(z)$ as

$$\tilde{h}(z) = \sum_{i=1}^{\tilde{N}} \tilde{h}_i z^{-i} = z^{-\tilde{N}}\,\tilde{h}_p(z) \tag{3.3a}$$

where $\tilde{h}_p(z) = \sum_{i=1}^{\tilde{N}} \tilde{h}_i z^{\tilde{N}-i}$ = a polynominal in z,

and $h(z) = z^{-N} h_p(z)$,

$$h_p(z) = \sum_{i=1}^{N} h_i z^{N-i} \tag{3.3b}$$

Then by straight forward manipulation, the closed loop characteristic polynominal is

$$\Phi_{cl,p}(z) = z^{N}(z-1)h_p(z) + z^{\tilde{N}}(1-\alpha)\,h_p(z) \tag{3.4}$$

For closed-loop stability, $\Phi_{cl,p}(z)$ must have all the roots strictly inside the unit disk $|z| = 1$. For perfect identification $\tilde{N} = N$, $\tilde{h}_p(z) = h_p(z)$, and $\Phi_{cl,p}(z) = z^{N}(z-\alpha)\tilde{h}_p(z)$. Of course the zeros of $\tilde{h}_p(z)$ and z^{N} will be cancelled eventually leaving the only closed loop pole at $z = \alpha$. However N, the order of the true plant, is usually unknown. In real-world situations, (3.4) can not be evaluated. The actual plant $h(z)$ must be considered as a perturbation of the nominal plant $\tilde{h}(z)$, and the stability conditions must be derived in terms of the nominal sequence $\{\tilde{h}_i\}$ and the perturbation $\Delta h_a(z)$ or $\Delta h_m(z)$. Let us assume that $\Delta h_a(z)$ and $\Delta h_m(z)$ can be expressed as in (3.3), i.e.,

$$\Delta h_a(z) = \sum_{i=1}^{N_a} h_{ai} z^{-i} = z^{-N}\,\Delta h_{ap}(z), \quad \Delta h_{ap}(z) = \text{a polynomial in } z \tag{3.4a}$$

$$\Delta h_m(z) = \sum_{i=1}^{N_m} \Delta h_{mi} z^{-i} = z^{-N_m}\,h_{mp}(z) \tag{3.4b}$$

although the following theorem can be developed without such an explicit form. Note that the index in (3.4b) must start from 0 to accomodate constant multiplicative perturbation. We have the following theorem on robustness:

<u>Theorem 2</u>:

(i) The system is closed-loop stable for all additive perturbations $\Delta h_a(z)$ satisfying

$$|\Delta h_{ap}(z)| < \sqrt{\frac{2-2\alpha\cos\omega+\alpha^2}{1-\alpha}}\;|h_p(z)|, \quad z = \exp(j\omega) \tag{3.5a}$$

(ii) The system is closed-loop stable for all multiplicative perturbations $\Delta h_m(z)$ satisfying

$$|\Delta h_{mp}(z)| < z^{N_m}\,\frac{z-\alpha}{1-\alpha} \tag{3.5b}$$

on the unit circle, where $\Delta h_{ap}(z)$ and $\Delta h_{mp}(z)$ are given by (3.4).

<u>Proof</u>: The proof is straightforward if we express $h(z)$ using the form (3.3) - (3.4), find the corresponding closed-loop characteristic polynomial, and finally use Rouche's theorem to prove (3.5) on the assumption that

the nominal loop is internally stable and hence $(z-\lambda)\,\tilde{h}_p(z)$ has all the roots strictly inside the unit disk $|z| = 1$.

The tests of the type (3.5) are sufficient conditions and generally tend to be conservative. Nevertheless we can make the following remarks:

(i) Both tests (3.5a) and (3.5b) are useful. For example when an actual known model h_i, $i=1,\ldots,N$ is truncated to obtain $\tilde{h}_i$, $i=1,\ldots,\tilde{N}$, $\tilde{N} \le N$, so that $\Delta h_{ai} = h_i$, $i = \tilde{N}, \tilde{N}+1,\ldots,N$ and $\Delta h_{ai} = 0$, $i \le \tilde{N}$, stability around $\tilde{h}_i$ can be obtained from (3.5a).

(ii) For constant multiplicative gain mismatch, i.e. $h_i = kh_i$ for all i, $\Delta h_{mi} = k$ when $i=0$ and $\Delta h_{mi} = 0$ when $i>0$, so that $\Delta h_{mp}(z) = kz^{N_m}$ and test (3.5b) yields that the system is stable for all k such that

$$k - 1 < \frac{\lambda-1}{1-\lambda}, \quad z = \exp(j\omega) \qquad (3.6)$$

But it is easy to see that $\min |\exp(j\omega) - \lambda| = 1-\lambda$ so that (3.6) becomes $k-1 < 1$ which implies $k \in (0,2)$. This clearly shows that these test are conservative. (See remarks (iv) of the previous section).

CLOSED-LOOP IDENTIFICATION

The results of identification of the plant under closed-loop control using MAC are described in this section. The major difficulty in closed-loop identification is that the future plant inputs are correlated with the past outputs due to the feedback. Many identification procedures assume the absence of such correlation, and produce biased estimates or have other difficulties in their presence. Maximum likelihood will handle such correlation, but can be computationally expensive especially if not provided with good initial estimates of the parameters.

For identification in this study, the canonical variate analysis method was used. This approach to stochastic realization was first proposed by Akaike (1975). A recent generalization (Larimore (1983)) extends the method to input-output identification in the presence of noise. The method is based upon a decomposition of the covariance matrix of the past $p(t)$ and future $f(t)$ of the plant input process $u(t)$ and output process $y(t)$ where

$$p^T(t) = (p^T(t), u^T(t), u^T(t-1), y^T(t-1), \ldots)$$

$$f^T(t) = (f^T(t-1), f^T(t+1), \ldots)$$

A canonical variate decomposition of the covariance of $p(t)$ and $f(t)$ determines the important linear combinations of $p(t)$ for prediction of $f(t)$. This is a state-space model is constructed of the form

$$x_{t+1} = \Phi x_t + Gu_t + w_t$$

$$\hat{y}_t = Hx_t + Au_t + Bw_t + v_t$$

where w and v are white noise processed that are independent with covariance matrices Q and R respectively. These white noise processes model the covariance structure of the error in predicting y from u. Computationally, a singular value decomposition of the sample covariance matrix between $p(t)$ and $f(t)$ is used. This decomposition is numerically very well conditioned and stable.

To demonstrate the identification algorithm, the feedback system under MAC control illustrated in Fig. 2 was considered where there is input white noise added prior to observing the plant input and output white noise added prior to observing the plant output with power spectral densities S_I and S_O respectively. The particular plant considered is the very lightly damped missile dynamics model (Mehra et. al.) (1980)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1.4868 & 1.00 \\ -149.43 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 281.11 \end{pmatrix} u$$

$$y = x_1$$

where the states are $x_1 = \alpha$ the angle of attack (rad), $x_2 = q$ the perturbed pitch rate (rad/s), input $u = \delta_e$ the elevator angle (rad), and output $y = \alpha$ the angle of attack (rad). An analysis of the dynamics gives a natural frequency of 12.24 r/s (1.95 Hz) and a damping ratio (ζ) of 0.061.

The canonical variate method was used to identify a second-order system while operating under MAC control with input and measurement noise. No other input nor change in the set point was present, and the system was in statistical steady-state. The presence of an input or varying set point would improve the ability of the algorithms to identify the plant. The plant was approximated by a discrete time system using the exponential transformation at a sample rate of 10 Hz. This was used for the actual plant in the discrete time simulation, and in the MAC control computations the discrete time impulse response was used out to 5 seconds and set to zero at longer times. The true and identified plant models are shown using sample sizes of 100 and 200 in Fig. 2. Note that the identified plant is close to the true even for a substantial amount of measurement noise.

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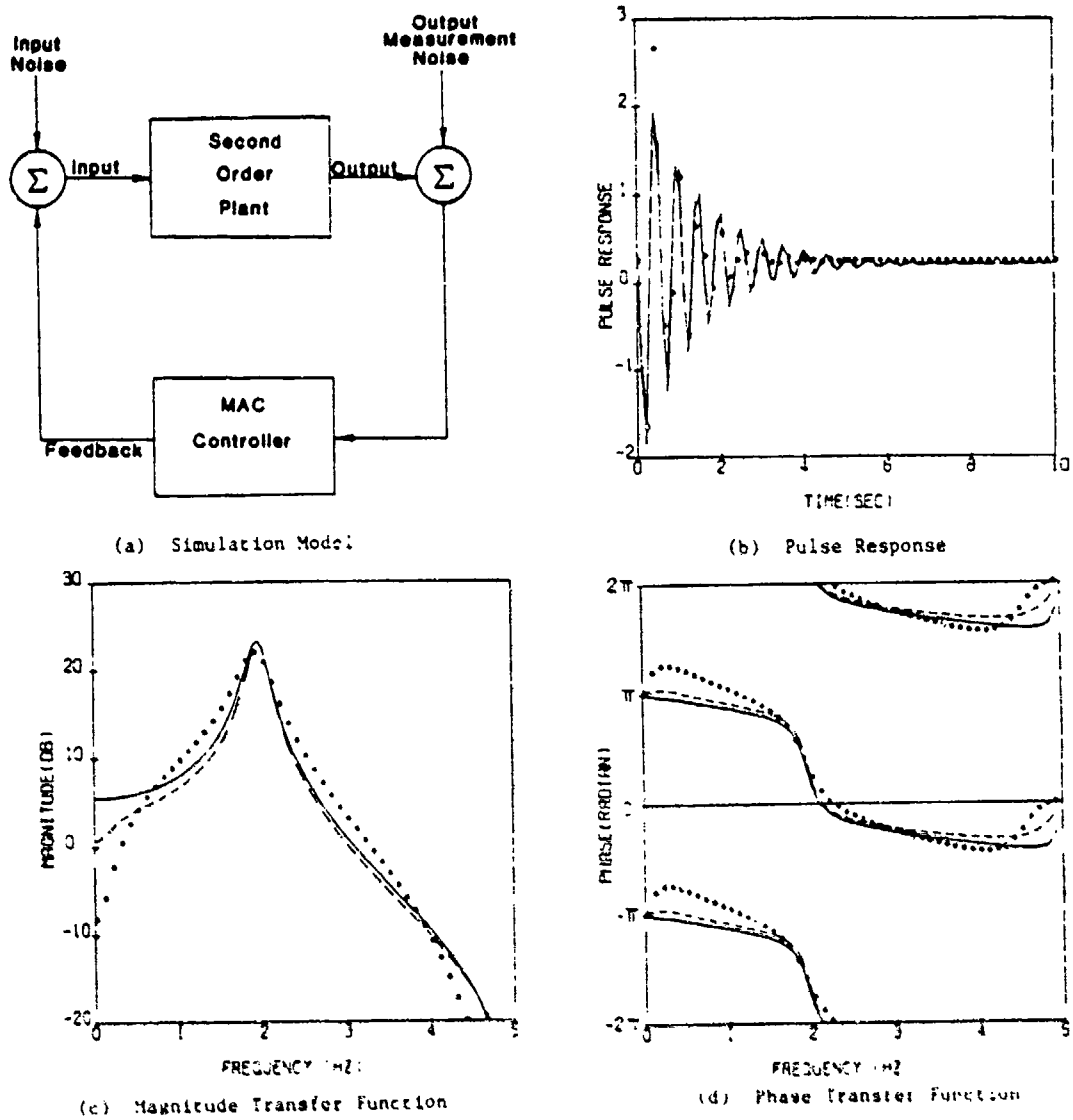

(a) Simulation Model

(b) Pulse Response

(c) Magnitude Transfer Function

(d) Phase Transfer Function

Fig. 2. Identification Under MAC Control. True Plant (Solid), Identified Plant for N=100, S_I=1, S_o=0.01 (Dashed), and for N=900, S_I=1, S_o= 0.5 (Dotted).

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APPENDIX D

MULTIVARIABLE ADAPTIVE MODEL ALGORITHMIC CONTROL

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Proceedings of 23rd Conference
on Decision and Control
Las Vegas, NV, December 1984

MULTIVARIABLE ADAPTIVE MODEL ALGORITHMIC CONTROL[1]

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ABSTRACT

In this paper the multivariable adaptive control problem is addressed using the Model Algorithmic Control (MAC) method in conjunction with the canonical variate identification method. Under some simplifying assumptions multivariable MAC is shown to be equivalent to a classical controller in a unit feedback configuration. Robustness of the MAC controller against unmodelled dynamics is assessed by perturbation analysis. The canonical variate identification method is described in terms of choosing a state of a given order based upon past information to optimally predict the future. The computation is a noniterative algebraic stochastic realization algorithm that involves primarily a singular value decomposition which is numerically very stable and accurate. The canonical variate method is shown to give an optimal choice of instrumental variables, and simulation results show it to be approximately maximum likelihood.

1. MULTIVARIABLE MAC AS A CLASSICAL CONTROLLER

MAC control strategy has been described and analyzed in earlier reports and publications (Mehra et al, 1977, 1979, 1980; Mereau, 1978). The following is an extended version for MIMO plants.

The MAC methodology generates a control sequence by on-line optimization of a cost functional, and the algorithm is suitable for implementation on microprocessors. One of the attractive features of MAC is the clear and transparent relationship between system performance and various design parameters embedded in the design procedure. We assume in the following that the input sequence $u(n)$ is m-dimensional and the output sequence $y(n)$ is p-dimensional. There are five basic elements in MAC:

(i) An actual stable plant, possibly not known exactly, with a pulse response sequence $\{H_n\}$, $n=1,2,\ldots,N$ where each H_n is a pxm dimensional matrix. (We assume for simplicity that the plant has no time delay element and is purely dynamic, i.e. it has no feedthrough term). Then the input sequence $u(n)$ and the output sequence $y(n)$, are related by

$$y(n) = H_1u(n-1) + H_2u(n-2) + \ldots + H_Nu(n-N) \qquad (1.1a)$$

$$\text{or, } Y(z) = H(z)U(z) \qquad (1.1b)$$

where $U(z)$, $Y(z)$ and $H(z)$ are z-transforms of $y(n)$, $u(n)$ and $\{H_n\}$ respectively. Here

$$H(z) = H_1z^{-1} + H_2z^{-2} + \ldots + H_Nz^{-N} = H_p(z)z^{-N}$$

where $H_p(z)$ is a pxm dimensional polynomial matrix in z and is given by

$$H_p(z) = H_1z^{N-1} + H_2z^{N-2} + \ldots + H_N \qquad (1.1c)$$

This is an "all-zero" model and $H_p(z)$ determines the zeros of the plant. The locations of non-minimum phase zeros impose restrictions on the achievable performance of MAC. We must remind the reader that the physical interpretation of a zero in the impulse response description of the plant is different from that of a transmission zero in a rational transfer function (RTF) model (or equivalently difference equation (DE) model) of the plant. Also the physical interpretation of poles of a RTF model as natural modes of a plant are lost in this description.

(ii) An internal model of the plant having the same input-output dimension pxm as that of the actual plant and the pulse response sequence $\{\hat{H}_n\}$, $n=1,2,\ldots,\hat{N}$. The input $u(n)$ is the same as that to the actual plant and therefore the output $\hat{y}(n)$ of the model is given by

$$\hat{y}(n) = \hat{H}_1u(n-1) + \hat{H}_2u(n-2) + \ldots + \hat{H}_{\hat{N}}u(n-\hat{N}) \qquad (1.2a)$$

$$\text{or, } \hat{Y}(z) = \hat{H}(z)U(z) \qquad (1.2b)$$

where, as before,

$$\hat{H}(z) = \hat{H}_p(z)\, z^{-\hat{N}} \qquad (1.2c)$$

and $\hat{H}_p(z)$ is a pxm dimensional polynomial matrix. $\{\hat{H}_n\}$ is generally different from $\{H_n\}$.

(iii) A p-dimensional reference trajectory $y_r(n)$, preferably smooth, initialized on the current output of the actual plant $y(n)$ that leads $y(n)$ to a possibly time varying p-dimensional set point c. If each of the reference trajectories $y_{ri}(n)$ has a first order dynamics with time constant a_i leading to set point c_i, $i=1,2,\ldots,p$ and if the trajectories do not interact with each other, then $y_r(n)$ evolves as

$$y_r(n+1) = A_a\, y_r(n) + (I-A_a)c, \quad y_r(n) = y(n) \qquad (1.3a)$$

$$\text{or, } zY_r(z) = A_aY(z) + (I-A_a)C(z) \qquad (1.3b)$$

where $A_a = \mathrm{diag}(a_i)$ and $C(z)$ is the z-transform of c.

(iv) A closed loop prediction scheme for predicting the future output of the plant according to the scheme

$$y_p(n+1) = \hat{y}(n+1) + y(n) - \hat{y}(n) \qquad (1.4a)$$

$$\text{or, } Y_p(z) = \hat{Y}(z) + z^{-1}[Y(z) - \hat{Y}(z)] \qquad (1.4b)$$

Here $y_p(n)$ is also p-dimensional.

[1] This work was supported by the Air Force Wright Aeronautical Laboratory under Contract No. F33615-82-C-3600

Reprinted from THE 23RD IEEE CONFERENCE ON DECISION AND CONTROL, December 12-14, 1984

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(v) A quadratic cost functional J based on the error between $y_p(n)$ and $y_r(n)$ over a finite horizon T_n (here T_n is an integer):

$$J = \sum_{k=1}^{T_n} [e^T(n+k)W(n+k)e(n+k) \tag{1.5a}$$

$$+ u^T(n+k-1)R(n+k-1)u(n+k-1)]$$

$$= \mathrm{Tr} \sum_{k=1}^{T_n} [W(n+k)e(n+k)e^T(n+k) \tag{1.5b}$$

$$+ R(n+k-1)u(n+k-1)u^T(n+k-1)]$$

where $W(\cdot)$ and $R(\cdot)$ are positive semidefinite time varying weights and $e(n+k) = y_p(n+k)-y_r(n+k)$. In most MAC applications $R(\cdot)$ is set to be zero.

Given (i)-(v), MAC finds an optimal control sequence $\{u^*(n+i-1), i=1,\ldots T_n\}$ by minimizing J over the admissible input sequence $\{u(n+i-1)\in\Omega(i), i=1\ldots T_n\}$. Once the optimal control sequence is computed, the first element of the sequence is applied to the actual plant and the process repeats all over again.

In general, there are no analytic solutions for the control sequence $\{u^*(n)\}$ - it is computed at each step using an algorithm known as IDCOM. Therefore in its most general form, MAC cannot be put into a classical control framework. However under the following simplifying assumptions MAC can be modelled as a unit feedback configuration:

(i) The actual plant $H(z)$ is minimum phase;
(ii) The plant model $\hat{H}(z)$ is minimum phase;
(iii) There are no input constraints, i.e. $\Omega(i)=R^m$ for all i;
(iv) $T_n=1$, i.e. the optimization is carried over one future step ahead. Under this condition MAC is a one-step ahead predictive controller.

In addition, if we assume that the plant model $\hat{H}(z)$ is exactly known, i.e. $\hat{H}(z)=H(z)$, the MAC is equivalent to an inverse control law. However, under the simplifying assumptions, (i)-(iv), it is sufficient to select $u^*(n)$ to satisfy

$$y_p(n+1) = y_r(n+1) \text{ for all } n\geq 0 \tag{1.6}$$

for a minimum of the cost function J. The assumptions (i)-(iii) ensure the existence of an optimum control $u^*(n)$ that satisfies (1.4) - the resulting optimal cost J^* is zero in this case. However $U^*(z)$ is then implicitly generated by $Y_p(z)=Y_r(z)$ so that

$$U^*(z) = [(z-1)\hat{H}(z)+(I-\Lambda_a)H(z)]^{-1}[I-\Lambda_a]C(z) \tag{1.7a}$$

$$Y(z) = H(z)[(z-1)\hat{H}(z)+(I-\Lambda_a)H(z)]^{-1}[I-\Lambda_a]C(z) \tag{1.7b}$$

Equations (1.7a) and (1.7b) relate the setpoint $C(z)$ with the optimal input sequence $U^*(z)$ and output sequence $Y(z)$. It is easy to see that this simplified form of MAC is equivalent to the following MIMO unit feedback configuration (we have dropped henceforth the the * superscript).

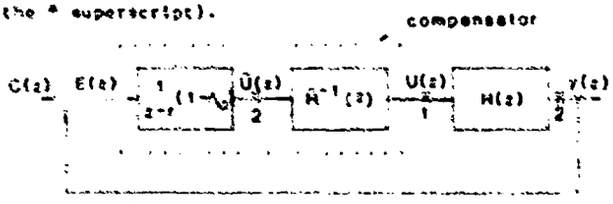

Figure 1. MAC as a Classical Controller

To see that the setup in Figure 1 indeed represents equation (1.7), note that at point 1 we have

$$U(z) = \frac{1}{z-1}\hat{H}^{-1}(z)(I-\Lambda_a)E(z)$$

$$= \frac{1}{z-1}\hat{H}^{-1}(z)(I-\Lambda_a)[C(z)-H(z)U(z)]$$

Multiplying both sides of this equation by $(z-1)\hat{H}(z)$ and rearranging we have

$$[(z-1)\hat{H}(z)+(I-\Lambda_a)H(z)]U(z)=(I-\Lambda_a)C(z)$$

from which (1.7a) and (1.7b) follow. The block within the dashed line in Figure 1 can be thought of as a dynamic controller of the classical type. The loop transfer function at the plant input (point 1) is given by

$$L(z)= \frac{1}{z-1}\hat{H}^{-1}(z)(I-\Lambda_a)H(z) \tag{1.8}$$

and determines the robustness of the feedback configuration at this point. When we have perfect identification, i.e. $H(z) = \hat{H}(z)$, then at point 2

$$\tilde{U}(z) = Y(z) = \frac{1}{z-1}(I-\Lambda_a)E(z)$$

$$\text{or} \quad \tilde{U}(z) = \frac{1}{z-1}(I-\Lambda_a)[C(z)-\tilde{U}(z)]$$

$$\text{or} \quad z\tilde{U}(z) = \Lambda_a\tilde{U}(z)+(I-\Lambda_a)C(z) \tag{1.9}$$

Equation (1.9) is equivalent to

$$\tilde{u}(n+1) = \Lambda_a\tilde{u}(n)+(I-\Lambda_a)c, \quad \tilde{u}(n) = y(n)$$

which shows that $\tilde{u}(n)$ is the reference trajectory sequence $y_r(n)$ as shown in equation (1.3a). This means that when the plant model is known exactly, the control sequence $U(z)$ is generated as

$$U(z) = \hat{H}^{-1}(z)\tilde{U}(z) = \hat{H}^{-1}(z)Y_r(z) \tag{1.10a}$$

Therefore the output of the actual plant is

$$Y(z) = H(z)U(z) = Y_r(z) \tag{1.10b}$$

which shows that in steady state the plant output $y(n)$ is identical to the reference trajectory $y_r(n)$ - perfect tracking has been achieved. Equation (1.10a) clearly shows the need for $H(z)$ to be minimum phase. This analysis has revealed another interesting property of MAC. Exact tracking could as well be achieved by inverting the plant to generate the sequence $u(n)$ in an open-loop configuration, but in MAC it does so in a closed-loop configuration. Therefore the additional benefits of a feedback configuration such as disturbance rejection, sensitivity reduction, etc, are also obtained while simultaneously achieving exact tracking.

2. ROBUSTNESS ANALYSIS OF MINI-MAC

In the following analysis we describe MAC using a rational transfer function or difference equation (DE) model. It can be shown that the simplified MAC using DE description of the dynamics is also equivalent to the unit feedback configuration in Figure 1. The advantage of using this description is that the robustness of the closed loop can be examined in terms of the recently developed criteria employing the loop transfer function and return difference function at appropriate points in the loop.

Let us analyze the loop transfer function $L_2(z)$ at the output of the plant, i.e. at point 2. $L_2(z)$ is given by

$$L_2(z) = \frac{1}{z-1} H(z)\hat{H}^{-1}(z)(I-A_a) \tag{2.1}$$

Here $\hat{H}(z)$ is the model of the actual plant. In a nominal design, the actual plant $H(z)$ is assumed to be equal to the model $\hat{H}(z)$, i.e. $H(z) = \hat{H}(z)$. However the return difference function $H_2(z)$ is then

$$H_2(z) = I+L_2(z) = \frac{1}{z-1}\left[(z-1)I+H(z)\hat{H}^{-1}(z)(I-A_a)\right] \tag{2.2}$$

The closed loop poles are given by the zeros of $\det(H_2(z))$. For the nominal loop, i.e. $H(z)=\hat{H}_2(z)$, the closed loop poles are given by the zeros of

$$\det(zI-A_a)=0 \tag{2.3}$$

which shows that the plant dynamics are cancelled and the overall behavior of the loop is governed by the reference trajectory dynamics as given by the poles a_1, $a_2 \ldots a_p$. In this case each of the p-outputs is identical to the corresponding reference trajectory $y_{ri}(n)$, $i=1, 2,\ldots p$.

This property of MAC is also obtained if we compute the overall transfer function $T_2(z)$ from the reference trajectory set point $C(z)$ to the output $Y(z)$, i.e. $Y(z) = T_2(z)C(z)$. Since $L_2(z) = (I-A_a)/(z-1)$ we have

$$T_2(z) = L_2(z)(I+L_2(z))^{-1} = (I-A_a)(zI-A_a)^{-1} \tag{2.3a}$$

Since A_a is diagonal, $T_2(z)$ is also diagonal. This shows that the overall transfer function is non-interacting: any change in the reference trajectory parameter in the i-th input channel affects the output in i-th channel only; the other output channels are not affected at all. This decoupling property of MAC has made it very popular in industries where the practicing engineer always prefers a decoupling control strategy. This characteristic of MAC as an output feedback controller is outstanding.

It is straightforward to compute the gain margin from (2.1) if we recall that it is the tolerence of the nominal loop to a multiplicative perturbation. In this case the gain margin is given by the range of values of a diagonal matrix $K=\mathrm{diag}(k_i)$ such that the perturbed loop remains stable. Here the perturbed loop $L_{2p}(z)$ is given by

$$L_{2p}(z) = \frac{1}{z-1} KH(z)\hat{H}^{-1}(z)(I-A_a) = \frac{1}{z-1} K(I-A_a) \tag{2.4}$$

The closed loop poles are given by

$$\det((z-1)[I+K-A_a])=0 \tag{2.5}$$

If A_a is diagonal and if z_i are the roots of equation (2.5), then the closed loop poles z_i are given by

$$z_i-k_i-1-k_i a_i=0 \text{ or } z_i=k_i a_i+1-k_i \tag{2.6}$$

Clearly the perturbed loop is stable if for all k_i, $|z_i|<1$. This immediately implies that

$$0<k_i<\frac{2}{1-a_i} \tag{2.7}$$

which agrees with the SISO results obtained earlier. This is a satisfactory result since MAC is an output-feedback controller and not a state-feedback one. Equation (2.7) shows that if any reference trajectory has a fast dynamics, i.e. $a_i \to 0$, then MAC has an upward gain margin of 6 db on that channel. As a matter of fact 6 db upward gain margin is a guaranteed one for each channel. However on the other hand by slowing down the reference trajectory, i.e. by making $a_i \to 1.0$, the upward gain margin at each channel can be increased to infinity - which is an unusual result for an output feedback control scheme.

Using the above analysis, the tolerence of the nominal loop to any perturbation K (not necessarily diagonal) can be obtained. If the perturbation is dynamic, the analysis is slightly complicated as shown in the following.

We select the internal model $\hat{H}(z)$ and therefore $\hat{H}(z)$ is completely known to us. On the otherhand the plant $H(z)$ is not known to us exactly. It is customary to think that the actual plant $H(z)$ lies in a neighborhood of $\hat{H}(z)$. If we define this neighborhood by an additive perturbation $\Delta H_a(z)$ that satisfies

$$\bar{\sigma}(\Delta H_a(e^{j\omega}))<a(\omega),\ 0<\omega<2\pi, \tag{2.8}$$

then we assume that $H(\cdot)$ lies among the class

$$\hat{H}(z)+\Delta H_a(z) \tag{2.9}$$

where $\Delta H_a(z)$ is given by (2.8). Here $\bar{\sigma}(X)$ denotes the maximum singular value of X and $a(\omega)$ is a frequency dependent function that is usually known to a designer from his apriori experience with the system. For convenience of the analysis we assume that $\Delta H_a(z)$ is analytic in $|z|>1$. To analyze The worst possible case, suppose the actual plant $H(z)$ lies on the boundary of the class of systems given in (2.9), i.e.

$$H(z) = \hat{H}(z)+\Delta H_a(z)$$

The perturbed loop $L_{2p}(z)$ is then

$$L_{2p}(z)=\frac{1}{z-1}\left[\hat{H}(z)+\Delta H_a(z)\right]\hat{H}^{-1}(z)(I-A_a) \tag{2.10}$$

$$=\frac{1}{z-1}\left[I+\Delta H_a(z)\hat{H}^{-1}(z)\right]\left[I-A_a\right] \tag{2.11}$$

The closed loop poles of the perturbed loop is given by the zeros of

$$\det(I+L_{2p}(z))=0 \tag{2.12}$$

and we want to find some condition on $\hat{H}(z)$ such that the zeros in (2.12) lie within the unit circle $|z|<1$. The nominal loop transfer function $L_2(z)$ and return difference function $H_2(z)$ are obtained from (2.1) as

$$L_2(z)=\frac{1}{z-1}(I-A_a) \tag{2.13a}$$

$$H_2(z)=I+L_2(z) = \frac{1}{z-1}(zI-A_a) \tag{2.13b}$$

Clearly the perturbed loop transfer function $L_{2p}(z)$ can be considered as a perturbation $\Delta L_2(z)$ of the nominal loop transfer function $L_2(z)$ where

$$\Delta L_2(z)=\frac{1}{z-1}\left[\Delta H_a(z)\hat{H}^{-1}(z)\right]\left[I-A_a\right] \tag{2.14}$$

and $L_{2p}(z)=L_2(z)+\Delta L_2(z)$.

Equation (2.12) then reduces to

$$\det(I+L_2(z)+\Delta L_2(z))=0. \tag{2.15}$$

The nominal loop is stable, and, as we saw earlier, it has p number of closed loop poles in

$|z|<1.0$. Then a sufficient condition that $\det(I+L_{2p}(z))$ in (2.15) has also p number of zeros in $|z|<1.0$ is (Sain (1981))

$$\bar{\sigma}(\Delta L_2(e^{j\omega}))<\underline{\sigma}(I+L_2(e^{j\omega})) \tag{2.16}$$

where $\sigma(X)$ denotes the smallest singular value of X. If $\Delta L_2(z)$ satisfies (2.16), the perturbed loop is stable. Equation (2.16) can further be simplified as follows. First note that $\bar{\sigma}(mX)<\underline{\sigma}(mY)$ implies $\bar{\sigma}(X)<\underline{\sigma}(Y)$. Then using (2.14), and (2.13) in (2.16), we have

$$\bar{\sigma}\,[\Delta H_a(e^{j\omega})\hat{R}^{-1}(e^{j\omega})(I-\Lambda_\alpha)]<\underline{\sigma}(e^{j\omega}I-\Lambda_\alpha) \tag{2.17}$$

which is implied by

$$\bar{\sigma}[\Delta H_a(e^{j\omega})]\,\underline{\sigma}[\hat{R}(e^{j\omega})]\,\bar{\sigma}[I-\Lambda_\alpha]<\underline{\sigma}(e^{j\omega}I-\Lambda_\alpha) \tag{2.18}$$

Let $\alpha_{min} = \min_i \alpha_i$ where each α_i is such that $0\le\alpha_i\le 1$. Then

$$\bar{\sigma}[I-\Lambda_\alpha]=1-\alpha_{min} \tag{2.19}$$

Equation (2.18) is satisfied if

$$\underline{\sigma}[\hat{R}(e^{j\omega})] < \frac{a(\omega)(1-\alpha_{min})}{\underline{\sigma}(e^{j\omega}I-\Lambda_\alpha)} \tag{2.19}$$

The RHS of (2.19) is precomputable. If the identified model $\hat{R}(z)$ satisfies (2.19), then the MAC control law is stable for all plants under the class given by (2.8). However, we are still looking at the physical interpretation of the condition given in (2.19). For SISO systems, the singular value is replaced by the magnitude function.

Similar relations can be drived for multiplicative perturbations and for modelling uncertainties at the plant input.

3. SYSTEM IDENTIFICATION USING CANONICAL VARIABLES

Proposed methods for multivariable parameter identification are plagued with problems of computational complexity and unreliability. For iterative optimization approaches such as maximum likelihood, there is no apriori bound on the number of iterations required for convergence unless a good initial estimate is available. The computations involved in many schemes become illconditioned if the parameter identifiability is illconditioned which occurs frequently in practice. The canonical variate method involves solution of an algebraic problem involving primarily a singular value decomposition which is numerically accurate and stable for any set of data. The system is identified in an implicit state space form which avoids the identifiability problem.

The approach to system identification using generalized canonical variables is described in some detail in Larimore (1983b). That approach involves consideration of the past $p(t)$ and future $f(t)$ of a vector process at a time t defined as

$$p^T(t) = (y^T(t),\ u^T(t),\ y^T(t-1),\ u^T(t-1),\ \ldots\ ) \tag{3.1}$$

$$f^T(t) = (y^T(t+1),\ y^T(t+2),\ \ldots\ ) \tag{3.2}$$

where $u(t)$ is the input and $y(t)$ is the output of an unknown system with state space structure of the form

$$x(t+1) = Ax(t) + Bu(t) + w(t) \tag{3.3}$$

$$y(t) = Hx(t) + Du(t) + v(t) \tag{3.4}$$

with $v(t)$ a measurement noise and $w(t)$ a process noise with respective cross spectral density matricies R and Q. From the theory of Markov processes and in particular the theory of stochastic realization, the minimal state vector defines the information from the past relevant to the future of the process and is called the predictor space (Akaike, 1974a).

The approach of canonical variables for system identification is to determine the optimal set of linear combinations $m(t)$ of the past $p(t)$ that best predict the future $f(t)$ in terms of minimizing the prediction error

$$E||f - \hat{f}||^2 = E[(f - \hat{f})^T \Sigma_{ff} (f - \hat{f})] \tag{3.5}$$

where Σ_{ff} is the covariance matrix of the future f and $\hat{f}$ is the best prediction of f based upon the memory $m(t)$. This optimization problem involves the optimal selection of the dimension of $m(t)$ as well as the optimal selection of the linear combinations of the past.

The problem of minimizing (4.5) is precisely a generalization of the classical canonical correlation analysis problem of mathematical statistics (Hotelling, 1936). Modern computational procedures use a generalized singular value decomposition (SVD) (Golub, 1969) involving the covariance matricies of the past and future. The generalized SVD determines transformations J and L and a diagonal matrix D such that

$$J\,\Sigma_{pf}\,L = \mathrm{Diag}(\gamma_1>\ldots>\gamma_h>0,\ldots,0) = D \tag{3.6}$$

$$J\,\Sigma_{pp}\,J = I\ ;\ L\,\Sigma_{ff}\,L = I \tag{3.7}$$

The transformations can be interpreted as defining a new set of coordinates for the past and future in which the covariances are D, I, and I respectively as given in the last equations (3.6) and (3.7). For a full order state model, the optimal memory or state $x(t)$ is related to the past $p(t)$ in terms of the first h canonical variables as $m(t) = (I,0)Jp(t)$, i.e. the first h components of the canonical predictor variables $Jp(t)$. A minimal order realization is obtained with this choice of state. The computation of the state space matricies is given in Larimore (1983b).

In system identification, the covariance matricies are not known but are estimated from the observations. The statistical determination of rank in the canonical variate analysis is given approximately using standard canonical correlation analysis methods (Akaike, 1976). A more refined comparison between the different order models is given by use of the Akaike information criterion (AIC) which is asymptotically optimal in minimizing entropy (Shibata, 1981). The use of entropy measures such as the AIC has a fundamental justification in terms of the basic statistical principles of sufficiency and repeated sampling (Larimore, 1983a).

The minimal order realization can be determined from the canonical correlation analysis with k=h. However with k<h when a reduced memory is selected, the approximate system does not in general minimize the prediction error for that order. This is because the reduced rank canonical variables are not in general recursively computable. However in the case of the statistical rank determination problem, there is an insignificant difference between the state of the realized system corresponding to the statistically optimum choice of order and the full rank canonical variables.

The instrumental variables method has a natural interpretation in terms of the generalized canonical variate problem. In the instrumental variables approach, the state equations (3.3) are considered as

unobserved structural relationships that are indirectly observed through the noisy measurement equations (3.4). A vector m(t) of instrumental variables is constructed which is hopefully close to the true state x(t). This is used in place of the true state in solving the problem. This apparently works well for an appropriate choice of the instrumental variables.

A more general problem is the optimal choice of instrumental variables as posed by Rao(1965, 1979). This is formulated as finding the optimal choice of k linear combinations of the past p(t) that predict the future f(t) as measured in terms of the squared error $(f - \hat{f})^T(f - \hat{f})$. This is precisely the generalized canonical variate problem (Larimore, 1983b) with weighting matrix $\Theta = I$. If k is chosen as full rank, then the memory and the state space realization are independent of the weighting matrix $\Theta=I$ replacing Σ_{ff} in (3.5) and (3.7) (Larimore, 1983b). However, for lower rank $k<h$, there can be a considerable difference between the state space and reduced order system for different weightings Θ (Larimore, 1983b). The squared error relates to energy while the canonical correlation analysis relates to the statistical significance of the problem. The canonical correlation analysis can be viewed as an optimal choice of the instrumental variables using the appropriate weighting (3.5) of the prediction errors for the determination of the statistically significant number of states.

Time recursive methods using instrumental variables and approximate maximum likelihood (IV-AML) are claimed to be approximately efficient parameter identification methods for large samples as shown in simulation examples (Young, 1979). This is shown by Monti Carlo simulation and by estimating the parameter by Monte Carlo simulation that the canonical correlation method also gives efficient identification of the system dynamics. This is done by evaluating the spectral estimation error.

4. EFFICIENCY OF CANONICAL CORRELATION ANALYSIS

The asymptotic efficiency of system identification using canonical correlation analysis is discussed in this section. An entropy measure of the error between the true and identified system is used to measure the error in estimating the spectrum.

To directly describe errors in the identified system, a recently developed entropy measure of the system identification errors involving the power spectrum is used. The entropy measure is a fundamental measure of the error in approximating a system using a model selection procedure that may include the choice of model order such as state space dimension. In a predictive inference setting, the entropy measure follows naturally from the fundamental statistical principles of sufficiency and repeated sampling (Larimore,1983a).

Consider a vector stationary Gaussian process $\ldots,y(-1),y(0),y(1),\ldots$, with power cross-spectral density matrix $S(\omega)$, and suppose that some parameter estimation or model fitting scheme is used to choose a model $\hat{S}(\omega)$ based upon a sample of N time observations. The negative entropy per unit time, or negentropy for brevity, for measuring the error between the true spectrum $S(\omega)$ and the model selection procedure which estimates the spectrum $\hat{S}(\omega)$ can be expressed as

$$N(S,\hat{S}) = E\,\frac{1}{2}\int_{-\pi}^{\pi}\{\log S(\omega)\hat{S}^{-1}(\omega) + \mathrm{tr}[I - S(\omega)\hat{S}^{-1}(\omega)]\}\,\frac{d\omega}{2\pi} \tag{4.1}$$

$$= E\,\frac{1}{4}\int_{-\pi}^{\pi}\mathrm{tr}\{S^{-1}(\omega)[\hat{S}(\omega) - S(\omega)]\}^2\,\frac{d\omega}{2\pi}$$

where expectation is taken with respect to the parameter estimates, and the approximation holds to second order in the elements of $\hat{S}-S$. The last expression is a generalization of the integrated squared relative error in estimating the power spectrum S, and there is an interpretation in the multivariable case in terms of the principle components of the power cross-spectral matrix (Larimore,1984).

In the case of ML estimation of the parameters θ, the estimates are asymptotically consistent and efficient achieving the Cramer-Rao lower bound $E(\theta-\hat{\theta})(\theta-\hat{\theta})^T \geq F^{-1}$. Using this lower bound, the lower bound on the entropy measure of spectral accuracy is derived as $E\{N(S,\hat{S})\} \leq k/2N$. This implies the lower bound (Larimore, 1982, 1984)

$$E\int_{-\pi}^{\pi}\mathrm{tr}\{S^{-1}(\omega)[\hat{S}(\omega) - S(\omega)]\}^2\,\frac{d\omega}{2\pi} \geq \frac{2k}{N} \tag{4.2}$$

on the expected integral of the relative square error. This is a fundamental bound on the achievable accuracy in spectral estimation.

To demonstrate the efficiency of the canonical variate method of system identification relative to MLE, the spectral accuracy of the method was compared with the lower bound (4.2). The autoregressive moving average (ARMA) process

$$\begin{aligned} y(t) = {} & 1.3136\,y(t-1) - 1.4401\,y(t-2) + 1.0919\,y(t-3) \\ & - 0.83527\,y(t-4) + w(t) + 0.17921\,w(t-1) \\ & + 0.82020\,w(t-2) + 0.26764\,w(t-3) \end{aligned} \tag{4.3}$$

of order (4,3) respectively for the AR and MA parts with the noise variance of w as Q = 1.72581E-2 was used to simulate samples of size N=800. This process was analyzed by Gersch and Sharp (1973) and Akaike (1974b) to show the increased accuracy of ARMA models over AR models. The canonical variate analysis was done on sample covariance matricies involving 16 lags of the past and future.

Figure 2(a) shows the power spectrum of the true and estimated models for 6 Monte Carlo trials of N=800 samples each. The estimated spectrum appears to be close to the true spectrum with a small bias at the peaks and troughs. Figure 2(b) gives the squared relative error of the variability, excluding the bias, in estimating the power spectrum at each frequency along with the lower bound for the expected squared relative error. The average of the errors over the 6 Monte Carlo trials is very close to the lower bound demonstrating the relative efficiency of the canonical variate method.

In Larimore et al (1983), the identification of a very lightly damped plant under closed-loop control using MAC is simulated in Monte Carlo trials. The adequacy of the identified model is demonstrated by comparing the fitted impulse response and transfer function with the known plant dynamics.

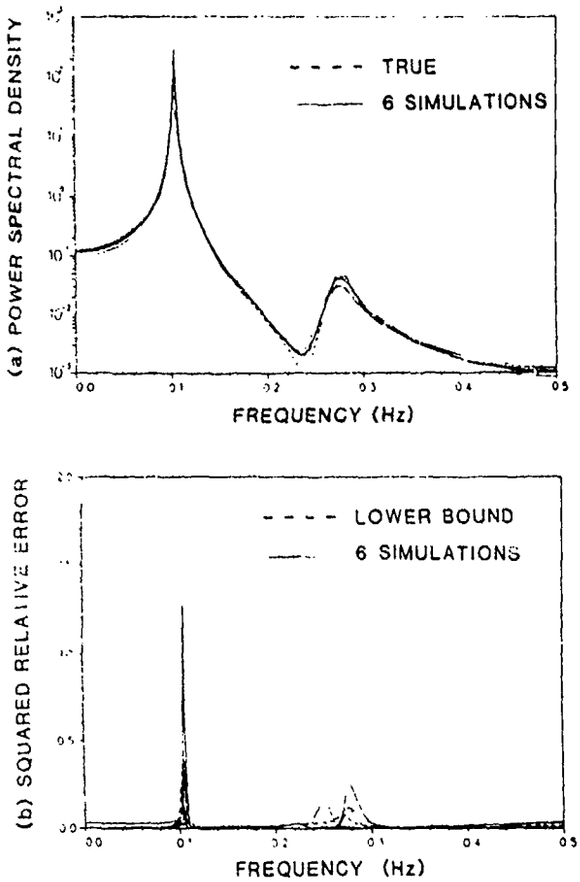

Figure 2. Variability of Monte Carlo Simulations in Terms of (a) Power Spectral Density and (b) Squared Relative Error for Simulations and Lower Bound Average

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APPENDIX E

SIMULTANEOUS CONFIDENCE BANDS FOR EFFICIENT PARAMETRIC SPECTRAL ESTIMATION

By: W.E. Larimore

SIMULTANEOUS CONFIDENCE BANDS FOR EFFICIENT PARAMETRIC MULTIVARIATE SPECTRAL ESTIMATION

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SUMMARY

Simultaneous confidence bands are derived for efficient parametric multivariate spectral estimation methods. Spectral estimation is considered for time series or random fields with a multidimensional 'time', and the presence of an exogenous input is included. The usual vector result for simultaneous confidence intervals in the Scheffe method is generalized to the case of a complex Hermitian matrix which is required in the multivariate spectral estimation problem. The main result is the derivation of simultaneous confidence bands for the matrix elements of arbitrary matrix functions such as transfer, power spectral, correlation, and impulse response functions. For spectral quantities, the computation is done entirely in the frequency domain. The results are applied to confidence bands on the relative squared error in spectral estimation.

Some key words: Achievable accuracy, Simultaneous confidence bands, Spectral estimation error.

Submitted to *Biometrika*

1. SPECTRAL ESTIMATION PROBLEM

The problem of determining the statistical accuracy in identifying a model for a stationary multiple time series is considered in this paper. The cases of the presence or absence of an exogenous input or additive measurement noise are included. Consider the general case where the vector $x(t)$ is the exogenous input and the vector $y(t)$ is the observed endogenous output of a system which may include other unknown excitations and measurement noise. Thus consider the jointly stationary gaussian vector time series $x(t)$ and $y(t)$, $t = \ldots,-1,0,1,\ldots$, with power cross-spectral matrices $S_{xx}(\omega,\theta)$, $S_{xy}(\omega,\theta)$, $S_{yy}(\omega,\theta)$ parameterized by θ, and denote the power cross-spectral matrix of the joint vector $(x^T(t),y^T(t))^T$ as $S(\omega,\theta)$.

Statistical inference is considered on a class of linear Gaussian processes parameterized by θ. Specifying a parametric model for the conditional process $y(t)$, $t \geq s$, given $x(t)$, $t < s$, implies a causal linear model of the form

$$y(t) = q(t) + \sum_{\tau=0}^{\infty} h(t-\tau;\theta)x(t) = q(t) + r(t)$$

where $h(t;\theta)$ is a causal linear system giving the response $r(t)$ in $y(t)$ due to the past exogenous input $x(t)$ and where $q(t)$ is the error in predicting $y(t)$ by $r(t)$. From linear prediction theory, the transfer function of $h(t;\theta)$ is $H(\omega;\theta) = S_{yx}(\omega,\theta)S_{xx}^{-1}(\omega,\theta)$, and the error $q(t)$ in predicting $y(t)$ is uncorrelated with $r(t)$ with power spectrum $S_{qq}(\omega;\theta) = S_{yy}(\omega,\theta) - H(\omega,\theta)S_{xx}(\omega,\theta)H^*(\omega,\theta)$. Note that any class of parameterized models $S(\omega,\theta)$ can be equivalently specified by the parameterized models $(S_{qq}(\omega,\theta),H(\omega,\theta))$ which will prove more convenient.

It is convenient to work entirely in the frequency domain and specify the probability distribution and likelihood functions in terms of the power cross-spectral

density matrices and Fourier coefficients. For simplicity the time series case with t a scalar is developed below, however the results generalize easily to the random field case of a vector t . Then asymptotically the log likelihood function is given following Whittle (1953) and Larimore (1977) with $Q(\omega) = Y(\omega) - R(\omega)$ and using the relationship $AQ(\omega)X^*(\omega) = 0$ by

$$\log p(x,\theta) = -\frac{N}{2}\log 2\pi - \frac{N}{2}\int_{-\pi}^{\pi}[\log|S_{qq}(\omega)| + Q^*(\omega)S_{qq}^{-1}(\omega)Q(\omega)]\frac{d\omega}{2\pi}$$

and the elements of the gradient vector $\partial \log p/\partial\theta$ and Fisher information matrix $F(\hat{\theta})$ are

$$\frac{\partial \log p}{\partial \theta_i} = -\frac{N}{2}\int_{-\pi}^{\pi} tr[\{I - S_{qq}^{-1}(\omega)Q(\omega)Q^*(\omega)\}S_{qq}^{-1}(\omega)\frac{\partial S_{qq}(\omega)}{\partial \theta_i}$$

$$- X(\omega)Q^*(\omega)S_{qq}^{-1}(\omega)\frac{\partial H(\omega)}{\partial \theta_i}]\frac{d\omega}{2\pi}$$

$$F_{ij}(\theta) = -E\{\frac{\partial^2 \log p}{\partial \theta_i \partial \theta_j}\}$$

$$= \frac{N}{2}\int_{-\pi}^{\pi} tr[S_{qq}^{-1}(\omega)\frac{\partial S_{qq}(\omega)}{\partial \theta_i}\{S_{qq}^{-1}(\omega)\frac{\partial S_{qq}(\omega)}{\partial \theta_j} + S_{xx}^{-1}(\omega)\frac{\partial H(\omega)}{\partial \theta_j}\}]\frac{d\omega}{2\pi} \qquad (1)$$

2. SIMULTANEOUS CONFIDENCE BANDS

Let $\gamma \in \Gamma$ be a variable such as frequency or time, and consider a p-dimensional complex vector $f(\gamma,\theta)$ with components that are functions of γ and θ having continuous second derivatives with respect to the parameters θ. For example, the elements of the vector function $f(\gamma,\theta)$ could be the elements of the spectral matrix S,

the squared magnitude coherences, the impulse response functions of a spectral factor, or the covariance functions of the process. Asymptotically

$$f(\gamma,\hat{\theta}) - f(\gamma,\theta) = f_\theta(\gamma,\hat{\theta})(\hat{\theta} - \theta) \tag{2}$$

where $f_\theta(\gamma,\hat{\theta})$ denotes the matrix of partials $\partial f(\gamma,\theta)/\partial\theta^T$ evaluated at $\theta = \hat{\theta}$. This expansion and the Scheffe´ method (Scheffe´, 1953, 1959, p.68-70) of simultaneous confidence intervals as applied in Newton & Pagano (1984) lead to simultaneous confidence bands in the univariate case. For multivariate processes, it is of considerable interest to extend these results to simultaneous confidence bands on vector and matrix functions of the parameters, e.g. the spectral matrix. The extension that we will consider is the quadratic form

$$\{f(\gamma,\hat{\theta}) - f(\gamma,\theta)\}^* P(\gamma)\{f(\gamma,\hat{\theta}) - f(\gamma,\theta)\}$$

which will be bounded as a function of γ. In the multivariate case, there is a choice to be made for P. For reasons of invariance and to obtain an equally tight confidence bound on any linear combination of $f(\gamma,\hat{\theta}) - f(\gamma,\theta)$, P is naturally chosen as the inverse of the covariance of (2).

In the sequel, a general P is used and then specialized to this natural choice. The basic mathematical result needed for such an extension is given in the Appendix and is used to prove the following theorem on simultaneous confidence intervals.

Theorem 1. Consider a parametric family of stationary Gaussian vector processes with power cross-spectral density matrices $S(\gamma,\theta)$ for $\theta\in\Theta$ satisfying regularity conditions (Whittle, 1953), and for which the parameters are locally identifiable so that the Fisher information matrix $F(\theta)$ as given by (1) is full rank. Let

$y(1), y(2),\ldots,y(N)$ be a sample realization and $\hat{\theta}$ be an asymptotically normal and efficient estimator of θ. Let $P(\gamma,\hat{\theta})$ be a Hermitian matrix. Then as $N \to \infty$, the probability is at least $1-\alpha$ that simultaneously for all $\gamma \in \Gamma$ the true p-vector function $f(\gamma,\theta)$ is bounded by

$$\{f(\gamma,\hat{\theta}) - f(\gamma,\theta)\}^* P(\gamma,\hat{\theta})\{f(\gamma,\hat{\theta}) - f(\gamma,\theta)\}$$

$$\leq X^2_{\alpha,q}\ tr\ f_\theta(\gamma,\hat{\theta})\, F^{-1}(\hat{\theta})\, f^*_\theta(\gamma,\hat{\theta}) P(\gamma,\hat{\theta})$$

where q is the dimension of the vector θ and where $X^2_{\alpha,q}$ is the upper α critical point of the chi-squared distribution on q degrees of freedom.

Proof: As shown by Rothenberg (1971), the parameters are locally identifiable if and only if the Fisher information is full rank. Let $f(\gamma)$ and $\hat{f}(\gamma)$ denote $f(\gamma,\theta)$ evaluated at θ and $\hat{\theta}$ respectively. The vector random variable $N^{1/2}\{\hat{f}(\gamma) - f(\gamma)\}$ is asymptotically distributed as the normal random vector $N^{1/2} f_\theta(\gamma,\hat{\theta})(\hat{\theta}-\theta)$. Asymptotically $(\hat{\theta}-\theta)^T F(\hat{\theta})(\hat{\theta}-\theta)$ is a $X^2_{\alpha,q}$ random variable, where $F(\theta)$ is proportional to sample size N as in (1). So the probability is $1-\alpha$ that the true θ satisfies $(\hat{\theta}-\theta)^T M(\hat{\theta}-\theta) \leq 1$ where $M = F(\hat{\theta})/X^2_{\alpha,q}$. From the Appendix, this inequality is satisfied if and only if $\| H(\hat{\theta}-\theta)\|^2 \leq trHM^{-1}H^*$ for all $p \times q$-dimensional matrices H. Since the set $\{H = P^{1/2}(\gamma,\hat{\theta}) f_\theta(\gamma,\hat{\theta})$ for $\gamma \in \Gamma\}$ is possibly a proper subset of all $p \times q$-dimensional matrices H, it follows that asymptotically with probability at least $1-\alpha$ the inequality

$$N\{\hat{f}(\gamma) - f(\gamma)\}^* P(\gamma,\hat{\theta})\{\hat{f}(\gamma) - f(\gamma)\}$$

$$= N\{f_\theta(\gamma,\hat{\theta})(\hat{\theta}-\theta)\}^* P(\gamma,\hat{\theta})\{f_\theta(\gamma,\hat{\theta})(\hat{\theta}-\theta)\}$$

$$\leq N X^2_{\alpha,q}\ tr\ f_\theta(\gamma,\hat{\theta})\, F^{-1}(\hat{\theta})\, f^*_\theta(\gamma,\hat{\theta}) P(\gamma,\hat{\theta}) \tag{3}$$

is satisfied simultaneously for all $\gamma \in \Gamma$.

For the natural choice of $P = \{f_\theta(\gamma,\hat{\theta})\, F^{-1}(\hat{\theta})\, f_\theta^*(\gamma,\hat{\theta})\}^\dagger$, using $\dagger$ to denote the pseudoinverse of the covariance of (2), the inequality (3) becomes

$$\{\hat{f}(\gamma) - f(\gamma)\}^* \{f_\theta(\gamma,\hat{\theta})\, F^{-1}(\hat{\theta})\, f_\theta^*(\gamma,\hat{\theta})\}^\dagger \{\hat{f}(\gamma) - f(\gamma)\} \le r X^2_{\alpha,q}$$

where $r = Rank(P)$.

The relative squared spectral error $tr[\hat{S}^{-1}(\omega)\{\hat{S}(\omega) - S(\omega)\}]^2$ is a fundamental quantity in measuring the accuracy of a spectral estimation procedure. The integral of this quantity is asymptotically the Kullback-Leibler information of negative entropy (Larimore, 1983) which is a fundamental statistical measure of model approximation error. The expected value of the integral is proportional to the number of estimated parameters divided by the sample size (Larimore, 1982). From Theorem 1, simultaneous confidence bands on the sample relative squared spectral error are given by the following theorem.

Theorem 2. Under the conditions of Theorem 1, as $N \to \infty$, the probability is at least $1 - \alpha$ that simultaneously for all $\omega \in \Omega$ the sample squared relative spectral error is bounded as

$$\begin{aligned} & tr[\hat{S}^{-1}(\omega)\{\hat{S}(\omega) - S(\omega)\}]^2 \\ & \le X^2_{\alpha,q}\; tr \sum_{k,l} S^{-1}(\omega,\theta) \frac{\partial S(\omega,\theta)}{\partial \theta_k}\, g_{kl}(\hat{\theta})\, S^{-1}(\omega,\theta) \frac{\partial S(\omega,\theta)}{\partial \theta_l} \\ & \approx X^2_{\alpha,q} E\; tr[\hat{S}^{-1}(\omega)\{\hat{S}(\omega) - S(\omega)\}]^2 \end{aligned} \tag{4}$$

where $\{g_{kl}(\theta)\} = G = F^{-1}(\theta)$.

Proof: Asymptotically $\hat{S}(\omega)$ and $S(\omega)$ are equal so that we may consider its inverse in (4) a constant denoted $\bar{S}(\omega)$. To apply Theorem 1, we consider the Hermitian matrix $A(\omega) = \bar{S}^{-1/2}(\omega)\{S(\omega) - \hat{S}(\omega)\}\bar{S}^{-1/2}(\omega)$ and express the squared relative error symmetrically as

$$tr\,[\hat{S}^{-1}(\omega)\{\hat{S}(\omega) - S(\omega)\}]^2 = tr\,[\bar{S}^{-1/2}(\omega)\{\hat{S}(\omega) - S(\omega)\}\bar{S}^{-1/2}(\omega)]^2$$

$$= tr\ AA = tr\ AA^* = \sum_{i,j} a_{ij}\ a_{ij}^* = f^*(\omega)f(\omega) \tag{5}$$

where $f(\omega) = vecA(\omega)$ is a vector containing the elements of the matrix $A(\omega)$. Application of Theorem 1 to the vector function $f(\omega)$ and rearrangement as in (5) proves the inequality. Expanding $S(\omega,\theta)$ as in (5), the equality follows from

$$E\ tr\,[\hat{S}^{-1}(\omega)\{\hat{S}(\omega) - S(\omega)\}]^2$$

$$= tr \sum_{k,l} S^{-1}(\omega,\theta)\frac{\partial S(\omega,\theta)}{\partial\theta_k}\ E(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T\ S^{-1}(\omega,\theta)\frac{\partial S(\omega,\theta)}{\partial\theta_l}$$

and using $E(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T = F^{-1}$ from the asymptotic efficiency of $\hat{\theta}$.

In principle any quadratic form in the components of the spectral matrix could be used as in Theorem 1 by introducing a weighting matrix $P(\omega,\theta)$. For confidence intervals on the spectral matrix, the weighting of the inverse covariance of the error in estimating the spectral matrix gives tightest confidence bands which can be expressed as

$$vec^*\{\hat{S}(\omega) - S(\omega)\}\{\sum_{k,l}\frac{\partial vecS(\omega)}{\partial\theta_k}\ g_{kl}(\hat{\theta})\frac{\partial vec^*S(\omega)}{\partial\theta_l}\}^{\dagger}vec\{\hat{S}(\omega) - S(\omega)\} \le \chi^2_{\alpha,d}$$

For a given confidence level α, this gives a simultaneous confidence band for all

frequencies $\omega \in \Omega$ as a quadratic form in the elements of $\hat{S}(\omega) - S(\omega)$.

ACKNOWLEDGEMENTS

Partial support for the research was provided by Air Force Wright Aeronautical Laboratories, Wright-Patterson AFB, Ohio.

APPENDIX: AN ELLIPSOIDAL INEQUALITY LEMMA

Lemma 1. Let ϕ be a real q-dimensional vector and for a particular p let $\mathcal{H}$ be the class of $p \times q$-dimensional complex matrices H, and let M be a symmetric positive definite matrix. Then ϕ satisfies $\phi^T M \phi \leq 1$ if and only if for every $H \in \mathcal{H}$ the following inequality holds

$$\| H\phi \|^2 \leq tr\ HM^{-1}H^*$$

Proof: Any Hermitian matrix A has an eigenvalue-eigenvector expansion

$$A = \sum_m \lambda_m x_m x_m^*$$

From the Schwartz inequality, $(\psi^* x)^2 \leq (\psi^* \psi)(x^* x)$ for any ψ and x with equality if and only if $\psi = cx$ for c a scalar. Let B be such that $BB^* = M^{-1}$. Setting $\psi = B^{-1}\phi$ and denoting the quantity in braces { } by A, we have for every $H \in \mathcal{H}$ and every ϕ

$$\| H\phi \|^2 = \| HB\psi \|^2 = \psi^* \{(HB)^*(HB)\}\psi$$

$$= \psi^* A \psi = \psi^* \sum_m \lambda_m x_m x_m^* \psi = \sum_m \lambda_m (\psi^* x_m)^2$$

$$\leq \psi^* \psi \sum_m \lambda_m \, (x_m^* \, x_m) = \phi^* M \phi \, trA = \phi^* M \phi \, trHM^{-1}H^*$$

From this the "only if" part of the lemma follows, and choosing $H^* = B^{*-1}(\psi, 0, \ldots, 0)$ gives $A = \psi\psi^*$ and strict equality which implies the "if" part of the lemma.

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U.S. GPO: 646-066