The proposal to AFOSR emphasized research on the effective solution of discrete elliptic equations. In particular, the proposal emphasized that the research would concentrate on four main areas: 1) Sharp Estimates for Multigrid Methods; 2) Preconditioned Conjugate Gradient methods with a particular emphasis on multigrid as a preconditioner; 3) Indefinite elliptic problems and singularly perturbed elliptic problems; 4) The implementation of these methods on parallel computers and multiprocessors. Significant progress was made on all the above areas with the exception of topic 4. The press of preliminary theoretical work and computational studies undertaken in support of that theoretical work just kept us too busy.
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I. Research Objectives and Status

A. Proposed Research Goals The proposal to AFOSR emphasized research on the effective solution of discrete elliptic equations. In particular the proposal emphasized that the research would concentrate on four main areas.

2. Preconditioned Conjugate Gradient methods with a particular emphasis on multigrid as a preconditioner.
3. Indefinite elliptic problems and singularly perturbed elliptic problems.
4. The implementation of these methods on parallel computers and multiprocessors.

B. Status of the Research: Significant progress was made on all the above areas with the exception of topic 4. The press of preliminary theoretical work and computational studies undertaken in support of that theoretical work just kept us too busy.

1. In the area of *sharp estimates for Multigrid Methods* there were three significant papers.

   (a) S.V. Parter, *Estimates for Multigrid Methods Based on Red-Black Gauss-Seidel Smoothings*, Numerische Mathematik, 52, 701-723 (1988). In this paper precise numerical values were obtained for a large class of multigrid methods. These red-black Gauss-Seidel smoothing methods are among the most frequently used multigrid methods. They are particularly attractive for use on parallel processors.


   Both of these papers are concerned with estimating the "rate of convergence" of multigrid methods (the V-cycle and the F-cycle) in the case were full $H_2$ elliptic regularity is not present. When there is such $H_2$ elliptic regularity these multigrid methods possess a "rate of convergence" which is independent of the number of grids. Without such regularity experimental evidence has shown that there is a decay in this rate of convergence as the number of grids increases. These papers provide asymptotic estimates of this behavior. The results may not be sharp, but the bounds provide valuable practical insight in how one should proceed.
Problems without $H_2$ elliptic regularity occur often in practice. Hence these studies are particularly relevant for the practical implementation of multigrid.

2. In the area of Preconditioned Conjugate Gradient with an emphasis on Multigrid as a Preconditioner there were several very important contributions. The thesis work of Ms. Naomi Decker was aimed at the study of a particular multigrid preconditioner (suggested in a Brookhaven National Laboratory Report by C.I. Goldstein) for singular perturbation problems. Therefore, her work combined our interest in topics 2 and 3. These results appeared in:

(a) The k-Grid Fourier Analysis of Multigrid-Type Iterative Methods. CS Technical Report #703, University of Wisconsin
(b) The Fourier Analysis of Multigrid-Type Iterative Methods. Thesis, August 1987

In the general area of preconditioning there are three lengthy reports. Two of which have been accepted for publication and will appear shortly. We have every reason to believe the third will also be accepted for publication. These reports represent several breakthroughs on the general topic: Given an elliptic operator $A$ and a discretization $A_h$, how does one determine appropriate choices of other elliptic operators $B$, with discretizations $B_h$, so that the preconditioned operators $B_h^{-1}A_h, A_hB_h^{-1}$ are well conditioned. In these works we have not only studied the necessary and sufficient conditions that

$$\| B_h^{-1}A_h \|_{L_2}, \quad \| A_hB_h^{-1} \|_{L_2}, \quad \| B_h^{-1}A_h \|_{H_1}$$

be uniformly bounded, independent of $h$, but we have also begun the study of the distribution of the singular values ($L_2$) of $B_h^{-1}A_h, A_hB_h^{-1}$. This last topic has usually been considered too delicate and untactable. Previous research has concentrated almost entirely on the first question of $\| B_h^{-1}A_h \|_{L_2}, \quad \| A_hB_h^{-1} \|_{L_2}$, and $\| B_h^{-1}A_h \|_{H_1}$. Indeed, most earlier work has been limited to the $H_1$ case. Unlike previous reports (by other authors) on this question we have not required the definiteness of either $A$ or $B$. Thus, once more, this work also has implications for Area (3). Further, the basic results carry over when $B_h^{-1}$ is replaced by $B_h^{-1}$, a multigrid cycle for $B_h^{-1}$. This is very important because in practice $B_h^{-1}$ is extremely difficult to obtain while $B_h^{-1}$ is quite accessible in many cases (e.g. when $B$ is definite).
The three reports are:


3. In the Area of Indefinite Elliptic problems and singularly perturbed elliptic problems there have been two papers—in addition to the work mentioned above which also impinges on this area. These are:


There have been several other publications. However, the above discussion is limited to the most important results.

II. Interactions

During the period of this grant, Seymour Parter, the Principal Investigator has given many talks on this research and collaborated with individuals in several other institutions. e.g. the Computational Mathematics Group at the University of Colorado at Denver, the Los Alamos National Laboratory and the Brookhaven National Laboratory.

The following is a list of most of the Professional Lectures given by the personnel supported under this grant during the grant period.
Naomi H. Decker:

Seymour V. Parter:
October 6, 1986, Northwestern University, Applied Math. Colloquium; “Another Look at the Rotating Disk Problem”

October 7, 1986, University of Chicago, Applied Math Seminar, “Another Look at the Rotating Disk Problem”


December 1986–Applied Math Seminar Tel-Aviv University, Tel-Aviv, Israel. “The Singular Values of Toeplitz Matrices”


November 1987–Applied Mathematics Colloquium, Columbia University, “Multigrid Convergence.”

January 1988–Applied Mathematics Seminar, Tel-Aviv University, Tel-Aviv. Israel, “Preconditioning and Boundary Conditions.”

September 1988–Seminar at Center for Supercomputing Research, University of Illinois. “Preconditioning and Boundary Conditions”

January 1989–Fifth Haifa Matrix Theory Conference, Haifa, Israel, “Preconditioning and Boundary Conditions.”

February 1989–Applied Mathematics Seminar, University of Maryland, College Park, MD. “Preconditioning and Boundary Conditions.”


III. Publications

The following papers, reports, and theses were prepared with the support or partial support of this Grant.


IV. Personnel

The following individuals were supported by this grant.

Seymour V. Parter (Principal Investigator)


Sze-Ping Wong (Research Assistant) A graduate student in the Department of Mathematics. His research work is concerned with preconditioning for Elliptic Boundary Value problems.