The Sixth
Clemson mini-Conference
on
Discrete Mathematics
Funded by the Office of Naval Research (ONR)

Conference Program and
Invited Talk View Graphs

October 3-4, 1991

Clemson University
Clemson, South Carolina
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Organizers:
S.T. Hedetniemi Department of Computer Science
R.C. Laskar Department of Mathematical Sciences
R.D. Ringeisen Department of Mathematical Sciences

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Schedule of talks
(All talks given in Student Senate Chambers)

Thursday, October 3

1:30 - 2:00  
Sign-in/Registration

2:00 - 2:05  
Welcoming Remarks by R.D. Ringeisen, S.T. Hedetniemi, and Dr. Bobby Wixson, Dean of College of Sciences

2:05 - 2:50  
Marc J. Lipman, Office of Naval Research
Mathematical Science Division, Arlington, VA

"Sphere-of-Influence Graphs"
An introduction to a class of geometrically defined objects, sphere-of-influence graphs (SIGs), used by the computer vision community to capture the low-level perceptual structure of a scene, that is, for pattern recognition. The mathematics of SIGs isn't yet mature.

2:55 - 3:35  
Roger Entringer, University of New Mexico
Dept. of Mathematics, Albuquerque, NM

"Two Extremal Problems In Graph Theory"
Two specific instances of the following general problems are addressed:

(i) How many edges can a graph $G$ of order $n$ have if $G$ must have a specific property?

(ii) If $G$ is to have order $n$ and a given number of edges, how are the edges arranged if a specific property must be optimized?

The instance of the second problem involves an attempt to shorten delivery time by the USPS.

3:35 - 4:00  
BREAK
"Containment Orders and Planar Graphs"

We will explore interesting relationships between the worlds of partially ordered sets (especially containment orders) and planar graphs.

Given any graph G = (V, E), let its vertex-edge incidence order, P(G), be the partially ordered set whose ground set is V ∪ E together with relations v < e exactly when v ∈ V, e ∈ E, and v is an end point of e. What graph properties of G can we deduce from poset properties of P(G)?

In this talk we will focus on order theoretic properties of P(G) which turn out to be equivalent to G being planar. We will phrase these properties in terms of geometric containment orders, which we now define.

Given a family Σ of objects, we call a partially ordered set P = (X, ≤) a Σ-order provided we can assign to each x ∈ X, an element S_x ∈ Σ, so that x ≤ y iff S_x ≤ S_y. In particular, if Σ is the set of disks (circles with their interiors) in the plane, then Σ-orders are also known as circle orders.

We will discuss a number of results, all of which have the following flavor.

Theorem. A graph G is planar if and only if its vertex-edge incidence order, P(G), is a circle order. □

Some of the theorems to be presented include joint work with Graham Brightwell, Ann Trenk and Daniel Ullman.

"On Finding Transmitter-Receiver Matchings"

The problem of finding a maximum transmitter-receiver matching (TRM) in communication networks is addressed. TRM remains NP-complete even for networks whose topologies correspond to chordal graphs. We address the problem for a subclass of chordal graphs, namely those graphs whose clique graphs are acyclic. Using several interesting properties of these graphs, we devise a linear time algorithm to solve the problem.
Friday, October 4

8:30 - 9:10  
**J. Chris Fisher, University of Regina, Canada**  
(Visiting Clemson University, Dept. of Math. Sci.)  

"The Jamison Method in Galois Geometries"

In a fundamental paper Robert E. Jamison showed, among other things, that any subset of the points of AG(2,q) — the affine plane of order q — that intersects all lines contains at least 2q-1 points. Here I shall discuss my recent work with Aiden Bruen in which we show that Jamison's method of proof can be applied to several other basic problems in finite geometries of a varied nature. These problems include the celebrated flock theorem and also the characterization of the elements of GF(q) as a set of squares in GF(q^2) with certain properties. This last result, due to A. Blokhuis, settled an important conjecture due to J.H. van Lint and the late J. MacWilliams.

9:15 - 9:55  
**Fred S. Roberts, Rutgers University**  
Dept. of Mathematics, Center of Operations Research (RUTCOR), and Center for Discrete Mathematics and Theoretical Computer Science (DIMACS)  
New Brunswick, NJ

"Elementary, Sub-Fibonacci, Regular, Van Lier and Other Interesting Sequences"

In the past five years, problems of the uniqueness of scales of measurement have been giving rise to a variety of interesting sequences of positive integers with fascinating combinatorial properties. Examples of such sequences are all non-decreasing sequences of positive integers x_1, x_2, ..., x_n so that x_1 = x_2 = 1. Such a sequence is called **elementary** if all k \leq n, x_k > 1 implies that x_k = x_i + x_j for some i \neq j. It is called **sub-Fibonacci** if x_k \leq x_{k-1} + x_{k-2}, k = 3, 4, ...  It is called **regular** if x_j \leq \sum_{i=1}^{j-1} x_i, j = 3, 4, ...  A regular sequence is called **Van Lier** if for all j < k \leq n, there is a subset A of {1, 2, ..., n} with j not in A and x_k - x_j = \sum_{i \in A} x_i. We discuss these and other sequences and some of their combinatorial properties.

9:55 - 10:20  
**BREAK**

10:20 - 11:00  
**Michael S. Jacobson, University of Louisville**  
Department of Mathematics, Louisville, KY

"Generating k-element Subsets of an n-element Set"

In this talk, a generalization of the idea of De Bruijn graphs will be used to establish sequences which generate all k-element subsets of an n-element set. In the case when n is odd, by using a result of Good, these sequences are shown to exist. When n is even, the technique shown will not generate an appropriate sequence. In fact the generalized De Bruijn graph is disconnected, and by a unique application of Polya's Theorem, the number of components of this graph is calculated.
E. Rodney Canfield, University of Georgia
Dept. of Computer Science, Athens, GA

"Matchings In the Partition Lattice"

Let \([n]\) be the set \([1, 2, \ldots, n]\). A partition of \([n]\) is a set of nonempty, pairwise disjoint subsets of \([n]\), called blocks, whose union is \([n]\). Partition \(\pi_1\) is a refinement of partition \(\pi_2\), denoted \(\pi_1 \leq \pi_2\), provided each block of \(\pi_1\) is contained in a block of \(\pi_2\). Under this ordering the set of partitions \(P_n\) forms a lattice. The subcollection of partitions \(P_{n,k} \subseteq P_n\) which have exactly \(k\) blocks has cardinality \(S(n,k)\), the Stirling number of the second kind. The Stirling numbers are unimodal, raising the question of decomposing \(P_n\) into disjoint chains, \(S(n, K_n)\) in number, \(S(n, K_n)\) being max \(S(n, k)\). Our topic in this talk: for what \(k\) is it possible to find a matching of \(P_{n,k}\) into \(P_{n,k+1}\)? That is, to find a one-to-one function \(\emptyset\) from \(P_{n,k}\) into \(P_{n,k+1}\) with the property that \(\pi\) and \(\emptyset(\pi)\) are comparable under the refinement relation "\(\leq\)".

LUNCH

Ronald C. Read, University of Waterloo
Dept. of Combinatorics and Optimization, Ontario, Canada

"Algorithms for Small Graphs"

The compilation of an "Atlas" of graph theory - a project that I am working on with R.J. Wilson - has called for the computation of many invariants (girth, connectivity, etc.) and properties (planarity, hamiltonicity, etc.) of large number of graphs; but the graphs themselves are quite small. Thus the usual concern about complexity of the algorithms is largely irrelevant, and the methods that will be used are often quite different from those that would be used for large graphs.

My talk describes some of this work. We shall see what graph theory algorithms look like through the wrong end of the telescope.

Nathaniel Dean, Bellcore
Morristown, NJ

"Characterization of Generalized Bicritical Graphs"

A recent theorem of Thomas and Yu states that every 4-connected projective planar graph is hamiltonian and, as a corollary, has a 2-factor. We extend this latter result by showing that the deletion of any vertex or two vertices of such a graph leaves a graph with a 2-factor. This result is in fact only an application of results we prove concerning \(f\) factors in graphs with removed elements and generalizes several notions in matching theory including bicritical graphs, i.e., where the deletion of any pair of vertices yields a graph with a perfect matching.

BREAK
"Extremal Problems Involving Neighborhood Numbers and Other Parameters"

Given a simple graph $G = (V, E)$, a subset $S$ of $V$ is called a neighborhood set provided $G$ is the union of the subgraphs induced by the closed neighborhoods of the vertices in $S$. The minimum and maximum cardinalities among all minimal neighborhood sets of $G$ are denoted by $\eta(G)$ and $N(G)$, respectively; $\eta(G)$ is called the neighborhood number of $G$. It is known, for instance, that $\gamma(G) \leq \eta(G) \leq \alpha(G)$, where $\gamma(G)$ and $\alpha(G)$ are the (vertex) domination and covering numbers, respectively.

My colleague, Y.H. Harris Kwong, and I have been investigating the problem of finding the maximum neighborhood number $\eta(p)$ among all connected graphs of order $p$. Our work so far has lead us to conjecture that

$$\eta(p) \leq \lfloor \frac{9p}{13} \rfloor$$

a result that holds for $2 \leq p \leq 15$. I will report on this work and, as time permits, a number of other extremal problems, including some recent work of David K. Garnick, Kwong, and Felix Lazebnik on the maximum number of edges among all graphs of order $p$ having girth at least 5.

"Random Graph Processes with Degree Restrictions"

Suppose that a process begins with $n$ isolated vertices, to which edges are added randomly one by one so that the maximum degree of the induced graph is always bounded above by $d$. We prove that if $n$ approaches infinity with $d$ fixed, then with probability tending to 1, the final result of this process is a graph with $\lfloor nd/2 \rfloor$ edges. For $d = 2$, the number of 1-cycles in this graph is shown to be asymptotically Poisson ($1 > 2$).
"Sphere-of-Influence Graphs"

Marc J. Lipman, Office of Naval Research
Mathematical Science Division, Arlington, VA
On Abstract Sphere-of-Influence Graphs

Frank Harary
Michael S. Jacobson
Marc J. Lipman
F. R. McMorris

"Perfection of means and confusion of goals seem to characterize our age."
Let $S$ be a finite set of at least two points in the Euclidean plane. For each $x \in S$, let $r_x$ be the smallest distance from $x$ to any other point in $S$.

Let $B_x$ be the open ball of radius $r_x$ centered at $x$.

Let $A_x$ be the closed ball of radius $r_x$ centered at $x$.

The **Sphere-of-Influence Graph** of $S$, $G(S)$, has vertex set $S$, and for $x, y \in S$, $x$ and $y$ are adjacent in $G(S)$ if $B_x \cap B_y \neq \emptyset$.

The **Closed-Sphere-of-Influence Graph** of $S$, $G^c(S)$, has vertex set $S$, and for $x, y \in S$, $x$ and $y$ are adjacent in $G^c(S)$ if $A_x \cap A_y \neq \emptyset$.
Thus: The graphs $G(S)$ and $G^*(S)$ for a set $S$ with $|S|\leq n$ can be computed in time $O(n \log n)$.

So what's the problem?

We don't know much about the graphs $G(S)$ and $G^*(S)$. Do they do what they appear to do?

Even more basic: Which graphs are $S16s$?
Which graphs are $C16s$?

Easy (important) result: The union of $S16s$ ($C16s$) is a $S16$ ($C16$).
Are there graphs which are not \( S(1) \)?

\( K_{1,3} \) is not a \( S(16) \).

Thus \([ E(1) = B(16) ]\): If \( |E| = n \), then \( G(s) \) has at most \( 17n \) edges.

So \( S(16) \) (and \( C(16) \)) aren't dense.

Some Complete Graphs are \( S(16) \).

\[ G(s) \equiv k_{\infty} \]
\[ G(CS) = K_4 \]

**Theorem:** The path or 3-cycle is not a CSIG.

If \( 1 \leq s \leq 3 \), then \( C^s(3) \cong K_3 \).

**Theorem:** Every path are CSIGs. Odd ones aren't.

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \\
\end{array}
\]

\( \rightarrow \text{ NOT } P_n (K^2) \)

**Theorem:** The \( \Delta \) and even cycles are CSIGs.
The others aren't.

\[
\begin{array}{c}
\circ \quad \circ \\
\circ \\
\end{array}
\]

\[
\begin{array}{c}
\circ \quad \circ \\
\circ \\
\end{array}
\]

**Theorem:** The class of SIGs (CSIGs) is not closed under taking subgraphs (or even taking induced subgraphs)!

\[ \rightarrow \text{ There is no forbidden subgraph characterization of SIGs (CSIGs).} \]

**Theorem:** Every tree is the induced subgraph of a SIG.

\[ \begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\circ \quad \circ \\
\circ \quad \circ \\
\end{array} \]

\[ \begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
\end{array} \]

Let us say that "x defines y" if \( x \) is a nearest neighbor of \( y \) (that is: \( y \in \bar{G}(x,y) \)) and of course \( yx \) is then an edge in the SIG.

**Theorem:** If \( G \) is a CSIG without a \( \Delta \), then \( G \) has a 1-factor of "defining edges."

(a 1-factor is a set of edges meeting every vertex exactly once.)
proof: If \( x \) defines \( y \) and \( z \), then \( \Delta \).

![Diagram]

Otherwise: suppose \( x_1 \) defines \( x_2 \),
\( x_3 \), defines \( x_4 \),
\( x_5 \), defines \( x_6 \).

Then:
\[
\begin{align*}
\rho_1 &= d(x_1, x) + \rho_2, \\
\rho_2 &= d(x_2, x) + \rho_3, \\
\rho_3 &= d(x_3, x) + \rho_4, \\
\rho_4 &= d(x_4, x) + \rho_5, \\
\rho_5 &= d(x_5, x) + \rho_6.
\end{align*}
\]

\[\Rightarrow \rho_1 + \rho_2 + \cdots + \rho_6 = d(x_1, x_6).\]

If \( k>2 \), then \( x_2 \) defines \( x_1 \) and \( x_2 \).

Therefore, \( k=2 \) and \( x_1 \) and \( x_2 \) define each other
(and no other points).

Therefore, the set of all "defining edges,"
pairs up the points as above -- and that
is the 1-factor.

**Note**: The theorem merely states that there
is a 1-factor. The proof shows that the full
set is the 1-factor.

**Cor**: If \( G \) is a CSIG without a \( A \),
then \( G \) has an even number of points.

However, if \( G \) is a tree, we get
a complete characterization:

**Theorem**: Suppose \( G \) is a tree. Then
\( G \) is a CSIG \( \iff \)
\( G \) has a 1-factor.

\( G \) is a OSIG \( \iff \)
\( G \) has a \( \{ x, x, 3 \} \)-factor.
This can't work in general, since every possible edge between \( K_n \) has too many edges to be a \( 51G \) but has a \( 1 \)-factor and is \( \Delta \)-free.

**Question:**

Since we are interested in computer vision, we really care about \( 51G \)s where the points have to show up in pixels.

Which \( 51G \)s show up here?

**Theorem:** If \( G \) is a O51G, then \( G \) has a representation in which every point has integer coordinates.

**Ex:**

\[
(0,0), (0,1), (1,0), (1,1), (0,2), (2,0), (0,3), (3,0), \ldots
\]
Idea: Suppose \( G \) has a representation with no circles tangent:

\[
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\]

Then \( \exists \) \( \alpha \) so that if any circle is moved \( \alpha \) no intersections are changed.

Move the circles one at a time to \textit{rational} points.

Then "put up" to integer points.

Oops: Sometimes you have to have tangencies: \( P_2 \), \( P_3 \) \( \alpha \) for \( \text{OS16} \).

Fix: Suppose only thus, that is, no "accidental" tangencies:

\[
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}
\]

\( P_2 \cup P_3 \)

\[
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\]

Then: move in "clumps."

If \( C_1 \) and \( C_2 \) are tangent at \( \alpha \),

\( \alpha \) division then! Then move all \( 3 \) circles together until \( \alpha \) has rational coordinates.

Then rotate \( {\alpha}^\circ \) and move.

along the \textit{xy} line to fix \( x \) and \( y \).

This works \( \frac{f}{\alpha} \) such a representation exists.
So:

1. We know a little about S16 and C516.
2. They seem to be useful for object separation in some contexts.
3. They may be useful for object identification in similar contexts.
4. The definitions generalize to three dimensions and different geometries.
5. Our ignorance exceeds our knowledge.
"Two Extremal Problems in Graph Theory"

Roger Entringer, University of New Mexico
Department of Mathematics, Albuquerque, NM
Q. How many edges can a graph of order $n$ have if it doesn't contain any cycle?
A. $n - 1$.

Q. What graphs have this many edges but don't contain a cycle?
A. Trees.

Q. How many edges can a graph of order $n$ have if it doesn't contain a subdivision of $K_3$?
A. $\text{ex}(n;K_3) = 2n - 3$.

Q. How many edges can a graph of order $n$ have if it doesn't contain a subdivision of $K_5$?
A. $\text{ex}(n;K_5) = 3n - 6$.

Fix the graph $F$. We define the subdivision threshold of $F$ to be the maximum number of edges, $\text{ex}(n;FS)$, a graph of order $n$ can have without containing a subdivision of $F$ as a subgraph.

We denote by $EX(n;FS)$ the family of those graphs of order $n$ that have $\text{ex}(n;FS)$ edges and do not contain a subdivision of $F$.

A result of Mader shows that for any graph $F$ there is a constant, $c_F$, such that $\text{ex}(n;FS) \leq c_F n$.

Theorem (Mader 1967). If a graph has order $n$ and size $\left(\begin{array}{c}n-1 \end{array}\right) + 1$ then it contains a subdivision of $K_{n+1}$. 

**UNAVOIDABLE SUBGRAPHS OF SPARSE GRAPHS**

Q. How many edges can a graph of order $n$ have if it doesn't contain a hamilton cycle?
A. (Ore 1961) $\binom{n - 1}{2} + 1$.

Q. What graphs have this many edges but don't contain a hamilton cycle?
A. $K_{n-1}$ with a pendant vertex, $n \neq 5$.

Let $P(n)$ be a property enjoyed by $K_n$.

Generic Extremal Problem: Determine the maximum number of edges, $\text{ex}(n;P(n))$, a graph of order $n$ can have if it doesn't satisfy property $P(n)$.

The graphs of order $n$ that have $\text{ex}(n;P(n))$ edges but do not satisfy property $P(n)$ are called the extremal graphs for $P(n)$.
Theorem. (Erdős and Pósa 1965) \( \text{ex}(n;FS) = 3n - 6 \). \( G \) is in \( EX(n;FS) \) iff \( G = K_3 + \overline{K}_{n-3} \).

Theorem. (BCDES) \( \text{ex}(n;FS) = \begin{cases} 2n - 2, & n \equiv 1 \mod 3 \\ 2n - 3, & n \equiv 1 \mod 3 \end{cases} \). \( G \) is in \( EX(n;FS) \) iff every block of \( G \), with at most one exception, \( B \), is isomorphic to \( K_5 \), and \( B = K_5 + \overline{K}_5 \), or \( B = K_{3,3} \) or \( B = K_5 \times K_5 \) or \( B \) is the nearly 3-regular graph of order 5.

Certain subgraphs of certain graphs. Candidates for graphs \( G \) satisfying \( \text{ex}(n;FS) < 3n - 6 \).

Problem. Find all graphs with subdivision threshold less than 3n - 6.

Properties of \( F \) when \( \text{ex}(n;FS) < 3n - 6 \):

(i) \( \Delta(F) \leq 6 \).

(ii) \( F \) has at most one vertex with degree \( \geq 6 \).

(iii) \( F \) has at most two vertices with degrees \( \geq 5 \).

(iv) \( F \) is planar.

(v) If \( F \) is connected, then it has order \( \leq 7 \).

(vi) If \( F \) is 2-connected, then it has order \( \leq 6 \).

(vii) \( F \) is a subgraph of \( K_5 + \overline{K}_{n-3} \).

Theorem. (Thomassen 1974) \( \text{ex}(n;FSR) = 2n - 3 \). \( G \) is in \( EX(n;FSR) \) iff \( G \) is a \((\ast,2)\)-cockade where each member \( \ast \) of the cockade is \( K_5 \) or \( K_{3,3} \).

Theorem. (Krusentjerna-Haæstman and Toft 1980) \( \text{ex}(n;FSR) = 2n - 3 \). \( G \) is in \( EX(n;FSR) \) iff \( G \) is a \((3,2)\)-cockade.
Theorem. (Erdős and Pósa 1965) \( ex(n;FS) = 3n - 6 \). \( G \) is in \( EX(n;FS) \) if \( G = K_3 + K_{n-3} \).

Theorem. (BCDEKS) \( ex(n;FSR) = 3n - 6 \). \( G \) is in \( EX(n;FSR) \) if \( G = K_3 + K_{n-3} \).

The load, \( L(v) \), of a vertex \( v \) of a tree \( T \) is the number of paths in \( T \) containing vertex \( v \) as an internal vertex. This has also been called the cutting number by Harary and Ostrand.

Suppose \( T \) has order \( n \) and that the branches of \( T \) at \( v \) have \( n_i \) edges, \( 1 \leq i \leq k \), (so that \( \sum n_i = n - 1 \)) then

\[
L(v) = \sum n_i n_j = \frac{1}{2} \left[ (n - 1)^2 - \sum n_i^2 \right].
\]

The load, \( L(T) \), of a tree \( T \) is the sum of the loads of the vertices.

The transmission, \( \sigma(v) \), of a vertex \( v \) in a connected graph \( G \) is the sum of the distances from \( v \) to each of the remaining vertices of \( G \).

The transmission of a graph \( G \) is defined by

\[
\sigma(G) = \frac{1}{2} \sum \sigma(v)
\]

Observation.

\[
L(T) = \sigma(T) - \frac{n}{2}
\]

(since a path joining \( u \) and \( v \) contributes 1 to the load of each of \( d(u,v) - 1 \) vertices.)

The transmission center of a graph is the set of vertices with minimum transmission and consists of one vertex or two adjacent vertices. The transmission center is the centroid (Zelinka).
Theorem. Of all trees of order $n$ with exactly $k$ end vertices, $S(n,k)$ has minimum transmission.

Theorem. Of all trees of order $n$ with exactly $k$ end vertices, $D(n,k)$ has maximum transmission.

Problem. Given a connected graph find a spanning tree with minimum transmission. For example, find a spanning tree of $\Omega$ with minimum transmission.

Question. What fraction of the vertices of a tree of order $n$ can have maximum load? In particular, does this fraction tend to 0?

Theorem. Let $G$ be a $k$-partite graph with smallest part $H$. The spanning tree of $G$ with minimum transmission contains two adjacent vertices, $u$ in $H$ and $v$, where $u$ is adjacent to all vertices of $G$ not in $H$ and $v$ is adjacent to all vertices of $G$ in $H$.

Question. What fraction of the vertices of a tree of order $n$ can have a relatively maximum load?
"Containment Orders and Planar Graphs"

Edward R. Scheinerman, Johns Hopkins University
Department of Mathematical Sciences, Baltimore, MD
Definitions...

Let $P$ be a finite poset. We call $P$ a circle order provided we can assign to each $x \in P$ a circle $C_x$ so that $x \leq y$ iff $C_x \subseteq C_y$.

\[ P \quad \begin{array}{c} \cap \\ \cap \\ \cap \\ x \\ y \\ z \\ C_x \\ C_y \\ C_z \end{array} \]
parabola order

parabola $P_x$ (upwards, filled)

$P_x \subseteq P_y$

space-time* order

"event" $P_x$ in space-time*

$P_x$ precedes $P_y$

RS2PD* order

real, symmetric, 2-by-2 matrix $M_x$

$M_y - M_x$ is positive

(semi) definite

$M_x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$M_y = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

$M_z = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

* two space coordinates, one time coordinate

What's the connection?

Theorem. The following statements about a finite poset $P$ are equivalent:

- $P$ is a circle order
- $P$ is a space*-time
- $P$ is a parabola order
- $P$ is a RS2PD order

* two space coordinates

Theorem. The following statements about a finite poset $P$ are equivalent:

- $P$ is a sphere order
- $P$ is a space*-time
- $P$ is a H2PD order

* three space coordinates

But! Both statements are false for infinite posets.

[Example & Statement]
Why it works
(Q/L Orders)

Q/L order on \( \mathbb{R}^{n+1} \)

\[ x \leq y \quad \text{if} \quad \begin{cases} Q(y-x) \geq 0 \\ L(y-x) \geq 0 \end{cases} \]

Theorem. The only possible Q/L orders on \( \mathbb{R}^{n+1} \) are:
- anti-chain
- disjoint union of chains
- space-time order (balls in \( \mathbb{R}^n \))
- \( n \) space coordinates

What can you deduce about \textit{graph} properties of \( G \) from \textit{order} properties of \( P(G) \)?

Posets from Graphs

Theorem: A graph \( G \) is planar iff \( P(G) \) is a triangle order. (Schneider)
Theorem: A graph $G$ is planar iff $P(G)$ is a circle order. [Scheinerman]

Key Ideas in the Proof

#1 O.K. to work with the dual.

#2 Thurston's Theorem.
Every planar graph has a representation by disks in the plane...

such that...

|externally tangent| disjoint|
G planar $\Rightarrow P(G)$ is a circle order

- Form Thurston circles for $G$. These will be the circles for $V(G)$.
- Points of tangency will be the circles (of radius 0) for $E(G)$.
- Notice: Every edge circle is contained in exactly its endpoints' circles.
G planar $\iff P(G)$ is a circle order

Draw the dual of $P(G)$ as a circle order...

...and this will give an embedding of $G$ in the plane!

Why 2-step paths might be needed

Circle/Sphere Orders at their extreme...

Point-Halfspace Orders

A "bipartite" poset is called a point-halfspace order if...

Theorem. Let $G$ be a graph. $P(G)$ is a point-halfspace order in $\mathbb{R}^3$ if and only if $G$ is planar or $K_5$. 

[Schiermeyer, Tuyl & Ueblin]
Summary

For any graph $G$...

- $P(G)$ is a circle order
- $G$ is planar
- $P(G)$ is a triangle* order
- $P(G)$ is a point-halfspace order

What about non-planar graphs?

Theorem. If $G$ is any graph, then $P(G)$ is a sphere order. [Schaefer]

Corollary. Let $G$ be a graph. The least $d$ such that $G$ embeds in $\mathbb{R}^d$ equals the least $d$ such that $P(G)$ is representable by balls in $\mathbb{R}^d$. [Schaefer]

Theorem. If $G$ is any graph, then $P(G)$ is a point-halfspace order in $\mathbb{R}^4$. [Schaefer, Trotter, Ullman]

Note: $P(G)$ can have arbitrarily high poset dimension.

Planar Maps

(bounded faces only)

$P(M)$

faces
edges
vertices

$F(M)$
$E(M)$
$V(M)$

Theorem. For any planar map, $\dim P(M) \leq 3$. (Brightwell & Trotter)

...but is $P(M)$ a circle order?

Double Thurston Theorem

Theorem. (Brightwell & Scheinerman, Pulleyblank & Rat) If $G$ is a 3-connected planar graph and $G^*$ is its dual, then we can make "Thurston Circles" for $G$ and $G^*$ simultaneously so that...

And they cross at 90°
Containment of Positive and Negative Circles

Main Theorem

Theorem. If $M$ is a 3-connected planar map then $P(M)$ is a circle order. (Brightwell & Scheinerman)

Proof...
Why $P(M)$ is a circle order

A Conjecture of Tutte...
Let $G$ be a 3-connected planar graph. Can one properly draw $G$ and its dual $G^*$ with straight line segments for edges, so that dual edges cross at $90^\circ$?

What about the unbounded faces?

For example...

$2^{\{1,2,3,4\}}$ is not a circle order (Jamison), which implies $P^*(M)$ is not a circle order.

but...
Theorem. If \( M \) is a 3-connected planar map, then \( P^+(M) \) is a "cap order".

(Brightwell & Scheinerman)

Note: \( P^+(M) \) is the face lattice of a convex polyhedra in \( \mathbb{R}^3 \).
"On Finding Transmitter-Receiver Matchings"

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On Finding Transmitter - Receiver Matchings

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Given: a communication network with bidirectional links
Goal: to simultaneously test as many links as possible
Constraints:
1) a transmitting site sends signals along all of the links emanating from it
2) a site cannot be both a transmitter and a receiver at the same time
3) two or more signals reaching a site at the same time interfere

AC Graphs

Graph G:

Clique Graph $G_c$
- one node for each maximal clique in $G$
- an edge between 2 nodes iff the corresponding cliques intersect

AC Graphs
- graphs whose clique graphs are acyclic

Terminology
- leaf node of $G$, leaf clique of $G$,
  parent clique, simplicial node of $G$
An AC Graph

The corresponding clique graph.

Useful Properties of AC Graphs

Lemma 2.1 -- Any node of an AC Graph $G$, belongs to at most two cliques.

Lemma 2.2 -- Any induced subgraph of an AC Graph is also an AC Graph.

Lemma 2.3 -- Every leaf clique contains at least one simplicial node.

Lemma 2.4 -- Every AC Graph is chordal.

Some graphs that are not AC graphs.

G:

Gc:

H:

Hc:

Conflicts

TR-pairs: $(x,y)$ and $(w,z)$ conflict iff at least one of the edges $(x,y)$ and $(w,z)$ is in $G$.

Edges: $(x,y)$ and $(w,z)$ conflict iff every orientation of the edges results in an AC.

Lemma 2.5 -- Two distinct edges $(x,y)$ and $(w,z)$ conflict iff the subgraph $S$ of $G$ induced on $(x,y,w,z)$ contains a 3-cycle.
Orientability

A matching $M$ is orientable iff there exists an orientation of $M$ with no conflicts.

$M$ - a matching

$V_M$ - the nodes that comprise $M$

$G_M$ - the subgraph of $G$ induced on $V_M$

Lemma 2.6 - $M$ is orientable iff $G_M$ does not contain a cycle that uses an edge in $M$.

Lemma 2.7 - If no pair of edges in $M$ conflict, then $M$ is orientable.

(Both results require that $G$ be chordal.)

Sketch of the Algorithm

With each clique $C_i$, store PossiblePair($C_i$)
(initial value is TRUE)

{Find a maximum cardinality orientable matching}

Until all nodes have been removed from $G$

(a) choose a leaf clique $C_i$

(b) if PossiblePair($C_i$)

choose a "good" pair from $C_i$

remove from $G$ all nodes that would conflict with the pair

set PossiblePair($C_i$) to FALSE

else remove $C_i$ from $G$

Orient the edges

Nodes in a Clique and Orientable Matchings

Lemma 3.1 - Let $I = C_i \cap V_M$ for clique $C_i$ and orientable matching $M$. Exactly one of following is true:

(a) $I = \emptyset$

(b) for $x, y \in I$, $f(x, y) \in M$ and $I - \{x, y\} = \emptyset$

(c) nodes in $I$ single endpoints pairs in $M$, $f$ in valid orientable labels are same

Lemma 3.3 - If for $x, y \in C_i$, $f(x, y) \in M$ then at most one node from each $C_i \cap C_j$ can be in $I$

Choosing a "good pair"

Lemma 3.1(b) ⇒ no matter what pair, no other nodes in $C_i$ can belong to $V_M$

Lemma 3.1 ⇒ if choose any node from $C_i \cap C_j$ then no pairs come from $C_j$

⇒ if choose two nodes from $C_i \cap C_j$ then no other nodes come from $C_j$

Stay away from $C_j$ as much as possible

Lemma 2.3 ⇒ 3 at least one simplicial node in $C_j$
Example -- Possible Pair (C) = TRUE
C contains > 1 simplicial node

Choose (x, y) and prune G.

Algorithm

1. Construct G_C = (V, E_C) (the clique graph of G)
2. M = ∅
3. for each clique C_i do PossiblePair(C_i) = true
4. while G contains two or more nodes do
   a. c_j = a leaf node of G
   b. C_j = the corresponding leaf clique of G
   c. C_j = the clique that intersects C_i
   d. if PossiblePair(C_j) then
      i. x = a simplicial node in C_j
      ii. if C_j - {x} has a simplicial node then
          y = a simplicial node in C_i - {x}
          P = (all nodes in C_i except one from each C_i ∩ C_j, i ≠ j and i ≠ p)
       else P = ∅
       if PossiblePair(C_j) then
         PossiblePair(C_j) = false
         P = (all nodes in C_i except one from each C_i ∩ C_j, i ≠ j and i ≠ p)
       else P = ∅
      else y = a node in C_j - {x}
      if PossiblePair(C_j) then
        PossiblePair(C_j) = false
        P = (all nodes in C_i except one from each C_i ∩ C_j, i ≠ j and i ≠ p)
      else P = ∅
    iii. Add (x, y) to M
    iv. Delete C_j ∪ P from G and c_j from G
    v. if |C_j - P| = 1 then Delete c_j from G
    else vi. Delete the simplicial nodes of C_i from G
    vii. Delete c_i from G
5. if G is not empty then
   a. z = a simplicial node in G's remaining clique
   b. y = a simplicial node distinct from x in G
   c. Add {x, y} to M
6. Orient the edges of M using AleOrient

Example -- Possible Pair (C) = TRUE
C contains one simplicial node

Choose (x, y) and prune G (Lemma 3.3).
Possible Pair (C) = FALSE

Orientation Step

Given the set M_j and G_M

---

13

---

14

---

15

---

16
AlgOrient

1. Find the connected components of $G_{M}$
2. Shrink non-matching portions of each component
3. Perform Breadth-First-Search on each component labeling nodes on adjacent levels opposite.

Linear Time of AlgTRM

1. finding the set of maximal cliques -- standard technique
2. constructing the clique graph -- use $O(kl^{2} + kel)$ space to construct adjacency lists in $O(kl^{2} + kel)$ time
3. for each leaf clique choose a pair and prune $G$ and $G_{c}$
   - each clique is "the leaf clique" at most once
   - each clique is pruned as a parent clique at most once
   - time required to process a leaf clique (or a parent clique) is proportional to the number of nodes in the clique
   - since $G$ is an $AC$-graph, each node of $G$ belongs to at most two cliques
4. orient the edges in $M$ -- similar to 2-deleting a bipartite graph

Final Remarks

TRM for Chordal Graphs is NP-Complete

Determining if $G$ is an AC-graph

Let $G$ be a chordal graph.
Let $G_{C}$ be its clique graph.

Property: If there is a cycle in $G_{C}$ then there is a node in $G$ that belongs to at least 3 cliques.

Lemma 21: Any node of an AC graph $G$ belongs to at most 2 cliques.

1. Check for chordality of $G$
2. Find maximal cliques of $G$
3. Check to see if any node in $G$ belongs to more than 2 cliques

$O(kl^{2} + kel)$ time
"The Jamison Method in Galois Geometries"

J. Chris Fisher, University of Regina, Canada
Visiting Clemson University, Department of Mathematical Sciences
JAMISON'S METHOD
Joint work with AIDEN BROWN

\[ T = \mathbb{A}(2, q) \] (Affine plane over \( \mathbb{GF}(q) \))

**Definition.**
"\( S \) BLOCKS THE LINES OF \( T \)" :
For every line \( \ell \in T \) there is a \( P \in S \) with \( P \perp \ell \).

**THEOREM (JAMISON, 1977)**
A BLOCKING SET OF \( T \) must contain at least \( 2q^2 - \) points.

**Proof**
Suppose to the contrary that \( |S| \leq 2q^2 - 2 \).

**Step 1**
**REPHRASE THE THEOREM AS A RELATIONSHIP INVOLVING SETS OF POINTS IN \( T \).**

**Original**
\( S \) contains \( 2q^2 \) points.
Every \( \ell \in T \) contains at least one \( P \in S \).

**Dual**
\( S^* \) has \( 2q^2 - 3 \) affine lines.
Every \( P \in T \setminus \{ \ell \} \) lies on at least one \( \ell \in S^* \).

i.e.
\( S^* \) has \( 2q^2 - 3 \) affine lines that cover the nonzero points of \( T \).

**Step 2**
FORMULATE THE THEOREM IN TERMS OF POLYNOMIALS OVER \( \mathbb{GF}(q^2) \).
Let \( S^* \) consist of the lines
\[ l_i = z^q + a_i z + b_i = 0 \quad b_i \neq 0 \]
Assumption:
\[ T \setminus \{ \ell \} \subseteq S^* \]
\[ \exists z^{q^2 - 1} \text{ divides } l_i \]
Or
\[ (z^q + a_i z + b_i) \equiv 0 \pmod{z^{q^2 - 1} - 1} \]

Cf. Projective planes:
\[ |S'| = q + 1 \] (since each pair of lines meet)
Or (when \( S' \) contains no line)
\[ |S'| = q + \sqrt{q} + 1 \]

**Facts needed for the proof:**
1. Points: \((x, y) \mapsto \bar{z} = x + iy \in \mathbb{GF}(q^2)\)
2. Elements of \( \mathbb{GF}(q^2) \setminus \mathbb{GF}(q) \)
3. Field Automorphism of \( \mathbb{GF}(q^2) \) of period 2:
\[ \overline{z} = z^q \] (recall \( z^q = z \))
4. Line:
\[ \overline{z} + a \overline{z} + b = 0 \] (\( = z + a z + b \))
The Flock Theorem.

**GIVEN AN ELLIPTIC QUADRIC IN PG(3,q)**

AND A SET OF \(q-1\) DISJOINT CONICS

PARTITIONING ALL BUT TWO OF ITS

POINTS, THEN THE \(q-1\) PLANES OF THOSE

CONICS MUST CONTAIN A COMMON LINE

THAT MISSES THE QUADRIC.

**Blokhuis’s Theorem.**

FOR \(q\) ODD, IF A \(q\)-ELEMENT SUBSET OF

\(\text{GF}(q^2)\) CONTAINING 0 AND 1 HAS THE

PROPERTY THAT THE DIFFERENCE OF ANY

TWO OF ITS ELEMENTS IS A SQUARE OF

\(\text{GF}(q^2)\), THEN IT IS \(\text{GF}(q)\).

**Step 2. Reformulate**

Set \(C_j = z^{qj+1}a_jz^{q} - a_jz^2 - qj\)

Claim:

\[
\prod_{j=1}^{q-1} C_j = z^{q^{q+1}} - 1 \quad \text{iff all } a_j = 0
\]

**Step 3. Calculate**

\[
\prod_{j=1}^{q-1} (z^{qj+1}a_jz^q - a_jz^2) + \text{at most } j^{(q-1)(q-2)}
\]

**Equate terms of degree \((q+1)(q-2)+1\)**

\[
\prod (z^{qj+1}a_jz^q) = z^{q^{q+1}} \quad \text{iff all } a_j = 0
\]

\[
\prod (z^{qj+1}) = z^{q^{q+1}} \quad \text{iff all } a_j = 0
\]

But \(\text{GF}(q^2)[z]\) is a unique

factorization domain.
"Elementary, Sub-Fibonacci, Regular, Van Lier and Other Interesting Sequences"

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ELEMENTARY, SUB-FIBONACCI, REGULAR, VAN LIER, AND OTHER INTERESTING SEQUENCES

BY

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MEASUREMENT THEORY:

THE THEORY OF MEASUREMENT IS CONCERNED WITH UNDERSTANDING THE CONDITIONS UNDER WHICH MEASUREMENT PROCESSES TAKE PLACE, WHAT KINDS OF SCALES OF MEASUREMENT ONE GETS, AND WHAT KINDS OF STATEMENTS WE CAN MAKE USING SCALES OF MEASUREMENT.

MEASUREMENT THEORY AND COMBINATORICS:

IN THE PAST FEW YEARS, PROBLEMS OF UNIQUENESS OF SCALES OF MEASUREMENT HAVE BEEN GIVING RISE TO A VARIETY OF INTERESTING SEQUENCES OF POSITIVE INTEGERS WITH FASCINATING COMBINATORIAL PROPERTIES.

THIS TALK:

IN THIS TALK, I MENTION SUCH SEQUENCES AND DISCUSS THEIR COMBINATORIAL PROPERTIES. BECAUSE OF THE SHORTNESS OF TIME, I CANNOT DESCRIBE THE MEASUREMENT THEORY MOTIVATION EXCEPT IN ONE CASE AND I SHALL CONCENTRATE ON JUST ONE COMBINATORIAL PROBLEM: COUNTING THE NUMBER OF SEQUENCES OF DIFFERENT KINDS.

REFERENCES


FISHERBUM AND ROBERTS, "ELEMENTARY SEQUENCES, SUB-FIBONACCI SEQUENCES," TECHNICAL REPORT 90-51, DIMACS CENTER.


FISHERBUM AND ROBERTS, "UNIQUENESS IN FINITE MEASUREMENT," IN P.S. ROBERTS (ED.), APPLICATIONS OF COMBINATORICS AND GRAPH THEORY TO THE BIOLOGICAL AND SOCIAL SCIENCES, IMA VOL. 17, SPRINGER-VERLAG, NEW YORK, 1990, 105-137. (SURVEY)


THE FIBONACCI SEQUENCE

\[ F_1, F_2 = - \]
\[ F_1 = F_2 = 1 \]
\[ F_K = F_{K-1} + F_{K-2} \quad K = 3, 4, \ldots \]

\[ F_1, F_2 = \text{is the sequence } 1, 1, 2, 3, 5, 8, \ldots \]

ELEMENTARY SEQUENCES


VARIATION ON THE FIBONACCI SEQUENCE

\[ X_1, X_2 = - \]

POSITIVE, NONDECREASING INTEGER SEQUENCE

\[ X_1 = X_2 = 1 \]
\[ X_K > 1 \quad \text{some } i \neq j \]

EXAMPLES: 1, 1, 2, 3, 5
1, 1, 2, 3, 4, 6

\[ \mathcal{E}_N = \text{collection of all elementary sequences of length } N \]

THEOREM (FISHBURN AND ROBERTS 1989):

\[ |\mathcal{E}_N| = \alpha^{N(1+o(1))}/2 \]

WHERE

\[ \alpha = (1+\sqrt{5})/2 = 1.61803 \quad \text{(THE GOLDEN SECTION)} \]

AND \( o(N) \) IS A FUNCTION OF \( N \) THAT APPROACHES 0 AS \( N \) APPROACHES \( \infty \).

COROLLARY: THE SAME ESTIMATE HOLDS FOR \( |\mathcal{E}_N| \)

WHERE \( \mathcal{E}_N \) IS THE SUBSET OF \( \mathcal{E}_N \) WHOSE ELEMENTS STRICTLY INCREASE FROM \( K = 2 \) ON.

OPEN QUESTION: FIND A SIMILAR RESULT FOR \( |\mathcal{E}_N| \).

THEOREM (FISHBURN AND ROBERTS 1989):

\[ |\mathcal{E}_N|/|\mathcal{E}_N| = 0 \quad \text{as } N \rightarrow \infty \]

AND HENCE \( |\mathcal{E}_N|/|\mathcal{E}_N| = 0 \) AS \( N \rightarrow \infty \).

SOME COUNTS:

\begin{align*}
N & \quad 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
|\mathcal{E}_N| & 2 & 4 & 10 & 31 & 120 & 578 & 3277 & 22504 & 205744 \\
|\mathcal{E}_N| & 2 & 4 & 10 & 31 & 120 & 578 & 3277 & 22504 & 205744 \\
|\mathcal{E}_N| & 1 & 1 & 2 & 6 & 27 & 177 & 1353 & 22504 & 451088 \\
\end{align*}

SO \( |\mathcal{E}_N| \) DOES NOT EXCEED \( |\mathcal{E}_N| \) UNTIL \( N = 11 \).

THE-FIBONACCI SEQUENCES

TERM INTRODUCED BY FISHBURN AND ROBERTS (1989)

\[ X_1, X_2 = - \]

POSITIVE, NONDECREASING INTEGER SEQUENCE

\[ X_1 = X_2 = 1 \]
\[ X_K = X_{K-1} + X_{K-2} \]

\[ \mathcal{R}_N = \text{collection of sub-fibonacci sequences of length } N \]

NOTE: \( \mathcal{E}_N \neq \mathcal{R}_N \) FOR ALL \( N \).

BY ENUMERATION, \( \mathcal{E}_N = \mathcal{R}_N \) FOR \( N = 0 \).

THE SMALLEST SUB-FIBONACCI SEQUENCE WHICH IS NOT ELEMENTARY HAS LENGTH 5: 1, 1, 2, 3, 4, 7

THIS IS NOT ELEMENTARY BECAUSE 7 IS NOT THE SUM OFANY TWO PRECEDING TERMS \( (9 \) HAS SEVEN SEQUENCES NOT IN \( \mathcal{E}_N \).

REGULAR SEQUENCES

MOTIVATION: "SUBJECTIVE PROBABILITY" MEASUREMENT INTRODUCED BY: FISHBURN AND ODLYZKO 1989

\[ X_1, X_2 = - \]

POSITIVE, NONDECREASING INTEGER SEQUENCE

\[ X_1 = X_2 = 1 \]
\[ X_K = \sum_{i=1}^{K-1} X_i \quad J = 3, 4, \ldots \]

EXAMPLES:

\begin{align*}
N & \quad 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
|\mathcal{R}_N| & 1 & 2 & 6 & 27 & 182 & 2280 & 67297 & \]
\end{align*}

SOME COUNTS:

\begin{align*}
N & \quad 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
|\mathcal{N}_N| & 1 & 2 & 6 & 27 & 182 & 2280 & 67297 & \]
\end{align*}
THE FOLLOWING THEOREM WILL BE USEFUL LATER.

**Theorem (Fisburn and Roberts 1963)** Suppose \( X_1, X_2, \ldots \) is a positive, nondecreasing integer sequence with \( X_1 = X_2 = 1 \). Then \( X_1, X_2, \ldots \) is a regular sequence if and only if for every \( i \), there is a set \( S \subset \{1, 2, \ldots, N\} \) such that

\[
X_j = \sum_{i \in S} X_i
\]

For example, in \( 1, 1, 2, 3 \):
- \( 2 \) is \( 1 + 1 \) and \( 3 \) is \( 1 + 2 \).

It is clear that condition \((*)\) implies regularity since it implies that \( X_1, X_2, \ldots \); the converse is harder.


Positive, nondecreasing integer sequence

\[
X_1 = X_2 = 1 \\
X_j = \sum_{i=1}^{j-1} X_i 
\]

For example: \( 1, 1, 2, 4 \)

\[
4 - 2 = 1 + 1 \\
4 - 1 = 1 + 2 \\
2 - 1 = 1 
\]

Example: \( 1, 1, 2, 3, 5, 6 \) (\( F_1, F_2, F_3, F_4, F_5, F_6 \))

\[
8 - 5 = 3, 8 - 3 = 5, 8 - 7 = 1 + 5, 8 - 1 = 2 + 5, \\
5 - 3 = 2, \text{ etc.}
\]

It is easy to see that every initial subsequence of the Fibonacci sequence is Van Lier.

The following result is quite a bit harder:

**Theorem (Fisburn, Roberts, & Marcus-Roberts 1963)** \( X_1, X_2, \ldots \) are every sub-Fibonacci sequence is Van Lier.

Not every regular sequence is Van Lier. The smallest regular sequence which is not Van Lier is \( 1, 1, 2, 4, 5 \).

\( 5 - 2 \) is not a sum of terms from \( (1, 1, 4) \).

\( X_k \) is the collection of Van Lier sequences of length \( k \).

**Open Question:** Find an asymptotic formula for \( |X_k| \).

**Conjecture (Fisburn and Roberts 1963)** \( |X_k|/|X_{k+1}| \to 0 \) as \( k \to \infty \).

We have shown that \( |X_k|/|X_{k+1}| \leq 1 \) for some constant \( A < 1 \) \( (A = 0.9 \text{ suffices}) \).

**Theorem (Fisburn, Roberts, and Marcus-Roberts 1963)** Every regular sequence without gaps is a Van Lier sequence.
Theorem (Dedekind, Roberts, and Marcus-Roberts):

Suppose \( X_1, X_2, \ldots, X_N \) is a Van Lier sequence.

Then:

1. Every one-term extension of \( X_1, X_2, \ldots, X_N \) to a regular sequence is Van Lier if and only if \( X_1, X_2, \ldots, X_N \) has no gaps.

2. If \( X_1, X_2, \ldots, X_N \) has a gap at \( X_j \), then \( X_1, X_2, \ldots, X_N \) \( Y = X_j + T + \sum_{i=j+2}^{N} X_i \)

Then \( X_1, X_2, \ldots, X_N, Y \) is a one-term regular extension which is not Van Lier.

Example 1: \( 1, 1, 2, 3, 6 \) is Van Lier and there is a gap at \( X_2 = 2 \). Then \( \sum_{i=1}^{N} X_i = 0 \), take \( T = 5 \) and \( Y = X_2 + T + \sum_{i=j+2}^{N} X_i = 2 + 5 + 0 = 7 \). The sequence \( 1, 1, 2, 3, 6, 7 \) is regular but not Van Lier. (Note that \( 7 - 3 \) is not a sum of other terms.)

Example 2: Suppose \( N = 4 \) and > is defined by the following:

\[
\begin{align*}
(1) & \rightarrow (1, 2) \\
(2) & \rightarrow (1, 3) \\
(3) & \rightarrow (1, 4)
\end{align*}
\]

An agreeing probability measure is given by

\[
\begin{align*}
P(\{1\}) &= 1/10, \\
P(\{2\}) &= 2/10, \\
P(\{3\}) &= 3/10, \\
P(\{4\}) &= 4/10
\end{align*}
\]

With the rest of \( P \) defined by finite additivity. This uniquely agrees because it is the unique solution to the system of equations

\[
\begin{align*}
P(\{1\}) &= P(\{1\}) + P(\{2\}) \\
P(\{2\}) &= P(\{1\}) + P(\{3\}) \\
P(\{3\}) &= P(\{1\}) + P(\{4\})
\end{align*}
\]

Aside: Connection to Measurement Theory

One of the most interesting problems in the theory of measurement concerns subjective judgements about probabilities. Let \( \mathcal{F} \) be the set of elements of the finite Boolean algebra consisting of all subsets of \( \{1, 2, \ldots, N\} \). Set \( \{1\} \) is called an atom of \( \mathcal{F} \).

Let \( > \) be a binary relation on \( \mathcal{F} \), with \( A > B \) interpreted to mean that \( A \) is judged subjectively more probable than \( B \). We say a (finitely additive) probability measure \( P \) on \( \mathcal{F} \) agrees with \( > \) if

\[
A > B \iff P(A) > P(B)
\]

For all \( A, B \) in \( \mathcal{F} \), it is a very old question of measurement theory to understand conditions on the binary relation \( (\mathcal{F}, >) \) under which it agrees with some probability measure. The measure is said to agree uniquely if it is the only agreeing probability measure.

Example 1: Suppose \( N = 3 \) and \( P \) is defined by

\[
\begin{align*}
P(\{1\}) &= 1/6, \\
P(\{2\}) &= 2/6, \\
P(\{3\}) &= 3/6
\end{align*}
\]

But this is not unique, since a second agreeing probability measure is given by

\[
\begin{align*}
P(\{1\}) &= 2/10, \\
P(\{2\}) &= 3/10, \\
P(\{3\}) &= 5/10
\end{align*}
\]

Example 2: Suppose \( N = 2 \) and \( P \) is defined by

\[
(1) > (2)
\]

Then there are infinitely many agreeing probability measures \( P \), with

\[
P(\{1\}) = a, \quad P(\{2\}) = 1-a
\]

For any \( a \) with \( 1/2 > a > 1/2 \).
Let us take the unique solution (*) in the first example. We can translate this into a sequence of positive integers \( 1, 2, 3, 4 \) (with no common divisor) by multiplying by the denominator 10.

Conversely, any finite sequence of positive integers can be thought of as a sequence of probabilities by normalizing, i.e., by dividing each element by the sum of elements in the sequence. Let us call a nondecreasing sequence of positive integers with no common divisor which arises from a binary relation \( \succ \) by finding a uniquely agreeing probability measure a unique probability sequence. Thus, \( 1, 2, 3, 4 \) is a unique probability sequence while \( 1, 2, 3 \) is not.

Fisheburn and Olszewski (1989) prove that all regular sequences are unique probability sequences. However, not all unique probability sequences are regular. For instance, \( 1, 2, 3, 4 \) is not regular; we do not have \( x_2 = 1 \).

This corresponds to the subjective probability constraints

\[
P(\{2\}) = P(\{3\})
\]
\[
P(\{1,2\}) = P(\{2,3\})
\]
\[
P(\{1,4\}) = P(\{2,3\})
\]

This sequence \( 1, 2, 3, 3 \) is again irregular since \( x_3 \neq 1 \). It turns out that \( 1, 2, 3, 4 \) and \( 1, 2, 3, 3 \) are the only irregular unique probability sequences of length 4.

However, there are 7 irregular unique probability sequences of length 5, including \( 1, 1, 3, 3, 3 \) and \( 2, 2, 2, 2, 2 \). The former is interesting. It is a unique probability sequence since it is the solution to the \( 5 - 1 = 4 \) linearly independent equations

\[
\begin{align*}
x_1 &= x_2 \\
x_3 &= x_4 \\
x_1 + x_2 + x_3 &= x_5 \\
x_1 + x_4 &= x_1 + x_5
\end{align*}
\]

It is not regular since \( x_5 \) is not less than or equal to the sum of the previous terms in the sequence. \( 1 + 3 \).

Theorem (Fisheburn and Olszewski 1989): A nondecreasing sequence \( x_1, x_2, \ldots, x_N \) of positive integers with no common divisor is a unique probability sequence if and only if it is the solution to \( N - 1 \) linearly independent equations of the form

\[
\begin{align*}
x_1 &= x_2 \\
x_3 &= x_1 + x_3 \\
x_4 &= x_1 + x_3 \\
x_5 &= x_1 + x_5
\end{align*}
\]

where \( 5, 7 \subseteq \{1, 2, \ldots, N\} \) and \( 5 \neq 7 \).

For instance, the sequence \( 1, 2, 2, 3 \) is a unique probability sequence because it is the solution to the \( 4 - 1 = 3 \) linearly independent equations

\[
x_3 = x_2 + x_3 \\
x_4 = x_1 + x_2 \\
x_2 + x_3 = x_1 + x_4
\]

Also, \( 1, 2, 2, 3 \) is a unique probability sequence since it is the solution to the \( 4 - 1 \) linearly independent equations

\[
x_3 = x_2
\]

Let \( \mathcal{R}_N \) be the collection of unique probability sequences of length \( N \). Recall that \( \mathcal{R}_N \) is the collection of regular sequences of length \( N \).

Theorem (Fisheburn and Olszewski 1989): \( |\mathcal{R}_N| / |\mathcal{R}_N| \rightarrow 0 \) as \( N ightarrow \infty \).

We do not know much about \( |\mathcal{R}_N| \). However, we have the following upper bound:

Theorem (Fisheburn and Olszewski 1989): \( |\mathcal{R}_N| \leq 3^N [1 + o(1)] \).

Where do regular and van der Waerden sequences arise?

The problem of finding conditions under which there is a finitely additive probability measure which agrees with a given binary relation "subjectively more probable than" \( (\mathcal{R}, \succ) \) is an old problem. Some necessary conditions were stated by Bruno de Finetti in 1931. Define \( A \succ B \) to mean that either \( A > B \) or \( A = B \).
DE FINETTI AXIOMS

AXIOM A1. \( \geq \) IS TRANSITIVE AND COMPLETE (\( A \geq B \) OR \( B \geq A \) FOR ALL \( A, B \) IN \( \mathcal{A}_0 \))

AXIOM A2. \( \mathcal{A}(1, 2, \ldots, N) > \dagger \)

AXIOM A3. \( A \geq \dagger \)

AXIOM A4. IF \( (A \cap B) \cap C = \dagger \), THEN \( A \geq B \) IFF \( A \cup C \geq B \cup C \)

IT IS EASY TO SEE THAT THESE FOUR AXIOMS ARE NECESSARY FOR THE EXISTENCE OF AN AGREEING PROBABILITY MEASURE. IT WAS SHOWN BY KRAFT, PRATT, AND SEIDENBERG IN 1969 THAT THEY ARE NOT SUFFICIENT.

VARIOUS CONDITIONS CAN BE ADDED TO THESE AXIOMS TO GIVE SUFFICIENT CONDITIONS. ONE SIMPLE CONDITION WAS ADDED BY KRAFT, PRATT, AND SEIDENBERG IN 1969. IT SAYS THAT IF \( A_1, A_2, \ldots, A_\lambda, B_1, B_2, \ldots, B_\mu \) ARE CHOSEN FROM \( \mathcal{A}_0 \), IF EVERY ATOM IS INCLUDED IN AS MANY \( A_j \) AS \( B_j \), AND IF \( A_j > B_j \) FOR \( j = 1, 2, \ldots, M \), THEN \( B_\mu > A_M \).


AXIOM UJ: suppose \( X \) IS AN ATOM SUCH THAT \( X > Y \) FOR SOME ATOM \( Y \). THEN THERE IS AN EVENT \( A(X) \) IN \( \mathcal{A}_0 \) SO THAT \( X \in A(X) \) AND \( X > Y \) FOR EVERY ATOM \( Y \) IN \( A(X) \).

(PUT ANOTHER WAY, THE CONCLUSION OF THIS AXIOM SAYS THAT THERE ARE ATOMS \( Y_1, Y_2, \ldots, Y_K \) SO THAT \( X = Y_1 \cup Y_2 \cup \ldots \cup Y_K \) AND \( X > Y_1 \) \( I = 1, 2, \ldots, K \).)

THIS IS RELATED TO THE FISHEBURN-ODLYZKO CHARACTERIZATION OF REGULAR SEQUENCES AS NONDECREASING SEQUENCES OF POSITIVE INTEGERS SUCH THAT \( X_1 = X_2 = 1 \) AND SUCH THAT EACH \( X_j \) IS A SUM OF OTHER \( X_i \) \( i \neq j \).

THEOREM (FISHEBURN AND ROBERTS 1969): GIVEN \( \mathcal{A}_0 \), THE DE FINETTI AXIOMS PLUS AXIOM U1 IMPLY THAT THERE IS AN AGREEING PROBABILITY MEASURE AND IT IS UNIQUE.

THEOREM (FISHEBURN AND ROBERTS 1969): A NONDECREASING SEQUENCE OF POSITIVE INTEGERS WITH NO COMMON DIVISOR DEFINES A REGULAR SEQUENCE IF AND ONLY IF IT IS A UNIQUE PROBABILITY SEQUENCE AGREEING WITH A BINARY RELATION \( \mathcal{A}_0 \), WHICH SATISFIES AXIOM U1.

ANOTHER AXIOM IS DUE TO VAN LIER [1969].

AXIOM U2: FOR EVERY \( i, j \in \{1, 2, \ldots, N\} \), IF \( i > j \), THERE IS \( C \in \mathcal{A}_0 \) SUCH THAT \( i \in C \cup B \).

THEOREM (VAN LIER 1969): GIVEN \( \mathcal{A}_0 \), THE DE FINETTI AXIOMS PLUS AXIOM U2 IMPLY THAT THERE IS AN AGREEING PROBABILITY MEASURE AND IT IS UNIQUE.

THEOREM (FISHEBURN AND ROBERTS 1969): A REGULAR SEQUENCE IS A VAN LIER SEQUENCE IF AND ONLY IF IT IS A UNIQUE PROBABILITY SEQUENCE AGREEING WITH A BINARY RELATION \( \mathcal{A}_0 \), WHICH SATISFIES AXIOM U2.

THEOREM (FISHEBURN AND ROBERTS 1969): A NONDECREASING SEQUENCE OF POSITIVE INTEGERS WITH NO COMMON DIVISOR DEFINES A REGULAR SEQUENCE IF AND ONLY IF IT IS A UNIQUE PROBABILITY SEQUENCE AGREEING WITH A BINARY RELATION \( \mathcal{A}_0 \), WHICH SATISFIES AXIOM U1.

TYPE AI TWO-STAGE GENERALIZED FIBONACCI SEQUENCES


THIS IS A SEQUENCE OF POSITIVE INTEGERS WHICH, IN CONTRAST TO ALL THE TYPES OF SEQUENCES SO FAR, MAY BE DECREASING.

START WITH A PAIR OF ADJACENT IS.

CONSTRUCT THE SEQUENCE INSIDE-OUT BY ADDING ONE TERM AT A TIME WHERE VALUE IS A SUM OF ONE OR MORE CONTIGUOUS TERMS IMMEDIATELY ADJACENT TO THE NEW TERM.

EXAMPLE: \( 4, 4, 1, 1, 2, 4 \)


AXIOM UJ: suppose \( X \) IS AN ATOM SUCH THAT \( X > Y \) FOR SOME ATOM \( Y \). THEN THERE IS AN EVENT \( A(X) \) IN \( \mathcal{A}_0 \) SO THAT \( X \in A(X) \) AND \( X > Y \) FOR EVERY ATOM \( Y \) IN \( A(X) \).

(PUT ANOTHER WAY, THE CONCLUSION OF THIS AXIOM SAYS THAT THERE ARE ATOMS \( Y_1, Y_2, \ldots, Y_K \) SO THAT \( X = Y_1 \cup Y_2 \cup \ldots \cup Y_K \) AND \( X > Y_1 \) \( I = 1, 2, \ldots, K \).)

THIS IS RELATED TO THE FISHEBURN-ODLYZKO CHARACTERIZATION OF REGULAR SEQUENCES AS NONDECREASING SEQUENCES OF POSITIVE INTEGERS SUCH THAT \( X_1 = X_2 = 1 \) AND SUCH THAT EACH \( X_j \) IS A SUM OF OTHER \( X_i \) \( i \neq j \).

THEOREM (FISHEBURN AND ROBERTS 1969): GIVEN \( \mathcal{A}_0 \), THE DE FINETTI AXIOMS PLUS AXIOM U1 IMPLY THAT THERE IS AN AGREEING PROBABILITY MEASURE AND IT IS UNIQUE.
THE COLLECTION OF (TYPE A) TWO-SIDED GENERALIZED FIBONACCI SEQUENCES OF LENGTH N.

**Theorem (Fishburn, Marcus-Roberts, and Roberts 1991 and Fishburn, Odlyzko, and Roberts 1992):**
\[ |a_N| = \frac{N^{(1+o(1))}}{2} \]

In fact, Fishburn, Odlyzko, and Roberts show that
\[ \alpha_N = K \frac{N^{(1+o(1))}}{x^{(1+o(1))}} \]

where
\[ K = e^{-1} - \int_0^1 \frac{\log(1+y)}{1+y} dy = 0.45439... \]

**TYPE B TWO-SIDED GENERALIZED FIBONACCI SEQUENCES**

**Motivation: "Difference" Measurement**

Same as type A two-sided generalized Fibonacci sequences with the new term being a sum of one or more contiguous terms, but not necessarily of terms immediately adjacent to the new term.

**Example:** 0, 1, 1, 2, 4

This is built up as:
1. 1, 2
2. 1, 1, 2, 4
3. 0, 1, 2, 4

This is not attainable if we insist that each new term is a sum of terms immediately adjacent to the new one.

\[ |\beta_N| = \frac{N^{(1+o(1))}}{x^{(1+o(1))}} \]

**TYPE C TWO-SIDED GENERALIZED FIBONACCI SEQUENCES**

**Motivation: "Difference" Measurement**

Same as type A two-sided generalized Fibonacci sequences with the new term being a sum of one or more previous terms, but not necessarily of contiguous terms and not necessarily of terms immediately adjacent to the new term.

**Example:** 3, 1, 1, 2, 6

This is built up as:
1. 1, 2
2. 1, 1, 2
3. 1, 1, 2, 6

It cannot be built up by adding contiguous terms each time, since 3 can only be obtained as 1 + 1 + 2.

\[ |\gamma_N| = \frac{N^{(1+o(1))}}{x^{(1+o(1))}} \]

**THE FOLLOWING COUNTS ARE KNOWN:**

\[
\begin{array}{cccccc}
N & 2 & 3 & 4 & 5 & 6 \\
|a_N| & 1 & 3 & 14 & 83 & 628 \\
|\beta_N| & 1 & 3 & 18 & 172 & 2433 \\
|\gamma_N| & 1 & 3 & 16 & 185 &
\end{array}
\]

All other values are still open.
**TIEFELERO SEQUENCES**

**MOTIVATION:** "CONJOINT" MEASUREMENT

INTRODUCED BY FISHERBURN AND ROBERTS (1985)

*THESE ARE TWO-BLOCK SEQUENCES OF POSITIVE INTEGERS *

\[
X_1, X_2, \ldots, X_M / Y_1, Y_2, \ldots, Y_N
\]

*THEY ARE BUILT UP BY STARTING WITH 1 IN EACH BLOCK AND ADDING ONE TERM AT A TIME (TO EITHER BLOCK) THAT IS ADJACENT TO THE TERMS ALREADY SPECIFIED FOR THE BLOCK AND WHOSE VALUE EQUALS A SUM OF TERMS ALREADY SPECIFIED FOR THE OTHER BLOCK.*

**EXAMPLE:**

\[
2, 3, 1, 7 / 6, 1, 2, 10
\]

*Built up as:

1 / 1
1, 1 / 1
1, 1 / 1, 2
3, 1, 1 / 1, 2
2, 3, 1, 1 / 1, 2
2, 3, 1, 1 / 6, 1, 2
2, 3, 1, 1, 7 / 6, 1, 2
2, 3, 1, 1, 7 / 6, 1, 2, 10

**INTERVAL-RESTRICTED BIREGULAR SEQUENCES**

**MOTIVATION:** "CONJOINT" MEASUREMENT

SAME AS BIREGULAR SEQUENCES BUT ADD THE REQUIREMENT THAT EACH TERM IS A SUM OF CONTIGUOUS TERMS ALREADY SPECIFIED IN THE OTHER BLOCK.

**NOTE:** THE LAST EXAMPLE WORKS UP UNTIL THE LAST STEP. IN THE LAST STEP, 10 CANNOT BE ADDED. HOWEVER, 14 COULD BE, GIVING US

\[
2, 3, 1, 7 / 6, 1, 2, 14
\]

\[
G_B(M,N) = \text{THE COLLECTION OF INTERVAL-RESTRICTED BIREGULAR SEQUENCES OF LENGTHS M AND N.}
\]

**THEOREM (FISHERBURN AND ROBERTS 1985):**

\[
|G_B(M,N)| = \left( \begin{array}{c} N+1 \\ M+1 \end{array} \right)
\]
"Generating k-element Subsets of an n-element Set"

Michael S. Jacobson, University of Louisville
Department of Mathematics, Louisville, KY
Generating \( k \)-element subsets of an \( n \)-element set with DeBruijn Graphs

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Abstract

This tour is brought to you by the following problem:

Find an efficient way to generate all \( k \)-element subsets of an \( n \)-element set.

What does efficient mean?

Generate all subsets of an \( n \)-element set.

Binary representation of \( 0 - (2^n - 1) \) yields a "computer understandable" way to generate these sets.

BUT

An excessive amount of work for the computer to go from

\[ 2^{n-1} - 1 = 011...1 \] to \[ 2^{n-1} = 100...0 \]

Is there a sequence which proceeds from subset to subset without many elements being exchanged?

Proceed thru the subsets with exactly one "bit" changing, either 0 to 1, or 1 to 0.

One element difference from subset to subset.

GRAY CODES

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
\end{array}
\]
Utilize the power of the computer!!

Can we generate all binary sequences of length $n$ by a "shift"??

001011101
010111010
101110101
011101010
110101011
101010111

Does there exist a sequence of length $2^n$ so that each sequence of length $n$ occurs as a consecutive subsequence exactly once (wrapping allowed)??

Efficient way to store all subsets of an $n$ element set.

Efficient way to generate all subsets.

Efficient way to generate all subsets many times.

Efficient way to generate a random subset.

Connected

Eulerian Digraph (Graph)
Starting at any point, trace thru the digraph (graph) traversing each edge exactly once.

Euler (1736) Good (1936) If $D$ is a connected digraph with $id(x) = od(x)$ for all vertices $x$ then $D$ is Eulerian.
Do there exist circular sequences of length $2^n$ with each $n$ sequence occurring exactly once?

\[ \text{Double Shift Registers} \]

\[ \begin{align*}
1101001101000 \\
010010100011 \\
0110001100101 \\
01010001100100 \\
00011010001 \\
\end{align*} \]

...?

Does there exist a circular sequence of order $2^k$ so that by double shifting all $k$ element subsets are generated?

... and now a message from the sponsor

"But we only want the $k$ element subsets!!??"

Generate all the subsets, and use only the ones you need.

Can we use shift registers??
For each vertex \( x \) in this digraph, either 
\[ id(x) = od(x) = 2 \text{ (# 1's is } k-1 \text{)} \] or 
\[ id(x) = od(x) = 1 \text{ (#1's is } k \text{ or } k-2 \text{)}. \]

Euler (1736) Good (1946) If \( D \) is a connected digraph with \( id(x) = od(x) \) for all vertices \( x \) then \( D \) is Eulerian.

Good's Thm says Eulerian provided it is connected ...

Consider a different graph ...

Hence, for \( n \) odd, both graphs are connected and the graphs are Eulerian and Hamiltonian, respectively.

For \( n \) even

\[
\begin{array}{cccccccccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Cycle thru from component to component
Count the components:

Polya's Thm (Burnside) Let $G$ be a group of permutations acting on $A$, and let $S$ be the equivalence relation on $A$ induced by $G$

$$\# E.C. = \frac{1}{|G|} \sum_{\pi \in G} \text{inv}(\pi)$$

$G = \mathbb{Z}_2$ -
$A = \text{"family" of k element subsets}
with (10 = 01),
Equivalence Classes = Components of $G$. 

Example
3-element subsets of an 8-element set.

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$$

One pair = 11 and one pair = 01
or
Three pairs = 01
$|A| = (12 + 4) = 16.$

$\# \text{ Components} = \frac{1}{3} (16 + 0 + 0 + 0) = 4$

What's really the story ??

"We need the $k$ - element subsets of an $n$ - element set !!"

When $n$ is odd, find the cycle, and generate the sets...

When $n$ is even, find the $k$ element subsets of an $n+1$ element set, throw out the subsets with $n+1$. 

Good's result is an existence Theorem.

How do you find the Eulerian Circuit ??
For DeBruijn Sequences...

Fredrickson has given a "linear" (in $n$) algorithm to generate the sequence.

For these generalized DeBruijn Sequences,

- worst case $O(n^k)$
- average case $O(\log n)$
- Best Possible ??

UPDATE!!

Hochberg, Hurlbert and Isaak have also discovered the idea of multiple shifting.

Carla Savage uses this idea to generate "new" Grey Codes...

The idea "works" to generate the $n!$ permutations of an $n$ set.
"Matchings in the Partition Lattice"

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Chain decompositions in the Boolean lattice $B_n$, $n=3$, base on the equation:

$$\{x\} = (\{\phi\}) + (\{\phi\}).$$

For example:

- $\{1,2,3\}$
- $\{1,2,\phi\}$
- $\{2,3\}$
- $\{2,\phi,3\}$
- $\{1,3\}$
- $\{1,\phi,3\}$
- $\{\phi\}$
- $\{1\}$

Partitioning of the lattice:

Red Gaps:

- $\{1,3\}$

White of Georgia:

- $\{1\}$

Theorem:

$$|T_{\alpha,\delta}| = S(n,k)$$

Proof:

- $\sum_{k=0}^{n} S(n,k) = \sum_{k=0}^{n} S(n,k) + S(n,k)$

Hence:

- $S(n,k) = \sum_{k=0}^{n} S(n,k)$

Corollary:

- $E \subseteq T_{\alpha,\delta}$

Theorem:

$$d(\alpha) = \sum_{i=1}^{m} \alpha_i$$

Partitioning $\pi$ into $n$ parts: $\pi = \{B_i\}$

$$\pi = \{B_i\}$$

$B_i \cap B_j = \emptyset$ for $i \neq j$
Construct chains using partition recursion. **Claim**: There is a sharp threshold
\[ L_n \text{ s.t. } \Pi_{n,k} \leftrightarrow \Pi_{n+1,k} \iff q < L_n. \]

First, recall the **Phillips Hall Criteria**
\[ X \leftrightarrow Y \iff \forall S, |S| \leq d_E(S) \]
\[ E \subseteq X \rightarrow Y \]

LMFH: \[ d_E(x) \geq d_E(y) \Rightarrow x \rightarrow y. \]

Stirling numbers of the second kind

Proof that: \( L_{n,k} = L_n + \left[ \frac{k}{2} \right] \).

\[ S \in \Pi_{n,k}, \quad k < L_n \]
\[ |S| = \sum_{j=0}^{k} |S_j| , \]
\[ S_j \subseteq \Pi_{n,k}, \]
\[ S_j \in \Pi_{n,k}, \]
\[ d_E(S) = \sum_{j=0}^{k} d_E(S_j) \]
\[ \geq \sum_{j=0}^{k} |S_j| \]
\[ = |S|. \]

That's half of the proof.
Claim \[ \exists R_n \text{ s.t. } \Pi_{n,k} \rightarrow \Pi_{n,k+1}, \]
\[ \Rightarrow k > R_n. \]

Proof Let \( A \subseteq \Pi_{n,k} \cup \Pi_{n,k+1} \) be an antichain, \( \omega \lor k > R_n + 1 \). Want to show \( |A| \leq S(n, k+1) \).

For each \( B \subseteq \{1, \ldots, n\} \), let \( A_B \) be those partitions \( A \) of \( \{1, \ldots, n\} \) s.t. \( \pi \cup \{ n + 1 \} \in A \).

\[ A_B \subseteq \Pi_{n,1, \ldots, 1, k} \cup \Pi_{n,1, \ldots, 1, k+1}, \]
and \( A_B \) is an antichain.

Hence, \( |A_B| \leq S(n, k+1) \), and
\[ |A| = \sum_B |A_B| \]
\[ \leq S(n, k+1) + \sum_{B \neq \emptyset} S(n, 1, \ldots, 1, k+1) \]
\[ = S(n, k+1) + (k+1) S(n, k+1) \]
\[ = S(n, k+1) \]

Theorem There exist monotonically increasing sequences \( L_n \) and \( R_n \) such that
\[ \Pi_{n,k} \rightarrow \Pi_{n,k+1} \iff k < L_n \]
\[ \Pi_{n,k} \rightarrow \Pi_{n,k+1} \iff k > R_n \]

As \( n \) increases by 1, each of \( L_n, R_n \) grows by at most 1.

Fact 1 Fix \( \varepsilon > 0 \). If \( n \) is sufficiently large and \( k \geq (1 + \varepsilon) n \log^2 \), then \( \Pi_{n,k} \rightarrow \Pi_{n,k+1} \). Hence, \( R_n \leq (1 + \varepsilon) n \log^2 / \log n \).

Proof Curiously, we throw away edges. Take \( \delta = \varepsilon / 2 \); define
\[ k = \left\lfloor (1 - \frac{\delta}{2}) \frac{\log n}{\log 2} \right\rfloor. \]

Let \( E = (y, z) \in \Pi_{n,k+1} \setminus \Pi_{n,k} \) s.t. \( x \) is obtained from \( y \) by splitting a block of size \( \leq 2k \).

Say \( x \in \Pi_{n,k} \); since
\[ k (k + 1) \geq \left( 1 + \frac{\delta}{2} \right) n, \]
\[ \left( 1 - \left( 1 + \frac{\delta}{2} \right) \right) k \geq \frac{2\delta}{3} k \] blocks of
\[ x \text{ are of size } k \leq l. \text{ If } \delta k \geq 15, \]
\[ d_E(x) \geq \frac{1}{45} \delta^2 k^2. \]

On the other hand,
\[ d_E(y) \leq k \delta^2 l \]
\[ \leq k n^{-l/2}. \]

Hence, if
\[ \frac{\delta^2}{45} (l + 8) \log 4 / \log 2 n \leq n^{-l/2}, \]
\[ d_E(z) \leq d_E(y). \]

How far from the truth might these two bounds be?

\[ \text{Then } \frac{\ln n}{\log 2 n} \rightarrow \log 2 \]
\[ \frac{\ln n}{\log 4 n} \rightarrow \log 4. \]

Further research:

1. For what \( n \) do we have
\[ |\{ L_n, K_n, R_n\}| = 2 \]
(\( \text{Let } n_0 = \text{smallest} \))

2. \( \max \{ |A| : A \in \mathcal{T}_n, \text{antisymmetric} \} \approx ? \)

3. other

Given sequence \((x; k)\) w. \( n \to \infty \),
\[ k = \beta n / \log n, \quad \beta > \log 2 : \]

\[ \text{exhibit } n, k, m \text{ s.t.} \]

\( (1) \) estimation procedure works

\( (2) \) some \( r \) usable for \( A, C, C_m \)

\( (3) \) all three \( \sigma^2 \)'s are \( \sim \)

Find
\[ \frac{|C| \cdot |C_m|}{|A|} \sim \frac{g(n)}{2^{z+1}} + \frac{g(n)^2}{(k-2) \cdot (k-4) \cdot \cdots \cdot \frac{k}{4}} \to 0. \]
Say \( k \leq (1 - \delta) n \log n / \log n \).

Let \( m = \lfloor (1 + \epsilon) \log n / \log n \rfloor \).

\[ S = \{ \pi \in \Pi_{r,k} : \]
\[ \text{maximize} (m) \leq \{ m/2m, 2m, 3m, \ldots \} \]
\[ \leq n^{1/2} \text{ blocks of size } 2m \]
\[ \leq n^{1/2} \text{ blocks of size } 3m \}

\[ \frac{1}{2} \left( \frac{2m}{m} \right) n^{-1/2} \text{ and } \left( \frac{2m}{m} \right) \Omega(m^2) \text{ are both } \text{maximize} (m) \].

Hence, \( \text{maximize} (S) \leq \Pi_{r,k} \).

\[ \text{and } \Pi_{r,k} \subseteq \Pi_{r,k+1}. \]

Knowledge about \( B_n \):

1928 Sperner: The subset lattice is Sperner.
1967 Rota: Is \( B_n \) Sperner?
1967 Harper: Asymptotic normality, \( K_n \sim n/\log n \)
1968 Lish: Log convexity
1969 Mullin: \( K_n \) is the critical part
1970 Graham, Harper: \( B_{1,n} \sim B_{1,n} \) via Philip Hall quotient reduction
1971 Dilworth, Greene: geometric \# Sperner (\( > 6,000 \) elements)
1971 Kleitman, Edelberg, Lubell: Antichains are symmetric
1974 Harper: \( B_n \) is LYM for \( n \leq 19 \)
1974 Spencer: \( B_n \) is not LYM for \( n \geq 20 \)
1977 Fishburn, Tuma: Every lattice can be embedded in \( B_n \) for some \( n \)
1977 Canfield: \( B_n \) is not Sperner, \( n \leq (7) 6.5 \times 10^4 \)
1979 Shearer: \( B_n \) is not Sperner, \( n \leq 3.7 \times 10^9 \)
1980 Canfdell: \( K_n \) = 
1980 Shearer: Maximum antichains intersect \( B_{n,\delta} \) for \( \delta \geq (1 - \delta) \log n / \log n \)
1984 Jiang, Kleitman: \( B_n \) is not Sperner, \( n \leq 3.4 \times 10^9 \)
1985 Harper: \( n \geq (7) (1 - \sqrt{5/6})^{-3/2} \log(n, K_n) \)
1990 Kung: \( B_{n,\delta} \sim B_{n,\delta+1} \) for \( \delta > n/2 \)
1991 Canfield: \( B_n \) and \( K_n \)
"Algorithms for Small Graphs"

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Algorithms for small graphs

Ron Read

University of Waterloo
Homeomorphically irreducible trees (continued)

4-regular graphs with 11 vertices (continued)

Self-complementary digraphs (continued)

2-regular digraphs
**Special graphs. 6**

**Parameters and Properties.**

- $p = \text{no. of vertices}$
- $q = \text{no. of edges}$
- $k = \text{no. of components}$
- Degree sequence
- $g = \text{girth}$ (Girth number)
- Circumference
- Diameter
- $K = \text{vertex connectivity}$
- $\lambda = \text{edge connectivity}$
- Order of automorphism group
- Eulerian?
- Hamiltonian?
- Planar?
- Bipartite?
- Tree?
- Chromatic number $\chi$
- Edge chromatic number $\chi'$
- Chromatic polynomial
- Characteristic polynomial
- Spectrum

**Characteristic polynomial**

Adjacency matrix $A$

$$\phi(x) = |xI - A|.$$  

Newton's method:

Let $q_j = \text{trace } A^j$

and $\phi(x) = \sum_{k=0}^{n} p_k x^k$.

Then

$$k p_k = \sum_{j=0}^{k} q_j p_{k-j}.$$  

**Example:**

$$
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
$$

$q_1 = 0, q_2 = 8, q_3 = 6, q_4 = 28$

$p_0 = 1, p_1 = 0, p_2 = -4$

$p_3 = -2, p_4 = 1$

$$\phi(x) = x^4 - 4x^3 - 2x + 1$$

**SPECTRUM**

The set of eigenvalues of the adjacency matrix $A$.

$\lambda = \text{set of roots of } \phi(x) = 0$

Roots are between $-(n-1)$ and $n-1$.
All roots are real.

Find integer roots first. Then use Newton's formula, starting with the largest.
As each root is found, take out the linear factor.

Newton's formula

If $x$ is an approximate root, then

$$x' = x - \frac{\phi(x)}{\phi'(x)}$$

is a better one (under appropriate conditions).
Chromatic Polynomial

\[ P_k(n) = \text{number of ways of coloring } G \text{ with } \lambda \text{ colors available.} \]

Chromatic reduction:

\[ P_k(n) = P_k(n) - P_{k-1}(n) \]

Stop when only 2 edges are left. This works even if G is not connected.

**Properties.**

If \( P_k(n) > 0 \), G is bipartite.

The lowest power of \( \lambda \) with non-zero coefficient is \( k \) — the number of components.

If \( g \) is the girth of \( G \), the coefficients (in absolute magnitude) are

\[ 1, \frac{\lambda}{g}, \left(\frac{\lambda}{g}\right)^2, \left(\frac{\lambda}{g}\right)^3, \ldots, \left(\frac{\lambda}{g}\right)^{g-1} \]

and the next coefficient is less than \( \left(\frac{\lambda}{g}\right)^g \)

by the number of cycles of length \( g \).

---

**Diameter**

Diameter = greatest distance between two vertices.

\[ = \max \{ \min (|u-v|) \} \]


For \( k = 1 \) to \( p \)
\[ \text{do: for } i = 1 \text{ to } p \text{ and } \neq k \]
\[ \text{for } j = 1 \text{ to } p \text{ and } \neq k \]
\[ d(i,j) = \min(d(i,j), d(i,k) + d(k,j)) \]
\[ (\text{All } d(i,j) \textrm{ initially } = \infty) \]
**Circumference**

Circumference = length of longest cycle

NP-complete (includes the Hamilton cycle problem)

**Method:** Start at some vertex $v$. Try to construct a Hamilton cycle, but keep track of the longest cycle found.

If this is $p$, circumference is $p$ - graph is Hamiltonian

If it is $p-1$ circumference $= p-1$ - graph is not Hamiltonian

Otherwise, repeat with the graph $G - v$.

**Short cuts:** Eliminate vertices of degree $< 2$.

A vertex of degree $2$, if included, implies inclusion of its incident edge.

\[ \ldots \rightarrow \]

The $q > 3p - 6$ criterion is obscured by the presence of vertices of degree 2.

**Method:** Delete all vertices of degree 1. "Smooth out" vertices of degree 2.

(continued if possible)

If $G$ is disconnected, keep components with $> 5$ vertices. (For $p > 7$ there will be only one at most)

Assume $G$ connected.

Perform the preliminary tests.

If planarity/nonplanarity is still not determined, perform a planarity test.

Which one?

---

**Planarity**

Hopcroft & Tarjan 1974 (linear).

Too elaborate!

**Preliminaries**

**Theorem** If $q > 3p - 6$ $G$ is nonplanar.

If $q - p = -1$ $G$ is a tree

If $q - p = 0$ $G$ is a unicycle

If $q - p = 1$ $G$ has two indent. cycles

If $q - p = 2$ $G$ has $3$ " $\pi"$'s

-all these must be planar.

The case of $K_{3,3}$ ($p = 6, q = 9$) shows that $q - p = 3$ does not imply planarity.

If $p \leq 4$, or $p = 5$ and $G \not\approx K_5$

then $G$ is planar.

---

The Fisher/Wing algorithm.

**Special case when $G$ is Hamiltonian**

Easy to compute whether the chords are compatible.

What if the graph is not Hamiltonian?

ASK
Connectivity (\(\kappa\) and \(\lambda\))

Information from the chromatic polynomial.

1. If the coefficient of \(\lambda\) is zero.
   - \(G\) is not connected.

2. If \(P_c(\lambda)\) is divisible by \((\lambda-1)^2\)
   then \(G\) has a cut vertex.

   Otherwise, \(G\) is at least 2-connected, but there seems to be no short cut to finding the exact value of \(\kappa\).

The value of \(\lambda\) can be found by using the max-flow-min-cut theorem; but no obvious short cuts.

---

Number of automorphisms

For graphs with \(p \leq 10\) this number was computed and recorded when the graph was generated.

For regular graphs

Example. 4-regular (quartic) graphs, \(p = 11\).

These were generated by extracting from the 10-vertex catalog these graphs with degree sequence \(4444443333\) and joining the four vertices of degree 3 to a new vertex of degree 4.

This gives each required graph at least once.

Now eliminate duplicates.

How?

---

Classify vertices by convenient criteria.

- e.g. number of triangles a vertex ison.
- (cubes of adjacency matrix).
- Number of 2-paths between vertices.
- (square of adjacency matrix).
- Run through permutations of vertices which permute sets of equivalent vertices.
"Characterization of Generalized Bicritical Graphs"

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Characterization of Generalized Bicritical Graphs

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Definitions

2-factor

Definitions

f-Factor

Definitions

4-connectivity
Applications

- Lower bound for TSP
- Degree sequences
  - Construct graphs
  - Prove theorems
- Easy proof of Tutte’s
  “4-conn., planar ⇒ Ham.” theorem?

1-Factor Theorems

Menger - 1927
Kenig & Egervary - 1931
Hall - 1935

A bipartite graph $G = (V_1, V_2; E)$ with $|V_1| = |V_2|$ has a 1-factor

$$\iff |N(S)| \geq |S| + S \subseteq V_1.$$ 

Tutte - 1947

$G$ has a 1-factor

$$\iff \delta(G-S) \leq |S| + S \subseteq V(G).$$

Bicritical Graphs

Even Halin Graphs

Wheels

Special Graphs

- Elementary Graph:
  Has a p.m., and the edges contained in a p.m. form a connected subgraph.

- Bicritical Graph:
  $G - u - v$ has a p.m. for all vertices $u, v$. 
What is a bicritical graph?

- Lovász: \( G \) is bicritical iff 
  \[ S \subseteq V(G) \text{ and } |S| \geq 2 \implies c_0(G - S) \leq |S| - 2. \]

- Bicritical \( \implies \) 2-connected with \( \delta \geq 3 \).

- \( G \) is bicritical iff 
  each split graph \( G_i \) of \( G \) is bicritical.

Corollary (Nishizeki-1978)

- Every even, 4-connected, projective-planar graph has a perfect matching.

- 3 even, 4-connected, toroidal graphs which are not bicritical. Example: \( C_{2m} \times C_{2n} \).

Examples - Lovász and Plummer

- Every even Halin graph is bicritical.

- \( G \) is connected, even, vertex-transitive
  \( \implies G \) is elementary bipartite or bicritical.

- \( G \) is cyclically \((k+1)\)-edge connected, even, \( k \)-regular
  \( \implies G \) is elementary bipartite or bicritical.

\[ c_0(G - S) > |S| \implies \text{no p.m.} \]
Characterization

Let $G$ be 4-connected and embeddable in the torus or the Klein bottle. Then $G$ is bicritical iff $G$ is even and

1. $G$ is projective-planar or
2. $G$ has no set $S \subseteq V(G)$

$G-S$ has no even components and contracting each component of $G-S$ yields a graph $G'$ with $|S| = |V(G)| - |S|$ and with

$G' = E[G[S])$ 4-regular.

Matching Extendability

- $n$-extendable: $p \geq 2n + 2$,
  $G$ has a p.m., and every matching of size $n$ is contained in a p.m.
- Matching extendability $\mu(\Sigma)$ of a surface $\Sigma$:
  smallest integer $n \geq 3$ no graph embeddable in $\Sigma$ is $n$-extendable.
- Plummer: $\mu(\Sigma) = ?$ for orientable $\Sigma$.
- Answer: $\Sigma \neq$ sphere $\Rightarrow$
  $\mu(\Sigma) = 2 + \lfloor \sqrt{4 - 2\chi} \rfloor$.

Telecommunications

- Irregularly shaped surface
- Nodes communicate along $\leq 1$ edge
- Can matching be extended to one where every node communicates?
- To what extent does the surface obstruct the extension?
Sufficient Conditions for Hamiltonian Cycle

Whitney (1931)
4-connected plane triangulation
Tutte (1956)
4-connected, planar
Duke (1972)
1) 2-conn., toroidal, \( s \geq 6 \)
2) 2-conn., toroidal, \( s \geq 4 \), triangle-free
Thomas & Yu (1991)
4-conn., projective planar

Conjectures

Grunbaum (1970) & Nash-Williams (1972)
4-conn., toroidal \( \Rightarrow \) Ham.
Molluzzo (1979) - Negami & Ota
6-conn., toroidal \( \Rightarrow \) Ham.-conn.
Grunbaum (1970) - Thomas & Yu
4-conn., projective planar \( \Rightarrow \) Ham.
Dean (1990)
4-conn., proj. planar \( \Rightarrow \) Ham.-conn.
5-conn., toroidal \( \Rightarrow \) Ham.-conn.?

Tutte's f-Factor Theorem

\[ \delta(f, s, T) = \sum_{x \in S} f(x) + \sum_{x \in T} d_0(x) - e(s, T) \]
\[ - \sum_{x \in T} f(x) - h(f, s, T) \]

where

\( h(f, s, T) = \) number of components \( C \) of \( G - S - T \) \( \forall C \)
\[ \sum_{x \in V(C)} f(x) + e(V(C), T) \] is odd.

\( G \) has an \( f \)-factor \( \iff \) \( \delta(f, s, T) \leq 0 \)
\( \forall T \) disjoint \( S, T \subseteq V(G) \).

f-Bicritical Graphs

Let \( f : V(G) \rightarrow \mathbb{Z} \).
\( G \) is \( f \)-bicritical if \( \forall u, v \in V(G), G - uv \) has an \( f_{uv} \)-factor where
\( f_{uv}(x) = f(x) - 1 \) if \( x \in \{ u, v \} \) and
\( f_{uv}(x) = f(x) \) otherwise.

Note:

\( \bullet \) \( G \) is \( f \)-bicritical \( \Rightarrow \) every edge lies in an \( f \)-factor.
\( \bullet \) \( G \) is 1-bicritical \( \iff \) \( G \) is bicritical
\( \text{i.e.,} \forall u, v \in V(G), G - u - v \) has a p.m.

Theorem. \( \delta(f, s, T) \geq 2 \), \( \forall S, T \subseteq V(G) \)
\( \text{with} S \cap T = \emptyset \) and \( S \cup T \neq \emptyset \) \( \Rightarrow \) \( f \)-bicritical.
**CONJECTURE**

Grunbaum - 1970
Nash-Williams - 1973

Every 4-connected, toroidal graph is hamiltonian.

---

**Surfaces**

- Sphere
- Torus, ...
- Projective plane,
  Klein bottle, ...

1. \( G \) is bipartite and embeddable in the P.P. or the K.B. or torus \( \Rightarrow |E(G)| \leq 2|V(G)|. \)

2. \( G \rightarrow \Sigma \) are closed under minors.

---

**Graph Minors Family \( \mathcal{Z}_k \)**

1. Closed under minors
2. Every bipartite member \( B \) satisfies \( |E(B)| \leq 2|V(B)| - k. \)

**Examples**

- \( \mathcal{Z}_0 = \{ \text{toroidal graphs} \} \)
- \( \mathcal{Z}_0 = \{ \text{Klein bottle graphs} \} \)
- \( \mathcal{Z}_2 = \{ \text{projective planar graphs} \} \)
- \( \mathcal{Z}_4 = \{ \text{planar graphs} \} \)

\( \mathcal{Z}_0 \supseteq \mathcal{Z}_1 \supseteq \mathcal{Z}_2 \supseteq \mathcal{Z}_3 \supseteq \mathcal{Z}_4 \)

---

**2-Factors in 4-Conn. G**

\( G \in \mathcal{Z}_0 \Rightarrow G \) has a 2-factor.

\( G \in \mathcal{Z}_1 \Rightarrow \)

1. \( G \) is 2-bicritical.
2. \( G-u \) has a 2-factor, \( \forall u \in V(G) \).
3. \( G-u-v \) has a 2-factor, \( \forall u, v \in V(G) \).
**Sample Theorem** (with K. Ota)

G is 4-connected and embeddable in the torus or the Klein bottle

⇒ G has a 2-factor.

Best possible: $K_{4,n}, n \geq 5$
- 4-connected
- not embeddable in torus
- not embeddable in Klein bottle
- no 2-factor

---

**Proof Strategy**

- Reduce G to the essentials
- Case $|SUT| \leq 3$
- Case $|SUT| \geq 4$
- Collect info on odd components - definitions & claims
- Substitute into formula for $\delta(S, T)$ to show $\geq 0$

---

**Tutte's 2-Factor Theorem**

$$\delta(S, T) \triangleq 2|S| + \sum_{x \in T} d_G(x) - e(S, T) - 2|T| - h(S, T)$$

Theorem.

G has a 2-factor $\iff$

$\delta(S, T) \geq 0 \iff$ disjoint $S, T \subseteq V(G)$.

Note: $\delta(\emptyset, \emptyset) = 0$. $\forall \emptyset$.  

---

**Reductions**

$G \rightarrow G_0$

$\delta(S, T) \triangleq 2|S| + \sum_{x \in T} d_G(x) - e(S, T) - 2|T| - h(S, T)$

Contract each odd component.
Case $|SUT| \leq 3$

$G_6$:

- $h(S, T) \leq 1$ since $G$ is 4-connected.
- $\delta(S, T) \geq 2|S| + 4|T| - e(S, T) - 2|T| - 1$
  \[
  = 2|SUT| - e(S, T) - 1
  \geq \begin{cases} 
    4 - 2 - 1, & \text{if } |SUT| = 2 \times 3 \\
    2 - 0 - 1, & \text{if } |SUT| = 1 \\
    \geq 1
  \end{cases}
  \]

Case $|SUT| \geq 4$

$H_1 \geq 3$

$H_2 \geq 3$

$H_3 \geq 3$

$h(S, T) = |H_1| + |H_2| + |H_3|$

Claim.

$\forall u \in H_2 \exists v \in T \ni e_6(x, C(u)) \geq 2.$

Connectivity for 2-Factors

$t(\Sigma) \equiv$ smallest integer $k \ni$

every $k$-connected graph embeddable in $\Sigma$ has a 2-factor.

$\chi(S_6) = 2 - 2h$

$\chi(N_k) = 2 - k$

Duke (1972)

$t(\Sigma) \leq 3 + \sqrt{9 - 3\chi},$ if $\Sigma \neq S_6.$

New Results:

$t(N_4) = 4 \leq 5$

$t(S_1) = t(N_2) = 4 \leq 6$
"Extremal Problems Involving Neighborhood Numbers and Other Parameters"

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Extremal Problems Involving Neighborhood Numbers
and Other Graph Parameters

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ABSTRACT

Given a simple graph $G = (V, E)$, a subset $S$ of $V$ is called a neighborhood set provided $G$ is the union of the subgraphs induced by the closed neighborhoods of the vertices in $S$. The minimum cardinality among all minimal neighborhood sets of $G$ is denoted by $n(G)$ and is called the neighborhood number of $G$. It is known, for instance, that $\gamma(G) \leq n(G) \leq \omega(G)$ for any $G$ without isolated vertices, where $\gamma(G)$ and $\omega(G)$ are the (vertex) domination and covering numbers, respectively.

My colleague, Y.H. Harris Kwong, and I have been investigating the problem of finding the maximum neighborhood number $n(p)$ among all connected graphs of order $p$. Our work so far has lead us to conjecture that

$$n(p) \leq \lfloor 9p/13 \rfloor$$

a result that holds for $2 \leq p \leq 18$. I will report on this work and, as time permits, some recent work of David K. Garnick, Kwong, and Felix Lazebnik on the maximum number of edges among all graphs of order $p$ having girth at least 5.

Observe that if $G$ is the disjoint union of graphs $G_1$ and $G_2$, then

$$\gamma(G) = \gamma(G_1) + \gamma(G_2), \quad n(G) = n(G_1) + n(G_2), \quad \omega(G) = \omega(G_1) + \omega(G_2)$$

We thus assume henceforth that $G = (V, E)$ is connected. Now, if $u$ and $v$ are nonadjacent vertices of $G$, then

$$\gamma(G + uv) \leq \gamma(G) \quad \text{and} \quad n(G + uv) \geq n(G)$$

However,

$$n(G) - 1 \leq n(G + uv) \leq n(G) + 1$$

For example, consider the graph $G = (\{u, v, w, x, y\}, \{(u, v, w, x, y)\})$. Note that $G$ is a 5-cycle, $n(G + uv) = 7$, $n(G) = 3$, and $n(G - uv) = 2$.

This leads us to consider the following extremal problem: find the maximum neighborhood number $n(p)$ among all connected graphs of order $p$.

Given a simple graph $G = (V, E)$, the set $N(v) = \{w \in V : \{v, w\} \in E\}$ is called the neighborhood of $v$ and $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood. A subset $S$ of $V$ is called a neighborhood set provided $G$ is the union of the subgraphs induced by the closed neighborhoods of the vertices in $S$. The minimum cardinality among all minimal neighborhood sets of $G$ is denoted by $n(G)$ and is called the neighborhood number of $G$. It was observed in the above mentioned paper that $n(G) = n(G')$ if $G$ is connected and has girth at least 4. However, Jayara Kwong, and Straight [Neighborhood sets in graphs, Indian J. Pure Appl. Math. 16 (1985), 105-132].

Two related parameters are the (vertex) domination and covering numbers. A sub $S$ of $V$ is a dominating set provided every vertex not in $S$ is adjacent to a vertex in $S$; it is a covering set provided every edge of $G$ has at least one of its incident vertices in $S$. Let $\gamma(G)$ denote the minimum cardinality of a dominating set and let $\omega(G)$ denote the minimum cardinality of a covering set; if $G$ has no isolated vertices, then any covering set is also a neighborhood set and any neighborhood set is also a dominating set. Thus, any graph $G$ without isolated vertices,

$$\gamma(G) \leq n(G) \leq \omega(G)$$

It is natural to wonder whether these parameters are independent, in some sense. It is easy to see that $\gamma(G) \leq 1$ if and only if $n(G) = 1$; the complete graph $K_p$ of order $p$ has $\gamma(K_p) = n(K_p) = 1$ and $\omega(K_p) = p - 1$. It was also observed in the above mentioned paper that $n(G) = n(G')$ if $G$ is connected and has girth at least 4. However, Jayara Kwong, and Straight [Neighborhood sets in graphs, Indian J. Pure Appl. Math. 16 (1985), 105-132].

To begin, we find that $n(2) = n(3) = 1$ and $n(4) = 2$. Next, $n(5) = 3$, with the unique extremal graph being the 5-cycle. At this point we make the following observations: 1. If $G$ be formed from the disjoint union of $G_1$ and $G_2$ by adding a single edge that joins a vertex of $G_1$ with a vertex of $G_2$. Then $n(G) \geq n(G_1) + n(G_2)$. As a consequence, let $G$ be a rational number between 0 and 1 and suppose there exists a graph $G_3$ of order $p$ having $n(G_3) = rp$. Then there exist infinitely many values of $p$ for which $n(p) \geq r$, Since we have a graph of order $p$ with neighborhood number 3, we tentatively conjecture that $n(p) \leq 3p/8$ for all $p > 1$.

Two more useful observations. The first is that

$$n(p + 1) \leq n(p) + 1$$

Secondly, suppose $G_1$ is a connected graph of order $p$ and $G_2$ is the complete graph of order 2. Form the graph $G$ as above. Then $G$ has order $p + 2$ and $n(G) = n(G_1) + 1$. Therefore, we find that

$$n(p) = n(p + 1) - n(p + 2) = n(p + 1) + 1$$

Continuing, we find that $n(6) = 3$ and $n(7) = 4$. But now consider the following graph of order 2:

$$J = \{(a, t, t, u, u, x, x, x, x, x, w, w, w, w, y, y, y, y, y)\}$$

It is not difficult to show that $n(J) = 5$. It follows that $n(8) = 5$, and we are forced to revise our tentative conjecture, namely, we now conjecture that $n(p) \leq 3p/8$. 3
Let $p$ be fixed and suppose the value of $n(p)$ is known. As a consequence of the observations made on the preceding page, if one claims that $n(p+1) = n(p) + 1$, then one must give an example of a graph $G$ of order $p+1$ having $n(G) = n(p) + 1$. On the other hand, if one claims that $n(p+1) = n(p)$, then one must prove that every connected graph of order $p + 1$ has neighborhood number at most $n(p)$.

We can show that every connected graph of order $0$ has neighborhood number at least $5$. This, together with $n(2) = 5$, gives us that $n(9) = 9$. Joining two disjoint $6$-cycles with an edge yields a connected graph of order $10$ having neighborhood number $6$; hence, $n(10) = 6$. At this point in our investigation we were able to find a graph of order $11$ with neighborhood number $7$. This disproves our tentative conjecture that $n(p) = 6$ for $p = 11$. However, if one writes down a table of values of $n(p)$ for $2 \leq p \leq 11$ the following revised conjecture strongly suggests itself.

Conjecture. For $p \geq 2$,
\[ n(p) = n(p+1) = n(p+2) = n(p+3) = n(p+4) = n(p+5) = n(p+6). \]

We derive bounds for $f(v)$, the maximum number of edges in a graph on $v$ vertices that contains neither three-cycles nor four-cycles. Also, we give the exact value of $f(v)$ for all $v$ up to $24$ and constructive lower bounds for all $v$ up to $200$.

$\fbox{5}$

In this section we present some theoretical results about $f(v)$ and the structure of extremal graphs. Many of them will be used in the subsequent sections. We call a graph $C$ of order $v$ extremal if $g(C) \geq 5$ and $v = n(C) = f(v)$. The following statement is a simple fact about extremal graphs.

Proposition 5.1. Let $C$ be an extremal graph of order $v$. Then

(a) $C$ is connected and the diameter of $C$ is at most $3$.

(b) If $d(v) = d(C) = 1$, then the graph $C - v$ has diameter at most $2$.

It turns out that the extremal graphs of diameter $2$ are very rare. In fact, it has been shown that the only graphs of order $v$ with no 4-cycles and of diameter $2$ are:

- The star $K_{1,v-1}$;
- Moore graphs $C_7$, Petersen graph (the only known 3-regular graph of order 10, diameter $2$ and girth $5$), Hoffman-Singleton graph (the only 7-regular graph of order 50, diameter $2$ and girth $8$), and a 57-regular graph of order 2280, diameter $2$ and girth $3$, if it exists;
- Polar graph.

Remark: The only graphs from the list above which contain no triangles are the Moore graphs. It is also known that a graph of diameter $d \geq 3$ with girth $2d + 1$ must be regular.

We now derive an upper bound on $f(v)$.

Theorem 5.2. Let $C$ be an extremal graph of order $v \geq 3$ and size $e$. Then
\[ f(v) = e \leq \frac{2}{3}v\sqrt{v-1}. \]

Furthermore, equality holds if and only if $G$ is a triangle or a cycle.

Corollary 5.2. Let $C$ be an extremal graph of order $v$, size $e$, and diameter $3$. If $C$ is regular, then
\[ f(v) = e \leq \frac{2}{3}v\sqrt{v-1} - \frac{2}{3}. \]

If, in addition, the average degree of $G$ is an integer, then
\[ f(v) = e \leq \frac{2}{3}v\sqrt{v-1} - \frac{1}{2}. \]
Now we derive a lower bound for \( f(v) \). Let \( q \) be a prime power, and let \( e_q = q + q + 1 \) and \( e = (q + 1)e_q \). By \( B_q \) we denote the point-line incidence bipartite graph of the projective plane \( PGL(2,q) \). More precisely, the partite sets of \( B_q \) represent the set of points and the set of lines of \( PGL(2,q) \), and the edges of \( B_q \) correspond to the pairs of incident points and lines. Then \( B_q \) is a \((q+1)\)-regular bipartite graph of order \( 2e_q \) and size \( e_q \). Also \( \theta(B_q) \geq 5 \): being bipartite \( B_q \) has no 3-cycles, and the existence of a 4-cycle in \( B_q \) would mean that in \( PGL(2,q) \) there are two distinct lines passing through two distinct points.

Theorem 2.4. Let \( G \) be an extremal graph of order \( n \) and size \( n + 1 \). Let \( q \) be the largest prime power such that \( 2e_q \leq n \). Then

\[
f(v) = s \geq 2e_q = 2(q - 2e_q) - 2e_q + (q - 3)e_q
\]

Proof. Consider the graph obtained from \( B_q \) by adding to \( V(B_q) \) a set of \( n - 2e_q \) isolated vertices and connecting each of these to two nonadjacent vertices of \( B_q \) taken from different partite sets (such two vertices always exist, since \( B_q \) is not a complete bipartite graph). If the chosen pairs of vertices of \( B_q \) are distinct for distinct isolated vertices, then we obtain a graph \( H \) of order \( n \), size \( e_q + 2(e_q - 2e_q) \) and girth at least 5. There are \( e_q \) pairs of disjoint vertices of \( B_q \) in which two vertices in the pair belong to different partite sets. We claim that \( e_q > 2e_q \). Indeed, if it is not the case and \( q > 2 \), then \( q > (q - 2e_q) > 8e_q \). According to Dirac’s Theorem, for any integer \( n \geq 1 \), there is at least one prime number \( p \) such that \( n < p < 2n \). Let \( n = q \), and let \( p \) be a prime satisfying the inequality \( q < p \leq 2e_q \). Then

\[
2e_q < 2e_q \leq 8e_q + 4q < 8e_q + 4q - 8e_q < q
\]

which contradicts our choice of \( q \) in the statement of the theorem. For \( q = 2 \), \( 2e_q = 14 \leq q \leq 20 \implies n = 14 < 12 < 22 \implies 2e_q < q \). Therefore, for all prime powers \( q \), \( q > 2e_q \) and the construction of \( H \) described above is possible.

Theorem 2.2 states that \( f(v) \leq \lfloor \sqrt{n/2} \rfloor \). For \( 1 \leq n \leq 10 \), we have constructed \( (G_1, G_2) \)-free graphs with \( \lfloor \sqrt{n/2} \rfloor \) edges. These graphs are shown in Figure 1. This yields the following theorem.

Theorem 3.1. For \( 1 \leq n \leq 10 \), \( f(v) = \lfloor \sqrt{n/2} \rfloor \).

![External graphs with \( n \leq 10 \)](image)

Figure 1: External graphs with \( n \leq 10 \).

Theorem 3.2. The values of \( f(v) \) for \( 11 \leq n \leq 30 \) are as follows:

- \( f(11) = 16 \)
- \( f(12) = 18 \)
- \( f(13) = 21 \)
- \( f(14) = 23 \)
- \( f(15) = 26 \)
- \( f(16) = 23 \)
- \( f(17) = 21 \)
- \( f(18) = 24 \)
- \( f(19) = 28 \)
- \( f(20) = 41 \)

We next define a restricted type of tree; many of the proofs in the next section rely on the presence of these trees in extremal graphs. Consider a vertex \( v \) of maximum degree \( \Delta \) in a \((G_1, G_2)\)-free graph \( G \). Let the neighborhood of \( v \) be \( n(v) = (v_1, v_2, \ldots, v_k) \). Clearly \( n(v) \) is an independent set of vertices. Furthermore, the sets of vertices \( n(v) \) are pairwise disjoint; otherwise there would be a quadrilateral in \( G \). This motivates the notion of an \((m,n)\)-star \( S_{m,n} \), which is defined to be the tree in which the root (center) has \( m \) children, and each of the root’s children has \( n \) children, all of which are leaves. The subtree containing a child of the root and all its \( n \) children is called a branch of \( S_{m,n} \).

For all \( (G_1, G_2) \)-free graphs, \( n \geq 1 + \Delta d \geq 1 + \Delta^2 \).

Proposition 2.6. For all \( (G_1, G_2) \)-free graphs, \( n \geq 1 + \Delta d \geq 1 + \Delta^2 \).

Proposition 2.7. For all \( (G_1, G_2) \)-free graphs \( G \) on \( n \geq 1 \) vertices and \( e \) edges, \( n = f(e - 1) \).

Proposition 2.8. For all \( e \geq 1 \), we have \( \geq (f(e)/e)(f(e) - f(e - 1)) \).

Proof. We look at several specific cases to illustrate the techniques involved.
It can be noted that $G_{12}$ is the Robertson graph - the unique $(4, 5)$-cage. It follows that $G_{14}$ and $G_{20}$ are the unique extremal graphs of orders 19 and 20, respectively.

**Theorem 3.1.** The values of $f(n)$ for $21 \leq n \leq 34$ are as follows:

\[
\begin{align*}
  f(21) &= 44 \\
  f(22) &= 47 \\
  f(23) &= 50 \\
  f(24) &= 54
\end{align*}
\]

**Proof.** Again, to illustrate the techniques involved, we give the proof that $f(22) = 47$.

First of all, a $(C_4, C_6)$-free graph of order 22 and size 47 is constructed, showing that $f(22) \geq 47$. Suppose there exists a $(C_4, C_6)$-free graph $G$ with $\omega = 22$, $\delta = 45$. By Proposition 2.7, if $\Delta \geq 4$ and $\Delta \geq 6$. Then $\Delta = 5$, for otherwise there is a $(5,3)$ star in $G$, and such a tree has more than 22 vertices; therefore, $G$ contains 14 vertices of degree 4 and 8 vertices of degree 6. This in turn implies that there are at least 8 edges among the degree 4 vertices. Since any graph with 14 vertices and at least 8 edges must contain a path of length 3, there must be at least one degree 4 vertex adjacent to two other degree 4 vertices; thus there is a $P_4$ in $G$ each of whose vertices has degree 4. However, if there is a path $P$ on 3 vertices of degree 4, then $G = V(P)$ is the Robertson graph. Each of the pendant vertices in $P$, $x$ and $y$, has 3 neighbors in the Robertson graph that are mutually distance 2 apart. But the Robertson graph has only one such set of 3 vertices. Therefore, $G$ cannot contain a $P_4$ of degree 4 vertices, giving us a contradiction. Therefore, $f(22) \leq 47$.

This paper gives exact values of $f(n)$ for $1 \leq n \leq 24$. It is also noted that $f(50) = 175$, the extremal graph being the Hoffman-Singleton graph. The paper also gives constructive lower bounds for $f(n)$ for $25 \leq n \leq 200$; these are found using an algorithm that combines hill-climbing and backtracking techniques. For instance, at one point a $(C_4, C_6)$-free graph of order 96 and size 307 was found; adding an isolated vertex gave a $(C_4, C_6)$-free graph of order 97 and size 307. Backtracking applied to this graph yielded a $(C_4, C_6)$-free graph of the same order with 403 edges. Hill-climbing from that point resulted in the addition of one more edge, giving a $(C_4, C_6)$-free graph $G$ of order 97 and size 404. Then, $f(97) \geq 404$.

Further hill-climbing rearranged the edges of $G$ so that some vertex had degree 6; removing this vertex then gave a $(C_4, C_6)$-free graph of order 96 and size 308, thereby improving the lower bound for $f(96)$.
"Random Graph Processes with Degree Restrictions"

Andrzej Rucinski, Emory University
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Random Graph Processes
with degree restrictions

3 major differences:
1° many final stages
2° the length varies
3° not an equiprobable space

Focus on final stage:
a maximal graph with \( \Delta(G) = d \)
d-maximal graph
saturated vts (degree \( d \))
unsaturated vts (degree \(< d\))

Random graph process

\[
\begin{align*}
(\binom{n}{2})^{-1} \\
\end{align*}
\]

Degree restriction \( \Delta(G) \leq d \)

\[
\begin{align*}
d = 2 & \quad \text{or} \quad \begin{cases} 
4 \quad \text{or} \\
5 \quad \text{or} \\
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
\frac{nd}{2} \leq d(d+1) \leq \frac{dn}{2} \\
\end{align*}
\]

Unsaturated vts form a clique, so there are \( \leq d \) of them.
Let \( U = U(d,n) = \# \) unsaturated vts at the end

Erdős asked

\[
\lim_{n \to \infty} P(U = x), x = 0, 1, \ldots, d ?
\]

Nontrivial even for \( d = 2 \)

\[
P(U=0) = \frac{11}{15} \to \frac{17}{27} \approx 0.63 \approx 0.63
\]

\[
P(U=0) \to 1 ?
\]
Theorem (R., Wormald, 1992) \( \forall d \) the process saturates a.s., i.e. \( P(U=0) \to 1 \) if \( nd \) even \\
\( P(U=1) \to 1 \) if \( nd \) odd \( \star \)

It contrasts with the equiprobable space of \( d \)-maximal graphs, where \( d \geq 2 \), an even,

\[
P(U=i) \to a_i > 0 \quad i=0,1,2,
\]

\[
a_0 + a_1 + a_2 = 1
\]

Structural results only for \( d=2 \)

\[
EC \leq \log n + 3
\]

\# \( C_e \) \( \sim \) Poisson

\[
E[#C_3] \sim \frac{1}{2} \int_0^\infty \left( \log (1+x) \right)^2 dx \approx 1.88735349357788850 \neq \frac{1}{6}
\]

The idea of proof:

- Study \( \# \) isolates \( I_t \)

\( d=2 \)

\[
I=\alpha(R) \Rightarrow I=0 \Rightarrow I=0 \Rightarrow 00
\]

\[
I=I_k \Rightarrow I=0
\]

\( d \geq 2 \) induction on \( d \)

Quite messy: 1° forbidden pairs at start 
2° nonuniform degree bounds

Lemma \( P(u,v) \) - probability that vertex \( 1 \) remains isolated until time \( v \), provided it was isolated at time \( u \). Then

\[
P(u,v) = O \left( \frac{n^2 - v}{n^2} \right)
\]

Proof

\[
P(u,v) = \sum_{t=0}^{u-1} P(t,t+1)
\]

\( H_t \) - the event that \( 1 \) is isolated in

\[
P(t,t+1) = E \left( E(\text{Ind}(H_{t+1}) | I_t) | H_t \right)
\]

\[
= E \left( 1 - \frac{U_t - 1}{U_t - F_t} \right) \leq E \left( \exp \left( -\frac{2}{n-2t} + O \left( \frac{1}{11t} \right) \right) \right)
\]
Lemma
\[ I_{\lfloor \log n \rfloor} - I_{n^{0.10}} = O\left( \frac{n^{0.10}}{\log n} \right) \]

Outline of proof:
\[ E(I_{t+1} - I_t | G_t) = ? \]

\[ I_t - 2I_t = \begin{cases} -1 & \text{if } U_t < I_t \\ 0 & \text{if } U_t = I_t \\ 1 & \text{if } U_t > I_t \end{cases} \]

\[ I_t - 2I_t \sim U_t - I_t \]

But
\[ a_n - 2t \geq a I_t + (U_t - I_t) \]
\[ \geq (d=2) \]

\[ U_t \leq a_n - 2t - (d-1) I_t \]
\[ \geq (d=2) \]

So
\[ E(I_{t+\Delta t} - I_t | G_t) \leq - \frac{2\Delta t I_t}{a_n - 2t - (d-1) I_t} \]

Define Doob's martingale
\[ X_i = E(I_{t+\Delta t} - I_t | G_t, E_{t+1}, \ldots, E_{t+i}) \]
\[ i = 0, \ldots, 4t \]

to show the sharp concentration
\[ I_{t+\Delta t} - I_t \sim E(I_{t+\Delta t} - I_t | G_t) \]

by Azuma's inequality

Define \( b = b(x), 0 \leq x \leq \frac{1}{2} \)

by
\[ (\ast) \quad b' = \frac{-2b}{d - 2x - (d-1)b}, \quad b(0) = 1 \]

\( b(x) \) should well approximate an upper bound on \( \frac{I_t}{n}, x = \frac{t}{n} \)

We justify this by partitioning
\[ 0, 4t, 2t, \ldots, \frac{a_n - 2t - n^{0.10}}{t} \]

and using induction
\[ \Delta \sim n^{1/4} \]
Finally, we have

\[
\frac{20}{n^{\log n}} = \frac{20}{a-1)\log n}
\]

For the asymptotic solution, as \( x \to \mathbb{E} \), is

\[
(\log(\frac{x}{2}))^{(a-2x)}
\]

max and min of

2)

over all \( a \)-regular \( G \)
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Sixth Discrete Mathematics mini-Conference
October 3-4, 1991
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### Sixth Discrete Mathematics mini-Conference

**October 3-4, 1991**

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