DIGITAL SIGNATURE STANDARD (DSS)

CATEGORY: COMPUTER SECURITY
# Digital Signature Standard (DSS) FIPS PUB 186-2 (Change Notice)

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**ABSTRACT (Maximum 200 Words):**
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Foreword

The Federal Information Processing Standards Publication Series of the National Institute of Standards and Technology (NIST) is the official series of publications relating to standards and guidelines adopted and promulgated under the provisions of Section 5131 of the Information Technology Management Reform Act of 1996 (Public Law 104-106), and the Computer Security Act of 1987 (Public Law 100-235). These mandates have given the Secretary of Commerce and NIST important responsibilities for improving the utilization and management of computer and related telecommunications systems in the Federal Government. The NIST, through its Information Technology Laboratory, provides leadership, technical guidance, and coordination of Government efforts in the development of standards and guidelines in these areas.

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William Mehuron, Director
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Key words: ADP security, computer security, digital signatures, public-key cryptography, Federal Information Processing Standards.
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Announcing the

DIGITAL SIGNATURE STANDARD (DSS)

Federal Information Processing Standards Publications (FIPS PUBS) are issued by the National Institute of Standards and Technology (NIST) after approval by the Secretary of Commerce pursuant to Section 5131 of the Information Technology Management Reform Act of 1996 (Public Law 104-106), and the Computer Security Act of 1987 (Public Law 100-235).

Name of Standard: Digital Signature Standard (DSS).


Explanation: This Standard specifies algorithms appropriate for applications requiring a digital, rather than written, signature. A digital signature is represented in a computer as a string of binary digits. A digital signature is computed using a set of rules and a set of parameters such that the identity of the signatory and integrity of the data can be verified. An algorithm provides the capability to generate and verify signatures. Signature generation makes use of a private key to generate a digital signature. Signature verification makes use of a public key which corresponds to, but is not the same as, the private key. Each user possesses a private and public key pair. Public keys are assumed to be known to the public in general. Private keys are never shared. Anyone can verify the signature of a user by employing that user's public key. Signature generation can be performed only by the possessor of the user's private key.

A hash function is used in the signature generation process to obtain a condensed version of data, called a message digest (see Figure 1). The message digest is then input to the digital signature (ds) algorithm to generate the digital signature. The digital signature is sent to the intended verifier along with the signed data (often called the message). The verifier of the message and signature verifies the signature by using the sender's public key. The same hash function must also be used in the verification process. The hash function is specified in a separate standard, the Secure Hash Standard (SHS), FIPS 180-1. FIPS approved ds algorithms must be implemented with the SHS. Similar procedures may be used to generate and verify signatures for stored as well as transmitted data.
**Signature Generation**

Message

Secure Hash Algorithm-1

Message Digest

Private Key

Digital ds Algorithm Sign Operation

Signature

**Signature Verification**

Received Message

Secure Hash Algorithm-1

Message Digest

Digital Key

ds Algorithm Verify Operation

Signature

Yes - Signature Verified or
No - Signature Verification Failed

**Approving Authority**: Secretary of Commerce.

**Maintenance Agency**: U.S. Department of Commerce, National Institute of Standards and Technology (NIST), Information Technology Laboratory (ITL).

**Applicability**: This standard is applicable to all Federal departments and agencies for the protection of sensitive unclassified information that is not subject to section 2315 of Title 10, United States Code, or section 3502(2) of Title 44, United States Code. This standard shall be used in designing and implementing public-key based signature systems that Federal departments and agencies operate or which are operated for them under contract. Adoption and use of this standard is available to private and commercial organizations.

**Applications**: A digital signature (ds) algorithm authenticates the integrity of the signed data and the identity of the signatory. A ds algorithm may also be used in proving to a third party that data was actually signed by the generator of the signature. A ds algorithm is intended for use in electronic mail, electronic funds transfer, electronic data interchange, software distribution, data storage, and
other applications that require data integrity assurance and data origin authentication. The techniques specified in ANSI X9.31 and ANSI X9.62 may be used in addition to the Digital Signature Algorithm (DSA) specified herein. (NIST editorial note: either DSA, RSA [ANSI X9.31], or ECDSA [ANSI X9.62] may be used; all three do not have to be implemented.)

**Implementations:** A ds algorithm may be implemented in software firmware, hardware or any combination thereof. NIST has developed a validation program to test implementations for conformance to DSA. Currently, conformance tests for ANSI X9.31 and ANSI X9.62 have not been developed. These tests will be developed and made available in the future. Information about the planned validation program can be obtained from the National Institute of Standards and Technology, Information Technology Laboratory, Attn: DSS Validation, 100 Bureau Drive Stop 8930, Gaithersburg, MD 20899-8930.

Agencies are advised that separate keys should be used for signature and confidentiality purposes when using the X9.31 standard. This is because the RSA algorithm can be used for both data encryption and digital signature purposes.

**Export Control:** Certain cryptographic devices and technical data regarding them are subject to Federal export controls. Applicable Federal government export controls are specified in Title 15, Code of Federal Regulations (CFR) Part 740.17; Title 15, CFR Part 742; and Title 15, CFR Part 774, Category 5, Part 2.

**Patents:** The algorithms in this standard may be covered by U.S. or foreign patents.

**Implementation Schedule:** This standard becomes effective July 27, 2000. A transition period from July 27, 2000 until July 27, 2001 is provided to enable all agencies to develop plans for the acquisition of equipment which implements the digital signature techniques adopted by FIPS 186-2. During the transition period, agencies may continue to use their existing digital signature systems and to acquire additional equipment that may be needed to interoperate with these legacy digital signature systems. Agencies without legacy digital signature systems should plan for the acquisition and use of equipment implementing the digital signature techniques that are adopted by FIPS 186-2. After the transition period, only equipment that implements FIPS 186-2 endorsed techniques should be acquired.

**Specifications:** Federal Information Processing Standard (FIPS) 186-2 Digital Signature Standard (affixed). Also see an important **change notice** at the end of this document.

**Cross Index:**

a. FIPS PUB 46-3, Data Encryption Standard.

c. FIPS PUB 140-1, Security Requirements for Cryptographic Modules.

d. FIPS PUB 171, Key Management Using ANSI X9.17.

e. FIPS PUB 180-1, Secure Hash Standard.


Qualifications: The security of a digital signature system is dependent on maintaining the secrecy of users' private keys. Users must therefore guard against the unauthorized acquisition of their private keys. While it is the intent of this standard to specify general security requirements for generating digital signatures, conformance to this standard does not assure that a particular implementation is secure. The responsible authority in each agency or department shall assure that an overall implementation provides an acceptable level of security. This standard will be reviewed every five years in order to assess its adequacy.

Waiver Procedure: Under certain exceptional circumstances, the heads of Federal agencies, or their delegates, may approve waivers to Federal Information Processing Standards (FIPS). The head of such agency may redelegate such authority only to a senior official designated pursuant to section 3506(b) of Title 44, United States Code. Waiver shall be granted only when:

a. Compliance with a standard would adversely affect the accomplishment of the mission of an operator of a Federal computer system; or

b. Cause a major adverse financial impact on the operator which is not offset by Government wide savings.

Agency heads may act upon a written waiver request containing the information detailed above. Agency heads may also act without a written waiver request when they determine that conditions for meeting the standard cannot be met. Agency heads may approve waivers only by a written decision which explains the basis on which the agency head made the required finding(s). A copy of each such decision, with procurement sensitive or classified portions clearly identified, shall be sent to: National Institute of Standards and Technology; ATTN: FIPS Waiver Decisions, 100 Bureau Drive Stop 8970, Gaithersburg, MD 20899-8970.

In addition, notice of each waiver granted and each delegation of authority to approve waivers shall be sent promptly to the Committee on Government Operations of the House of Representatives and the Committee on Governmental Affairs of the Senate and shall be published promptly in the Federal Register.
When the determination on a waiver applies to the procurement of equipment and/or services, a notice of the waiver determination must be published in the Commerce Business Daily as a part of the notice of solicitation for offers of an acquisition or, if the waiver determination is made after that notice is published, by amendment to such notice.

A copy of the waiver, any supporting documents, the document approving the waiver and any supporting and accompanying documents, with such deletions as the agency is authorized and decides to make under 5 U.S.C. Sec. 552(b), shall be part of the procurement documentation and retained by the agency.

**Where to Obtain Copies of the Standard:** Copies of this publication are for sale by the National Technical Information Service, U.S. Department of Commerce, Springfield, VA 22161. When ordering, refer to Federal Information Processing Standards Publication 186-2 (FIPSPUB186-2), and identify the title. When microfiche is desired, this should be specified. Prices are published by NTIS in current catalogs and other issuances. Payment may be made by check, money order, deposit account or charged to a credit card accepted by NTIS.
Specifications for the

DIGITAL SIGNATURE STANDARD (DSS)

1. INTRODUCTION

This publication prescribes three algorithms suitable for digital signature (ds) generation and verification. The first algorithm, the Digital Signature Algorithm (DSA), is described in sections 4 - 6 and appendices 1 - 5. The second algorithm, the RSA ds algorithm, is discussed in section 7 and the third algorithm, the ECDSA algorithm, is discussed in section 8 and recommended elliptic curves in appendix 6. An important change notice has been appended to this document.

2. GENERAL

When a message is received, the recipient may desire to verify that the message has not been altered in transit. Furthermore, the recipient may wish to be certain of the originator's identity. Both of these services can be provided by a ds algorithm. A digital signature is an electronic analogue of a written signature in that the digital signature can be used in proving to the recipient or a third party that the message was, in fact, signed by the originator. Digital signatures may also be generated for stored data and programs so that the integrity of the data and programs may be verified at any later time.

This publication prescribes two algorithms suitable for digital signature generation and verification.

3. USE OF A DIGITAL SIGNATURE (ds) ALGORITHM

A ds algorithm is used by a signatory to generate a digital signature on data and by a verifier to verify the authenticity of the signature. Each signatory has a public and private key. The private key is used in the signature generation process and the public key is used in the signature verification process. For both signature generation and verification, the data which is referred to as a message,
M, is reduced by means of the Secure Hash Algorithm (SHA-1) specified in FIPS 180-1. An adversary, who does not know the private key of the signatory, cannot generate the correct signature of the signatory. In other words, signatures cannot be forged. However, by using the signatory's public key, anyone can verify a correctly signed message. A means of associating public and private key pairs to the corresponding users is required. That is, there must be a binding of a user's identity and the user's public key. This binding may be certified by a mutually trusted party. For example, a certifying authority could sign credentials containing a user's public key and identity to form a certificate. Systems for certifying credentials and distributing certificates are beyond the scope of this standard. NIST intends to publish separate document(s) on certifying credentials and distributing certificates.

4. DSA PARAMETERS

The DSA makes use of the following parameters:

1. \( p \) = a prime modulus, where \( 2^{L-1} < p < 2^L \) for \( 512 \leq L \leq 1024 \) and \( L \) a multiple of 64
2. \( q \) = a prime divisor of \( p - 1 \), where \( 2^{159} < q < 2^{160} \)
3. \( g = h^{(p-1)/q} \mod p \), where \( h \) is any integer with \( 1 < h < p - 1 \) such that \( h^{(p-1)/q} \mod p > 1 \) (\( g \) has order \( q \) mod \( p \))
4. \( x \) = a randomly or pseudorandomly generated integer with \( 0 < x < q \)
5. \( y = g^x \mod p \)
6. \( k \) = a randomly or pseudorandomly generated integer with \( 0 < k < q \)

The integers \( p \), \( q \), and \( g \) can be public and can be common to a group of users. A user's private and public keys are \( x \) and \( y \), respectively. They are normally fixed for a period of time. Parameters \( x \) and \( k \) are used for signature generation only, and must be kept secret. Parameter \( k \) must be regenerated for each signature.

Parameters \( p \) and \( q \) shall be generated as specified in Appendix 2, or using other FIPS approved security methods. Parameters \( x \) and \( k \) shall be generated as specified in Appendix 3, or using other FIPS approved security methods.

5. DSA SIGNATURE GENERATION

The signature of a message \( M \) is the pair of numbers \( r \) and \( s \) computed according to the equations below:
r = (g^k \mod p) \mod q \quad \text{and} \quad s = (k^{-1}(\text{SHA}-1(M) + xr)) \mod q.

In the above, $k^{-1}$ is the multiplicative inverse of $k$, mod $q$; i.e., $(k^{-1}k) \mod q = 1$ and $0 < k^{-1} < q$. The value of $\text{SHA}-1(M)$ is a 160-bit string output by the Secure Hash Algorithm specified in FIPS 180-1. For use in computing $s$, this string must be converted to an integer. The conversion rule is given in Appendix 2.2.

As an option, one may wish to check if $r = 0$ or $s = 0$. If either $r = 0$ or $s = 0$, a new value of $k$ should be generated and the signature should be recalculated (it is extremely unlikely that $r = 0$ or $s = 0$ if signatures are generated properly).

The signature is transmitted along with the message to the verifier.

6. DSA SIGNATURE VERIFICATION

Prior to verifying the signature in a signed message, $p$, $q$ and $g$ plus the sender's public key and identity are made available to the verifier in an authenticated manner.

Let $M'$, $r'$, and $s'$ be the received versions of $M$, $r$, and $s$, respectively, and let $y$ be the public key of the signatory. To verify the signature, the verifier first checks to see that $0 < r' < q$ and $0 < s' < q$; if either condition is violated the signature shall be rejected. If these two conditions are satisfied, the verifier computes

\[ w = (s')^{-1} \mod q \]
\[ u_1 = ((\text{SHA}-1(M'))w) \mod q \]
\[ u_2 = ((r')w) \mod q \]
\[ v = (((g)^{u_1} (y)^{u_2}) \mod p) \mod q. \]

If $v = r'$, then the signature is verified and the verifier can have high confidence that the received message was sent by the party holding the secret key $x$ corresponding to $y$. For a proof that $v = r'$ when $M' = M$, $r' = r$, and $s' = s$, see Appendix 1.

If $v$ does not equal $r'$, then the message may have been modified, the message may have been incorrectly signed by the signatory, or the message may have been signed by an impostor. The message should be considered invalid.

7. RSA DIGITAL SIGNATURE ALGORITHM
The RSA ds algorithm is a FIPS approved cryptographic algorithm for digital signature generation and verification. This is described in ANSI X9.31.

8. ELLIPTIC CURVE DIGITAL SIGNATURE ALGORITHM (ECDSA)

The ECDSA ds algorithm is a FIPS approved cryptographic algorithm for digital signature generation and verification. ECDSA is the elliptic curve analogue of the DSA. ECDSA is described in ANSI X9.62. The recommended elliptic curves for Federal Government use are included in Appendix 6.
This appendix is for informational purposes only and is not required to meet the standard.

The purpose of this appendix is to show that in the DSA, if \( M' = M, r' = r \) and \( s' = s \) in the signature verification then \( v = r' \). We need the following easy result.

**Lemma.** Let \( p \) and \( q \) be primes so that \( q \) divides \( p - 1 \), \( h \) a positive integer less than \( p \), and \( g = h^{(p-1)/q} \mod p \). Then \( g^q \mod p = 1 \), and if \( m \mod q = n \mod q \), then \( g^m \mod p = g^n \mod p \).

**Proof:** We have
\[
g^q \mod p = (h^{(p-1)/q} \mod p)^q \mod p
= h^{(p-1)} \mod p
= 1
\]

by Fermat's Little Theorem. Now let \( m \mod q = n \mod q \), i.e., \( m = n + kq \) for some integer \( k \). Then
\[
g^m \mod p = g^{n+kq} \mod p
= (g^n g^{kq}) \mod p
= ((g^n \mod p) (g^q \mod p)^k) \mod p
= g^n \mod p
\]
since \( g^q \mod p = 1 \). □

We are now ready to prove the main result.

**Theorem.** If \( M' = M, r' = r \), and \( s' = s \) in the signature verification, then \( v = r' \).

**Proof:** We have
\[
w = (s')^{-1} \mod q = s^{-1} \mod q
\]
\[
u_1 = ((\text{SHA}-1(M'))w) \mod q = ((\text{SHA}-1(M))w) \mod q
\]
\[
u_2 = ((r')w) \mod q = (rw) \mod q.
\]
Now \( y = g^x \mod p \), so that by the lemma,

\[
v = ((g^{u_1} y^{u_2}) \mod p) \mod q
\]

\[
= ((g^{\text{SHA-1}(M)w} y^{rw}) \mod p) \mod q
\]

\[
= ((g^{\text{SHA-1}(M)w} g^{xrw}) \mod p) \mod q
\]

\[
= ((g^{\text{SHA-1}(M)+xrw}) \mod p) \mod q.
\]

Also

\[
s = (k^{-1} (\text{SHA-1}(M) + xr)) \mod q.
\]

Hence

\[
w = (k(\text{SHA-1}(M) + xr)^{-1}) \mod q
\]

\[
(\text{SHA-1}(M) + xr)w \mod q = k \mod q.
\]

Thus by the lemma,

\[
v = (g^k \mod p) \mod q
\]

\[
= r
\]

\[
= r'. \blacksquare
\]
APPENDIX 2. GENERATION OF PRIMES FOR THE DSA

This appendix includes algorithms for generating the primes p and q used in the DSA. These algorithms require a random number generator (see Appendix 3), and an efficient modular exponentiation algorithm. Generation of p and q shall be performed as specified in this appendix, or using other FIPS approved security methods.

2.1. A PROBABILISTIC PRIMALITY TEST

In order to generate the primes p and q, a primality test is required.

There are several fast probabilistic algorithms available. The following algorithm is a simplified version of a procedure due to M.O. Rabin, based in part on ideas of Gary L. Miller. [See Knuth, The Art of Computer Programming, Vol. 2, Addison-Wesley, 1981, Algorithm P, page 379.] If this algorithm is iterated n times, it will produce a false prime with probability no greater than $1/4^n$. Therefore, $n \geq 50$ will give an acceptable probability of error. To test whether an integer is prime:

Step 1. Set $i = 1$ and $n \geq 50$.

Step 2. Set $w$ = the integer to be tested, $w = 1 + 2^a m$, where $m$ is odd and $2^a$ is the largest power of 2 dividing $w - 1$.

Step 3. Generate a random integer $b$ in the range $1 < b < w$.

Step 4. Set $j = 0$ and $z = b^m \mod w$.

Step 5. If $j = 0$ and $z = 1$, or if $z = w - 1$, go to step 9.

Step 6. If $j > 0$ and $z = 1$, go to step 8.

Step 7. $j = j + 1$. If $j < a$, set $z = z^2 \mod w$ and go to step 5.

Step 8. $w$ is not prime. Stop.

Step 9. If $i < n$, set $i = i + 1$ and go to step 3. Otherwise, $w$ is probably prime.

2.2. GENERATION OF PRIMES

The DSA requires two primes, p and q, satisfying the following three conditions:

a. $2^{159} < q < 2^{160}$

b. $2^{L-1} < p < 2^L$ for a specified L, where $L = 512 + 64j$ for some $0 \leq j \leq 8$
This prime generation scheme starts by using the SHA-1 and a user supplied SEED to construct a prime, q, in the range $2^{159} < q < 2^{160}$. Once this is accomplished, the same SEED value is used to construct an X in the range $2^{L-1} < X < 2^L$. The prime, p, is then formed by rounding X to a number congruent to 1 mod 2q as described below.

An integer x in the range $0 \leq x < 2^g$ may be converted to a g-long sequence of bits by using its binary expansion as shown below:

\[ x = x_1 \cdot 2^{g-1} + x_2 \cdot 2^{g-2} + ... + x_{g-1} \cdot 2 + x_g \rightarrow \{ x_1, ..., x_g \}. \]

Conversely, a g-long sequence of bits \{ x_1, ..., x_g \} is converted to an integer by the rule

\{ x_1, ..., x_g \} \rightarrow x_1 \cdot 2^{g-1} + x_2 \cdot 2^{g-2} + ... + x_{g-1} \cdot 2 + x_g.

Note that the first bit of a sequence corresponds to the most significant bit of the corresponding integer and the last bit to the least significant bit.

Let $L - 1 = n \cdot 160 + b$, where both b and n are integers and $0 \leq b < 160$.

Step 1. Choose an arbitrary sequence of at least 160 bits and call it SEED. Let g be the length of SEED in bits.

Step 2. Compute

\[ U = \text{SHA-1}[\text{SEED}] \text{ XOR } \text{SHA-1}[(\text{SEED}+1) \mod 2^g]. \]

Step 3. Form q from U by setting the most significant bit (the $2^{159}$ bit) and the least significant bit to 1. In terms of boolean operations, $q = U \text{ OR } 2^{159} \text{ OR } 1$. Note that $2^{159} < q < 2^{160}$.

Step 4. Use a robust primality testing algorithm to test whether q is prime\(^1\).

Step 5. If q is not prime, go to step 1.

Step 6. Let counter = 0 and offset = 2.

Step 7. For k = 0, ..., n let

\[ V_k = \text{SHA-1}[(\text{SEED} + \text{offset} + k) \mod 2^g]. \]

\(^1\)A robust primality test is one where the probability of a non-prime number passing the test is at most $2^{-80}$. 

Step 8. Let $W$ be the integer

$$W = V_0 + V_1 \times 2^{160} + \ldots + V_{n-1} \times 2^{(n-1) \times 160} + (V_n \mod b) \times 2^n \times 160$$

and let $X = W + 2^{L-1}$. Note that $0 \leq W < 2^{L-1}$ and hence $2^{L-1} \leq X < 2^L$.

Step 9. Let $c = X \mod 2q$ and set $p = X - (c - 1)$. Note that $p$ is congruent to 1 mod $2q$.

Step 10. If $p < 2^{L-1}$, then go to step 13.


Step 12. If $p$ passes the test performed in step 11, go to step 15.

Step 13. Let counter = counter + 1 and offset = offset + n + 1.

Step 14. If counter $\geq 2^{12} = 4096$ go to step 1, otherwise (i.e. if counter $< 4096$) go to step 7.

Step 15. Save the value of SEED and the value of counter for use in certifying the proper generation of $p$ and $q$. 
APPENDIX 3. RANDOM NUMBER GENERATION FOR THE DSA

Any implementation of the DSA requires the ability to generate random or pseudorandom integers. Such numbers are used to derive a user's private key, x, and a user's per message secret number, k. These randomly or pseudorandomly generated integers are selected to be between 0 and the 160-bit prime q (as specified in the standard). They shall be generated by the techniques given in this appendix, or using other FIPS approved security methods.

One FIPS approved pseudorandom integer generator is supplied in Appendix C of ANSI X9.17, "Financial Institution Key Management (Wholesale)."

Other pseudorandom integer generators are given in this appendix. These permit generation of pseudorandom values of x and k for use in the DSA. The algorithm in section 3.1 may be used to generate values for x. An algorithm for k and r is given in section 3.2. The latter algorithm allows most of the signature computation to be precomputed without knowledge of the message to be signed.

The algorithms employ a one-way function \( G(t,c) \), where \( t \) is 160 bits, \( c \) is \( b \) bits (160 ≤ \( b \) ≤ 512) and \( G(t,c) \) is 160 bits. One way to construct \( G \) is via the Secure Hash Algorithm (SHA-1), as defined in the Secure Hash Standard (SHS). The 160-bit message digest output of the SHA-1 algorithm when message M is input is denoted by SHA-1(M). A second method for constructing \( G \) is to use the Data Encryption Standard (DES). The construction of \( G \) by these techniques is discussed in sections 3.3 and 3.4 of this appendix.

In the algorithms in sections 3.1 and 3.2, a secret \( b \)-bit seed-key is used. The algorithm in section 3.1 optionally allows the use of a user provided input. If \( G \) is constructed via the SHA-1 as defined in section 3.3, then \( b \) is between 160 and 512. If DES is used to construct \( G \) as defined in section 3.4, then \( b \) is equal to 160.

### 3.1. ALGORITHM FOR COMPUTING \( m \) VALUES OF x

Let \( x \) be the signer's private key. The following may be used to generate \( m \) values of \( x \):

Step 1. Choose a new, secret value for the seed-key, XKEY.

Step 2. In hexadecimal notation let

\[
    t = 67452301 \text{ EFCDBAB9} 98BADCDFE 10325476 \text{ C3D2E1F0}.
\]

This is the initial value for \( H_0 \| H_1 \| H_2 \| H_3 \| H_4 \) in the SHS.

Step 3. For \( j = 0 \) to \( m - 1 \) do
3.2. ALGORITHM FOR PRECOMPUTING ONE OR MORE $k$ AND $r$ VALUES

This algorithm can be used to precompute $k$, $k^{-1}$, and $r$ for $m$ messages at a time. Note that implementation of the DSA with precomputation may be covered by U.S. and foreign patents.

Algorithm:

Step 1. Choose a secret initial value for the seed-key, $KKEY$.

Step 2. In hexadecimal notation let

$$t = \text{EFCDAB89 98BADCFE 10325476 C3D2E1F0 67452301.}$$

This is a cyclic shift of the initial value for $H_0 \parallel H_1 \parallel H_2 \parallel H_3 \parallel H_4$ in the SHS.

Step 3. For $j = 0$ to $m - 1$ do

a. $k = G(t,KKEY) \mod q$.

b. Compute $k_j^{-1} = k^{-1} \mod q$.

c. Compute $r_j = (g^k \mod p) \mod q$.

d. $KKEY = (1 + KKEY + k) \mod 2^b$.

Step 4. Suppose $M_0, \ldots, M_{m-1}$ are the next $m$ messages. For $j = 0$ to $m - 1$ do

a. Let $h = \text{SHA-1}(M_j)$.

b. Let $s_j = (k_j^{-1}(h + xr_j)) \mod q$.

c. The signature for $M_j$ is $(r_j,s_j)$.

Step 5. Let $t = h$.

Step 6. Go to step 3.
Step 3 permits precomputation of the quantities needed to sign the next m messages. Step 4 can begin whenever the first of these m messages is ready. The execution of step 4 can be suspended whenever the next of the m messages is not ready. As soon as steps 4 and 5 have completed, step 3 can be executed, and the results saved until the first member of the next group of m messages is ready.

In addition to space for KKEY, two arrays of length m are needed to store \( r_0, \ldots, r_{m-1} \) and \( k_0^{-1}, \ldots, k_{m-1}^{-1} \) when they are computed in step 3. Storage for \( s_0, \ldots, s_{m-1} \) is only needed if the signatures for a group of messages are stored; otherwise \( s_j \) in step 4 can be replaced by \( s \) and a single space allocated.

### 3.3. CONSTRUCTING THE FUNCTION G FROM THE SHA-1

\( G(t,c) \) may be constructed using steps (a) - (e) in section 7 of the Specifications for the Secure Hash Standard. Before executing these steps, \( \{H_j\} \) and \( M_1 \) must be initialized as follows:

i. Initialize the \( \{H_j\} \) by dividing the 160 bit value \( t \) into five 32-bit segments as follows:

\[
  t = t_0 \parallel t_1 \parallel t_2 \parallel t_3 \parallel t_4
\]

Then \( H_j = t_j \) for \( j = 0 \) through 4.

ii. There will be only one message block, \( M_1 \), which is initialized as follows:

\[
  M_1 = c \parallel 0^{512-b}
\]

(The first \( b \) bits of \( M_1 \) contain \( c \), and the remaining \( (512-b) \) bits are set to zero).

Then steps (a) through (e) of section 7 are executed, and \( G(t,c) \) is the 160 bit string represented by the five words:

\[
  H_0 \parallel H_1 \parallel H_2 \parallel H_3 \parallel H_4
\]

at the end of step (e).

### 3.4. CONSTRUCTING THE FUNCTION G FROM THE DES

Let \( \text{XOR} \) denote the bitwise exclusive-or of bit strings \( a \) and \( b \). Suppose \( a_1, a_2, b_1, b_2 \) are 32-bit strings. Let \( b_1' \) be the 24 least significant bits of \( b_1 \). Let \( K = b_1' \parallel b_2 \) and \( A = a_1 \parallel a_2 \). Define

\[
  \text{DES}_{b_1,b_2}(a_1,a_2) = \text{DES}_K(A)
\]

In the above, \( \text{DES}_K(A) \) represents ordinary DES encryption of the 64-bit block \( A \) using the 56-bit
key K. Now suppose t and c are each 160 bits. To compute G(t,c):

Step 1. Write

\[ t = t_1 \| t_2 \| t_3 \| t_4 \| t_5 \]
\[ c = c_1 \| c_2 \| c_3 \| c_4 \| c_5 \]

In the above, each \( t_i \) and \( c_i \) is 32 bits.

Step 2. For \( i = 1 \) to 5 do

\[ x_i = t_i \text{ XOR } c_i \]

Step 3. For \( i = 1 \) to 5 do

\[ b1 = c_{(i+3 \mod 5) + 1} \]
\[ b2 = c_{(i+2 \mod 5) + 1} \]
\[ a1 = x_i \]
\[ a2 = x_{(i \mod 5) + 1} \text{ XOR } x_{((i+3 \mod 5) + 1} \]
\[ y_{i,1} \| y_{i,2} = \text{DES}_{b1,b2}(a1,a2) \quad (y_{i,1}, y_{i,2} = 32 \text{ bits}) \]

Step 4. For \( i = 1 \) to 5 do

\[ z_i = y_{i,1} \text{ XOR } y_{((i+1) \mod 5)+1,2} \text{ XOR } y_{((i+2) \mod 5)+1,1} \]

Step 5. Let

\[ G(t,c) = z_1 \| z_2 \| z_3 \| z_4 \| z_5 \]
APPENDIX 4. GENERATION OF OTHER QUANTITIES FOR THE DSA

This appendix is for informational purposes only and is not required to meet the standard.

The algorithms given in this appendix may be used to generate the quantities $g$, $k^{-1}$, and $s^{-1}$ used in the DSA.

To generate $g$:

Step 1. Generate $p$ and $q$ as specified in Appendix 2.

Step 2. Let $e = (p - 1)/q$.

Step 3. Set $h$ = any integer, where $1 < h < p - 1$ and $h$ differs from any value previously tried.

Step 4. Set $g = h^e \mod p$.

Step 5. If $g = 1$, go to step 3.

To compute the multiplicative inverse $n^{-1} \mod q$ for $n$ with $0 < n < q$, where $0 < n^{-1} < q$:

Step 1. Set $i = q$, $h = n$, $v = 0$, and $d = 1$.

Step 2. Let $t = i \div h$, where $\div$ is defined as integer division.

Step 3. Set $x = h$.

Step 4. Set $h = i - tx$.

Step 5. Set $i = x$.

Step 6. Set $x = d$.

Step 7. Set $d = v - tx$.

Step 8. Set $v = x$.

Step 9. If $h > 0$, go to step 2.

Step 10. Let $n^{-1} = v \mod q$.

Note that in step 10, $v$ may be negative. The $v \mod q$ operation should yield a value between 1 and $q - 1$ inclusive.
This appendix is for informational purposes only and is not required to meet the standard. Let $L = 512$ (size of $p$). The values in this example are expressed in hexadecimal notation. The $p$ and $q$ given here were generated by the prime generation standard described in appendix 2 using the 160-bit SEED:

$d5014e4b$ $60ef2ba8$ $b6211b40$ $62ba3224$ $e0427dd3$

With this SEED, the algorithm found $p$ and $q$ when the counter was at 105. $x$ was generated by the algorithm described in appendix 3, section 3.1, using the SHA-1 to construct $G$ (as in appendix 3, section 3.3) and a 160-bit XKEY:

\[
XKEY = \begin{array}{c}
bd029bbe \\
7f51960b \\
61f06f0f \\
6b5a38b6
\end{array}
\]

\[
t = 67452301\ EFCDAB89\ 98BADCFE\ 10325476\ C3D2E1F0
\]

\[
x = G(t,XKEY) \mod q
\]

$k$ was generated by the algorithm described in appendix 3, section 3.2, using the SHA-1 to construct $G$ (as in appendix 3, section 3.3) and a 160-bit KKEY:

\[
KKEY = \begin{array}{c}
687a66d9 \\
0648f993 \\
867e121f \\
4dd9f9db
\end{array}
\]

\[
t = EFCDAB89\ 98BADCFE\ 10325476\ C3D2E1F0\ 67452301
\]

\[
k = G(t,KKEY) \mod q
\]

Finally:

\[
h = 2
\]

\[
p = 8df2a494\ 492276aa\ 3d25759b\ b06869cb\ eac0d83a\ fb8d0cf7\ cbb8324f\ 0d7882e5\ d0762fc5\ b7210eaf\ c2e9adac\ 32ab7aac
\]
\begin{align*}
\text{q} &= \text{c773218c 737ec8ee 993b4f2d ed30f48e dace915f} \\
\text{g} &= \text{626d0278 39ea0a13 413163a5 5b4cb500 299d5522 956cefc} \\
&\quad \text{b 3bff10f3 99ce2c2e 71cb9de5 fa24babf 58e5b795 21925c9c} \\
&\quad \text{c42e9f6f 464b088c c572af53 e6d78802} \\
\text{x} &= \text{2070b322 3dba372f delcOffc 7b2e3b49 8b260614} \\
\text{k} &= \text{358dad57 1462710f 50e254cf 1a376b2b deaadfbf} \\
\text{k}^\text{-1} &= \text{0d516729 8202e49b 4116ac10 4fc3f415 ae52f917} \\
\text{M} &= \text{ASCII form of "abc" (See FIPS PUB 180-1, Appendix A)} \\
\text{(SHA-1)(M)} &= \text{a9993e36 4706816a ba3e2571 7850c26c 9cd0d89d} \\
\text{y} &= \text{19131871 d75b1612 a819f29d 78d1b0d7 346f7aa7 7bb62a85} \\
&\quad \text{9bfe6c56 75da9d21 2d3a36ef 1672ef66 0b8c7c25 5cc0ec74} \\
&\quad \text{858fba33 f44c0669 9630a76b 030ee333} \\
\text{r} &= \text{8baclab6 6410435c b7181f95 b16ab97c 92b341c0} \\
\text{s} &= \text{41e2345f 1f56df24 58f426d1 55b4ba2d b6dcd8c8} \\
\text{w} &= \text{9df4ec5 826be95f ed406d41 b43edc0b 1c18841b}
\end{align*}
\[ u_1 = \]
\[
\begin{array}{c}
bf655bd0 46f0b35e c791b004 804afcb \ b \ 8ef7d69d
\end{array}
\]
\[ u_2 = \]
\[
\begin{array}{c}
821a9263 12e97ade \ abcc8d08 \ 2b527897 \ 8a2df4b0
\end{array}
\]
\[ g^{u_1} \mod p = \]
\[
\begin{array}{c}
51b1bf86 \ 7888e5f3 \ af6fb476 \ 9dd016bc \ fe667a65 \ aafc2753 \\
9063bd3d \ 2b138b4c \ e02cc0c0 \ 2ec62bb6 \ 7306c63e \ 4db95bbf \\
6f96660a \ 1987a21b \ e4ec1071 \ 010b6069
\end{array}
\]
\[ y^{u_2} \mod p = \]
\[
\begin{array}{c}
8b510071 \ 2957e950 \ 50d6b8fd \ 376a668e \ 4b0d633c \ 1e46e665 \\
5c611a72 \ e2b28483 \ be52c74d \ 4b30de61 \ a668966e \ dc307a67 \\
c19441f4 \ 22bf3c34 \ 08ae1baf \ 0a4dbec7
\end{array}
\]
\[ v = \]
\[
\begin{array}{c}
8bac1ab6 \ 6410435c \ b7181f95 \ b16ab97c \ 92b341c0
\end{array}
\]
This collection of elliptic curves is recommended for Federal government use and contains choices of private key length and underlying fields.

1. Parameter Choices

1.1 Choice of Key Lengths

The principal parameters for elliptic curve cryptography are the elliptic curve \( E \) and a designated point \( G \) on \( E \) called the base point. The base point has order \( r \), a large prime. The number of points on the curve is \( n \approx fr \) for some integer \( f \) (the cofactor) not divisible by \( r \). For efficiency reasons, it is desirable to take the cofactor to be as small as possible.

All of the curves given below have cofactors 1, 2, or 4. As a result, the private and public keys are approximately the same length. Each length is chosen to correspond to the cryptovariable length of a common symmetric cryptologic. In each case, the private key length is, at least, approximately twice the symmetric cryptovariable length.

1.2 Choice of Underlying Fields

For each cryptovariable length, there are given two kinds of fields.

- A prime field is the field \( GF(p) \) which contains a prime number \( p \) of elements. The elements of this field are the integers modulo \( p \), and the
field arithmetic is implemented in terms of the arithmetic of integers modulo $p$.

- A binary field is the field $GF(2^m)$ which contains $2^m$ elements for some $m$ (called the degree of the field). The elements of this field are the bit strings of length $m$, and the field arithmetic is implemented in terms of operations on the bits.

The following table gives the sizes of the various underlying fields. By $\|p\|$ is meant the length of the binary expansion of the integer $p$.

<table>
<thead>
<tr>
<th>Symmetric</th>
<th>Example</th>
<th>Prime Field</th>
<th>Binary Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>CV Length</td>
<td>Algorithm</td>
<td>$|p| = 192$</td>
<td>$m = 163$</td>
</tr>
<tr>
<td>80</td>
<td>SKIPJACK</td>
<td></td>
<td></td>
</tr>
<tr>
<td>112</td>
<td>Triple-DES</td>
<td>$|p| = 224$</td>
<td>$m = 233$</td>
</tr>
<tr>
<td>128</td>
<td>AES Small</td>
<td>$|p| = 256$</td>
<td>$m = 283$</td>
</tr>
<tr>
<td>192</td>
<td>AES Medium</td>
<td>$|p| = 384$</td>
<td>$m = 409$</td>
</tr>
<tr>
<td>256</td>
<td>AES Large</td>
<td>$|p| = 521$</td>
<td>$m = 571$</td>
</tr>
</tbody>
</table>

### 1.3 Choice of Basis

To describe the arithmetic of a binary field, it is first necessary to specify how a bit string is to be interpreted. This is referred to as choosing a basis for the field. There are two common types of bases: a polynomial basis and a normal basis.
A polynomial basis is specified by an irreducible polynomial modulo 2, called the field polynomial. The bit string \((a_m \ldots a_2 a_1 a_0)\) is taken to represent the polynomial
\[
a_{m-1} t^{m-1} + \ldots + a_2 t^2 + a_1 t + a_0
\]
over GF(2). The field arithmetic is implemented as polynomial arithmetic modulo \(p(t)\), where \(p(t)\) is the field polynomial.

A normal basis is specified by an element \(\theta\) of a particular kind. The bit string \((a_0 a_1 a_2 \ldots a_{m-1})\) is taken to represent the element
\[
a_0 \theta + a_1 \theta^2 + a_2 \theta^2^2 + a_{m-1} \theta^{2^{m-1}}.
\]
Normal basis field arithmetic is not easy to describe or efficient to implement in general, but is for a special class called Type T low-complexity normal bases. For a given field degree \(m\), the choice of \(T\) specifies the basis and the field arithmetic (see Appendix 6.2).

There are many polynomial bases and normal bases from which to choose. The following procedures are commonly used to select a basis representation.

**Polynomial Basis:** If an irreducible trinomial \(t^m + t^k + 1\) exists over GF (2), then the field polynomial \(p(t)\) is chosen to be the irreducible trinomial with the lowest-degree middle term \(t^k\). If no irreducible trinomial exists, then one selects instead a pentanomial \(t^m + t^a + t^b + t^c + 1\). The particular pentanomial chosen has the following properties: the second term \(t^a\) has the lowest degree \(m\); the third term \(t^b\) has the lowest degree among all irreducible pentanomials of degree \(m\) and second term \(t^a\); and the fourth term \(t^c\) has the lowest degree among all irreducible pentanomials of degree \(m\), second term \(t^a\), and third term \(t^b\).
• **Normal Basis:** Choose the Type T low-complexity normal basis with the smallest $T$.

For each binary field, the parameters are given for the above basis representations.

### 1.4 Choice of Curves

Two kinds of curves are given:

- **Pseudo-random** curves are those whose coefficients are generated from the output of a seeded cryptographic hash. If the seed value is given along with the coefficients, it can be verified easily that the coefficients were indeed generated by that method.

- **Special curves** whose coefficients and underlying field have been selected to optimize the efficiency of the elliptic curve operations.

For each size, the following curves are given:

→ A pseudo-random curve over $GF(p)$.

→ A pseudo-random curve over $GF(2^m)$.

→ A special curve over $GF(2^m)$ called a *Koblitz curve* or *anomalous binary curve*.

The pseudo-random curves are generated via the SHA-1 based method given in the ANSI X9.62 and IEEE P1363 standards. (The generation and verification processes are given in Appendices 6-4 through 6-7.)

### 1.5 Choice of Base Points
Any point of order $r$ can serve as the base point. Each curve is supplied with a sample base point $G = (G_x, G_y)$. Users may want to generate their own base points to ensure cryptographic separation of networks.

2. Curves over Prime Fields

For each prime $p$, a pseudo-random curve

$$E : y^2 = x^3 - 3x + b \pmod{p}$$

of prime order $r$ is listed\(^1\). (Thus, for these curves, the cofactor is always $f = 1$.)

The following parameters are given:

- The prime modulus $p$
- The order $r$
- the 160-bit input seed $s$ to SHA-1 based algorithm
- The output $c$ of the SHA-1 based algorithm
- The coefficient $b$ (satisfying $b^2 c \equiv -27 \pmod{p}$)
- The base point $x$ coordinate $G_x$
- The base point $y$ coordinate $G_y$

The integers $p$ and $r$ are given in decimal form; bit strings and field elements are given in hex.

---

\(^1\) The selection $a = -3$ for the coefficient of $x$ was made for reasons of efficiency; see IEEE P1363.
Curve P-192

\[ p = \begin{array}{c}
62771017353866807638357894232076664160839087 \\
00390324961279
\end{array} \]

\[ r = \begin{array}{c}
62771017353866807638357894231760590137671947 \\
73182842284081
\end{array} \]

\[ s = \begin{array}{c}
3045ae6f c8422f64 ed579528 d38120ea e12196d5
\end{array} \]

\[ c = \begin{array}{c}
3099d2bb
\end{array} \]

\begin{array}{c}
bfcb2538 542dcd5f b078b6ef 5f3d6fe2 c745de65
\end{array} \]

\[ b = \begin{array}{c}
64210519
\end{array} \]

\begin{array}{c}
e59c80e7 0fa7e9ab 72243049 feb8deec c146b9b1
\end{array} \]

\[ G_x = \begin{array}{c}
188da80e
\end{array} \]

\begin{array}{c}
b03090f6 7cbf20eb 43a18800 f4ff0afd 82ff1012
\end{array} \]

\[ G_y = \begin{array}{c}
07192b95
\end{array} \]

\begin{array}{c}
ffc8da78 631011ed 6b24cdd5 73f977a1 1e794811
\end{array} \]
Curve P-224

\[ p = 26959946667150639794667015087019630673557916\backslash
260026308143510066298881 \]

\[ r = 26959946667150639794667015087019625940457807\backslash
714424391721682722368061 \]

\[ s = \text{bd713447 99d5c7fc dc45b59f a3b9ab8f 6a948be5} \]

\[ c = \text{5b056c7e 11dd68f4 0469ee7f 3c7a7d74 f7d12111 6506d031 218291fb} \]

\[ b = \text{b4050a85 0c04b3ab f5413256 5044b0b7 d7b8d8ba 270b3943 2355ff8b4} \]

\[ G_x = \text{b70e0cb6 6bb4bf7f 321390b9 4a03c1d3 56c21122 343280d6 115c1d21} \]

\[ G_y = \text{bd376388 b5f723fb 4e22dfe6 cd4375a0 5a074764 44d58199 85007e34} \]
Curve P-256

\[ p = \begin{align*}
11579208921035624876269744694940757353008614 \\
3415290314195533631308867097853951
\end{align*} \]

\[ r = \begin{align*}
11579208921035624876269744694940757352999695 \\
5224135760342422259061068512044369
\end{align*} \]

\[ s = \begin{align*}
c49d3608 86e70493 6a6678e1 139d26b7 819f7e90
\end{align*} \]

\[ c = \begin{align*}
7efba166 2985be94 03cb055c
75d4f7e0 ce8d84a9 c5114abc af317768 0104fa0d
\end{align*} \]

\[ b = \begin{align*}
5ac635d8 aa3a93e7 b3ebbd55
769886bc 651d06b0 cc53b0f6 3bce3c3e 27d2604b
\end{align*} \]

\[ G_x = \begin{align*}
6b17d1f2 e12c4247 f8bce6e5
63a440f2 77037d81 2deb33a0 f4a13945 d898c296
\end{align*} \]

\[ G_y = \begin{align*}
4fe342e2 fe1a7f9b 8ee7eb4a
7c0f9e16 2bce3357 6b315ece cbb64068 37bf51f5
\end{align*} \]
Curve P-384

\[ p = 39402006196394479212279040100143613805079739\backslash
27046544666794829340424572177149687032904726\backslash
6088258938001861606973112319 \]

\[ r = 39402006196394479212279040100143613805079739\backslash
27046544666794690527962765939911326356939895\backslash
6308152294913554433653942643 \]

\[ s = \text{a335926a a319a27a 1d00896a 6773a482 7acdac73} \]

\[ c = \text{79d1e655 f868f02f} \]

\[ \text{ff48dcde e14151dd b80643c1 406d0ca1 0dfe6fc5} \]
\[ \text{2009540a 495e8042 ea5f744f 6e184667 cc722483} \]

\[ b = \text{b3312fa7 e23ee7e4} \]

\[ \text{988e056b e3f82d19 181d9c6e fe814112 0314088f} \]
\[ \text{5013875a c656398d 8a2ed19d 2a85c8ed d3ec2aef} \]

\[ G_x = \text{aa87ca22 be8b0537} \]
\[ \text{8eb1c71e f320ad74 6e1d3b62 8ba79b98 59f741e0} \]
\[ \text{82542a38 5502f25d bf55296c 3a545e38 72760ab7} \]

\[ G_y = \text{3617de4a 96262c6f} \]
\[ \text{5d9e98bf 9292dc29 f8f41dbd 289a147c e9da3113} \]
\[ \text{b5f0b8e0 0a60b1ce 1d7e819d 7a431d7c 90ea0e5f} \]
Curve P-521

\[ p = 68647976601306097149819007990813932172694353 \]
\[ 00143305409394463459185543183397656052122559 \]
\[ 64066145455497729631139148085803712198799971 \]
\[ 6643812574028291115057151 \]

\[ r = 68647976601306097149819007990813932172694353 \]
\[ 00143305409394463459185543183397655394245057 \]
\[ 74633321719753296399637136332111386476861244 \]
\[ 0380340372808892707005449 \]

\[ s = \text{d09e8800 291cb853 96cc6717 393284aa a0da64ba} \]

\[ c = 0b4 8bfa5f42 \]
\[ \text{0a349a95 39d2bd1f 264e3eeeb 077688e4 4fb0ad8} \]
\[ \text{f6d0edb3 7bd6b533 28100051 8e19f1b9 ffbe0fe9} \]
\[ \text{ed8a3c22 00b8f875 e5238686c 70c1e5bf 55bad637} \]

\[ b = 051953eb961 \]
\[ \text{8e1c9a1f 929a21a0 b68540ee a2da725b 99b315f3} \]
\[ \text{b0b48991 8ef109e1 56193951 ec7e937b 1652c0bd} \]
\[ \text{3bb1bf07 3573df88 3d2c34f1 ef451fd4 6d503f00} \]

\[ G_x = \text{c6 858e06b7} \]
\[ \text{0404e9cd 9e3eb66 2395b442 9e648139 053fb521} \]
\[ \text{f828af60 6b4d3dba a14b5e77 efe75928 fe1dc127} \]
\[ \text{a2ffa8de 3348b3c1 856a429b f97e7e31 c2e5bd66} \]

\[ G_y = 118 \text{ 39296a78} \]
\[ 9a3bc004 5c8a5fb4 2c7d1bd9 98f54449 579b4468 \]
\[ 17afbd17 273e662c 97ee7299 5ef42640 c550b901 \]
3. Curves over Binary Fields

For each field degree $m$, a pseudo-random curve is given, along with a Koblitz curve. The pseudo-random curve has the form

$$E: y^2 + x y = x^3 + x^2 + b,$$

and the Koblitz curve has the form

$$E_a: y^2 + x y = x^3 + ax^2 + 1$$

where $a = 0$ or 1.

For each pseudorandom curve, the cofactor is $c = 2$. The cofactor of each Koblitz curve is $c = 2$ if $a = 1$ and $c = 4$ if $a = 0$.

The coefficients of the pseudo-random curves, and the coordinates of the base points of both kinds of curves, are given in terms of both the polynomial and normal basis representations discussed in 1.3.

For each $m$, the following parameters are given:

**Field Representation:**
- The normal basis type $T$
- The field polynomial (a trinomial or pentanomial)

**Koblitz Curve:**
- The coefficient $a$
- The base point order $r$
- The base point $x$ coordinate $G_x$
- The base point $y$ coordinate $G_y$

**Pseudo-random curve:**
- The base point order $r$
Pseudo-random curve (Polynomial Basis representation):

- The coefficient $b$
- The base point $x$ coordinate $G_x$
- The base point $y$ coordinate $G_y$

Pseudo-random curve (Normal Basis representation):

- The 160-bit input seed $s$ to the SHA-1 based algorithm
- The coefficient $b$ (i.e., the output of the SHA-1 based algorithm)
- The base point $x$ coordinate $G_x$
- The base point $y$ coordinate $G_y$

Integers (such as $T$, $m$, and $r$) are given in decimal form; bit strings and field elements are given in hex.
Degree 163 Binary Field

\[ T = 4 \]
\[ p(t) = t^{163} + t^7 + t^6 + t^3 + 1 \]

Curve K-163

\[ a = 1 \]
\[ r = 5846006549323611672814741753598448348329118574063 \]

Polynomial Basis:
\[ G_x = 2 \text{ fe}13\text{e}053 \text{ 7}7\text{bc}11\text{ac} \text{ aa}07\text{d}793 \text{ de}4e6d5e \text{ 5c}94\text{ee}8 \]
\[ G_y = 2 \text{ 89}070\text{fb}0 \text{ 5d}38\text{ff}58 \text{ 32}1\text{f}2\text{e}80 \text{ 05}36\text{d}538 \text{ cc}da3\text{d}9 \]

Normal Basis:
\[ G_x = 0 \text{ 56}79\text{b}353 \text{ caa}46825 \text{ fea}2d371 \text{ 3ba}450\text{da} \text{ 0c}2a4541 \]
\[ G_y = 2 \text{ 35}7e671 \text{ 00}506899 \text{ 06bac}3\text{d}9 \text{ dec}76\text{a}83 \text{ 55}91\text{ed}b2 \]

Curve B-163

\[ r = 5846006549323611672814742442876390689256843201587 \]

Polynomial Basis:
\[ b = 2 \text{ 0a}601907 \text{ b8e}953ca \text{ 14}81\text{eb}10 \text{ 51}2f7874 \text{ 4a}3205\text{fd} \]
\[ G_x = 3 \text{ f0e}ba162 \text{ 86a}2d57e \text{ a0}991168 \text{ d4}994637 \text{ e8}343e36 \]
\[ G_y = 0 \text{ d5}1\text{fbc}6c \text{ 71a}0094f \text{ a2cdd}545 \text{ b11c}5c0c \text{ 79}7324f1 \]

Normal Basis:
\[ s = 85e25\text{b}fe \text{ 5c}86226c \text{ db}12016f \text{ 75}53f\text{9d}0 \text{ e6}93a268 \]
\[ b = 6 \text{ 64}5f3\text{cac} \text{ f1}638e13 \text{ 9c}6cd13e \text{ f6}1734fb \text{ e9}3d9\text{fb} \]
\[ G_x = 0 \text{ 31}1103c1 \text{ 71}67564a \text{ ce}77\text{ec}b0 \text{ 9c}681f88 \text{ 6ba}54\text{ee}8 \]
\[ G_y = \text{33ac13c6 447f2e67 613bf700 9daf98c8 7bb50c7f} \]
Degree 233 Binary Field

\[ T = 2 \]
\[ p(t) = t^{233} + t^{74} + 1 \]

**Curve K-233**

\[ a = 0 \]
\[ r = \text{34508731733952818937173779311385127605709409888622521} \backslash \text{26328087024741343} \]

**Polynomial Basis:**

\[ G_x = \]
\[ \text{172 32ba853a 7e731af1} \]
\[ \text{29f22ff4 149563a4 19c26bf5 0a4c9d6e efad6126} \]

\[ G_y = \]
\[ \text{1db 537dece8 19b7f70f} \]
\[ \text{555a67c4 27a8cd9b f18aeb9b 56e0c110 56fae6a3} \]

**Normal Basis:**

\[ G_x = \]
\[ \text{0fd e76d9dcd 26e643ac} \]
\[ \text{26f1aa90 1aa12978 4b71fc07 22b2d056 14d650b3} \]

\[ G_y = \]
\[ \text{064 3e317633 155c9e04} \]
\[ \text{47ba8020 a3c43177 450ee036 d6335014 34cac978} \]
Curve B-233

\[ r = 690174634679056378743475586227702555558398127373450135855379383634485463 \]

**Polynomial Basis:**

\[ b = \begin{array}{c}
066 647ede6c 332c7f8c \\
0923bb58 213b333b 20e9ce42 81fe115f 7d8f90ad
\end{array} \]

\[ G_x = \begin{array}{c}
0fa e9dfcbbc 8313bb21 \\
39f1bb75 5fe65bc391f8b36 f8f8eb73 71fd558b
\end{array} \]

\[ G_y = \begin{array}{c}
106a08a419 03350678 \\
e8528be bf8a0bef f867a7ca 36716f7e 01f81052
\end{array} \]

**Normal Basis:**

\[ s = \begin{array}{c}
74d59ff0 7f6b413d 0ea14b34 4b20a2db 049b50c3 \\
1a0 03e0962d 4f9a8e40
\end{array} \]

\[ G_x = \begin{array}{c}
7c904a95 38163adb 82521260 0c7752ad 52233279 \\
18b 863524b3 cdefeb94 \\
f2784e0b 116faac5 4404bc91 62a363ba b84a14c5
\end{array} \]

\[ G_y = \begin{array}{c}
049 25df77bd 8b8ff1a5 \\
ff519417 822bfedf 2bbd7526 44292c98 c7af6e02
\end{array} \]
Degree 283 Binary Field

\[ T = 6 \]
\[ p(t) = t^{283} + t^{12} + t^7 + t^5 + 1 \]

Curve K-283

\[ a = 0 \]
\[ r = 38853377844514581418389238136470378132848117337930613 \]
\[ 24295874997529815829704422603873 \]

Polynomial Basis:
\[ G_x = \]
\[ 503213f78ca44883f1a3b8162f188e5 \]
\[ 53cd265f23c1567a16876913b0c2ac2458492836 \]
\[ G_y = \]
\[ 1ccda380f1e9e318d90f95d07e5426fe87e45c0e8184698e45962364e34116177dd2259 \]

Normal Basis:
\[ G_x = \]
\[ 3ab9593f8db09fc188f1d7c4ac9fcc3e57fcd3bdb15024b212c70229de5fcd92eb0ea60 \]
\[ G_y = \]
\[ 2118c4755e7345c98f603ef93b98b106fe8854ffeb9a3bb04634cc83a0e759f0c2686b1 \]
Curve B-283

\[ r = 77706755689029162836778476272940756265696259243769048\]
\[ 89109196526770044277787378692871 \]

Polynomial Basis:

\[ b = \quad 27b680a \, c8b8596d \, a5a4af8a \, 19a0303f \]
\[ \quad \text{ca97fd76} \, 45309fa2 \, a581485a \, f6263e31 \, 3b79a2f5 \]
\[ G_x = \quad 5f93925 \, 8db7dd90 \, e1934f8c \, 70b0dfec \]
\[ \quad \text{2eed25b8} \, 557eac9c \, 80e2e198 \, f8cde4cd \, 86b12053 \]
\[ G_y = \quad 3676854 \, fe24141c \, b98fe6d4 \, b20d02b4 \]
\[ \quad 516ff702 \, 350eddb0 \, 826779c8 \, 13f0df45 \, \text{be8112f4} \]

Normal Basis:

\[ s = \quad 77e2b073 \, 70eb0f83 \, 2a6dd5b6 \, 2dfc88cd \, 06bb84be \]
\[ b = \quad 157261b \, 894739fb \, 5a13503f \, 55f0b3f1 \]
\[ \quad 0c560116 \, 66331022 \, 01138cc1 \, 80c0206b \, \text{dafbc951} \]
\[ G_x = \quad 749468e \, 464ee468 \, 634b21f7 \, f61cb700 \]
\[ \quad 701817e6 \, \text{bc36a236} \, 4cb8906e \, \text{940948ea} \, \text{a463c35d} \]
\[ G_y = \quad 62968bd \, 3b489ac5 \, e9b859da \, 68475c31 \, 5bafecdc4 \]
\[ \quad \text{ccd0dc90} \, 5b70f624 \, 46f49c05 \, 2f49c08c \]
Degree 409 Binary Field

\[ T = 4 \]
\[ p(t) = t^{409} + t^{87} + 1 \]

Curve K-409

\( a = 0 \)
\[ r = 33052798439512429947595765401638551991420234148214060\]
\[ 96423243950228807112892491910506732584577774580140963\]
\[ 66590617731358671 \]

Polynomial Basis:
\[ G_x = 060f05f 658f49c1 \text{ ad3ab189} \]
\[ 0f718421 0efd0987 \text{ e307c84c 27acefb8 f9f67cc2} \]
\[ c460189e \text{ b5aaaa62 ee222eb1 b35540cf e9023746} \]
\[ G_y = 1e36905 0b7c4e42 \text{ acba1dac} \]
\[ bf04299c 3460782f 918ea427 e6325165 e9ea10e3 \]
\[ da5f6c42 e9c55215 aa9ca27a 5863ec48 d8e0286b \]

Normal Basis:
\[ G_x = 1b559c7 \text{ cba2422e 3affe133} \]
\[ 43e808b5 5e012d72 6ca0b7e6 a63aeafb c1e3a98e \]
\[ 10ca0fcf 98350c3b 7f89a975 4a8e1dc0 713ce4c4a \]
\[ G_y = 16d8c42 052f07e7 713e7490 \]
\[ eff318ba 1abd6fefe 8a5433c8 94b24f5c 817aeb79 \]
\[ 852496fb ee803a47 bc8a2038 78ebf1c4 99afd7d6 \]
 Curve B-409

\[ r = 66105596879024859895191530803277103982840468296428121\]  
\[ 92846487983041577748273748052081437237621791109659798\]  
\[ 67288366567526771\]

Polynomial Basis:

\[ b = 021a5c2 c8ee9feb 5c4b9a75\]  
\[ 3b7b476b 7fd6422e f1f3dd67 4761fa99 d6ac27c8\]  
\[ a9a197b2 72822f6c d57a55aa 4f50ae31 7b13545f\]  
\[ G_x = 15d4860 d088dbd3 496b0c60\]  
\[ 64756260 441cde4a f1771d4d b01ffe5b 34e59703\]  
\[ dc255a86 8a118051 5603aeab 60794e54 bb7996a7\]  
\[ G_y = 061b1cf ab6be5f3 2bbfa783\]  
\[ 24ed106a 7636b9c5 a7bd198d 0158aa4f 5488d08f\]  
\[ 38514f1f df4b4f40 d2181b36 81c36ba 0273c706\]

Normal Basis:

\[ s = 4099b5a4 57f9d69f 79213d09 4e4bcd4d 4262210b\]  
\[ b = 124d065 1c3d3772 f7f5a1fe\]  
\[ 6e715559 e2129bdf a04d52f7 b6ac7c53 2cf0ed06\]  
\[ f610072d 88ad2f6c c50c6fde 72843670 f8b3742a\]  
\[ G_x = 0eceabc 9f475767 d8e69f3b\]  
\[ 5dfab398 13685262 bcacf22b 84c7b6dd 981899e7\]  
\[ 318c96f0 761f77c6 02e016ce d7c548de 830d708f\]  
\[ G_y = 199d64b a8f089c6 db0e0b61\]  
\[ e80bb959 34afd0ca f2e8be76 d1e5e9af fc7476df\]  

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Degree 571 Binary Field

\[ T = 10 \]
\[ p(t) = t^{571} + t^{10} + t^5 + t^2 + 1 \]

Curve K-571

\[ a = 0 \]
\[ r = 19322687615086291723476759454659936721494636648532174/\]
\[ 9932861762572579571144780212268133978522706711834706/\]
\[ 71280082535146127367497406661731192968242161709250355/\]
\[ 5733685276673 \]

Polynomial Basis:
\[ G_x = 26eb7a8 59923fbc 82189631 \]
\[ f8103fe4 ac9ca297 0012d5d4 60248048 01841ca4 \]
\[ 43709584 93b205e6 47da304d b4ceb08c bbd1ba39 \]
\[ 494776fb 988b4717 4dca88c7 e2945283 a01c8972 \]

\[ G_y = 349dc80 7f4f4aeade \]
\[ 3bca9531 4dd58cec 9f307a54 ffc61efc 006d8a2c \]
\[ 9d4979c0 ac44aea7 4fbeb3bb9 f772aedc b620b01a \]
\[ 7ba7af1b 320430c8 591984f6 01cd4c14 3ef1c7a3 \]

Normal Basis:
\[ G_x = 04bb2db a418d0db 107adaae0 \]
\[ 03427e5d 7cc139ac b465e593 4f0bea2a b2f3622b \]
\[ c29b3d5b 9aa7a1fd fd5d8be6 6057e100 8e71e484 \]
\[ bcd98f22 bf847642 37673674 29ef2ec5 bc3ebcf7 \]

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\[ G_y = 44cbb57 \text{ de20788d 2e952d7b} \\
56cf39bd 3e89b189 84bd124e 751ceff4 369dd8da \\
c6a59e6e 745df44d 8220ce22 aa2c852c fcbbef49 \\
ebaa98bd 2483e331 80e04286 feaa2530 50caff60 \]
Curve B-571

\[ r = 38645375230172583446953518909319873442989273297064349\]
98657235251451519142289560424536143999389415773083133\]
881121926944486246872462816813070234528288303332441139\]
3191105285703

Polynomial Basis:

\[ b = 2f40e7e2221f295 \text{ de297117} \]
\[ \text{b7f3d62f} \text{ 5c6a97ff} \text{ cb8ceff1} \text{ cd6ba8ce} \text{ 4a9a18ad} \]
\[ \text{84ffabbd} \text{ 8efa5933} \text{ 2be7ad67} \text{ 56a66e29} \text{ 4afdf185a} \]
\[ \text{78ff12aa} \text{ 520e4de7} \text{ 39baca0c} \text{ 7ffeeff7f} \text{ 2955727a} \]

\[ G_x = 303001d34b856296c16c0d4 \]
\[ \text{0d3cd775} \text{ 0a93d1d2} \text{ 955fa80a} \text{ a5f40fc8} \text{ db7b2abd} \]
\[ \text{bde53950} \text{ f4c0d293} \text{ cdd711a3} \text{ 5b67fb14} \text{ 99ae6003} \]
\[ \text{8614f139} \text{ 4abfa3b4} \text{ c850d927} \text{ e1e7769c} \text{ 8ec2d19} \]

\[ G_y = 37bf27342da639b6dceffe \]
\[ \text{b73d69d7} \text{ 8c6c27a6} \text{ 009cbba} \text{ 1980f853} \text{ 3921e8a6} \]
\[ \text{84423e43} \text{ bab08a57} \text{ 6291af8f} \text{ 461bb2a8} \text{ b3531d2f} \]
\[ \text{0485c19b} \text{ 16e2f151} \text{ 6e23dd3c} \text{ 1a4827af} \text{ 1b8ac15b} \]

Normal Basis:

\[ s = 2aa058f73a0e33ab486b0f610410c53a7f132310 \]
\[ b = 3762d0d47116006179da356 \]
\[ \text{88eeaccf} \text{ 591a5cde} \text{ a7500011} \text{ 8d9608c5} \text{ 9132d434} \]
\[ \text{26101a1d} \text{ fb377411} \text{ 5f586623} \text{ f75f0000} \text{ 1ce61198} \]
\[ \text{3c1275fa} \text{ 31f5bc9f} \text{ 4be1a0f4} \text{ 67f01ca8} \text{ 85c74777} \]

\[ G_x = 0735e035def5925cc33173e \]

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\[ G_y = \begin{array}{c}
\text{b2a8ce77 67522b46 6d278b65 0a291612 7dfea9d2 d361089f 0a7a0247 a184e1c7 0d417866 e0fe0feb 0ff8f2f3 f9176418 f97d117e 624e2015 df1662a8}
\text{04a3642 0572616c df7e606f ccadaecf c3b76dab 0eb1248d d03fdfe 9cd3242c}
\text{4726be57 9855e812 de7ec5c5 00b4576a 24628048 b6a72d88 0062eed0 dd34b109 6d3acbb6 b01a4a97}
\end{array} \]
APPENDIX 6.1: IMPLEMENTATION OF MODULAR ARITHMETIC

The prime moduli in the above examples are of a special type (called generalized Mersenne numbers) for which modular multiplication can be carried out more efficiently than in general. This appendix provides the rules for implementing this faster arithmetic, for each of the prime moduli appearing in the examples.

The usual way to multiply two integers (mod m) is to take the integer product and reduce it (mod m). One therefore has the following problem: given an integer A less than $m^2$, compute

$$B := A \mod m.$$  

In general, one must obtain B as the remainder of an integer division. If m is a generalized Mersenne number, however, then B can be expressed as a sum or difference (mod m) of a small number of terms. To compute this expression, one can evaluate the integer sum or difference and reduce the result modulo m. The latter reduction can be accomplished by adding or subtracting a few copies of m.

The prime moduli $p$ for each of the five example curves is a generalized Mersenne number.
Curve P-192:

The modulus for this curve is \( p = 2^{192} - 2^{64} - 1 \). Every integer \( A \) less than \( p^2 \) can be written

\[
A = A_5 \cdot 2^{320} + A_4 \cdot 2^{256} + A_3 \cdot 2^{192} + A_2 \cdot 2^{128} + A_1 \cdot 2^{64} + A_0,
\]

where each \( A_i \) is a 64-bit integer. The expression for \( B \) is

\[
B := T + S_1 + S_2 + S_3 \mod p;
\]

where the 192-bit terms are given by

\[
T = A_2 \cdot 2^{128} + A_1 \cdot 2^{64} + A_0
\]

\[
S_1 = A_3 \cdot 2^{64} + A_3
\]

\[
S_2 = A_4 \cdot 2^{128} + A_4 \cdot 2^{64}
\]

\[
S_3 = A_5 \cdot 2^{128} + A_5 \cdot 2^{64} + A_5.
\]
Curve P-224:

The modulus for this curve is \( p = 2^{224} - 2^96 + 1 \). Every integer \( A \) less than \( p^2 \) can be written

\[
A = A_{13} \cdot 2^{416} + A_{12} \cdot 2^{384} + A_{11} \cdot 2^{352} + A_{10} \cdot 2^{320} + A_9 \cdot 2^{288} + A_8 \cdot 2^{256} + A_7 \cdot 2^{224} +
A_6 \cdot 2^{192} + A_5 \cdot 2^{160} + A_4 \cdot 2^{128} + A_3 \cdot 2^{96} + A_2 \cdot 2^{64} + A_1 \cdot 2^{32} + A_0,
\]

where each \( A_i \) is a 32-bit integer. As a concatenation of 32-bit words, this can be denoted by

\[
A = (A_{13} \| A_{12} \| \cdots \| A_0).
\]

The expression for \( B \) is

\[
B := T + S_1 + S_2 - D_1 - D_2 \mod p,
\]

where the 224-bit terms are given by

\[
T = (A_6 \| A_5 \| A_4 \| A_3 \| A_2 \| A_1 \| A_0),
\]

\[
S_1 = (A_{10} \| A_9 \| A_8 \| A_7 \| 0 \| 0 \| 0 \| 0),
\]

\[
S_2 = (0 \| A_{13} \| A_{12} \| A_{11} \| 0 \| 0 \| 0 \| 0),
\]

\[
D_1 = (A_{13} \| A_{12} \| A_{11} \| A_{10} \| A_9 \| A_8 \| A_7),
\]

\[
D_2 = (0 \| 0 \| 0 \| 0 \| A_{13} \| A_{12} \| A_{11}).
\]
**Curve P-256:**

The modulus for this curve is \( p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1 \). Every integer \( A \) less than \( p^2 \) can be written

\[
A = A_{15} \cdot 2^{480} + A_{14} \cdot 2^{448} + A_{13} \cdot 2^{416} + A_{12} \cdot 2^{384} + A_{11} \cdot 2^{352} + \\
A_{10} \cdot 2^{320} + A_9 \cdot 2^{288} + A_8 \cdot 2^{256} + A_7 \cdot 2^{224} + A_6 \cdot 2^{192} + A_5 \cdot 2^{160} + \\
A_4 \cdot 2^{128} + A_3 \cdot 2^{96} + A_2 \cdot 2^{64} + A_1 \cdot 2^{32} + A_0,
\]

where each \( A_i \) is a 32-bit integer. As a concatenation of 32-bit words, this can be denoted by

\[
A = (A_{15} \parallel A_{14} \parallel \cdots \parallel A_0).
\]

The expression for \( B \) is

\[
B := T + 2S_1 + 2S_2 + S_3 + S_4 - D_1 - D_2 - D_3 - D_4 \mod p,
\]

where the 256-bit terms are given by

\[
T = (A_7 \parallel A_6 \parallel A_5 \parallel A_4 \parallel A_3 \parallel A_2 \parallel A_1 \parallel A_0),
\]

\[
S_1 = (A_{15} \parallel A_{14} \parallel A_{13} \parallel A_{12} \parallel A_{11} \parallel 0 \parallel 0 \parallel 0),
\]

\[
S_2 = (0 \parallel A_{15} \parallel A_{14} \parallel A_{13} \parallel A_{12} \parallel 0 \parallel 0 \parallel 0),
\]

\[
S_3 = (A_{15} \parallel A_{14} \parallel 0 \parallel 0 \parallel 0 \parallel A_{10} \parallel A_9 \parallel A_8),
\]

\[
S_4 = (A_8 \parallel A_{13} \parallel A_{15} \parallel A_{14} \parallel A_{13} \parallel A_{11} \parallel A_{10} \parallel A_9),
\]

\[
D_1 = (A_{10} \parallel A_8 \parallel 0 \parallel 0 \parallel 0 \parallel A_{13} \parallel A_{12} \parallel A_{11}),
\]

\[
D_2 = (A_{11} \parallel A_9 \parallel 0 \parallel 0 \parallel A_{15} \parallel A_{14} \parallel A_{13} \parallel A_{12}),
\]

\[
D_3 = (A_{12} \parallel 0 \parallel A_{10} \parallel A_9 \parallel A_8 \parallel A_{15} \parallel A_{14} \parallel A_{13}),
\]

\[
D_4 = (A_{13} \parallel 0 \parallel A_{11} \parallel A_{10} \parallel A_9 \parallel 0 \parallel A_{15} \parallel A_{14}).
\]
Curve P-384:

The modulus for this curve is $p = 2^{384} - 2^{128} - 2^{96} + 2^{32} - 1$. Every integer $A$ less than $p^2$ can be written

$$A = A_{23} \cdot 2^{736} + A_{22} \cdot 2^{704} + A_{21} \cdot 2^{672} + A_{20} \cdot 2^{640} + A_{19} \cdot 2^{608} +$$

$$A_{18} \cdot 2^{576} + A_{17} \cdot 2^{544} + A_{16} \cdot 2^{512} + A_{15} \cdot 2^{480} + A_{14} \cdot 2^{448} + A_{13} \cdot 2^{416} + A_{12}$$

$$\cdot 2^{384} + A_{11} \cdot 2^{352} + A_{10} \cdot 2^{320} + A_9 \cdot 2^{288} + A_8 \cdot 2^{256} + A_7 \cdot 2^{224} +$$

$$A_6 \cdot 2^{192} + A_5 \cdot 2^{160} + A_4 \cdot 2^{128} + A_3 \cdot 2^{96} + A_2 \cdot 2^{64} + A_1 \cdot 2^{32} + A_0,$$

where each $A_i$ is a 32-bit integer. As a concatenation of 32-bit words, this can be denoted by

$$A = (A_{23} \ || A_{22} \ || \cdots \ || A_0).$$

The expression for $B$ is

$$B := T + 2S_1 + S_2 + S_3 + S_4 + S_5 + S_6 - D_1 - D_2 - D_3 \mod p,$$

where the 384-bit terms are given by

$$T = (A_{11} \ || A_{10} \ || A_9 \ || A_8 \ || A_7 \ || A_6 \ || A_5 \ || A_4 \ || A_3 \ || A_2 \ || A_1 \ || A_0),$$

$$S_1 = ( 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || A_{23} \ || A_{22} \ || A_{21} \ || 0 \ || 0 \ || 0 \ || 0 ),$$

$$S_2 = (A_{23} \ || A_{22} \ || A_{21} \ || A_{20} \ || A_{19} \ || A_{18} \ || A_{17} \ || A_{16} \ || A_{15} \ || A_{14} \ || A_{13} \ || A_{12}),$$

$$S_3 = (A_{20} \ || A_{19} \ || A_{18} \ || A_{17} \ || A_{16} \ || A_{15} \ || A_{14} \ || A_{13} \ || A_{12} \ || A_{23} \ || A_{22} \ || A_{21}),$$

$$S_4 = (A_{19} \ || A_{18} \ || A_{17} \ || A_{16} \ || A_{15} \ || A_{14} \ || A_{13} \ || A_{12} \ || A_{20} \ || 0 \ || A_{23} \ || 0 ),$$

$$S_5 = ( 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || A_{23} \ || A_{22} \ || A_{21} \ || A_{20} \ || 0 \ || 0 \ || 0 \ || 0 \ || 0 ),$$

$$S_6 = ( 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || A_{23} \ || A_{22} \ || A_{21} \ || 0 \ || 0 \ || 0 \ || A_{20}),$$

$$D_1 = (A_{22} \ || A_{21} \ || A_{20} \ || A_{19} \ || A_{18} \ || A_{17} \ || A_{16} \ || A_{15} \ || A_{14} \ || A_{13} \ || A_{12} \ || A_{23}),$$

$$D_2 = ( 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || A_{23} \ || A_{22} \ || A_{21} \ || A_{20} \ || 0 ),$$

$$D_3 = ( 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || 0 \ || A_{23} \ || A_{23} \ || 0 \ || 0 \ || 0 ).$$
Curve P-521:

The modulus for this curve is \( p = 2^{521} - 1 \). Every integer \( A \) less than \( p^2 \) can be written

\[
A = A_1 \cdot 2^{521} + A_0,
\]

The expression for \( B \) is

\[
B := A_0 + A_1 \mod p
\]
APPENDIX 6.2: NORMAL BASES

The elements of $GF(2^m)$ are expressed in terms of the type $T$ normal basis $2^B$ for $GF(2^m)$, for some $T$. Each element has a unique representation as a bit string

$$(a_0 \ a_1 \ldots \ a_{m-1})$$

The arithmetic operations are performed as follows.

**Addition**: addition of two elements is implemented by bitwise addition modulo 2. Thus, for example,

$$(1100111) + (1010010) = (0110101).$$

**Squaring**: if

$$\alpha = (a_0 \ a_1 \ldots \ a_{m-1})$$

then

$$\alpha^2 = (a_{m-1} \ a_0 \ a_1 \ldots \ a_{m-2})$$

**Multiplication**: to perform multiplication, one first constructs a function $F(u, v)$ on inputs

$$u = (u_0 \ u_1 \ldots \ u_{m-1})$$

and

$$v = (v_0 \ v_1 \ldots \ v_{m-1})$$

as follows.

1. Set $p \leftarrow Tm + 1$
2. Let $u$ be an integer having order $T$ modulo $p$

---

2 It is assumed in this section that $m$ is odd and $T$ is even, since this is the only case considered in this standard.
3. Compute the sequence $F(1); F(2),\ldots,F(p-1)$ as follows:

3.1 Set $w \leftarrow 1$

3.2 For $j$ from 0 to $T-1$ do
   
   Set $n \leftarrow w$
   
   For $i$ from 0 to $m-1$ do
   
   Set $F(n) \leftarrow i$
   
   Set $n \leftarrow 2n \mod p$
   
   Set $w \leftarrow uw \mod p$

4. Output the formula

$$F(u, v) := \sum_{k=1}^{p-2} u_{F(k+1)} v_{F(0-k)}.$$ 

This computation need only be performed once per basis.

Given the function $F$ for $B$, one computes the product

$$(c_0 \ c_1 \ldots \ c_{m-1}) = (a_0 \ a_1 \ldots \ a_{m-1}) \times (b_0 \ b_1 \ldots \ b_{m-1})$$

as follows.

1. Set $(u_0 \ u_1 \ldots \ u_{m-1}) \leftarrow (a_0 \ a_1 \ldots \ a_{m-1})$

2. Set $(v_0 \ v_1 \ldots \ v_{m-1}) \leftarrow (b_0 \ b_1 \ldots \ b_{m-1})$

3. For $k$ from 0 to $m-1$ do
   
   3.1 Compute
   
   $$c_k := F(u, v)$$
   
   3.2 Set $u \leftarrow \text{LeftShift}(u)$ and $v \leftarrow \text{LeftShift}(v)$, where $\text{LeftShift}$
   
   denotes the circular left shift operation.

4. Output $c := (c_0 \ c_1 \ldots \ c_{m-1})$
EXAMPLE. For the type 4 normal basis for $GF(2^7)$, one has $p = 29$ and $u = 12$ or 17. Thus the values of $F$ are given by

\[
\begin{align*}
F(1) &= 0 & F(8) &= 3 & F(15) &= 6 & F(22) &= 5 \\
F(2) &= 1 & F(9) &= 3 & F(16) &= 4 & F(23) &= 6 \\
F(3) &= 5 & F(10) &= 2 & F(17) &= 0 & F(24) &= 1 \\
F(4) &= 2 & F(11) &= 4 & F(18) &= 4 & F(25) &= 2 \\
F(5) &= 1 & F(12) &= 0 & F(19) &= 2 & F(26) &= 5 \\
F(6) &= 6 & F(13) &= 4 & F(20) &= 3 & F(27) &= 1 \\
F(7) &= 5 & F(14) &= 6 & F(21) &= 3 & F(28) &= 0
\end{align*}
\]

Therefore

\[
F(u; v) = u_0 v_1 + u_1 (v_0 + v_2 + v_5 + v_6) + u_2 (v_1 + v_3 + v_4 + v_5) + u_3 (v_2 + v_5) + u_4 (v_2 + v_6) + u_5 (v_1 + v_2 + v_3 + v_6) + u_6 (v_1 + v_4 + v_5 + v_6).
\]

Thus, if

\[
a = (1 0 1 0 1 1 1) \text{ and } b = (1 1 0 0 0 0 1),
\]

then

\[
c_0 = F((1 0 1 0 1 1 1), (1 1 0 0 0 0 1)) = 1,
\]

\[
c_1 = F((0 1 0 1 1 1 1), (1 0 0 0 0 1 1)) = 0,
\]

\::

\[
c_6 = F((1 1 0 1 0 1 1); (1 1 1 0 0 0 0)) = 1,
\]

so that $c = ab = (1 0 1 1 0 0 1)$. 

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APPENDIX 6.3: SCALAR MULTIPLICATION ON KOBLITZ CURVES

This appendix describes a particularly efficient method of computing the scalar multiple \(nP\) on the Koblitz curve \(E_a\) over \(GF(2^m)\).

The operation \(\tau\) is defined by

\[
\tau(x, y) = (x^2, y^2)
\]

When the normal basis representation is used, then the operation \(\tau\) is implemented by performing right circular shifts on the bit strings representing \(x\) and \(y\).

Given \(m\) and \(a\), define the following parameters:

- \(C\) is some integer greater than 5.
- \(\mu := (-1)^{1-a}\)
- For \(i = 0\) and \(i = 1\), define the sequence \(s_i(m)\) by

\[
s_i(0) = 0, \quad s_i(1) = 1 - i,
\]

\[
s_i(m) = \mu \cdot s_i(m - 1) - 2 s_i(m - 2) + (-1)^i
\]

- Define the sequence \(V(m)\)

\[
V(0) = 2, \quad V(1) = \mu
\]

\[
V(m) = \mu \cdot v(m - 1) - 2V(m - 2).
\]
For the example curves, the quantities $s_i(m)$ and $V(m)$ are as follows.

**Curve K-163:**

$s_0(163) = 2579386439110731650419537$
$s_1(163) = -7553600644476226375461594$
$V(163) = -4845466632539410776804317$

**Curve K-233:**

$s_0(233) = -27859711741434429761757834964435883$
$s_1(233) = -44192136247082304936052160908934886$
$V(233) = -137381546011108235394987299651366779$

**Curve K-283:**

$s_0(283) = -665981532109049041108795536001591469280025$
$s_1(283) = 1155860054909136775192281072591609913945968$
$V(283) = 7777244870872830999287791970962823977569917$

**Curve K-409:**

$s_0(409) = -1830751045600238213781031719875646137859054248755686$
$9338419259$
$s_1(409) = -8893048526138304097196653241844212679626566100996606$
$444816790$
$V(409) = 1045728873731562592744768538704832073763879695768757$
$5791173829$

**Curve K-571:**
\[ s_0(571) = -373731944687646369242938589247611556714729396459613 \]
\[ 1024123406420235241916729983261305 \]
\[ s_1(571) = -3191857706446416099583814595948959674131968912148564 \]
\[ 65861056511758982848515832612248752 \]
\[ V(571) = -148380926981691413899619140297051490364542574180493 \]
\[ 936232912339534208516828973111459843 \]

The following algorithm computes the scalar multiple \( nP \) on the Koblitz curve \( E_a \) over \( GF(2^m) \). The average number of elliptic additions and subtractions is at most \( 1 + (m/3) \), and is at most \( m/3 \) with probability at least \( 1 - 2^{5\cdot C} \).

For \( i = 0 \) to 1 do
\[
\begin{align*}
n' &\left\lceil n / 2^{a\cdot C + (m-9)/2} \right\rceil \\
g' &\left\lceil s_i(m) \cdot n' \right\rceil \\
h' &\left\lfloor g' / 2^m \right\rfloor \\
j' &V(m) \cdot h' \\
l' &\text{Round}\left((g' + j') / 2^{(m+5)/2}\right) \\
\lambda_i &l' / 2^C \\
f_i &\text{Round}(\lambda_i) \\
\eta_i &\lambda_i - f_i \\
h_i &0 \\
\eta &2 \eta_0 + \mu \eta_1 \end{align*}
\]
If \( \eta \geq 1 \)
\[
\text{then}
\]
if $\eta_0 - 3 \mu \eta_1 < -1$
  
  then set $h_1 \leftarrow \mu$

else set $h_0 \leftarrow 1$

else

  if $\eta_0 + 4 \mu \eta_1 \geq 2$

  then set $h_1 \leftarrow \mu$

If $\eta < -1$

then

  if $\eta_0 - 3 \mu \eta_1 \geq 1$

  then set $h_1 \leftarrow -\mu$

else set $h_0 \leftarrow -1$

else

  if $\eta_0 + 4 \mu \eta_1 < -2$

  then set $h_1 \leftarrow -\mu$

$q_0 \leftarrow f_0 + h_0$

$q_1 \leftarrow f_1 + h_1$

$r_0 \leftarrow n - (s_0 + \mu s_1) q_0 - 2 s_1 q_1$

$r_1 \leftarrow s_1 q_0 - s_0 q_1$

Set $Q \leftarrow O$

$P_0 \leftarrow P$

While $r_0 \neq 0$ or $r_1 \neq 0$

  If $r_0$ odd then

  set $u \leftarrow 2 - (r_0 - 2 r_1 \text{ mod } 4)$

  set $r_0 \leftarrow r_0 - u$
if $u = 1$ then set $Q \leftarrow Q + P_0$
if $u = -1$ then set $Q \leftarrow Q - P_0$

Set $P_0 \leftarrow \tau P_0$

Set $(r_0, r_1) \leftarrow (r_1 + \mu r_0 / 2, -r_0 / 2)$

Endwhile

Output $Q$
APPENDIX 6.4: GENERATION OF PSEUDO-RANDOM CURVES (PRIME CASE)

Let $l$ be the bit length of $p$, and define

$$v = \left\lfloor \frac{(l - 1)}{160} \right\rfloor$$

$$w = l - 160v - 1$$

1. Choose an arbitrary 160-bit string $s$.
2. Compute $h := \text{SHA-1}(s)$.
3. Let $h_0$ be the bit string obtained by taking the $w$ rightmost bits of $h$.
4. Let $z$ be the integer whose binary expansion is given by the 160-bit string $s$.
5. For $i$ from 1 to $v$ do:
   5.1 Define the 160-bit string $s_i$ to be binary expansion of the integer $(z + i) \mod (2^{160})$.
   5.2 Compute $h_i := \text{SHA-1}(s_i)$.
6. Let $h$ be the bit string obtained by the concatenation of $h_0, h_1, \ldots, h_v$ as follows:

$$h = h_0 \parallel h_1 \parallel \ldots \parallel h_v$$

7. Let $c$ be the integer whose binary expansion is given by the bit string $h$.
8. If $c = 0$ or $4c + 27 \equiv 0 \pmod{p}$, then go to Step 1.
9. Choose integers $a, b \in GF(p)$ such that

$$c \cdot b^2 \equiv a^3 \pmod{p}.$$  

(The simplest choice is $a = c$ and $b = c$. However, one may want to choose differently for performance reasons.)
10. Check that the elliptic curve $E$ over $GF(p)$ given by $y^2 = x^3 + ax + b$ has suitable order. If not, go to Step 1.
APPENDIX 6.5: VERIFICATION OF CURVE
PSEUDO-RANDOMNESS (PRIME CASE)

Given the 160-bit seed value \( s \), one can verify that the coefficient \( b \) was
obtained from \( s \) via the cryptographic hash function SHA-1 as follows.

Let \( l \) be the bit length of \( p \), and define

\[
\nu = \left\lfloor \frac{(l - 1)}{160} \right\rfloor
\]

\[
w = l - 160\nu - 1
\]

1. Compute \( h := \text{SHA-1}(s) \).
2. Let \( h_0 \) be the bit string obtained by taking the \( w \) rightmost bits of \( h \).
3. Let \( z \) be the integer whose binary expansion is given by the 160-bit string \( s \).
4. For \( i \) from 1 to \( \nu \) do
   4.1 Define the 160-bit string \( s_i \) to be binary expansion of the integer
      \((z + i) \mod (2^{160})\).
   4.2 Compute \( h_i := \text{SHA-1}(s_i) \).
5. Let \( h \) be the bit string obtained by the concatenation of \( h_0, h_1, \ldots, h_\nu \) as follows:
      \[ h = h_0 \parallel h_1 \parallel \ldots \parallel h_\nu. \]
6. Let \( c \) be the integer whose binary expansion is given by the bit string \( h \).
7. Verify that \( b^2 c \equiv -27 \pmod{p} \).
APPENDIX 6.6: GENERATION OF
PSEUDO-RANDOM CURVES (BINARY CASE)

Let:

\[ v = \lceil (m - 1) / B \rceil \]
\[ w = m - Bv \]

1. Choose an arbitrary 160-bit string \( s \).
2. Compute \( h := \text{SHA-1}(s) \)
3. Let \( h_0 \) be the bit string obtained by taking the \( w \) rightmost bits of \( h \).
4. Let \( z \) be the integer whose binary expansion is given by the 160-bit string \( s \).
5. For \( i \) from 1 to \( v \) do:
   5.1 Define the 160-bit string \( s_i \) to be binary expansion of the integer
       \((z + i) \mod (2^{160})\).
   5.2 Compute \( h_i := \text{SHA-1}(s_i) \).
6. Let \( h \) be the bit string obtained by the concatenation of \( h_0, h_1, \ldots, h_v \) as follows:
   \[ h = h_0 || h_1 || \ldots || h_v. \]
7. Let \( b \) be the element of \( GF(2^m) \) which binary expansion is given by the bit string \( h \).
8. Choose an element \( a \) of \( GF(2^m) \).
9. Check that the elliptic curve \( E \) over \( GF(2^m) \) given by \( y^2 + xy = x^3 + ax^2 + b \) has suitable order. If not, go to Step 1.
APPENDIX 6.7: VERIFICATION OF CURVE PSEUDO-RANDOMNESS (BINARY CASE)

Given the 160-bit seed value $s$, one can verify that the coefficient $b$ was obtained from $s$ via the cryptographic hash function SHA-1 as follows.

Define

$$v = \left\lfloor \frac{(m - 1)}{160} \right\rfloor$$

$$w = m - 160v$$

1. Compute $h := \text{SHA-1}(s)$
2. Let $h_0$ be the bit string obtained by taking the $w$ rightmost bits of $h$.
3. Let $z$ be the integer whose binary expansion is given by the 160-bit string $s$.
4. For $i$ from 1 to $v$ do

   4.1 Define the 160-bit string $s_i$ to be binary expansion of the integer $(z + i) \mod (2^{160})$

   4.2 Compute $h_i := \text{SHA-1}(s_i)$.

5. Let $h$ be the bit string obtained by the concatenation of $h_0$, $h_1$, ..., $h_v$ as follows:

$$h = h_0 \| h_1 \| \ldots \| h_v.$$  

6. Let $c$ be the element of $GF(2^m)$ which is represented by the bit string $h$.
7. Verify that $c = b$.  

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APPENDIX 6.8: POLYNOMIAL BASIS TO NORMAL BASIS CONVERSION

Suppose that $\alpha$ an element of the field $GF(2^m)$. Denote by $p$ the bit string representing $\alpha$ with respect to a given polynomial basis. It is desired to compute $n$, the bit string representing $\alpha$ with respect to a given normal basis. This is done via the matrix computation

$$p \Gamma = n$$

Where $\Gamma$ is an $m$-by-$m$ matrix with entries in $GF(2)$. The matrix $\Gamma$, which depends only on the bases, can be computed easily given its second-to-last row. The second-to-last row for each conversion is given in the table below.

**Degree 163:**

```
3 e173baf 3a86434d 883a2918 a489ddbd 69fe84e1
```

**Degree 233:**

```
0be 19b89595 28bbc490
038f4bc4 da8bdfe1 ca36bb05 853fd0ed 0ae200ce
```

**Degree 283:**

```
3347f17 521fdabc 62ec1551 acf156fb
0bce855 f174d4c1 7807511c 9f745382 add53bc3
```

**Degree 409:**

```
0eb00f2 ea95fd6c 64024e7f
0b68b81f 5ff8a467 acc2b4c3 b9372843 6265c7ff
```

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Given the second-to-last row $r$ of $\Gamma$, the rest of the matrix is computed as follows. Let $\beta$ be the element of $GF(2^m)$ whose representation with respect to the normal basis is $r$. Then the rows of $\Gamma$, from top to bottom, are the bit strings representing the elements

$$\beta^m, \beta^{m-2}, \ldots, \beta^2, \beta, 1$$

with respect to the normal basis. (Note that the element 1 is represented by the all-1 bit string.)

Alternatively, the matrix is the inverse of the matrix described in Appendix 6.9.

More details of these computations can be found in Annex A.7 of the IEEE P1363 standard.
APPENDIX 6.9: NORMAL BASIS TO POLYNOMIAL BASIS CONVERSION

Suppose that $\alpha$ an element of the field $GF(2^m)$. Denote by $\mathbf{n}$ the bit string representing $\alpha$ with respect to a given normal basis. It is desired to compute $\mathbf{p}$, the bit string representing $\alpha$ with respect to a given polynomial basis. This is done via the matrix computation

$$\mathbf{n} \Gamma = \mathbf{p}$$

where $\Gamma$ is an $m$-by-$m$ matrix with entries in $GF(2)$. The matrix $\Gamma$, which depends only on the bases, can be computed easily given its top row. The top row for each conversion is given in the table below.

**Degree 163:**

$$7 \; 15169c10 \; 9c612e39 \; 0d347c74 \; 8342bcd3 \; b02a0bef$$

**Degree 233:**

$$149 \; 9e398ac5 \; d79e3685 \; 59b35ca4 \; 9bb7305d \; a6c0390b \; cf9e2300 \; 253203c9$$

**Degree 283:**

$$31e0ed7 \; 91c3282d \; c5624a72 \; 0818049d \; 053e8c7a \; b8663792 \; bc1d792e \; ba9867fc \; 7b317a99$$

**Degree 409:**

$$0dfa06b \; e206aa97 \; b7a41fff \; b9b0c55f \; 8f048062 \; fbe8381b \; 4248adf9 \; 2912ccc8 \; e3f91a24 \; e1cfb395 \; 0532b988 \; 971c2304 \; 2e85708d$$

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Given the top row $r$ of $\Gamma$, the rest of the matrix is computed as follows. Let $\beta$ be the element of $GF(2^m)$ whose representation with respect to the polynomial basis is $r$. Then the rows of $\Gamma$, from top to bottom, are the bit strings representing the elements

$$\beta, \beta^2, \beta^3, \ldots, \beta^{2^{m-1}}$$

with respect to the polynomial basis.

Alternatively, the matrix is the inverse of the matrix described in Appendix 6.8.

More details of these computations can be found in Annex A.7 of the IEEE P1363 standard.
Federal Information Processing Standard (FIPS) 186-2, Digital Signature Standard, specifies the Digital Signature Algorithm (DSA) that may be used in the generation and verification of digital signatures for sensitive, unclassified applications. FIPS 186-2 also allows the use of the digital signature techniques specified in American National Standards Institute (ANSI) X9.31 (Digital Signatures Using Reversible Public Key Cryptography for the Financial Services Industry (rDSA)) and ANSI X9.62 (Public Key Cryptography for the Financial Services Industry: The Elliptic Curve Digital Signature Algorithm (ECDSA)). The standard also specifies a transition period for the use of existing (legacy) digital signature systems. Reversible public key algorithms, such as the RSA or Rabin-Williams algorithms, are often used in these legacy systems.

FIPS 186-2 is used in conjunction with the hash function specified in FIPS 180-1, Secure Hash Standard (SHS), and includes specifications for the size of the prime modulus $p$, and algorithms for the generation of a user’s private key, $x$, and a user’s per message secret number, $k$.

This change notice provides changes for the continued use of DSA as specified in FIPS 186-2 about the size of the prime modulus $p$, modifications for the random number generation techniques specified in Appendix 3 of FIPS 186-2, and provides instructions for the use of these techniques when used in contexts other than the generation of DSA keys. This change notice also provides guidance for the use of the reversible public key algorithms within legacy systems.

Questions regarding this change notice may be directed to FIPS186@nist.gov or to Elaine Barker (ebarker@nist.gov, 301-975-2911).

The Size of the Prime Modulus

Section 4 of FIPS 186-2 specifies that the prime modulus $p$ of DSA is defined for the range of prime integers $2^{L-1} < p < 2^L$, where $512 \leq L \leq 1024$ and $L$ is a multiple of 64. This change notice specifies that $L$ should assume only the value 1024 for DSA as specified in FIPS 186-2, i.e., the prime modulus $p$ should be defined in the range $2^{1023} < p < 2^{1024}$.

The RSA and Rabin-Williams algorithms used within legacy systems are defined with a modulus $n$ and prime factors $p$ and $q$ of $n$. This change notice specifies that $n$ should be at least 1024 bits.
Random Number Generation

FIPS 186-2 includes algorithms for the generation of a user's private key, \( x \), and a user's per message secret number, \( k \). These values must be generated randomly or pseudorandomly and must have values between 0 and the 160-bit prime \( q \) (as specified in the standard). Techniques for generating \( x \) and \( k \) are provided in Appendix 3 of the standard.

Recently, an unpublished attack on DSA3 was found that relies on the non-uniformity of the pseudorandom number generators (PRNGs) specified in Appendix 3 of the standard. The attack has a workfactor of \( 2^{64} \) and requires \( 2^{22} \) known signatures. This attack can be defended against by either limiting the number of signatures created using a specific key pair to no more than 2 million signatures while using the PRNGs specified in FIPS 186-2, or by modifying the PRNGs.

If the PRNGs currently defined in FIPS 186-2 are used, the user should be provided with clear guidance about the limitation to the number of signatures that should be created.

Alternatively, the following modifications of the PRNGs may be used in lieu of those PRNGs specified in FIPS 186-2. These modifications reduce the non-uniformity of the PRNGs and do not affect interoperability.

The two algorithms described below use a one-way function \( G(t,c) \), where \( t \) is 160 bits, \( c \) is \( b \) bits and \( G(t,c) \) is 160 bits. Two methods for constructing \( G \) are defined in FIPS 186-2: using SHA-1 as defined in FIPS 180-1, and using the Data Encryption Standard (DES) as defined in FIPS 46-3. If \( G \) is constructed using SHA-1, \( b \) is between 160 and 512 bits (160 \( \leq \) \( b \) \( \leq \) 512); if \( G \) is constructed using DES, \( b \) is equal to 160 bits.

1. Revised Algorithm for Computing \( m \) values of \( x \) (Appendix 3.1 of FIPS 186-2)

Let \( x \) be the signer's private key. The following may be used to generate \( m \) values of \( x \):

   Step 1. Choose a new, secret value for the seed-key, \( XKEY \).

   Step 2. In hexadecimal notation let

   \[
   t = 67452301 \text{ EFCDBAB89 98BADCFE 10325476 C3D2E1F0.}
   \]

   This is the initial value for \( H_0 \parallel H_1 \parallel H_2 \parallel H_3 \parallel H_4 \) in the SHS [FIPS 180-1].

---

3 The attack was discovered by Dr. Daniel Bliechenbacher of Lucent Technologies, Bell Labs, Murray Hill, NJ. See a February 25, 2001 press article at http://www.lucent.com/press/0201/010205.bla.html.
Step 3. For \( j = 0 \) to \( m - 1 \) do

3.1 \( XSEED_j = \) optional user input

3.2 For \( i = 0 \) to \( 1 \) do
   a. \( WAL = (XKEY + XSEED_j) \mod 2^b \)
   b. \( w_i = G(t, WAL) \).
   c. \( XKEY = (1 + XKEY + w_i) \mod 2^b \)

3.3 \( x_j = (w_0 \| w_1) \mod q \)

2. Revised Algorithm for Precomputing one or More \( k \) and \( r \) Values (Appendix 3.2 of FIPS 186-2)

This algorithm can be used to precompute \( k, k^1 \), and \( r \) for \( m \) messages at a time. Note that implementation of the DSA with precomputation may be covered by U.S. and foreign patents.

Step 1. Choose a secret initial value for the seed-key, \( KKEY \).

Step 2. In hexadecimal notation let

\[
t = \text{EFCDAB89 98BADC}F\text{E} \text{10325476} \text{C3D2E1F0} \text{67452301}.
\]

This is a cyclic shift of the initial value for \( H_0 \| H_1 \| H_2 \| H_3 \| H_4 \) in the SHS.

Step 3. For \( j = 0 \) to \( m - 1 \) do

3.1 For \( i = 0 \) to \( 1 \) do
   a. \( w_i = G(t, KKEY) \)
   b. \( KKEY = (1 + KKEY + w_i) \mod 2^b \)

3.2 \( k = (w_0 \| w_1) \mod q \)

3.3 Compute \( k_j^{-1} = k^{-1} \mod q \)

3.4 Compute \( r_j = (g^k \mod p) \mod q \)
Step 4. Suppose $M_0, \ldots, M_{m-1}$ are the next $m$ messages. For $j = 0$ to $m - 1$ do

a. Let $h = \text{SHA-1}(M_j)$.

b. Let $s_j = (k_j^{-1}(h + x r_j)) \mod q$

   c. The signature for $M_j$ is $(r_j, s_j)$.

Step 5. Let $t = h$

Step 6. Go to step 3.

Step 3 permits pre-computation of the quantities needed to sign the next $m$ messages. Step 4 can begin whenever the first of these $m$ messages is ready. The execution of step 4 can be suspended whenever the next of the $m$ messages is not ready. As soon as steps 4 and 5 have completed, step 3 can be executed, and the results saved until the first member of the next group of $m$ messages is ready.

In addition to space for $KKEY$, two arrays of length $m$ are needed to store $r_0, \ldots, r_{m-1}$ and $k_0^{-1}, \ldots, k_{m-1}^{-1}$ when they are computed in step 3. Storage for $s_0, \ldots, s_{m-1}$ is only needed if the signatures for a group of messages are stored; otherwise $s_j$ in step 4 can be replaced by $s$, and a single space allocated.

**General Purpose Random Number Generation**

Several of the FIPS require the use of an Approved (i.e., FIPS-approved or NIST recommended) random number generator (RNG). The RNG specified as algorithm 1 above or the algorithm specified in Appendix 3.1 of FIPS 186-2 may be used in addition to any other Approved RNG. However, when the RNG is used for the generation of random numbers other than for DSA keys, the “mod $q$” term should be omitted. This will result in the following changes to the specification:

FIPS 186-2, Appendix 3.1, Step 3 c:
Change “$x_j = G(t, X \!\/\! XAL) \mod q$” to “$x_j = G(t, X \!\/\! XAL)$”.

Algorithm 1 of this change notice, Step 3, substep 3.2:
Change “$x_j = (w_0 \parallel w_1) \mod q$” to “$x_j = (w_0 \parallel w_1)$”.

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