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   Cun-Quan Zhang, Mathematics Dept.
   West Virginia University, Morgantown, WV
   26506

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   Cun-Quan Zhang and Yong-Jin Zhu

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LONG PATH CONNECTIVITY OF REGULAR GRAPHS

Cun-Quan Zhang*
Department of Mathematics
West Virginia University
Morgantown, WV 26506

Yong-Jin Zhu
Institute of System Science
Academia Sinica
Beijing, China

ABSTRACT: Any pair of vertices in a 4-connected non-bipartite k-regular graph are joined by a Hamilton path or a path of length at least 3k-6.

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The topics about Hamilton cycles, circumferences and Hamiltonian connectivities of regular graphs have been interesting many mathematicians in recent years ([2],[1],[4],[7],[3],[6]).

In this paper, we will investigate the length of a longest path joining any pair of vertices of regular graphs and establish the following theorem.

**THEOREM 1**

Let \( G \) be a 4-connected non-bipartite \( k \)-regular graph. Then any pair of distinct vertices of \( G \) are joined by a Hamilton path or a path of length at least \( 3k-6 \).

In a sense, this theorem is a generalization of the following results.

(i) (Bollobas and Hobbs [1]) Any 2-connected \( k \)-regular graph of order at most \( \frac{9}{4}k \) contains a Hamilton cycle.

(ii) (Jackson [4]) Any 2-connected \( k \)-regular graph of order at most \( 3k \) contains a Hamilton cycle.

(iii) (Zhu, Liu and Yu [7]) Any 2-connected \( k \)-regular graph of order at most \( 3k+3 \) contains a Hamilton cycle.

(iv) (Fan [3]) The length of a longest cycle in a 3-connected \( k \)-regular graph of order \( n \) is at least \( \min\{n,3k\} \).

(v) (Zhang and Zhu [6]) Any pair of vertices of a 3-connected non-bipartite \( k \)-regular graph of order at most \( 3k-4 \) are joined by a Hamilton path.

The condition of 4-connectivity in the theorem cannot be reduced. A 3-connected \( k \)-regular graph of order \( 3k+3 \) containing no path of length at least \( 2k+3 \) joining a pair of vertices can be constructed as follows. Let \( k=3h \). Let \( G_1, \ldots, G_9 \) be nine disjoint copies of complete graph \( K_h \) and
$v_1, v_2, v_3$ be three distinct vertices. Join an edge between each pair of
vertices in $G_{3i+1}$ for $i=0,1,2$, and join an edge between $v_j$ and each vertex of $G_{3i+j}$ for $i=0,1,2$ and $j=1,2,3$. The induced graph contains $9h+3$ vertices and is $3h$-regular $3$-connected, in which $v_i$ and $v_j$ are not joined by any path of length longer than $6h+2$ for $i,j \in \{1,2,3\}$.

(See fig. 1).

Actually, we can establish a result stronger than Theorem 1.

**THEOREM 2.** Let $G$ be a $4$-connected graph and $x,y$ be a pair of distinct vertices of $G$ such that

(i) $d(v)=k$ for any vertex $v \in V(G) \setminus \{x,y\}$,

(ii) $d(x), d(y) < k$.

Then the length of a longest path joining $x$ and $y$ is at least

(i) $\min \{ |V(G)| - 1, 3k-6 \}$ if $G$ is not a bipartite graph, or $G$ is a bipartite graph and $x, y$ belong to different parts of the bipartition of $G$;

(ii) $\min \{ |V(G)| - 2, 3k-6 \}$ if $G$ is a bipartite graph and $x,y$ belong to the same part of the bipartition of $G$.

Let $G=(V,E)$ be a graph with vertex set $V$ and edge set $E$. Let $P=\mathcal{P}_0 \cdots \mathcal{P}_p$ be a path of $G$. For $0 \leq i, j \leq p$, the segment $\mathcal{P}_i \cdots \mathcal{P}_j$ of $P$ is denoted by $\mathcal{P}_i \mathcal{P}_j$ if $i < j$ or $\mathcal{P}_i \mathcal{P}_j$ if $i \geq j$. The length of a path $P$ is the number of edges in $P$ and is denoted by $\ell(P)$. Let $H$ be a subgraph of $G$. Let $w,w'$ be two vertices of $H$. The length of a longest
path of $H$ joining $w,w'$ is denoted by $L_H(w,w')$. Let $v$ be a vertex of $G$. The set of vertices of $H$ adjacent to $v$ is denoted by $N_H(v)$ and the number of vertices of $N_H(v)$ is denoted by $d_H(v)$. When $V(H)=V(G)$, we simply write $d(v)$ and $N(v)$ instead of $d_G(v)$ and $N_G(v)$. Let $P=u_0\cdots u_p$ be a path of $G$ and $X$ be a subset of $V(P)$. Denote
\[
X_+^1=\{u_{i+1}=u_i\in X\}
\]
and \[
X_-^1=\{u_{i-1}=u_i\in X\}.
\]

Let $E(H,H')$ be the set of all ordered pairs of vertices $(x,y)$ such that $(x,y)\in E(G)$ and $x\in V(H)$, $y\in V(H')$. And let $|E(H,H')|=e(H,H')$. Note that if $V(H)\cap V(H')\neq\emptyset$, each edge $(x,y)$ in the induced subgraph $G(V(H)\cap V(H'))$ will counted twice in $e(H,H')$ since the ordered pairs $(x,y)$ and $(y,x)$ are considered deferent in $E(H,H')$. Thus $d(v)=e(v,G)$ for any vertex $v$ of $G$ and \[
\sum_{v\in V(H)}d(v)=e(H,G)
\]
for subgraph $H$ of $G$.

**PROOF OF THEOREM 2**

The theorem will be proved by contradiction. Suppose that the length of a longest path $P=v_0\cdots v_p$ joining $x=v_0$ and $y=v_p$ is less than $3k-6$ and $G\setminus V(P)$ is not empty.

**PART ONE.** In this part, we will show that $G\setminus V(P)$ is an independent set of $G$. The following lemmas will be applied in this part.

**LEMMA 1.1.** (Lemma 4, [3]) Let $H$ be a 2-connected graph and $Q=u_0\cdots u_q$ be a longest path of $H$. Then
\[
L_H(x,y)\geq\min\{d(u_0),d(u_q)\}
\]
for any pair of distinct vertices $x$ and $y$ in $H$. 

```
Let $C$ be a set and $\{A_1, \ldots, A_a\}, \{B_1, \ldots, B_h\}$ be partitions of $C$ such that $a \geq 2$ and $|A_u \cap B_j| \leq 1$ for any $u \in \{1, \ldots, a\}$ and any $j \in \{1, \ldots, h\}$. If $B_1 \cap A_1 \neq \emptyset$, $B_j \cap A_0 \neq \emptyset$ and $B_{i+1} = \cdots = B_{j-1} = \emptyset$

for some $u, \emptyset \in \{1, \ldots, a\}$ and $u \neq \emptyset$, then $\{i, \ldots, j\}$ is called a closed extendible interval of $\{B_1, \ldots, B_h\}$.

**Lemma 1.2** (Lemma 3.2, [6]) Let $C$ be a set, $\{A_1, \ldots, A_a\}$ and $\{B_1, \ldots, B_h\}$ be partitions of $C$ defined as above. If $s$ is an integer such that $a \geq s$ and $|A_u| \geq s$ for each $u \in \{1, \ldots, a\}$, then $\{B_1, \ldots, B_h\}$ has at least $s-1$ closed extendible intervals.

Suppose that $G \setminus V(P)$ is not an independent set and let $W_0$ be a component of $G \setminus V(P)$ which contains at least two vertices. Let $T_1, \ldots, T_t$ be all end-blocks of $W_0$. (An end-block of $W_0$ is a block of $W_0$ which contains at most one cut-vertex of $W_0$).

1. We claim that there exists a longest path $Q_i = x_i^1 \ldots x_i^q$ in each $T_i$ such that

\[(i) \ d_{W_0}(x_i^1) \leq d_{W_0}(x_i^q) \text{ and } x_i^1 \text{ is not a cut-vertex of } W_0, \text{ and} \]

\[(ii) \ d_{W_0}(x_i^q) \text{ is as big as possible.} \]

Let $R = y_1, \ldots, y_r$ be a longest path in $T_i$ such that $d_{W_0}(y_1) \leq d_{W_0}(y_r)$. 


(a) If \( y_1 \) is a cut-vertex of \( W_0 \) and \( d_{T_1}(y_1) \geq 2 \), then there is another longest path \( y_u \overrightarrow{R} y_{u+1} \overrightarrow{R} y_r \) or \( y_r \overrightarrow{R} y_{u+1} y_1 \overrightarrow{R} y_u \) satisfying (i) for any \( y_{u+1} \in N_R(y_1) \setminus \{y_2\} \). Of all longest paths in \( T_1 \) satisfying (i), let
\[
Q_1 = x^1_1 \cdots x^1_q \] be the one with the largest \( d_{W_0}(x^1_1) \).

(b) If \( y_1 \) is a cut-vertex of \( W_0 \) and \( d_{T_1}(y_1) = 1 \), then \( |T_1| = 2 \) and \( R = y_1 y_2 \) since \( T_1 \) is a block. Hence \( d_{W_0}(y_2) = 1 \) and \( d_{W_0}(y_1) > 1 \) because \( y_1 \) is a cut-vertex of \( W_0 \). It contradicts the assumption that \( d_{W_0}(y_1) \leq d_{W_0}(y_r) \).

II. Let \( d = \max\{d_{W_0}(x^1_i): i = 1, \ldots, t\} \). Without loss of generality, let
\[
d = d_{W_0}(x^1_1).\]

(i) When \( d \geq 2 \) and \( N^{-1}_{Q_1}(x^1_1) \cap \{\text{cut-vertices of } W_0\} = \emptyset \), let \( Z = N^{-1}_{Q_1}(x^1_1). \)

(ii) When \( d \geq 2 \) and \( x^1_0 \) is a vertex of \( N^{-1}_{Q_1}(x^1_1) \cap \{\text{cut-vertices of } W_0\} \).

Let \( Z = [N^{-1}_{Q_1}(x^1_1) \setminus \{x^1_0\}] \cup \{x^2_2\}. \)

In both cases (i) and (ii), we have that \( |Z| = |N^{-1}_{Q_1}(x^1_1)| - d_{W_0}(x^1_1) - d \), and by Lemma 1.1,
for each pair of distinct vertices, \( z, z' \in V(T_1) \). If \( z \in Z \cap V(T_1) \) and 
\( z' \in Z \setminus T_1 \) we have that \( z' = x_1^2 \) and 
\[
L_{W_0} (z, z') \geq L_{W_0} (z, x_1^1) + L_{W_0} (x_1^1, x_2^2)
\]
\[
\geq L_{T_1} (z, x_1^1)
\]
\[
\geq d
\]

By the choice of \( Q_1 \) and \( x_1^1 \), it follows that
\[
d = d_{W_0} (x_1^1) \geq d_{W_0} (z)
\]

for each \( z \in Z \).

(iii) When \( d = 1 \), \( T_1 \) is a single edge \((x_1^1, x_2^1)\). Hence, \( x_1^1 \) is a degree one vertex of \( W_0 \) and \( x_2^1 \) is either a cut-vertex of \( W_0 \) if \( W_0 \cap T_1 \), or a degree one vertex of \( W_0 \) if \( W_0 \cap T_1 \). If \( W_0 \cap T_1 \), then let \( Z = \{ x_1^1, x_2^1 \} \). If
by the choice of \( x_1^1 \), we must have that \( d_{W_0}(x_1^2) \leq d_{W_0}(x_1^1) \) and

\( x_1^2 \) is a degree one vertex of \( W_0 \). Then let

\[ Z = \{ x_1^1, x_1^2 \}. \]

Thus in either case, \( d_{W_0}(z) = 1 \) for any \( z \in Z \).

So we always have that

\[ |Z| = \max\{d, 2\}, \quad \cdots \cdots \quad (1) \]

\[ d_{W_0}(z, z') \geq d, \quad \cdots \cdots \quad (2) \]

\[ d_{W_0}(z) \leq d \text{ and } d_P(z) \geq k-d \quad \cdots \cdots \quad (3) \]

for each pair of distinct vertices \( z \) and \( z' \) of \( Z \). And

\[ |T_1| \geq d+1 \quad \cdots \cdots \quad (4) \]

since \( d = d_{W_0}(x_1^1) = d_{T_1}(x_1^1) \).

III. We claim that \( 1 \leq d \leq k-4 \).

Suppose that \( d \geq k-3 \). Since \( G \) is \( k \)-connected, there are four intermediately disjoint paths \( P_1 = v_{i_1} \cdots v_{i_\mu} \) joining \( T_1 \) and \( P \) for

\[ \mu = 1, \cdots, \mu \] where \( \{ v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4} \} \) are distinct vertices of \( P \),

\[ 0 \leq i_1 < i_2 < i_3 < i_4 \leq p, \{ x_1, x_2, x_3, x_4 \} \text{ belong to } T_1 \text{ and} \]

\[ |\{ x_1, \cdots, x_4 \}| = \min\{ |T_1|, 4 \}. \]
Let $R_u$ be a path joining $x_u$ and $x_{u+1}$ in $T_1$ such that $R_u$ is of length at least $d$ if $x_u \neq x_{u+1}$ (by Lemma 1.1), or $R_u = x_u$ if $x_u = x_{u+1}$. Then

$$\ell(v_i P v_{i+1}) \geq \ell(v_i P R x u R x_{u+1} P + 1 v_i) \geq d + 2$$

if $x_u \neq x_{u+1}$, or

$$\ell(v_i P v_{i+1}) \geq \ell(v_i P x P + 1 v_i) \geq 2$$

if $x_u = x_{u+1}$ since $P$ is a longest path joining $v_0$ and $v_p$.

If $|T_1| \geq 4$, then $\{x_1, x_2, x_3, x_4\}$ are a set distinct vertices and

$$\ell(P) \geq \frac{3}{\mu = 1} \ell(v_i P v_{i+1})$$

$$\geq 3(d + 2)$$

$$\geq 3k - 3 \quad \text{(by $d \geq k - 3$).}$$

It contradicts the assumption that $\ell(P) < 3k - 6$. Therefore $|T_1| \leq 3$ and some $x_i$ and $x_j$ of $\{x_1, x_2, x_3, x_4\}$ are the same vertex. However,

$$3k - 7 \geq \ell(P) \geq \frac{3}{\mu = 1} \ell(v_i P v_{i+1})$$

$$\geq \sum_{\mu \neq u} \ell(v_i P v_{i+1}) + \sum_{\mu = u} \ell(v_i P v_{i+1})$$

$$\geq (d + 2)(|T_1| - 1) + 2(4 - |T_1|)$$

$$= d(|T_1| - 1) + 6$$

$$\geq d^2 + 6 \quad \text{(by (4))}$$

$$\geq k^2 - 6k + 15 \quad \text{(by $d \geq k - 3$).}$$
Thus $0 \geq k^2 - 9k + 22$. But the value of $k^2 - 9k + 22$ is always positive for any $k$. It leads to a contradiction and follows our claim.

IV. Now we wish to show the following inequality

$$l(P) \geq (k-d-1)(d+2)$$ \hspace{1cm} (5)

Let $z, z'$ be a pair of distinct vertices of $Z$. We have known that

$d_p(z), d_p(z') \geq k - d$ and $\ell_{w_0}(z, z') \geq d$ (by (2) and (3)). Let

$$N_p(z) \cap N_p(z') = \sigma(z, z')$. Since $P$ is a longest path joining $v_0$ and $v_p$, $N_p(z) \cup N_p(z')$ does not contain two consecutive vertices of $P$. Let

$$\{v_1, \cdots, v_r\} = N_p(z) \cup N_p(z')$. Then $[v_1, \cdots, v_r] \cap [N_p(z) \cup N_p(z')]$ contains $r - 1$ open segments. A segment $v_i P v_{i+1}$ is called extendible with respect to $\{v, z', v_{i+1}\}$ if either $v_i \in N(z)$ and $v_{i+1} \notin N(z)$ or $v_i \notin N(z)$ and $v_{i+1} \in N(z)$.

Otherwise, it is called unextendible. It is not very hard to see that $P$ has at least $\sigma(z, z') - 1$ extendible segments with respect to $[z, z']$. Since $P$ is a longest path joining $v_0$ and $v_p$ and $\ell_{w_0}(z, z') \geq d$, each extendible segment is of length at least $d + 2$ and each unextendible segment is of length at least two.

(i) If there is a pair of distinct vertices $\{z_1, z_2\}$ of $Z$ such that $P$ has $\sigma(z_1, z_2)$ or $\sigma(z, z') - 1$ extendible segments with respect to $\{z_1, z_2\}$ then one of $\{N_p(z_1), N_p(z_2)\}$ must be a subset of another one and
\( \chi(z_1, z_2) \leq \min \{ |N_p(z_1)|, |N_p(z_2)| \} \geq k-d. \)

So

\[
I(P) \geq \text{(total length of all extendible segments)} \\
\geq (d+2)(\sigma(z_1, z_2) - 1) \\
\geq (d+2)(k-d-1). \quad \text{(since } \sigma(z_1, z_2) \geq k-d) \]

Thus we have established the inequality (5) in this case, and therefore we will assume that \( P \) has at least \( \alpha(z, z') + 1 \) extendible segments with respect to any pair of distinct vertices \( \{z, z'\} \) of \( Z \).

(ii) Case 1. \( d \leq \frac{k}{2} \)

Let \( \sigma = \max \{ \sigma(z, z') \mid z, z' \text{ are a pair of distinct vertices of } Z \} \).
Choose a pair of distinct vertices \( z_1 \) and \( z_2 \) of \( Z \) such that \( \sigma(z_1, z_2) = \sigma \)
and let \( r = |N_p(z_1) \cup N_p(z_2)| \). It is clear that

\[
r + \sigma = |N_p(z_1)| + |N_p(z_2)| \geq 2(k-d) \quad \ldots \ldots \quad (6)
\
r = |N_p(z_1)| \geq k - d \quad \ldots \ldots \quad (7)

Since \( P \) has at least \( \sigma + 1 \) extendible segments with respect to \( \{z_1, z_2\} \), we have that

\[
I(P) \leq \text{(total length of all extendible segments with} \\
\text{respect to } \{z_1, z_2\} + \\
\text{(total length of all unextendible segments with} \\
\text{respect to } \{z_1, z_2\}) \\
\geq (d+2)(\sigma+1) + 2[(r-1)-(\sigma+1)] \\
= 2r + 2d + d - 2 \\
\geq 2[2(k-d)-\sigma] + 2d + d - 2 \quad \text{(since } r \geq 2(k-d) - \sigma \text{ by (6)})
\]
\[ \begin{align*}
&= 4k - 4d - 2a + ad + d - 2 \\
&= (4k - 2d) - 2d + (a + 1)(d - 2) \\
&\geq 3k - 2d + (a + 1)(d - 2) \\
&\quad \text{ (since } d \leq \frac{k}{2})
\end{align*} \]

Thus \[ 3k - 7 \geq l(P) \geq 3k - 2d + (a + 1)(d - 2) \] \hspace{1cm} (8)

if \( a \geq 1 \), by (8), we have that
\[ 3k - 7 \geq 3k - 2d + 2(d - 2) \]
\[ = 3k - 4. \]

It is a contradiction and hence we have that \( a = 0 \). If \( d \leq 4 \), by (8), we have that
\[ 3k - 7 \geq l(P) \geq 3k - 2d + (d - 2) \]
\[ \quad \text{ (since } a = 0) \]
\[ \geq 3k - 6 \]
\[ \quad \text{ (since } d \leq 4). \]

It is also a contradiction and therefore we must have that \( d \geq 5 \). Note that
\[ |Z| \geq d \geq 5 \]
let \( z, z', z'' \) be three distinct vertices of \( Z \). By the definition of \( a \) and \( a = 0 \), the subsets \( N_p(z) \), \( N_p(z') \) and \( N_p(z'') \) of \( V(P) \) are pairwise disjoint. Hence
\[ |N_p(z) \cup N_p(z') \cup N_p(z'')| \geq 3(k - d) \]
and \( P \) has at least \( 3(k - d) - 1 \) segments each of which is of length at least two. So
\[ l(P) \geq 2[3(k - d) - 1] \]
\[ = 6k - 6d - 2 \]
\[ \geq 3k - 2 \]
\[ \quad \text{ (since } d \leq \frac{k}{2}). \]

It contradicts that \( l(P) \leq 3k - 7 \).
(iii) Case 2. \( d \geq \frac{k}{2} \).

Let \( C=E(Z,P) \) be a set and

\[ \{ A_z = E(z,P) : \text{for each } z \in Z \} \]

and

\[ \{ B_i = E(Z,v_i) : \text{for each } v_i \in V(P) \} \]

be partitions of \( C \). Note that \( |A_z| = |Z| - d \geq k-d \) and \( |A_z| = dp(z) \geq k-d \) for any \( z \in Z \) (by (3)), \( |A_z \cap B_i| \leq 1 \) for any \( z \in Z \) and \( v_i \in V(P) \). We can apply Lemma 1.2 on \( C \) and these two partitions of \( C \). Thus \( P \) has at least \( k-d+1 \) extendible segments each of which is of length at least \( d+2 \) and therefore

\[ \ell(P) \geq (\text{total length of all extendible segments}) \]

\[ \geq (d+2)(k-d-1) \]

and the inequality (5) holds for all cases.

V. Since \( 1 \leq d \leq k-4 \), the minimum value of \( (d+2)(k-d-1) \) is \( 3k-6 \) it contradicts that \( \ell(P) < 3k-6 \) and therefore, \( C \setminus V(P) \) is an independent set.

Part two.

It has been shown in part one that \( W=C \setminus V(P) \) is an independent set. Let \( w \in W \). Following [5], put \( Y_0 = \emptyset \) and for \( i \geq 1 \), put

\[ X_i = N(Y_{i-1} \cup \{W\}) \]

and

\[ Y_i = \{ v_j \in V(P) : v_{j-1} \in X_i \text{ and } v_{j+1} \in X_i \} \].

Thus \( N(w) \subseteq X_1 \subseteq X_2 \cdots \) and \( \emptyset = Y_0 \subseteq Y_1 \subseteq Y_2 \cdots \).
Put $X = \bigcup_{i=1}^{\infty} X_i$ and $Y = \bigcup_{i=1}^{\infty} Y_i$. The follow lemma has been proved in [6] and will be applied in this part of the proof.

**LEMMA 2.1.**

(i) (direct conclusion of the definition) $Y \subseteq V(P) \setminus \{v_0, v_p\}$ and $Y = (X \cap P)^{-1} \cup (X \cap P)^{-1}$.

(ii) (Lemma 4.4. [6]) $X$ does not contain two consecutive vertices of $P$.

(iii) (Lemma 4.4. [6]) $X \cap Y = \emptyset$.

(iv) (Lemma 4.7. [6]) $Y \cup W$ is an independent set of $G$, $N(Y) \subseteq V(P)$ and $N(Y \cup \{w\}) = X \subseteq V(P)$.

(v) $e(X, Y \cup \{w\}) = k(|Y| + 1)$ and $e(V', Y \cup \{w\}) = 0$ for any subset $V'$ of $V(G) \setminus X$.

**Proof.** We only need to prove (v). By (i) $v_0, v_p \not\in Y \cup \{w\}$, it follows that $d(v_i) = k$ for any $v_i \in Y \cup \{w\}$. Since $X = N(Y \cup \{w\})$, $e(Y \cup \{w\}, X) = e(Y \cup \{w\}, G) = k |Y \cup \{w\}|$ and $N(Y \cup \{w\}) \cap V' = \emptyset$ for any subset $V'$ of $V(G) \setminus X$.

Put $|X| = x$ and $|Y| = y$. Then $P \setminus XVY$ is a union of at most $x^2y^2$ segments of $P$. Let $S_1, \ldots, S_{t-1}$ be the segments of $P \setminus XVY$ not containing $v_0$ and $v_p$. Let $S_0$ (or $S_t$) be the segment of $P \setminus XVY$ containing $v_0$ (or $v_p$, respectively) if $v_0$ (or $v_p$, respectively) does not belong to $X$.

Obviously, $S_0 = \emptyset$ (or $S_t = \emptyset$) if $v_0 \not\in X$ (or $v_p \not\in X$, respectively). It is easy
to see that $|S_i| \geq 2$ for $1 \leq i \leq t-1$ and $t=x-\psi$. Let $S = \bigcup_{i=0}^{t} S_i$. Here $V(P) = X \cup Y \cup S$, by (i) and (iv) of Lemma 2.1.

Here $V(P) = X \cup Y \cup S$, by (i) and (iv) of Lemma 2.1.

Case 1. $S \neq \emptyset$.

Let $Z_i = S_i \cap (X^t \cup X^{-t})$ and $Z = \bigcup_{i=0}^{t} Z_i$. We have that

**LEMMA 2.2** (Lemma 4.8, [6])

$$e(Z,S) \leq (t-\lambda)(|S| - t + 3)$$

where $\lambda = 0$ if $S_0 \cup S_t \neq \emptyset$ and $\lambda = 1$ if $S_0 \cup S_t = \emptyset$.

and

**LEMMA 2.3** ( Lemma 4.9, [6])

$$e(X,W \setminus \{w\}) \geq e(Z,W \setminus \{w\})$$

Now we can prove our theorem in this case. Since

$$kX \geq e(X,G) \geq e(X,Z) + e(X,Y \cup \{w\}) = e(X,W \setminus \{w\})$$

and

$$k |Z| = e(Z,G) = e(Z,X) + e(Z,Y \cup \{w\}) + e(Z,S) + e(Z,W \setminus \{w\}),$$

we have that

$$kX - e(X,Y \cup \{w\}) - e(X,W \setminus \{w\}) \geq e(X,Z)$$

$$= e(Z,X)$$

$$= k |Z| - e(Z,S) - e(Z,W \setminus \{w\}) - e(Z,Y \cup \{w\}).$$

Thus

$$kX - k(\psi + 1) - e(X,W \setminus \{w\})$$

$$\geq k |Z| - e(Z,S) - e(Z,W \setminus \{w\})$$

by (v) of Lemma 2.1. Note that $x-\psi = t$ and

$$e(X,W \setminus \{w\}) \geq e(Z,W \setminus \{w\})$$

(by Lemma 2.3), it follows that
\[ e(Z, S) \geq -kt+k+|Z|. \]

When \( S_0 \cup S_t \neq \emptyset, |Z| \geq 2t-1. \) By Lemma 2.2,

\[ t(|S|-t+3) \geq -kt+k+2k(t-1). \]

Simplifying the above inequality, we have that

\[ |S| \geq t-3+k. \]

When \( S_0 \cup S_t = \emptyset, |Z| = 2(t-1). \) By Lemma 2.2,

\[ (t-1)(|S|-t+3) \geq -kt+k+2k(t-1). \]

Simplifying the above inequality, we obtain the inequality (9) again. Since \( V(P) = S \cup X \cup Y, \) and \( t+4 = x \geq |N(w)| = k, \)

\[ \ell(P) + 1 = |V(P)| - |S| \cup X \cup Y | \]

\[ \geq (t-3+k) + x + \psi \]

\[ = k + 2x - 3 \]

\[ \geq 3k-3 \]

It contradicts that \( \ell(P) < 3k-6 \) and therefore the path joining \( v_0 \) and \( v_p \)

is of length at least \( 3k-6 \) in the case of \( S \neq \emptyset. \)

Case two. \( S = \emptyset. \) In this case, we must have \( p=\ell(P) \) is even and

\( X = \{ v_{2i} : i=0, \cdots, \frac{p}{2} \}, Y = \{ v_{2i-1} : i=1, \cdots, \frac{p}{2} \}. \) Thus \( |Yu[w]| = |X|. \) We claim

that \( X \) is also an independent set and \( N(X) \subseteq Yu[w]. \) By (v) of Lemma 2.1, we have that

\[ e(Yu[w], X) = k \mid Yu[w] \mid = k \mid X \mid. \]

Since the maximum degree of \( G \) is \( k, \) all neighbors of every vertex of \( X \)

are contained in \( Yu[w]. \)

Moreover, by (iv) of Lemma 2.1, both \( X \) and \( Yu[w] \) are independent sets and

\[ E(X,Yu[w]) = E(X,G) = E(G,Yu[w]). \]
The connectivity of $G$ implies that $V(G)=X\cup Y\cup \{w\}$. Thus $(X,Y \setminus \{w\})$ is a bipartition of $G$ and $v_0,v_p$ are joined by a path of length $|V(G)|-2$.

REFERENCES


[6]. Zhang, C. Q. and Zhu, Y. J., Hamiltonian connectivity and factorization.

fig. 1.