

## CHAPTER 5

### MATHEMATICAL MODELS OF RELIABILITY

#### INTRODUCTION

In a general sense, we can say that the science of statistics attempts to quantify our uncertainty about the outcome of an experiment or test. Before we conduct an experiment, we are aware of the possible outcomes, but we are not sure which of the outcomes will actually result. The mathematical model, a logical extension of our assumptions about the experiment, quantifies our uncertainty and provides us with a tool for analysis of the outcome.

Suppose that a system is to be put into operation. We know that after a certain amount of operating time it will experience a failure. We would like to know when the system will fail. Generally, any prediction about the actual time of failure will not be accurate enough to be worthwhile. However, we can address more confidently questions such as: "Will the system operate free of failure for at least 500 hours?" A model that describes the experiment in question will help us answer these questions with a certain amount of assurance.

In this chapter, we consider three models that are pertinent to RAM considerations: the binomial, the Poisson, and the exponential. The binomial and Poisson are discrete models in that they essentially count numbers of failures. The exponential model is continuous in that it describes times between failures. Although the Poisson model addresses discrete events, **it is** a continuous time model. It counts failures as they occur over a period of time, i.e., the possible outcomes of the experiment, conducted on a continuous time basis, are enumerated as numbers of failures. This distinction will become clearer as we contrast the Poisson model and the binomial model.

In its most basic form, a mathematical model of a statistical experiment is a mathematical expression (function) that defines the probability associated with each of the outcomes of the experiment. For our purposes, we will discuss the two basic types of models: discrete and continuous. The type of model--discrete or continuous--is defined by the type of outcome that the experiment provides.

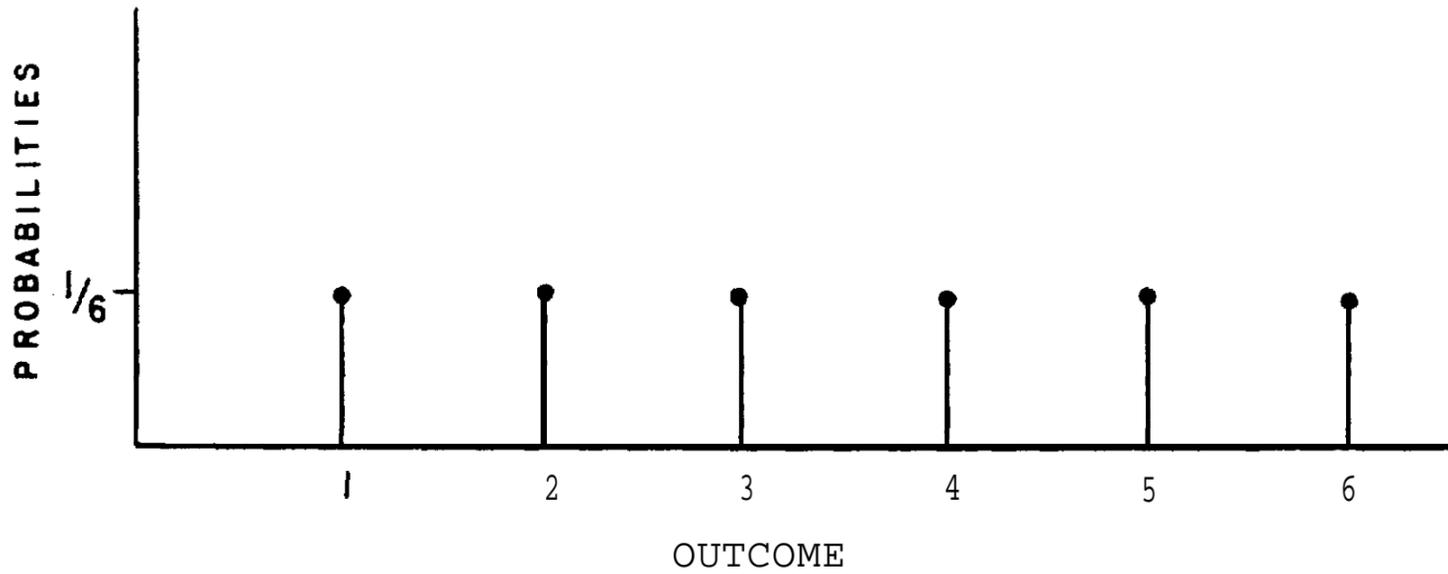
#### DISCRETE MODELS

A discrete model is appropriate when the possible outcomes of an experiment can be enumerated or counted. In its most basic form, the discrete model is a mathematical expression (function) that defines the probability of each individual outcome. A simple example is the following. Suppose a die is to be tossed once, and the outcome of interest is the number of dots facing up when the die comes to rest. The outcomes are {1, 2, 3, 4, 5, 6}. If we assume that the die is "fair," the probabilities can be expressed as follows :

$$p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = 1/6.$$

Graphically, we display the probabilities in Figure 5-1.

FIGURE 5-1

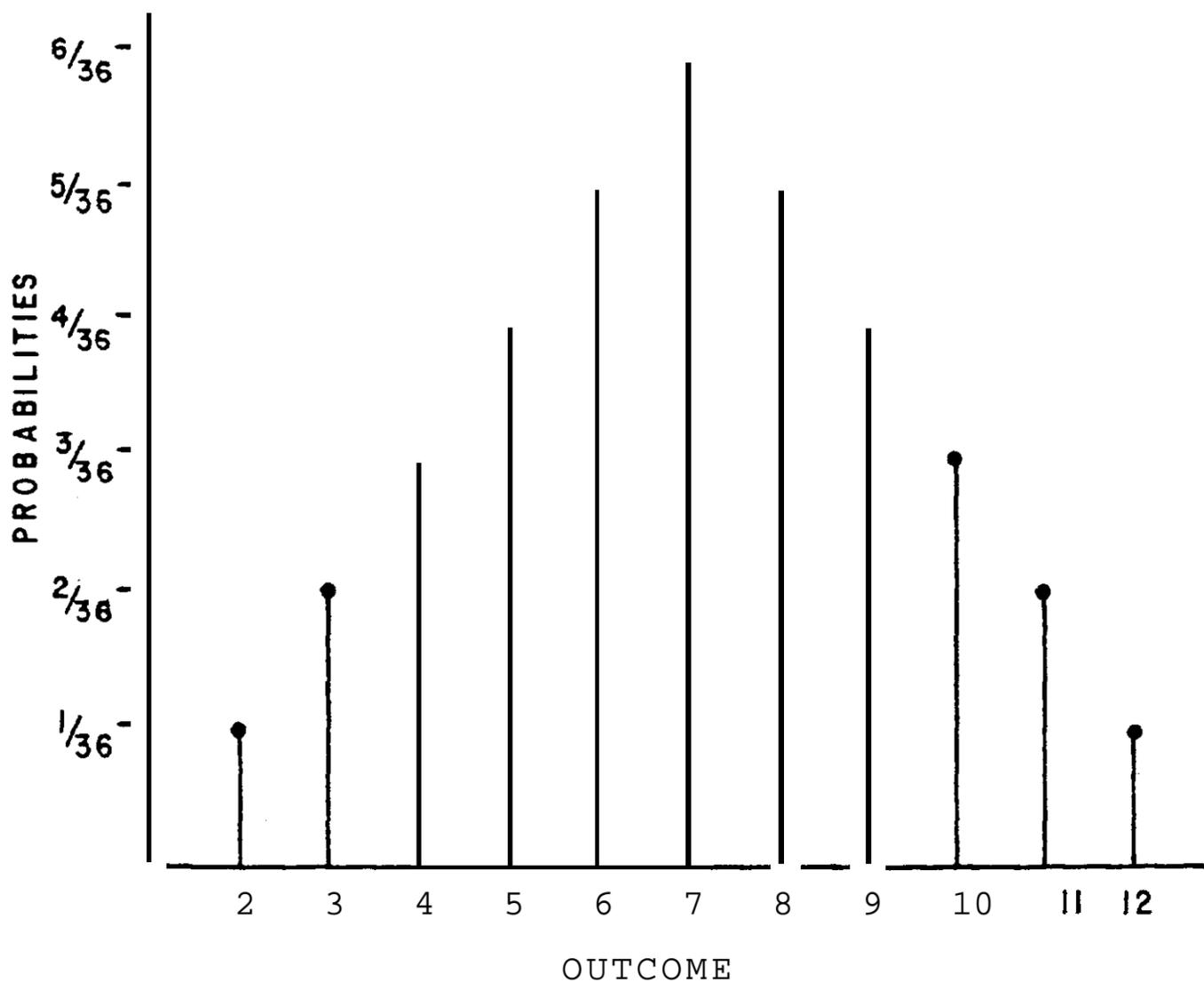


Suppose our experiment is to toss two dice, and the outcome of interest is the sum of the dots facing up. The set of all possible outcomes is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. The probabilities can be expressed as follows:

$$\begin{array}{ll}
 P(2) = p(12) = 1/36 & p(5) = p(9) = 4/36 \\
 P(3) = p(11) = 2/36 & P(6) = p(8) = 5/36 \\
 P(4) = p(10) = 3/36 & p(7) = 6/36.
 \end{array}$$

Graphically, we display the probabilities in Figure 5-2.

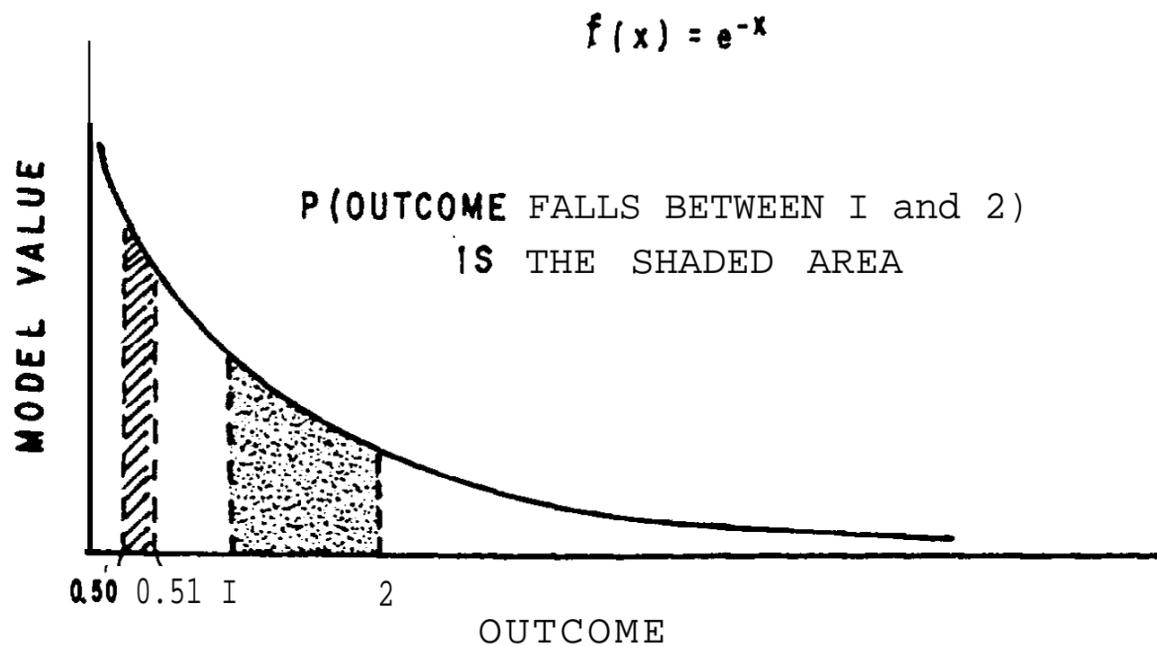
FIGURE 5-2



## CONTINUOUS MODELS

A statistical experiment often results in an outcome that is measured on a continuous scale. Time and distance are perhaps the most common continuous variables. In its most basic form, the continuous model is a mathematical expression (function) useful in computing probabilities of certain outcomes. It differs from probabilities for discrete models in that it does not define probabilities directly. Generally, for a continuous model, it only makes sense to consider the probability of an outcome within a range of values or a certain interval--between 100 and 150 miles, more than 10 hours. The probability of an outcome falling within a given range is the area which lies beneath the continuous model curve over that range. Consider the examples below.

FIGURE 5-3 GRAPH OF THE EXPONENTIAL MODEL



The probability that an outcome is between 1 and 2 is defined by the area under the curve between the values 1 and 2. Therefore,

P(outcome falls between 1 and 2)

$$= \int_1^2 e^{-x} dx = -e^{-x} \Big|_1^2 = e^{-1} - e^{-2} = 0.233.$$

The probability that an outcome is between 0.50 and 0.51 is

$$\int_{0.50}^{0.51} e^{-x} dx$$

which is  $e^{-0.50} - e^{-0.51}$  or 0.006.

FIGURE 5-4 GRAPH OF UNIFORM MODEL

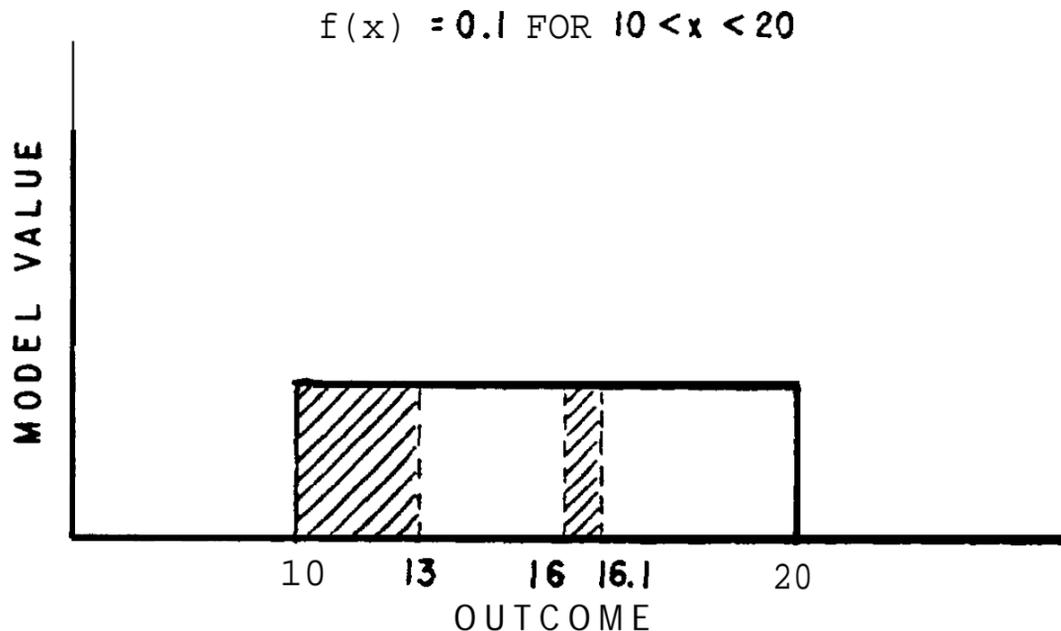


Figure 5-4 illustrates another possible continuous model with the function being  $f(x) = 0.1$  defined on the interval  $10 < x < 20$ .

The probability that an outcome is less than 13 is

$$\int_{10}^{13} 0.1 \, dx,$$

which is  $1.3 - 1.0$ , or  $0.3$ . The probability that an outcome is between 16 and 16.1 is

$$\int_{16}^{16.1} 0.1 \, dx,$$

which is  $1.61 - 1.60$ , or  $0.01$ .

#### BINOMIAL MODEL

The model that is used most commonly to describe the outcomes of success/failure test programs is the binomial model. In order for a testing program to be a binomial experiment, four conditions are required. They are:

- The test period consists of a certain number ( $n$ ) of identical trials.
- At any individual time unit (trial), the test results in a success or failure.
- The outcome at any individual time unit is independent of the outcomes of all other time units.
- The reliability (probability of success) of the system remains unchanged for each trial.

In a lot acceptance sampling test program, the third and fourth conditions are not precisely satisfied. In such a case, the hypergeometric distribution provides the exact analysis and must be used for small lot sizes. However, if the sample size ( $n$ ) is small compared to the lot size ( $N$ ), say  $n/N < 0.05$ , then the binomial model provides a reasonable basis for analysis. We do not present a discussion of the hypergeometric distribution in this text. The reader can refer to any number of moderate level statistics textbooks for information on the use of hypergeometric distribution.

Examples of binomial experiments are:

- firing a missile (is it launched successfully?);
- firing a missile (does it hit the intended target?);
- operating any system over time (does it achieve its durable life?).

For the remainder of this section on binomial models, the following notation is used.

$n$ : the number of trials or test units.

$p$ : the probability of failure for any trial. (We use  $p$  here as the probability of failure instead of the more classical probability of success because of the failure-oriented approach used by the Poisson model) .

$\binom{n}{k}$ : binomial coefficient, which by definition is equal to  $n!/[k!(n-k)!]$ , where  $k$  must be some integer between 0 and  $n$  inclusive. By definition,  $n! = n(n-1)(n-2) \dots 1$  and  $0! = 1$ .

$b_{n,p}(k)$ : the probability of  $k$  failures out of  $n$  trials with  $p$  the probability of failure on any one trial.

$B_{n,p}(k)$ : the probability of  $k$  or fewer failures out of  $n$  trials with  $p$  the probability of failure on any one trial.

Any binomial experiment (test program) is completely characterized by the number of trials and the probability of failure for any given trial. The probability of failure ( $p$ ) is generally an unknown value about which testing requirements are stated.

In Chapter 7, the binomial is used in this reversed role: the test results will have been observed and we will make inferences about the probability of success based on the binomial model.

For the binomial model an exact mathematical formula is used to compute probabilities. We denote the probability of exactly  $k$  failures out of  $n$  trials for a fixed probability of failure,  $p$ , by  $b_{n,p}(k)$ , and

$$b_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}. \quad (5.1)$$

The probability of  $k$  or fewer failures, termed the binomial cumulative distribution function, is  $B_{n,p}(k)$ , where

$$\begin{aligned}
 B_{n,p}(k) &= \sum_{i=0}^k \binom{n}{i} p^i (1-p)^{n-i} \\
 &= \sum_{i=0}^k b_{n,p}(i).
 \end{aligned}
 \tag{5.2}$$

Figure 5-5 presents graphs of three binomial models. The graphs chosen portray models for which the values of  $p$  are equal to 0.2, 0.5, and 0.7 with  $n$ , the number of trials, equal to 10 in each case.

For the case  $p = 0.2$ , the graph shows that a large number of failures--more than 5--is unlikely. Note that the most likely number of failures is 2, which corresponds to a percentage of failures equal to 0.2. For the case  $p = 0.7$ , the graph shows that a small number of failures--fewer than 5--is unlikely. Once again, the most likely number of failures, 7, corresponds to a percentage of failures equal to 0.7. For the case  $p = 0.5$ , the graph shows that a moderate number of failures--between 3 and 7--is likely, with 5 failures (half of the trials) being most likely.

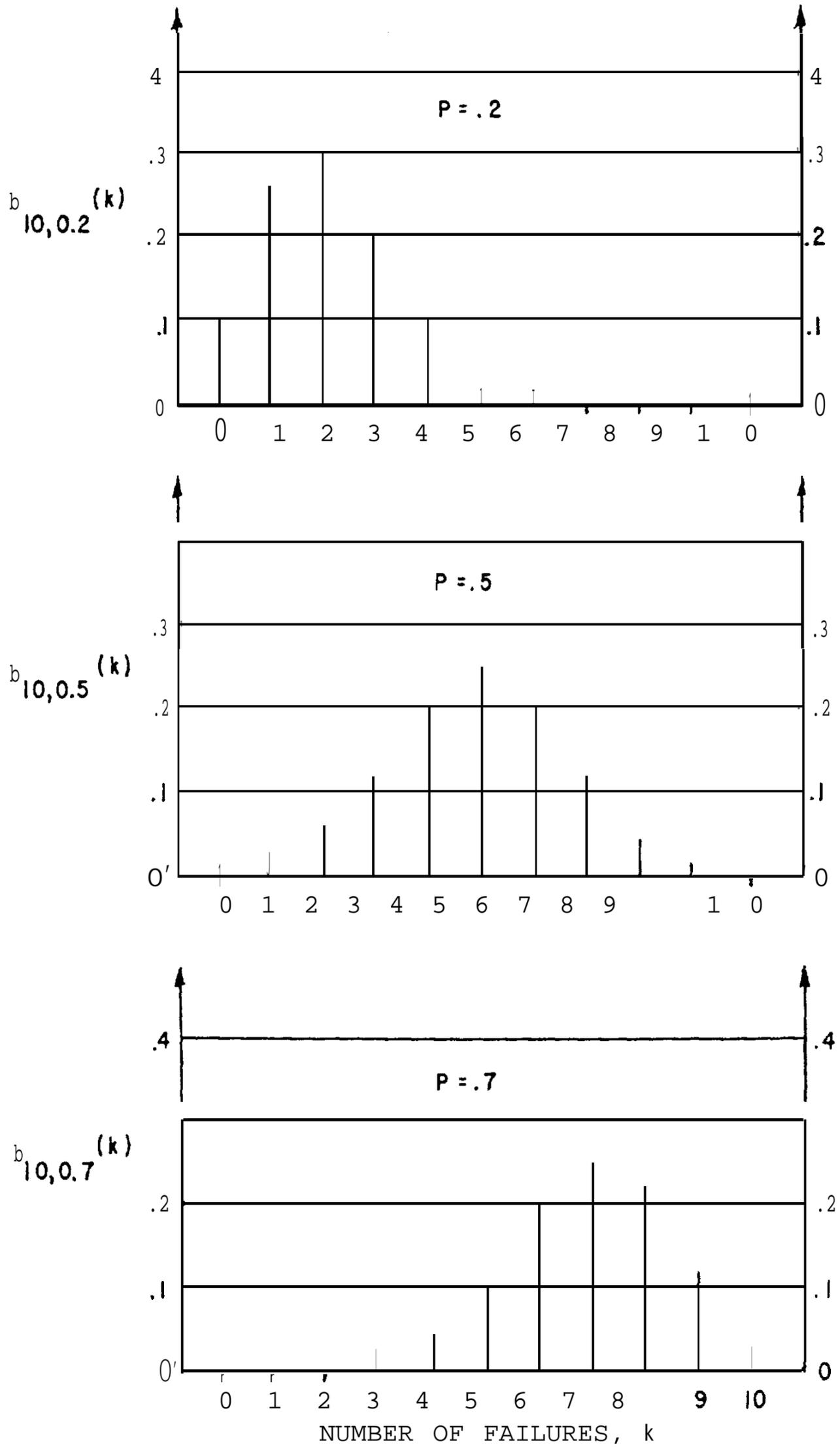
Computation of binomial probabilities using  $b_{n,p}$  or  $B_{n,p}$  is cumbersome when  $n$  is large. Three alternative methods for determining binomial probabilities are:

- use of a statistics package for a calculator,
- use of binomial tables, and
- use of an approximating distribution (Poisson or normal).

Tables of  $b_{n,p}$  or  $B_{n,p}$  for  $n$  less than or equal to 20 are generally published in elementary statistics textbooks. More extensive tables are available but are not as easy to locate. A table of cumulative binomial probabilities for selected values of  $n$ ,  $k$ , and  $p$  is given in Appendix B, Table 1. When  $n$  is larger than 20, either the Poisson distribution ( $p > 0.8$  or  $p < 0.2$ ) or the normal distribution ( $0.2 < p < 0.8$ ) provides reasonable approximations to binomial probabilities. (See Appendix A for details on these procedures.)

In Case Studies 5-1, 5-2 and 5-3, we demonstrate the application of the binomial model and compute probabilities associated with the model.

FIGURE 5-5 BINOMIAL PROBABILITIES



## POISSON MODEL

The most commonly used model for describing the outcomes of a continuous time testing program is the Poisson model. In order for a testing program to be a Poisson experiment, it is required that no more than one event (failure) can occur at the same time and that the number of events (failures) is related directly to the amount of test time. Examples of Poisson experiments are:

number of failures of a system during a test cycle;

number of unscheduled maintenance actions required during a given time period; and,

number of misfires of an automatic weapon firing over a given time period.

For the remainder of the section on Poisson models and the succeeding section on exponential models, the following notation is used.

$\lambda$ : Failure rate or average number of failures per unit time.

T: The length of the interval (hours, miles, etc.) of interest (e.g., mission duration, test exposure).

$g_{\lambda,T}(k)$ : The probability of exactly k failures during a test period of length T when the failure rate is A.

$G_{\lambda,T}(k)$ : The probability of k or fewer failures during a test period of length T when the failure rate is A.

Any Poisson experiment (test program) is completely characterized by the length of time on test and the mean value function. The mean value function for a specific test length is the average number of failures to occur during the specified length. When the system has a constant failure rate for the entire interval T, this function is simply AT. A discussion of constant failure rate assumptions can be found in Chapter 7. The value A is the failure rate of the system on test. A more familiar parameter is the mean time between failures (MTBF) which is the reciprocal of A. System requirements are generally stated in terms of the mean time between failures.

As with the binomial model, Poisson probabilities can be computed using an exact mathematical formula. We denote the probability of exactly k failures during a test period of length T where the failure rate is A, by  $g_{\lambda,T}(k)$ , and

$$g_{\lambda,T}(k) = \frac{(\lambda T)^k e^{-\lambda T}}{k!} \quad (5.3)$$

The number of failures may be any integer value including 0 and  $0! = 1$ . The probability of k or fewer failures, termed the Poisson cumulative distribution

function, is  $G_{\lambda, T}(k)$ , where

$$\begin{aligned} G_{\lambda, T}(k) &= \sum_{i=0}^k \frac{(\lambda T)^i e^{-\lambda T}}{i!} \\ &= \sum_{i=0}^k g_{\lambda, T}(i). \end{aligned} \tag{5.4}$$

Figure 5-6 presents graphs of three Poisson models. Values for  $\lambda T$  of 2, 5, and 7 were chosen to demonstrate the effect of time  $T$  on the numbers of failures likely to be seen. When  $\lambda$ , the failure rate, is fixed, the number of failures will, in all likelihood, increase with the amount of operating time  $T$ .

Alternative methods for computing Poisson probabilities include:

- use of a statistics package for a calculator,
- use of Poisson tables or charts, and
- use of an approximating function.

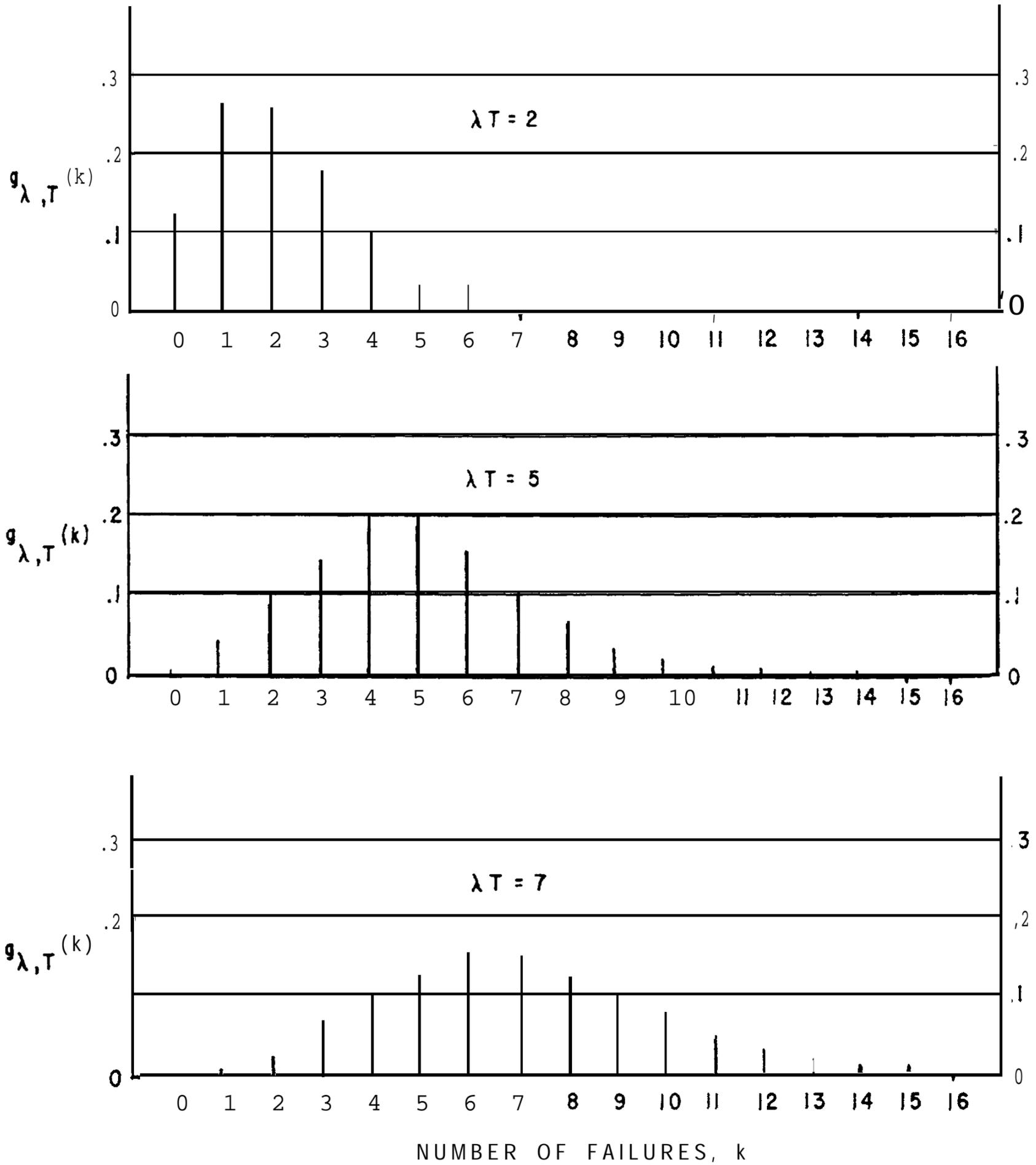
In Case Studies 5-5 and 5-6 we demonstrate the application of the Poisson model and compute probabilities associated with the model. Tables of  $g_{\lambda, T}(k)$  or  $G_{\lambda, T}(k)$  are available in many textbooks. A table of cumulative Poisson probabilities is given in Appendix B, Table 3. Appendix B, Chart 1 is also useful in determining cumulative Poisson probabilities. When the product  $\lambda T$  is greater than 5, the normal distribution provides reasonable approximations to Poisson probabilities. (See Appendix A for details on this procedure.)

#### EXPONENTIAL MODEL

Generally, it is more informative to study times between failures, rather than numbers of failures, for a continuous time testing program. The most **commonly** used model for describing the times between failures for a continuous time testing program is the exponential model. In order for a testing program to qualify as an exponential experiment, the following conditions are required: (1) the system is as good as new after each repair, and (2) the **probability** of failure in any given interval of time is the same no matter how old a system is and no matter how many failures it has experienced. The second condition is an intuitive description of the so-called **memoryless** property. The system cannot "remember" how old it is, nor can it "remember" how often it has failed. The examples listed for Poisson experiments can **serve also** as examples of exponential experiments.

For the exponential model, there is an exact mathematical formula used for computing probabilities that a certain amount of time will pass before the

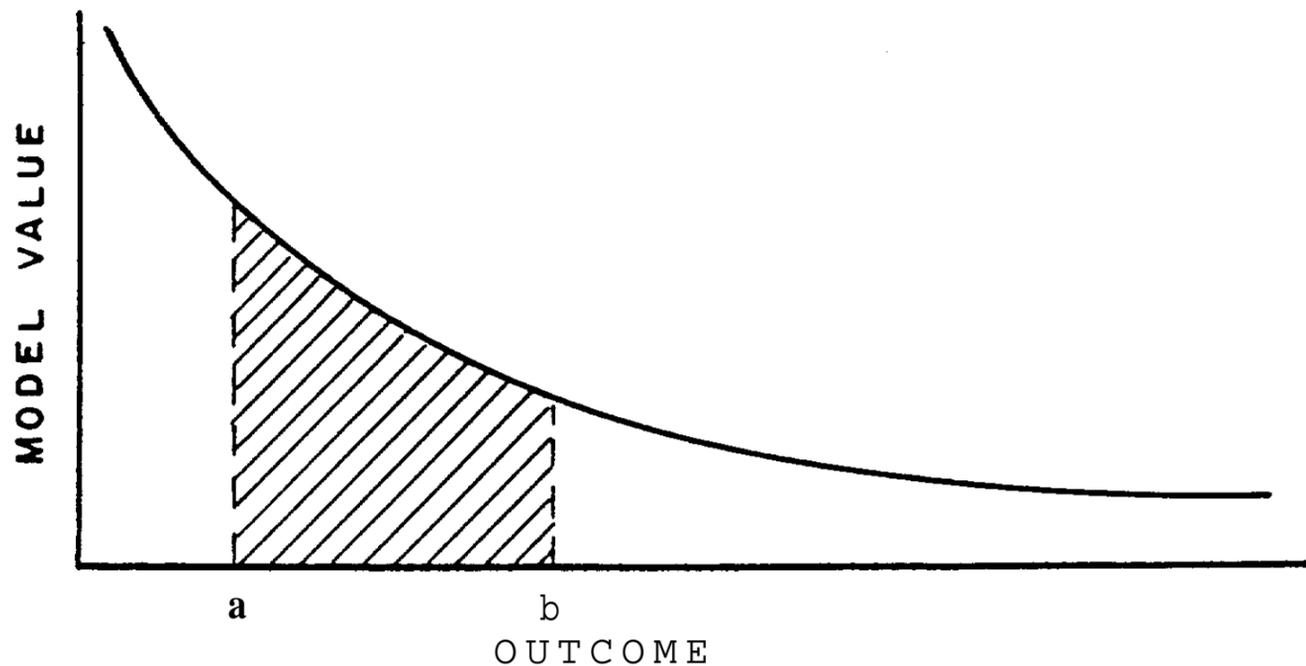
FIGURE 5-6 POISSON PROBABILITIES



next failure. The probability that a failure will occur in some future interval of time (a, b) for a system with failure rate  $\lambda$  is

$$\int_a^b \lambda e^{-\lambda x} dx. \quad (5.5)$$

FIGURE 5-7 GENERAL EXPONENTIAL MODEL:  $f(x) = \lambda e^{-\lambda x}$



The exponential cumulative distribution function,  $F_\lambda(t)$ , defines the probability that the system will fail before time  $t$ . By definition,

$$F_\lambda(t) = \int_0^t \lambda e^{-\lambda x} dx = 1 - e^{-\lambda t}. \quad (5.6)$$

A function of more interest to us is **the** reliability function,  $R(t)$ , which defines the probability that the system will **operate** for  $t$  units of time without failure. By definition,

$$R_\lambda(t) = \int_t^\infty \lambda e^{-\lambda x} dx = e^{-\lambda t}. \quad (5.7)$$

The reliability function,  $R(t)$ , can also be expressed as  $e^{-(t/\theta)}$ , where  $\theta$ , the reciprocal of  $\lambda$ , is the mean time between failures (MTBF). The reliability function  $R(t)$  translates the effectiveness parameters  $\lambda$ , the failure rate, or  $\theta$ , the MTBF, into reliability. Reliability is the probability of failure-free operation for a specified length of time,  $t$ .

We referred to the variable  $t$  as a time variable in the above discussion. Measure of life units which can be appropriate are hours, miles, **cycles and** rounds.

As an example, suppose that the mission profile for a system requires a mission duration (MD) of 40 hours and the system has a mean time between

operational mission failure (MTBOMF) of 400 hours. Then the probability that the system successfully completes a mission can be evaluated using the reliability function,  $R(t)$ . Now

$$R(t) = e^{-At} = e^{-t/\theta} = e^{-(MD/MTBOMF)}$$

Since  $MD = 40$  hours and  $MTBOMF = 400$  hours, the mission reliability is  $e^{-40/400}$ , which reduces to 0.905. In other words, the system has a 90.5% chance of completing the 40-hour mission.

NOTE : A useful approximation for the reliability function is :

$$e^{-t/\theta} \cong 1 - t/e \text{ for } t/\theta \leq 0.1.$$

For the above example,  $t/E1 = 0.1$  so that the approximation yields 0.90.

In Case Study 5-4, we demonstrate the application of the exponential model with computations based on both models. See Case Studies 5-4, 5-5, and 5-6, for more illustrations of this computation and other computations associated with the Poisson/exponential model.

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The reliability function  $e^{-At}$  may also be symbolized as  $\exp(-\lambda t)$  or  $\ln^{-1}(-At)$ . That is, it is the exponential function evaluated at the point,  $-At$ , or it is the inverse of the natural logarithm function evaluated at that point. Some calculators evaluate the exponential function as the inverse natural logarithm function.

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## CASE STUDY NO. 5-1

### Background

A sensor device has an operational mission reliability of 0.90 for a 4 hour mission. At least 3 sensors are required to locate targets. Failed sensors will not be repaired during a mission.

### Determine

1. If a system employs 3 sensors, what is the probability of successfully completing the mission?
2. If a system employs 4 sensors, what is the probability of successfully completing the mission?

### Solution

1. We use the mathematical formula for the binomial model given in equation 5.1, with

$$R = 1 - p = 0.9$$

$$p = 1 - 0.9 = 0.1$$

$$n = 3, \text{ and}$$

$$k = 0$$

Applying equation 5.1, the probability of 0 failures is:

$$b_{3,0.1}(0) = \binom{3}{0} (0.1)^0 (0.9)^3$$

The binomial coefficient is:

$$\binom{3}{0} = \frac{3!}{0!3!} = 1$$

The probability of 0 failures is:

$$(1)(1)(0.9)^3 = 0.729.$$

2. We use the binomial cumulative distribution function given in equation 5.2, with

$$R = 1 - p = 0.9$$

$$p = 1 - 0.9 = 0.1$$

$$n = 4, \text{ and}$$

$$k = 1$$

Applying equation 5.2, the probability of 1 or fewer failures is:

$$\begin{aligned} B_{4,0.1}(1) &= \sum_{k=0}^1 \binom{4}{k} (0.1)^k (0.9)^{4-k} \\ &= \binom{4}{0} (0.1)^0 (0.9)^4 + \binom{4}{1} (0.1)^1 (0.9)^3 \end{aligned}$$

The binomial coefficients are:

$$\binom{4}{0} = \frac{4!}{0!4!} = 1, \text{ and } \binom{4}{1} = \frac{4!}{1!3!} = 4.$$

The probability of 1 or fewer failures is:

$$\begin{aligned} &(1)(1)(0.9)^4 + (4)(0.1)^1(0.9)^3 \\ &= 0.656 + 0.292 = 0.948. \end{aligned}$$

#### Commentary

For the second problem, we use the binomial cumulative distribution function since we are required to compute the probability of 0 failures or 1 failure, i.e., the cumulative probabilities of both outcomes.

## CASE STUDY NO. 5-2

### Background

A lot of 500,000 rounds of ammunition is available for sale. The buyer will purchase the lot if he has reasonable assurance that the lot is no more than .15% defective.

### Determine

1. If the true proportion of defects is 0.15, what is the probability a sample of size 10 will yield fewer than 2 defects? More than 3 defects?
2. If the **true** proportion of defects is 0.05, what is the probability a sample of size 20 will yield fewer than 3 defects? More than 1 defect?
3. If the true proportion of defects is 0.02, what is the probability a sample of size 100 will yield fewer than 5 defects?
4. If the true proportion of defects is 0.15, what is the probability a sample of size 50 will yield more than 10 defects? Between 5 and 10 defects, inclusive?

### Solutions

This is a lot acceptance sampling problem. Although the binomial model is not technically correct, it will provide very good approximations in this case because the ratio of sample size  $n$  to lot size  $N$  ( $N = 500,000$ ) is not more than 0.0002. A further explanation of the use of the binomial model for lot acceptance sampling problems is found on page 5-5.

1. The probability of failure,  $p$ , is 0.15. For fewer than 2 defects, we look in the tables for  $n = 10$ , the column  $p = 0.15$ , and the row  $c = 1$ . The probability is 0.5443.

The probability of more than 3 defects is the difference between 1 and the probability of fewer than 4 defects. The probability of fewer than 4 defects is 0.9500, so the probability of more than 3 defects is 0.05.

2. The probability of fewer than 3 defects out of 20 is obtained directly in the tables for  $n = 20$ , the column  $p = 0.05$ , and the row  $c = 2$ . The probability is 0.9245.

The probability of more than 1 defect is the difference between 1 and the probability of fewer than 2 defects. The probability of fewer than 2 defects is 0.7358, so the probability of more than 1 defect is 0.2642.

3. A binomial table for  $n = 100$  is not given in the Appendix and would be difficult to locate in other sources. However, because the sample size,  $n$ , is large, we may use an approximation fairly confidently. Recall that there are **two** approximations (**Poisson** and normal) to the binomial presented in the text of Chapter 5. The procedures for using these approximations are detailed in

Appendices A-1 and A-2. Using the normal approximation (Appendix A-1) , the probability of 4 or fewer defects out of 100 trials is

$$1 - P(z \geq (4 + 0.5 - (100)(0.02))/\sqrt{100(0.02)(0.98)}) ,$$

which reduces to

$$1 - P(Z \geq 1.79) = 0.9633 .$$

Using the Poisson approximation (Appendix A-2) , we set  $m = 2$ ,  $c = 4$ , and obtain the probability directly from Appendix B, Table 3 as 0.947.

The exact value for the probability of fewer than 5 defects, obtained from the formula, is 0.9491, Note that, although each approximation is reasonably close to the exact value, the Poisson has provided the better approximation. As noted on page 5-6, the Poisson is more appropriate when  $p$  is very large or very small (in our case,  $p = 0.02$ ).

4. The probability of more than 10 defects is the difference between 1 and the probability of fewer than 11 defects. The probability of fewer than 11 defects using the binomial tables for  $n = 50$ , the column  $p = 0.15$ , and the row  $c = 10$ , is 0.8801. The probability of more than 10 defects is thus 0.1199. The probability of between 5 and 10 defects inclusive is the probability of fewer than 11 defects less the probability of fewer than 5 defects. These two numbers are 0.8801 and 0.1121, and the difference is 0.7680. The normal distribution is appropriate for approximating this probability. Using this approximation, we find that the probability of more than 10 (11 or more) defects is

$$P(Z \geq (11 - 0.5 - (50)(0.15))/\sqrt{450(0.15)(0.85)}) ,$$

which reduces to

$$P(z \geq 1.19) = 0.1170 .$$

The approximate probability of between 5 and 10 defects inclusive is

$$\begin{aligned} & P(z \geq (5 - 0.5 - 7.5)/\sqrt{6.38}) - P(z \geq (10 + 0.5 - 7.5)/\sqrt{6.38}) \\ & = P(Z \geq -1.19) - P(Z \geq 1.19) = 0.7660 . \end{aligned}$$

#### Commentary

We have calculated probabilities for certain outcomes which could result from an inspection of a sample from the lot of 500,000. To calculate these values it is necessary to assume that we know the true proportion of defects in the entire lot. Of course, we do not, in fact, know this true proportion of defects, but we perform this exercise in order to develop a rational plan for sampling from the lot in order to determine whether we should accept or reject the lot. Consider, for example, the solution to the second part of question 1. Namely, the probability of 4 or more defects out of 10 trials when the lot contains 15% defects is 0.05. Consequently, if the sample of 10 should yield 4 or more defects, the buyer has reasonable assurance that the

lot contains more than 15% defects and should be rejected. Consider now part 2 of question 4. In a preliminary step to the solution, we determined that the probability of 4 or fewer failures when the lot contains 15% defects is 0.1121. Consequently, if a sample of 50 yields 4 or fewer defects, the buyer has reasonable assurance that the lot contains fewer than 15% defects and should be accepted. We discuss in Chapter 8 methods for preparing test plans which provide reasonable assurance to both producer and consumer. Our purpose in presenting the above discussion is to introduce the reader to the probabilistic analysis that takes place in the preparation of test plans or lot acceptance sampling plans .

## CASE STUDY NO. 5-3

### Background

A mechanized infantry battalion has 36 armored personnel carriers (APC), 20 weapon stations (WS), and 5 command posts (CP). The mission reliability of an APC is 0.70, of a WS is 0.85, and of a CP is 0.95. The entire battalion can perform satisfactorily if at least 24 APCs, 17 WSS, and 5 CPS are operable throughout the mission.

### Determine

1. What is the probability each type of system has enough units operating to complete the mission?
2. What is the probability the mission will be successful?
3. How small do we need to make  $p$ , the probability of failure, in order to ensure that each set of systems has a probability 0.90 of performing satisfactorily?
4. If each of the probabilities of failure is fixed, then how many more of each type of system is required to achieve the goal mentioned in number 3?

### Solutions

The probability that a sufficient number of APCs operate throughout the mission is the probability of 12 or fewer failures out of 36 trials. Note that  $p$ , the probability of failure, is equal to 0.30. The probability that a sufficient number of WSS operate throughout the mission is the probability that 2 or fewer failures occur out of 20 trials. The probability that all CPS operate throughout the mission is the probability of no failures.

1a. The probability of 12 or fewer failures out of 36 trials where  $p = 0.30$ .

- i. Use Appendix B, Table 1. On page B-20 for  $n = 36$ , we look in the column labeled 0.300 and the row  $r = 12$ . The value is 0.7365.
- ii. Use Normal Approximation. (See Appendix A-1.) Since  $np = (36)(0.3) = 10.8$  and  $np(1-p) = (36)(0.3)(0.7) = 7.56$ , the approximate probability is

$$\begin{aligned} & 1 - P(Z \geq (12.5 - np) / \sqrt{np(1-p)}) \\ &= 1 - P(Z \geq (12.5 - 10.8) / \sqrt{7.56}) \\ &= 1 - P(Z \geq 0.62) \\ &= 0.7324 \end{aligned}$$

We obtain this value using Appendix B, Table 2, page B-42 in the row marked  $z_{\alpha} = 0.62$  and the column labeled  $P(Z \geq z_{\alpha})$ .

- iii. Use Poisson Approximation. (See Appendix A-2.) We make the identification  $m = np = (36)(0.3) = 10.8$  and use Appendix B, Table 3. On page B-47 we obtain the value from the column labeled 12 and interpolate between rows labeled 10.5 and 11.0. The value is 0.718.

Note that since  $0.2 < p < 0.8$ , the normal yields a better approximation than the Poisson.

1b. The probability of 3 or fewer failures out of 20 trials where  $p = 0.15$ .

- i. Use Appendix B, Table 1. On page B-7 for  $n = 20$ , we look in the column labeled 0.150 and the row  $r = 3$ . The value is 0.6477.
- ii. Use Normal Approximation. (See Appendix A-1.) Since  $np = (20)(0.25) = 5$  and  $np(1-p) = (20)(0.15)(0.85) = 2.55$ , the approximate probability is

$$\begin{aligned} & 1 - P(Z \geq (3.5 - np) / \sqrt{np(1-p)}) \\ & = 1 - P(z \geq (3.5 - 3) / \sqrt{2.55}) \\ & = 1 - P(Z \geq 0.32) = 1 - 0.3745 \\ & = 0.6255 \end{aligned}$$

We obtain this value using Appendix B, Table 2, page B-42 in the row marked  $z_{\alpha} = 0.32$  and the column labeled  $P(Z \geq z_{\alpha})$ .

- iii. Use Poisson Approximation. (See Appendix A-2.) We make the identification  $m = np = (20)(0.15) = 3$  and use Appendix B, Table 3. On page B-46 we obtain the value from the column labeled 3 and the row labeled 3.00. The value is 0.647.

Note that since  $p < 0.2$ , the Poisson yields a better approximation than the normal.

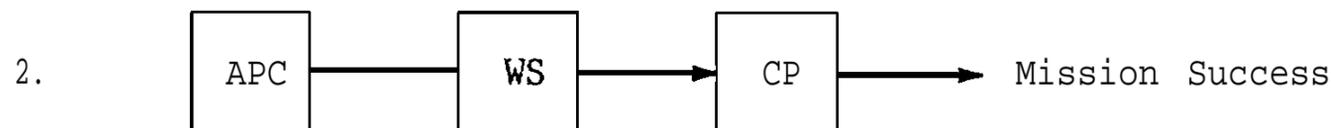
1.c. The probability of 0 failures out of 5 trials where  $p = 0.05$ . This probability is given by equation 5.1.

$$\begin{aligned} b_{n,p}(k) &= \binom{n}{k} p^k (1-p)^{n-k} \\ \binom{5}{0} (0.05)^0 (0.95)^5 &= (0.95)^5 = 0.774 \end{aligned}$$

Note the value of  $n$  is much too small to use an approximation. See discussion of this concept on page 5-6.

<u>System</u>	<u>Author- ized</u>	<u>Minimum Required</u>	<u>Max. # of Allowable Failures</u>	<u>System Mission Reliability</u>	<u>Probability of Mission Success</u>		
					<u>Binomial</u>	<u>Normal</u>	<u>"Poisson</u>
APC	36	24	12	0.70	0.7365	0.7324	0.718
WS	20	17	3	0.85	0.6477	0.6255	0.647
CP	5	5	0	0.95	0.774	-	

The term System Mission Reliability is the probability that an individual APC, WS, or CP will successfully complete a mission. The term Probability of Mission Success is the probability that for the individual systems at least the required minimum number of vehicles will successfully complete the mission.



We use a series model to evaluate the probability of mission success. This probability is the product of the probabilities of success for APC's (0.737), WS'S (0.648), and CP'S (0.774). The product is 0.37.

(See Chapter 2's section entitled "Series and Redundancy Models.")

3. In part 1 above, we determined that the mission success probabilities for APC'S, WS'S, and CP'S, are approximately 0.73, 0.65, and 0.77, respectively. If we assume that the authorized number of units of each type system remains fixed, then we must improve the mission reliability of each type system to achieve a success probability of 0.90 for each type system. Generally, statistics books do not address procedures for solving this type problem but the solutions are straightforward and logical.

- a. For APC'S, the probability of 12 or fewer failures must be at least 0.90.
  - i. Use Appendix B, Table 1. Note that for  $n = 36$  and  $c = 12$ , the values in the body of the table increase as  $p$ , the probability of failure decreases. This fact is actually true for any values of  $n$  and  $c$ . As we move in the body of the table from right to left, we find that the first time the probability exceeds 0.90 is for a  $p$  of 0.25. Consequently a  $p$  of 0.25 or less will achieve the goal.
  - ii. The normal approximation provides a method for determining the desired probability,  $p$ . The approximate probability of 12 or fewer failures out of 36 trials for any  $p$  is

$$1 - P(Z \geq (12.5 - 36p)/\sqrt{36p(1-p)}) .$$

The procedure is to set the above quantity equal to 0.90 and solve for  $p$ . To accomplish this, note that we can reduce the equation to

$$P(Z \geq (12.5 - 36p)/\sqrt{36p(1-p)}) = 0.10 .$$

Since  $P(Z \geq 1.28) = 0.10$ , the above equality occurs when

$$(12.5 - 36p)/\sqrt{36p(1-p)} = 1.28 .$$

To solve the equation for  $p$ , we multiply both sides by

$$\sqrt{36p(1-p)}$$

and square both sides to obtain the quadratic equation

$$156.25 - 900p + 1296p^2 = 58.98p(1-p), \text{ or}$$

$$1354.98p^2 - 958.98p + 156.25 = 0 .$$

We used the quadratic formula to find  $p = 0.254$ .

- iii. The Poisson approximation provides a method for determining the desired probability,  $p$ . Note that for  $c = 12$  in the Poisson tables, the probabilities increase as the value of  $m$  decreases. Searching the body of the table in the column labeled  $c = 12$ , we find that the probability exceeds 0.90 for the first time at  $M = 8.6$ . Recall that  $m = np$  for the Poisson approximation. Now  $n = 36$  and  $m = 8.6$ , so  $p = 8.6/36 = 0.24$ .
- b. For **WS's**, the probability of 3 or fewer failures must be at least 0.90. The procedures for determining the probability of failure,  $p$ , are identical to those in part a above.
- i. Appendix B, Table 1. For  $c = 3$ , as  $p$  decreases, the value in the table exceeds 0.90 for the first time at  $p = 0.09$ .
- ii. Normal Approximation. The equation to solve is
- $$(3.5 - 20p)/\sqrt{20p(1-p)} = 1.28 ,$$
- and the value of  $p$  that solves the equation is  $p = 0.092$ .
- iii. Poisson Approximation. For  $c = 3$ , as  $m$  decreases, the value in the table exceeds 0.90 for the first time at  $m = 1.7$ . Consequently,  $p = 1.7/20 = 0.085$ .
- c. For **CP's**, the probability of 0 failures must be at least 0.90. Using a direct computation, we solve the following equation for  $p$ :

$$\binom{5}{0} p^0 (1-p)^5 = 0.90 .$$

Equivalently,

$$(1-p)^5 = 0.90 \text{ or } 1-p = (0.90)^{0.2} .$$

The solution to the above equation is a p of 0.02.

4. If the mission reliability of the different systems cannot be improved, then in order to achieve a success probability of 0.90 for each system, the number of authorized units must be increased- This allows for more units to fail during the mission, while maintaining the required strength to achieve mission success.

a. For APC'S, the probability of at least 24 units performing successfully must be at least 0.90.

i. Appendix B, Table 1.

n = 36	P(12 or fewer failures) = 0.7365
n = 37	P(13 or fewer failures) = 0.8071
n = 38	P(14 or fewer failures) = 0.8631
n = 39	P(15 or fewer failures) = 0.9056

The requirement is satisfied for an n of 39.

ii. Normal Approximation. Since the number of failures allowed increases by one for every increase in allocation while the number of successes remains constant at 34, we can formulate the solution to this question more easily in terms of successes . The approximate probability of 24 or more successes out of n trials, for any n, when p = 0.7 is

$$P(Z \geq (23.5 - 0.7n)/\sqrt{0.21n}) .$$

We set the above term equal to 0.90 and solve for n. Since  $P(Z \geq -1.28) = 0.90$ , the value of n required is the one which solves the equation

$$(23.5 - 0.7n)/\sqrt{0.21n} = -1.28 .$$

We perform the same manipulations as in part 3 above to obtain a quadratic equation in n, one of whose solutions is n = 38.8.

iii. Poisson Approximation. Let us use Chart 6 to approximate the solution. The abscissa of the graph is labeled O/T which is the reciprocal of  $m = np$ . Consider the points where the curves cross the 0.90 ordinate line.

c	O/T	$n = (0.36/2')^{-1}$	<u># Successes</u>
12	0.12	27.8 $\cong$ (28)	16 or more
13	0.108	30.9 $\cong$ (31)	18 or more
14	0.097	34.4 $\cong$ (35)	21 or more
15	0.09	37.0 $\cong$ (38)	23 or more
16	0.083	40.2 $\cong$ (41)	25 or more

This method shows that 38 units are not enough and that 41 units are too many. Either 39 or 40 units are appropriate.

b. For WS'S, the probability of at least 17 units performing **successfully** must be at least 0.90.

i. Appendix B, Table 1.

n = 20	P(3 or fewer failures) = 0.6477
n = 21	P(4 or fewer failures) = 0.8025
n = 22	P(5 or fewer failures) = 0.9001

The requirement is satisfied for an n of 22.

ii. Normal Approximation. The equation to solve is

$$(16.5 - 0.85n)/\sqrt{0.1275n} = -1.28 .$$

The solution is n = 21.9.

iii. Poisson Approximation. Use Chart 6.

c	O/T	$n = (0.150/T)^{-1}$	<u># Successes</u>
4	0.42	15.9 (16)	12
5	0.33	20.2 (21)	16
6	0.26	25.6 (26)	20

Clearly, this method indicates that either 22 or 23 should be sufficient.

c. For CP's, the probability of at least 5 units performing satisfactorily must be at least 0.90. The number of units is so small that approximations are not appropriate. We can solve the problem very easily using the formula for computation of binomial probabilities .

$$\begin{aligned}
 {}^b_n P^{(k)} &= \binom{n}{k} p^k (1-p)^{n-k} \\
 &= \frac{n!}{[k!(n-k)!]} p^k (1-p)^{n-k}
 \end{aligned}$$

**n**            Probability of 5 or More Successes

$$5 \quad \binom{5}{0} (0.05)^0 (0.95)^5 = 0.774$$

$$6 \quad \binom{6}{1} (0.05)^1 (0.95)^5 + \binom{6}{0} (0.05)^0 (0.95)^6 = 0.96$$

**An n** of 6 is sufficient.

CASE STUDY NO. 5-4

Background

A new combat assault vehicle is being proposed. The mean miles between mission failure (MMBMF) is to be 320 miles. The mean miles between durability failures (MMBDF) is to be 5771 miles. Historically, both mission failures and durability failures have occurred at a fairly constant rate.

Determine

1. What is the mission reliability for a 50-mile mission?
2. What is the probability of operating for 4000 miles without experiencing a durability failure?
3. What MMBDF is required to have an 80% probability of operating for 4000 miles without a failure?

Solutions

1. We use the reliability function,

$$R(t) = e^{-t/\theta},$$

where  $t$  is mission duration and  $\theta$  is mission MMBF. Since  $t = 50$  miles and  $\theta = 320$  miles, the mission reliability is

$$e^{-50/320} = 0.855 .$$

2. Durability is defined as the probability of completing 4000 miles of operation without suffering a power train durability failure. Again we use the reliability function,

$$R(t) = e^{-t/\theta},$$

where  $t$  is the durability requirement and  $\theta$  is the power train MMBDF. Since  $t = 4000$  miles and  $\theta = 5771$  miles, the mission reliability is

$$e^{-4000/5771} = 0.50 .$$

3. Once again we use the reliability function,

$$R(t) = e^{-t/\theta},$$

but , in this case,  $t = 4000$  miles and  $R(4000) = 0.80$ . By solving the above equation for  $\theta$ , we have

$$\theta = \frac{t}{-\log_e R(t)}$$

Consequently,

$$\theta = \frac{4000}{-\log_e 0.80}$$
$$= 17,926 \text{ miles .}$$

Commentary

Usually durability requirements are imposed on specific subsystems such as the power train. When individual subsystems are considered, durability failures are likely to occur with an increasing failure rate due to physical wearout of components. In such a case, the use of the exponential model would not be appropriate. Other distributional models of life length (such as the Weibull distribution) would be appropriate for use in analyzing durability.

CASE STUDY NO. 5-5

Background

A heavy truck uses 10 road wheels. Each tire's mean miles between failures (MMBF) is 10,000 miles. Assume a series model for the 10 tire system.

Determine

1. What is the failure rate of each tire.
2. What is the probability that a single tire will experience no failure during a 1000 mile mission?
3. Assuming the tires operate independently, what is the probability of no tire failures in a 500 mile mission?
4. What is the MMBF of the entire system of 10 tires.
5. How many spares should be kept so that a 500 mile mission can be completed at least 90% of the time? 99% of the time?

Solutions

1. The failure rate ( $\lambda$ ) is the reciprocal of the mean miles between failures (MMBF). Consequently,

$$\lambda = \frac{1}{\text{MMBF}} = \frac{1}{10,000} = 0.0001$$

2. Using the formula for computing Poisson probabilities (equation 5.3) with the following parameters

Mission Length (T) = 1000 miles  
Failure Rate (A) = 0.0001,

we have

$$g_{\lambda, T}(k) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}$$

$$\begin{aligned} P(\text{no failures}) &= \frac{(\lambda T)^0 e^{-\lambda T}}{0!} = e^{-\lambda T} = e^{-(0.0001)(1000)} \\ &= e^{-0.1} = 0.905 \end{aligned}$$

This value could have been obtained using Appendix B, Table 3, using  $m = \lambda T = 0.1$  and  $c = 0$ . In addition, we could have used Appendix B, Chart 1 using  $0/T = 1/\lambda T = 10$  and  $c = 0$ .

3. For a single tire the probability of no failures in a 500 mile mission is

$$\begin{aligned}
 g_{\lambda, T}(k) &= \frac{(\lambda T)^k e^{-\lambda T}}{k!} \\
 &= \frac{[(0.0001)(500)]^0 e^{-(0.0001)(500)}}{0!} \\
 &= 0.951 .
 \end{aligned}$$

Since all tires are in series, have the same failure rate and operate independently, the probability of no failures among all 10 tires is

$$(0.951)^{10} = 0.605 .$$

4. In order to determine the MMBF of the system of 10 tires, we use the reliability function

$$R(T) = e^{-T/\theta},$$

where T is mission length and  $\theta$  is MMBF. Solving this equation for  $\theta$ , we obtain

$$\theta = \frac{T}{-\log_e R(T)}$$

In problem 3, we determined that  $R(500) = 0.605$  for the 10-tire system. Thus,

$$\begin{aligned}
 \theta &= \frac{500}{-\log_e 0.605} \\
 &= 1000 \text{ miles.}
 \end{aligned}$$

5. Mission Length (T) = 500 miles.  
 System Failure Rate ( $\lambda$ ) = 0.001.  
 (Note: MMBF = 1000 from question 4)

Using Appendix B, Table 3 with  $m = \lambda T = 0.5$ , we have

$$\begin{aligned}
 P(\text{no failures}) &= 0.607 \\
 P(\text{less than or equal to 1 failures}) &= 0.910 \\
 P(\text{less than or equal to 2 failures}) &= 0.986 \\
 P(\text{less than or equal to 3 failures}) &= 0.998
 \end{aligned}$$

For a **90%** chance of mission completion, one spare **will** suffice. For a 99% chance of mission completion, 3 spares are required. However, the improvement in reliability from 0.986 (2 spares) to **0.998** (3 spares) does not appear to warrant the extra spare.

## CASE STUDY NO. 5-6

### Background

A new vehicle is under development which is required to have a MTBF of at least 200 hours ( $\lambda$  no more than 0.005) and a mean time to repair (MTTR) of no more than 40 man-hours. Assume that the specifications (200 hours MTBF and 40 hours MTTR) have been met by contractor A.

### Determine

1. What is the probability that a development test of 1000 hours will show at least 10 failures?
2. What is the probability of no failures during the first 400 hours of the test?
3. How many man-hours of repair time are expected during the 1000-hour test?
4. How much calendar time in days--operating and maintenance--based upon an 8-hour day should be programmed so that at least 10 total repairs will be performed by a team of 4 mechanics working on 10 identical systems? We desire that there be at least 10 failures with probability 0.90. We desire to take only a 10% risk that repair time allotted will be insufficient. Assume that each repair takes 40 hours, the mean repair time. Rework the problem assuming each repair is completed within 60 hours.

### Solutions

1. The "background to this study indicates that this is a continuous time testing situation. In addition, we are interested in numbers of failures and not in times between failures, so the Poisson model is used for this analysis. Test time (T) is 1000 hours and MTBF is 200 hours ( $\lambda = 0.005$ ). The unknown value is the probability of 10 or more failures.

Use Appendix B, Table 3, with  $\lambda T = (0.005)(1000) = 5.0$ . The probability of 10 or more failures =  $1 - P(9 \text{ or fewer failures})$ .

For  $\lambda T = m = 5.0$ , we see in the column labeled  $c = 9$  that the probability of 9 or fewer failures is 0.968. Consequently, the probability of 10 or more failures is 0.032.

2. Test time (T) is 400 hours and MTBF is 200 hours ( $\lambda = 0.005$ ). The unknown value is the probability of 0 failures.

Use Appendix B, Table 3, with  $\lambda T = (0.005)(400) = 2.0$ .

For  $\lambda T = m = 2.0$  we see in the column labeled  $c = 0$  that the probability of zero failures is 0.135.

3. Test time (T) is 1000 hours and MTBF is 200 hours ( $\lambda = 0.005$ ). MTTR is 40 hours ( $\lambda = 0.025$ ). Each repair takes 40 hours on the average. There are

1000(0.005) or 5 failures expected. The expected repair time is 5(40) or 200 man-hours.

4. Let  $T_o$  be operating time needed and  $T_r$  be repair time needed. We desire that  $T_o$  be large enough that the probability of seeing 10 or more failures be at least 0.9. However we must allow repair time for that number of failures ( $n_{10}$ ) which represents all but the upper 10% of the distribution of numbers of failures, i.e., the probability that more than  $n_{10}$  failures occurring is 0.10.

- a. Determine  $T_o$ : Probability of at least 10 failures must be at least 0.90.

Note: The probability of 10 or more failures = 1 - the probability of 9 or fewer failures.

The probability of nine or fewer failures must be no more than 0.10. Under the column labeled  $c = 9.0$  of Table 3, we find by interpolation that a 0.10 probability of 9 or fewer failures is achieved when  $m = 14.2$ .

Since  $m = AT$  and  $m = 14.2$

$$T = T_o \text{ and}$$

$$A = 0.005$$

We solve the following equation for  $T_o$ :

$$T_o = m/A = 14.2/0.005 = 2,840 \text{ hours.}$$

- b. Determine  $n_{10}$ : Allow for 19 failures since the probability of 20 or more failures is just under 0.10.

- c. Determine  $T_r$ :  $(19)(40) = 760$  man-hours  
 $T_r = 760/4 = 190$  clock hours.

- d. Determine number of days needed:

Operating hours: 2,840

Ten systems operating hours: 284 for each system

Maintenance hours: 190

Total hours: 474

Days Required: 59.25

- e. If each **repair** takes 60 man-hours. then 1140 man-hours are required. This **corresponds** to 285 clock hours. Consequently, the total **number** of hours is 569 which represents 71.1 working days.

#### Commentary

For number 4, we have allowed 40 man-hours for each repair. If the average repair time is indeed 40 man-hours, then 760 total man-hours should be a

reasonably close estimate. Computation of the risk involved with this approach is complicated and beyond the scope of this work. One procedure to adopt is to increase the allotted time per repair as we have noted in 4e.