### SOME FACETS OF THE TRI-INDEX ASSIGNMENT POLYTOPE(U)

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END
MICROCOPY RESOLUTION TEST CHART
NATIONAL INSTITUTE OF STANDARDS PHYSICS

1.0  1.1  1.25  1.4  1.6

0.25  0.20  0.18

0.22  0.20

0.17  0.18
SOME FACETS OF THE THREE-INDEX ASSIGNMENT POLYTOPE

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Abstract

Given three disjoint n-sets and the family of all weighted triplets that contain exactly one element of each set, the 3-index assignment (or 3-dimensional matching) problem asks for a minimum-weight subcollection of triplets that covers exactly (i.e., partitions) the union of the three sets. Unlike the common (2-index) assignment problem, the 3-index problem is NP-complete. In this paper we examine the facial structure of the 3-index assignment polytope (the convex hull of feasible solutions to the problem) with the aid of the intersection graph of the coefficient matrix of the problem’s constraint set. In particular, we describe the cliques of the intersection graph as belonging to three distinct classes, and show that cliques in two of the three classes induce inequalities that define facets of our polytope. Furthermore, we give an O(n^4) procedure (note that the number of variables is n^3) for finding a facet-defining clique-inequality violated by a given noninteger solution to the linear programming relaxation of the 3-index assignment problem, or showing that no such inequality exists.
1. Introduction

The (axial) three-index assignment problem, to be denoted AP3, also known as the (axial) three-dimensional matching problem, can be stated as follows: given three disjoint n-sets, I, J, and K, and a weight $c_{ijk}$ associated with each ordered triplet $(i,j,k) \in I \times J \times K$, find a minimum-weight collection of n disjoint triplets $(i,j,k) \in I \times J \times K$.

An alternative interpretation of AP3 is as follows. A graph is complete if all of its nodes are pairwise adjacent. A maximal complete subgraph of a graph is a clique. A graph is $k$-partite if its nodes can be partitioned into k subsets such that no two nodes in the same subset are joined by an edge. It is complete $k$-partite, if every node is adjacent to all other nodes except those in its own subset. The complete $k$-partite graph with $n_i$ nodes in its $i^{th}$ part (subset) is denoted $K_{n_1, n_2, \ldots, n_k}$.

Consider now the complete tri-partite graph $K_{n,n,n}$ with node set $S = I \cup J \cup K$, $|I| = |J| = |K| = n$. Figure 1 shows $K_{n,n,n}$ for $n=2$ and $n=3$. $K_{n,n,n}$ has $3n$ nodes and $n^3$ cliques, all of which are triangles containing exactly one node from each of the three sets I, J, K. Let $(i,j,k)$ denote the clique induced by the node set $(i,j,k)$. If a weight $c_{ijk}$ is associated with each clique $(i,j,k)$, then AP3 is the problem of finding a minimum-weight exact clique cover of the nodes of $K_{n,n,n}$, where an exact clique cover is a set of cliques that partitions the node set $S$. 
AP3 can be stated as a 0-1 programming problem as follows:

$$\text{max } \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} c_{ijk} x_{ijk}$$

s.t. $$\sum_{j \in J} \sum_{k \in K} x_{ijk} = 1 \text{ } \forall \text{ } i \in I$$
$$\sum_{i \in I} \sum_{k \in K} x_{ijk} = 1 \text{ } \forall \text{ } j \in J$$
$$\sum_{i \in I} \sum_{j \in J} x_{ijk} = 1 \text{ } \forall \text{ } k \in K$$
$$x_{ijk} \in \{0, 1\} \text{ } \forall \text{ } i,j,k$$

where I, J and K are disjoint sets with $|I| = |J| = |K| = n$. The coefficient matrix of AP3 for the case $n=3$ is shown in figure 2.
One interpretation of the constraints is in terms of an \( n \times n \times n \) cube in three-space made up of \( n^3 \) unit cubes, each to be assigned a value of zero or one so that the following conditions hold: if the cube is viewed as a set of \( j,k \)-planes stacked up in the direction of the \( i \)-axis, the first set of constraints requires that the total value of the variables in each such plane be exactly one; and a similar interpretation holds for \( i,k \)-planes (the second group of constraints, corresponding to the \( j \)-axis) as well as for \( i,j \)-planes (the third group, corresponding to the \( k \)-axis). Figure 3 illustrates this for \( n=3 \).
We will denote by \( \text{AP}_3^n \) the (axial) 3-index assignment problem of order \( n \) (i.e., defined for \( n \)-sets), by \( A_n \) the coefficient matrix of its constraint set, and by \( I_n, J_n, K_n \) the 3 associated index sets. The row and column index sets of \( A_n \) will be denoted by \( R_n \) and \( S_n \) respectively. Clearly, \(|R_n| = |I_n| + |J_n| + |K_n| = 3n\) and \( S_n = |I_n| \times |J_n| \times |K_n| = n^3 \).

In terms of \( K_{n,n,n} \), \( A_n \) is the incidence matrix of nodes versus cliques (triangles): it has a row for every node and a column for every clique of \( K_{n,n,n} \).

As usual, the support of a (row or column) vector is understood to mean the index set of its nonzero components. Each element of \( S \) (that indexes a column of \( A_n \) and a clique of \( K_{n,n,n} \)) will also be used to denote the support of the given column of \( A_n \) and the node set of the given clique (triangle) of \( K_{n,n,n} \). Thus, if \( a^S \) has support \((i,j,k)\) (i.e., if clique \( s \) of \( K_{n,n,n} \) has node set \([i,j,k]\)), we will write \( s = (i,j,k) \) or \( a^S = a^{ijk} \), meaning that column \( a^S \) has ones in positions \( i \in I, j \in J \) and \( k \in K \).

\( \text{AP}_3^n \) is a special case of the (axial) 3-dimensional transportation problem, in which the right-hand sides of the constraints can be any positive integers, the sets \( I,J,K \) are not necessarily equal in size, and the integrality constraints are relaxed. This is in turn a generalization of the well-known transportation problem, a special case of which is the simple assignment problem.

Our problem is called axial to distinguish it from another 3-dimensional assignment problem, called planar, which can be formulated as follows:

\[
\begin{align*}
\text{max} \quad & \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} c_{ijk} x_{ijk} \\
\text{s.t.} \quad & \sum_{i \in I} x_{ijk} = 1 \quad \forall j \in J, k \in K \\
& \sum_{j \in J} x_{ijk} = 1 \quad \forall i \in I, k \in K \\
& \sum_{k \in K} x_{ijk} = 1 \quad \forall i \in I, j \in J
\end{align*}
\]
\[ x_{ijk} \in \{0, 1\} \quad \forall \ i, j, k \]

The coefficient matrix of the planar problem for \( n=3 \) is shown in figure 4.

![Coefficient Matrix](image)

**Figure 4**
This problem is a special case of the planar 3-dimensional transportation problem. If we view the cube as a plane of $n^2$ columns of unit cubes, then the constraints of the planar problem require that the sum of the values of variables in each column be exactly one, in each of the three possible orientations. This is illustrated in Figure 5.

The 3-dimensional transportation problem (TR3) in these and other formulations was first studied by Schell [20]. The literature on this problem includes the references \{2,4,5,9,10,12,13,14,15,18,19,20,21\}. The original motivation for considering this model was a problem in the transportation of
goods of several types from multiple sources to multiple destinations. Applications of AP3 mentioned in the literature include the following (Pierskalla [18,19]).

- In a rolling mill with |I| soaking pits (temperature stabilizing baths), schedule |K| ingots through the pits so as to minimize idle-time for the rolling mill (the next stage in the process).
- Find a minimum cost schedule of a set of capital investments (e.g., warehouses or plants) in different locations at different times.
- Assign troops to locations over time to maximize a measure of capability.
- Launch a number of satellites in different directions at different altitudes to optimize coverage or minimize cost.

AP3 is known to be an NP-complete problem [11]. Obviously, AP3 is a special case of the set partitioning problem (SPP):

\[
\begin{align*}
\max \ c^T x \\
\text{s.t.} \ B x &= e \\
x &\in \{0,1\}^q
\end{align*}
\]

where \( B = (b_{rs}) \) is a matrix of zeroes and ones and \( e \) is a vector of ones. The set being partitioned in this case is \( R = I \cup J \cup K \), with the rows of \( B \) corresponding to the elements \( r \in R \). The subsets \( s = \{i,j,k\} \) from which the partition is to be selected are those subsets (of cardinality three) which contain one element from each of the sets \( I, J \) and \( K \). Each column \( b^s \) of \( B \) is the incidence vector of one such subset (i.e., \( b^s \) has a one in each row corresponding to an element in the subset and zeroes elsewhere). The set packing relaxation (SP) of (1) is the program:

\[
\begin{align*}
\max \ c^T x \\
\text{s.t.} \ B x &\leq e
\end{align*}
\]
x ∈ \{0,1\}^q.

For properties of SPP and SP see the survey [3].

Let \( P_I \) denote the convex hull of feasible solutions to \( AP_3 \), i.e.,

\[
P_I = \text{conv} \{ x \in \{0,1\}^n : A_n x = e \}.
\]

The intersection graph \( G_A = (V,E) \) of a 0-1 matrix \( A \) has a node \( s \) for every column \( a^s \) of \( A \), and an edge \((s,t)\) for every pair of columns \( a^s, a^t \) such that \( a^s \cdot a^t \neq 0 \). The intersection graph \( G_{A_n} \) of \( A_n \) is the clique-intersection graph of \( K_{n,n,n} \), i.e., \( G_{A_n} \) has a node for every clique (triangle) of \( K_{n,n,n} \), and an edge for every pair of triangles that share some node of \( K_{n,n,n} \). The graph \( G_{A_n} \) for \( n=2 \) is shown in figure 6.

\[\text{Figure 6}\]
Although the 3-index assignment problem has a sizeable literature, no work has been done on describing the polytope $P_I$. In this paper we apply the tools of polyhedral combinatorics to $AP_3^n$ and obtain a partial characterization of the facial structure of $P_I$. In particular, in section 2 we identify three classes of cliques of the intersection graph of $A_n$ and show that they are exhaustive. These cliques are known to define facets of the polytope

$$P_I = \text{conv} \{ x \in \{0,1\}^n | A_n x \leq e \},$$

the set packing relaxation of the set partitioning polytope $P_I$. In section 3 we show that two of the 3 classes of cliques also define facets of $P_I$ (whereas the cliques in the third class define improper faces), and that these facets are all distinct. Finally, in section 4 we give an $O(n^4)$ procedure for detecting a clique inequality violated by some solution to the linear programming relaxation of $P_I$, or showing that no such inequality exists.

2. The Cliques of $G_A$

In this section we identify all the cliques of $G_A$, the intersection graph of $A$.

For any subset $V \subseteq S$ of the node set of $G_A$, we will denote by $<V>$ the subgraph induced by $V$. For $r \in R$, we will denote by $S^r$ the support of row $r$ of $A$, i.e., $S^r = \{ s \in S | a_{rs} = 1 \}$.

**Proposition 2.1.** For each $r \in R$, the node set $S^r$ induces a clique (of cardinality $n^2$) in $G_A$. 
Proof. The subgraph $<S^r>$ is obviously complete. To see that it is maximal, assume w.l.o.g. that $r \in I$ (an analogous reasoning holds if $r \in J$ or $r \in K$). Now let $s \in S \setminus S^r$ be arbitrarily chosen, and let $s = (i_0, j_0, k_0)$. Since $s \not\in S^r$, $r \not= i_0$; and since $S^r$ contains all triplets whose first element is $r$, there exists $t \in S^r$, $t = (r, j, k)$, such that $r \not= i_0$, $j \not= j_0$, $k \not= k_0$. Hence $S^r \cup \{s\}$ does not induce a complete subgraph of $G_A$; and since this is true of any $s \in S \setminus S^r$, the subgraph of $G_A$ induced by $S^r$ is maximally complete, i.e., a clique. Furthermore, $|S^r| = n^2$ for all $r \in R$.}

The set of cliques defined by Proposition 2.1 will be called class 1 and denoted $Q_1$. Clearly, $|Q_1| = 3n$. In terms of $K_{n,n,n}$, the clique of class 1 corresponding to row $r$ of $A$ contains those nodes of the intersection graph $G_A$, whose associated triangles in $K_{n,n,n}$ share node $r$ of $K_{n,n,n}$.

**Proposition 2.2.** For every $s \in S$, let

$$T(s) = \{t \in S \setminus \{s\} | a^s \cdot a^t = 2\}.$$  

Then the node set $\{s\} \cup T(s)$ induces a clique of size $3n - 2$ in $G_A$.

Proof. Let $s = (i_0, j_0, k_0)$, and let $t_1, t_2 \in T(s)$ be chosen arbitrarily, with $t_1 \neq t_2$. Since each of $t_1$ and $t_2$ contains two of the three elements $i_0, j_0, k_0$, $t_1$ and $t_2$ must have at least one element in common. Hence the node set $\{s\} \cup T(s)$ induces a complete subgraph in $G_A$. Now let $u \in S \setminus (\{s\} \cup T(s))$. Then the triplet $u = (i, j, k)$ contains at most one element of $s$. If $a^u \cdot a^s = 0$, we are done. Assume now that $a^u \cdot a^s = 1$, with $i = i_0$ (a similar reasoning holds if $j = j_0$ or $k = k_0$). Then $j \neq j_0$ and $k \neq k_0$. Now by definition, $T(s)$ contains some $t = (i_*, j_0, k_0)$ such that $i_* \neq i_0 (=i)$. But then $a^u \cdot a^t = 0$, i.e., $\{u\} \cup \{s\} \cup T(s)$ does not define a complete subgraph of $G_A$. Since the choice of $u$ was arbitrary, the subgraph defined by $\{s\} \cup T(s)$ is maximal complete.
For each $s \in S$ and for each of the three pairs of the triplet $s = (i_0, j_0, k_0)$, there are $n-1$ other triplets in $S$ containing the same pair; hence $|T(s)| = 3(n-1)$, and thus $(s) \cup T(s)$ has $3n-2$ elements.\|

The set of cliques defined in Proposition 2.2 will be called \textbf{class 2} and denoted $Q_2$. There is exactly one clique of class 2 for every column of $A$, and there is no double counting; hence $|Q_2| = n^3$. In terms of $K_{n,n,n}$, the clique of class 2 corresponding to column $s = (i_0, j_0, k_0)$ of $A$ contains the node of $G_A$ corresponding to the clique $(i_0, j_0, k_0)$ of $K_{n,n,n}$, along with the $3(n-1)$ nodes of $G_A$ corresponding to those cliques of $K_{n,n,n}$ that share an edge (a pair of nodes) with the clique $(i_0, j_0, k_0)$.

Proposition 2.3. For every ordered pair $s,t \in S$ such that $a^s \cdot a^t = 0$, let $t_1, t_2, t_3 \in S \setminus \{s,t\}$ be the (uniquely defined) triplets such that

\[ a^s \cdot a^{t_i} = 1, \quad a^t \cdot a^{t_i} = 2, \quad i = 1,2,3. \]

Then the node set $(s,t_1,t_2,t_3)$ induces a (4-)clique in $G_A$.

\textbf{Proof.} Let $s,t \in S$, with $a^s \cdot a^t = 0$, and let $s = (i_s,j_s,k_s)$, $t = (i_t,j_t,k_t)$. Then $t_1 = (i_s,j_t,k_t)$, $t_2 = (i_t,j_s,k_t)$ and $t_3 = (i_t,j_t,k_s)$ are the only 3 triplets in $S \setminus \{s,t\}$ that satisfy the requirements of the Proposition, i.e., they exist and are unique. Further, $a^s \cdot a^{t_i} = 1$ for $i=1,2,3$ and $a^t \cdot a^{t_j} = 2$ for all $i,j \in \{1,2,3\}$; hence $(s,t_1,t_2,t_3)$ induces a complete subgraph in $G_A$. To see that this subgraph is maximal, note that any triplet $u \in S \setminus \{s\}$ that contains an element of $s$, either contains two elements of $t$ (and hence is identical to one of the triplets $t_1$, $t_2$ or $t_3$), or else contains at most one element of $t$. But then $a^u \cdot a^{t_i} = 0$, where $t_i \in \{t_1,t_2,t_3\}$ is the triplet containing those two elements of $t$ not contained in $u$ (besides the element of $s$). Thus $(s,t_1,t_2,t_3)$ induces a maximal complete subgraph, hence a 4-clique in $G_A$.\}
The set of cliques described in Propositions 2.3 will be called class 3 and denoted \( Q_3 \). In terms of \( K_{n,n,n} \), every class 3 clique of \( G_A \) is associated with an ordered pair \((s,t)\) of disjoint triangles of \( K_{n,n,n} \), and its node set contains (a) the node of \( G_A \) corresponding to the triangle \( s \), and (b) the 3 nodes of \( G_A \) corresponding to those triangles \( t_1, t_2, t_3 \) of \( K_{n,n,n} \) that share 1 node with \( s \) and 2 nodes with \( t \).

As to the cardinality of \( Q_3 \), every ordered pair \((s,t)\) such that \( a_s^t = 0 \) gives rise to a clique of class 3. Since \(|S| = n^3 \) and for every \( s \in S \) that are \((n-1)^3 \) indices \( t \in S \) such that \( a_s^t = 0 \), the number of ordered pairs \((s,t)\) with \( a_s^t = 0 \) is \( n^3(n-1)^3 \).

To determine the number of cliques of class 3 we also need to know how many different ordered pairs give rise to the same clique. Let \( s = (i_s,j_s,k_s), t = (i_t,j_t,k_t), t_1 = (i_s,j_t,k_t), t_2 = (i_t,j_s,k_t), t_3 = (i_t,j_t,k_s) \), and denote by \( C(s,t) \) the node set of the clique (of class 3) corresponding to the ordered pair \((s,t)\), i.e. let \( C(s,t) := \{s,t_1,t_2,t_3\} \). Further, let \( \xi_1 = (i_t,j_s,k_s), \xi_2 = (i_s,j_t,k_s), \xi_3 = (i_s,j_s,k_t) \). Then we have

**Proposition 2.4.** \( C(s,t) = C(t_i, \xi_i) \) for \( i = 1,2,3 \).

**Proof.** Consider the ordered pair \((t_1, \xi_1)\). From the definitions, the 4 triplets of the set \( C(t_1, \xi_1) \) are \((i_s,j_t,k_t) = t_1, (i_s,j_s,k_s) = s, (i_t,j_t,k_s) = t_3, \) and \((i_t,j_s,k_t) = t_2; \) thus \( C(t_1, \xi_1) = C(s,t) \). By symmetry, \( C(t_i, \xi_i) = C(s,t) \) for \( i = 2,3 \). ||

**Corollary 2.5.** The number of cliques of class 3 is \( n^3(n-1)^3/4 \).

**Proof.** Every clique of class 3 arises from 4 distinct ordered pairs, and the number of the latter is \( n^3(n-1)^3 \). ||

Next we show that \( G_A \) has no other cliques than the ones described above. But first we need a property of \( G_A \).
Proposition 2.6. \( G_A \) is regular of degree \( 3n(n-1) \) and has \( \frac{3}{2} n^4(n-1) \) edges.

Proof. Let \( a^S \) be an arbitrary column of \( A \). There are \( (n-1)^3 \) columns \( a^t \) of \( A \) such that \( a^S \cdot a^t = 0 \), hence there are \( n^3 - 1 - (n-1)^3 = 3n(n-1) \) columns \( a^U \) of \( A \) such that \( a^S \cdot a^U \neq 0 \). Thus the degree of node \( s \) in \( G_A \) is \( 3n(n-1) \), and by symmetry this is true of all \( s \in S \). Since the number of edges of a graph is one half of the sum of the degrees of its nodes, \( G_A \) has \( \frac{1}{2} \cdot n^3 \cdot 3n(n-1) = \frac{3}{2} n^4(n-1) \) edges. \( || \)

Let \( Q \) denote the set of cliques of \( G_A \).

Theorem 2.7. The only cliques of \( G_A \) are those of classes 1, 2 and 3; i.e., \( Q = Q_1 \cup Q_2 \cup Q_3 \).

Proof. We will use induction on \( n \). For \( n = 2 \), the statement is found to be true by listing all the cliques of \( G_A \) (see Fig. 4). In fact, because of symmetry, it is sufficient to list the cliques containing a given node, say 1, and they are \( [1,2,3,4], [1,2,5,6], [1,3,5,7] \) (of class 1), \( [1,2,3,5], [1,2,4,6], [1,3,4,7], [1,5,6,7] \) (of class 2), and \( [1,4,6,7] \) (of class 3).

Suppose now that the statement is true for \( n = 2, \ldots, k \) and let \( n = k+1 \geq 3 \). Consider the relationship between \( G_{A_n-1} \) and \( G_{A_n} \). Note first that \( K_{n,n,n} \) is obtained from \( K_{n-1,n-1,n-1} \) by (a) adding three nodes \( i_*, j_*, k_* \), to the three sets \( I, J, K \), respectively, so that

\[
I_n = I_{n-1} \cup \{i_*\}, \quad J_n = J_{n-1} \cup \{j_*\} \quad \text{and} \quad K_n = K_{n-1} \cup \{k_*\};
\]

and (b) adding new edges \( (i_*, i) \) for all \( i \in J_n \cup K_n \), \( (j_*, i) \) for all \( i \in I_n \cup K_n \), and \( (k_*, i) \) for all \( i \in I_n \cup J_n \).

This creates \( 3n(n-1)+1 \) new triangles, namely:

(i) one new triangle \( (i_*, j_*, k_*) \) that shares no node with any triangle of \( K_{n-1,n-1,n-1} \);

(ii) \( 3(n-1) \) new triangles of the form \( (i, j_*, k_*) \) for all \( i \in I_{n-1} \),
(i, i, k) for all \( i \in J_{n-1} \), and (i, j, \( \star \)) for all \( i \in K_{n-1} \); and

(iii) \( 3(n-1)^2 \) new triangles of the form \((p, q, k)\) for all \((p, q) \in I_{n-1} \times J_{n-1}\), \((p, j, q)\) for all \((p, q) \in I_{n-1} \times K_{n-1}\), and \((i, p, q)\) for all \((p, q) \in J_{n-1} \times K_{n-1}\).

In terms of the coefficient matrix of AP3, \( A_n \) is obtained from \( A_{n-1} \) by adding 3 new rows, one to each of the sets \( I_{n-1}, J_{n-1}, K_{n-1} \), with zero entries in the positions indexed by \( S_{n-1} \); and adding \( 3n(n-1)+1 \) new columns (of dimension \( |R_n| = |R_{n-1}|+3 \)), with supports corresponding to the new triangles of \( K_{n,n,n} \) described under (iii) above.

It then follows that \( G_A \) is obtained from \( G_{A_{n-1}} \) by adding \( 3n(n-1)+1 \) new nodes corresponding to the triangles described under (i), (ii), (iii), and a new edge for every pair of nodes \((s, t)\) of \( G_{A_{n-1}} \) such that (a) at most one of \( s \) and \( t \) is a node of \( G_{A_{n-1}} \), and (b) the two triangles of \( K_{n,n,n} \) corresponding to \( s \) and \( t \) have at least one node (of \( K_{n,n,n} \)) in common. It also follows that \( G_{A_{n-1}} \) is an induced subgraph of \( G_A \).

Consider now the node set \( C \) of an arbitrary clique of \( G_{A_n} \). The restriction \( C \) of \( C \) to \( G_A \) induces a clique of \( G_{A_{n-1}} \), hence by the induction hypothesis \( <C> \) belongs to one of the three classes described in Propositions 2.1, 2.2, 2.3. With an argument analogous to the one used in the proof of the corresponding Proposition to show that a complete subgraph of the given class is maximal, we will show for each of the 3 cases that \( <C> \) belongs to the same class as \( <C> \). Since \( <C> \) was chosen arbitrarily, this will prove that the three classes of cliques are exhaustive for \( G_{A_n} = G_{A_{k+1}} \), thus completing the induction.

Suppose first that \( <C> \in Q_1 \). Then \( C \) is of the form \( C = S^j_{n-1} = \{ j \in S_{n-1} | a_{ij} = 1 \} \) for some \( i \in R_{n-1} \). W.l.o.g. we may assume that \( i \in I_{n-1} \) (an analogous reasoning holds if \( i \in J_{n-1} \), or \( i \in K_{n-1} \)).
Now suppose $\mathcal{C} \notin Q_1$. Then $\mathcal{C}$ has a node $s = \{i_0, j_0, k_0\}$ such that $i_0 \neq i$. Since $S_{n-1}^i$ contains all the triplets of $S_{n-1}$ whose first element is $i$, it certainly contains some $t = (i, j, k)$ such that $j \neq j_0$ and $k \neq k_0$. But then $a^s \cdot a^t = 0$ (where $a^s$ and $a^t$ are columns of $A_n$), hence we obtain the contradiction that $\mathcal{C}$ is not a clique.

Suppose next that $\mathcal{C} \in Q_2$. Then $\mathcal{C}$ is of the form $\{s\} \cup T(s)$, where $s \in S_{n-1}$ and $T(s) = \{t \in S_{n-1} \cap \{s\} | a^s \cdot a^t = 2\}$ ($a^s$ and $a^t$ being columns of $A_{n-1}$). Let $s = \{i_0, j_0, k_0\}$. Now suppose $\mathcal{C} \notin Q_2$. Then $\mathcal{C}$ has a node $u = (i, j, k)$, such that $a^s \cdot a^u \leq 1$ (where $a^s$ and $a^u$ are columns of $A_n$). Since $\mathcal{C}$ is a clique, $a^s \cdot a^u \neq 0$, hence $a^s \cdot a^u = 1$. W.l.o.g., assume that the common element of the triplets $s$ and $u$ is $i_0$ (a similar reasoning holds if the common element is $j_0$ or $k_0$). Then $j \neq j_0$ and $k \neq k_0$. By definition, $T(s)$ contains some $t = (i, j_0, k_0)$ such that $i \neq i_0(=i)$, and $t \in T(s)$ implies $t \in \mathcal{C}$. But then $a^u \cdot a^t = 0$, contradicting the assumption that $\mathcal{C}$ is a clique.

Finally, suppose $\mathcal{C} \in Q_3$. Then $\mathcal{C}$ is of the form $\{s, t_1, t_2, t_3\}$, where $s, t_1, t_2, t_3 \in S_{n-1}$ are distinct and such that

$$a^s \cdot a^{t_i} = 1, \quad a^t \cdot a^{t_i} = 2, \quad i = 1, 2, 3,$$

for some $t \in S_{n-1}$ such that $a^s \cdot a^t = 0$. (Here $a^s$, $a^t$ and all $a^{t_i}$ are columns of $A_{n-1}$). Let $s = (i_s, j_s, k_s)$, $t = (i_t, j_t, k_t)$, and $t_1 = (i_s, j_t, k_t)$, $t_2 = (i_t, j_s, k_t)$, $t_3 = (i_t, j_t, k_s)$. Now suppose $\mathcal{C} \notin Q_3$. Then $\mathcal{C}$ has a node $u \in S_n \setminus S_{n-1}$, say $u = (i, j, k)$, such that $a^u \cdot a^s > 0$, $a^u \cdot a^{t_i} > 0$, $i = 1, 2, 3$ (here $a^u$, $a^s$ and all $a^{t_i}$ are columns of $A_n$). Since $u \notin S_{n-1}$, at least one of the elements $i, j, k$ is not contained in any of the 4 triplets $s, t_1, t_2, t_3$. W.l.o.g. assume $k$ is such an element. Then each of $s, t_1, t_2, t_3$ must contain either $i$ or $j$. There are four cases: $i = i_s$, $j = j_s$; $i = i_t$, $j = j_t$; $i = i_s$, $j = j_t$; and $i = i_t$, $j = j_s$. In each case at least one of the four
triplets \( s, t_1, t_2, t_3 \) does not contain either \( i \) or \( j \), which contradicts the assumption that \( <C> \) is a clique.

Thus every clique of \( G_{A_n} = G_{A_k+1} \) belongs to either \( Q_1 \), or \( Q_2 \), or \( Q_3 \), and the induction is complete.

3. Facets of \( P_I \) Induced by Cliques of \( G_A \).

If \( C \) is the vertex set of a clique of \( G_A \), then obviously every \( x \in P \) satisfies the inequality

\[
(3.1) \quad \sum_{s \in C} x_s \leq 1.
\]

Such inequalities are known to define facets of \( P_I \), the set packing polytope associated with \( P_I \) [17]; but since \( P_I \) itself is a face of \( P_I \), it is an open question whether an inequality (3.1) also defines a facet of \( P_I \). In this section we answer this question exhaustively.

First, some definitions and basic concepts. For any polyhedron \( P \), let \( \dim P \) denote the dimension of \( P \) (defined as the dimension of the affine hull of \( P \), i.e. of the smallest subspace containing \( P \)). An inequality \( x \leq \pi_0 \) is said to define a facet of \( P \), if it is satisfied by every \( x \in P \) and the polyhedron \( P^* = \{ x \in P | x = \pi_0 \} \) has dimension \( \dim P - 1 \). If \( x = \pi_0 \) for all \( x \in P \), the inequality \( x \leq \pi_0 \) is said to define an improper face of \( P \). In this case of course \( \dim P^* = \dim P \). To show that \( x \leq \pi_0 \) does not define an improper face, it is sufficient to exhibit a point \( x \in P \) such that \( x < \pi_0 \). Once this is ascertained to be the case, \( \dim P^* \leq \dim P - 1 \), since

(a) \( \dim P \) is the number of variables in the system defining \( P \), minus the rank of the equality system of \( P \) (i.e. of the system of linear equations satisfied
by all x ∈ P); and (b) the addition of the equation πx = π₀, not implied by the system defining P, increases the rank of the equality system by at least 1. Thus showing that πx ≤ π₀ defines a facet of P essentially amounts to showing that the dimension of Pᵀ, known to be bounded by dim P - 1, is actually equal to this bound. The most commonly used procedure for doing this is to exhibit dim P affinely independent points x ∈ Pᵀ. Another way of doing it is to show that the addition of πx = π₀ to the constraints defining P increases the rank of the equality system of P by exactly one; in other words, that any equation satisfied by all x ∈ Pᵀ is a linear combination of the equations in the system defining Pᵀ. In this paper we will take the latter approach, and will use it also to establish the dimension of Pᵢ itself. We will implement this approach via a technique similar to that used by Maurras [16], as well as by Cornuejols and Pulleyblank [5], (see also Cornuejols and Thizy [6]).

We first establish the dimension of Pᵢ.

Let P denote the feasible set of the linear programming relaxation of Pᵢ, i.e.

$$P = \{ x ∈ R^n | Ax = e, x ≥ 0 \}.$$

**Lemma 3.1.** The rank of the system Ax = e is 3n-2.

**Proof.** The rank of Ax = e is at most 3n-2, since equation 2n is the sum of the first n equations, minus the sum of equations n+1,...,2n-1; and equation 3n is the sum of the first n equations minus the sum of equations 2n+1,...,3n-2. On the other hand, the rank of Ax = e is at least 3n-2, since we can exhibit 3n-2 affinely independent columns of A. Consider the three sets of columns indexed by the following triplets:
The first two sets contain \( n-1 \) columns each, the last one contains \( n \) columns. The matrix formed by these columns (in the order of their listing), after deletion of the first row of set \( \mathcal{I} \) and the first row of set \( \mathcal{J} \), becomes a square lower triangular (hence nonsingular) matrix of order \( 3n-2 \), with each diagonal element equal to 1 and all elements above the diagonal equal to 0.

**Corollary 3.2.** \( \dim \mathcal{P} = n^3 - 3n + 2 \).

**Proof.** The dimension of \( \mathcal{P} \) is the number of variables in its defining system \( (n^3) \), minus the rank of its equality system \( Ax = e \) \( (3n-2) \).

We are interested in \( \dim \mathcal{P}_I \). Since \( \mathcal{P}_I \subset \mathcal{P} \), \( \dim \mathcal{P}_I \leq n^3 - 3n + 2 \), and strict inequality holds if and only if there exists an equation \( \alpha x = \alpha_0 \) satisfied by all \( x \in \mathcal{P}_I \), that is not implied by (not a linear combination of) the equations \( Ax = e \). We will show that no such equation exists.

**Theorem 3.3.** Suppose every \( x \in \mathcal{P}_I \) satisfies \( \alpha x = \alpha_0 \) for some \( \alpha \in \mathbb{R}^{n^3} \), \( \alpha_0 \in \mathbb{R} \). Then there exist scalars \( \lambda_i, \forall i \in \mathcal{I}, u_j, \forall j \in \mathcal{J}, \) and \( v_k, \forall k \in \mathcal{K} \), such that

\[
\alpha_{ijk} = \lambda_i + u_j + v_k, \forall (i,j,k) \in \mathcal{I} \times \mathcal{J} \times \mathcal{K}
\]

and

\[
\alpha_0 = \sum_{i \in \mathcal{I}} \lambda_i + \sum_{j \in \mathcal{J}} u_j + \sum_{k \in \mathcal{K}} v_k.
\]

**Proof.** Let \( x \in \mathcal{P}_I, x \in (0,1)^{n^3} \) be such that \( x_{i_1 j_1 k_1} = x_{i_3 j_2 k_2} = 1 \), and let \( x' \in (0,1)^{n^3} \) be defined by

\[
x'_{i_3 j_1 k_1} = x'_{i_1 j_2 k_2} = 1,
\]

\[
x'_{i_1 j_1 k_1} = x'_{i_3 j_2 k_2} = 0,
\]
and \( x'_{ijk} = x_{ijk} \) for all other \((i,j,k)\). It is easy to see that \( x \in P_I \)
implies \( x' \in P_I \), and since \( \alpha x = \alpha x' (= \alpha_0) \) by assumption, we have

\[
(3.2) \quad \alpha_{i_1 j_1 k_1} + \alpha_{i_3 j_2 k_2} = \alpha_{i_3 j_1 k_1} + \alpha_{i_1 j_2 k_2}.
\]

Next, let \( \bar{x} \in P_I, \bar{x} \in (0,1)^n \) be such that \( \bar{x}_{i_2 j_1 k_1} = \bar{x}_{i_2 j_2 k_2} = 1 \), and
let \( \bar{x}' \in (0,1)^n \) be defined by

\[
\bar{x}'_{i_3 j_1 k_1} = \bar{x}'_{i_2 j_2 k_2} = 1,
\]

\[
\bar{x}'_{i_2 j_1 k_1} = \bar{x}'_{i_3 j_2 k_2} = 0,
\]

and \( \bar{x}'_{ijk} = \bar{x}_{ijk} \) for all other \((i,j,k)\). Again, \( \bar{x} \in P_I \) implies \( \bar{x}' \in P_I \),
and since \( \alpha \bar{x} = \alpha \bar{x}' \), we have

\[
(3.3) \quad \alpha_{i_2 j_1 k_1} + \alpha_{i_3 j_2 k_2} = \alpha_{i_3 j_1 k_1} + \alpha_{i_2 j_2 k_2}.
\]

Subtracting (3.3) from (3.2) gives

\[
(3.4) \quad \alpha_{i_1 j_1 k_1} - \alpha_{i_2 j_1 k_1} = \alpha_{i_1 j_2 k_2} - \alpha_{i_2 j_2 k_2}.
\]

If in the definition of \( x, x' \) and of \( \bar{x}, \bar{x}' \) we replace the pairs \((j_2,k_2)\)
by arbitrary \((j,k) \in J \times K\), we obtain (3.4) with \((j_2,k_2)\) on the right hand
side replaced by \((j,k)\). Thus the meaning of (3.4) is that two components of
\( \alpha \) whose index-triplets differ only in their first element, differ by an
amount depending only on the values of those first elements, i.e.

\[
(3.5) \quad \alpha_{i_1 j_1 k} - \alpha_{i_2 j_1 k} = \text{const.}, \quad \forall (j,k) \in J \times K.
\]
By symmetry we have equations analogous to (3.5) for components of $a$ whose index-triplets differ only in their second or only in their third element.

Now let us denote

\[ a_{ij}^{12} = a_{ij}^{1} - a_{ij}^{2}, \]
\[ a_{ij}^{12} = a_{ij}^{1} - a_{ij}^{2}, \]
\[ a_{ij}^{12} = a_{ij}^{1} - a_{ij}^{2}. \]

(3.6)

Then we have

\[ a_{ijk} - a_{ij}^{1}k_{1} = a_{ijk} - a_{ij}^{1}k_{1} + a_{ij}^{1}k_{1} - a_{ij}^{1}j_{1}k_{1} \]
\[ = a_{ij}^{1} + a_{ij}^{1} + k_{1}, \]
\[ or \]
\[ a_{ijk} = a_{ij}^{1} + a_{ij}^{1} + k_{1} + a_{ij}^{1}j_{1}k_{1}, \]
\[ \forall (i,j,k) \in I \times J \times K. \]

(3.7)

Next we fix $u_{j_{1}}$ and $v_{k_{1}}$ arbitrarily, we let $\lambda_{i_{1}} = a_{ij}^{1}j_{1}k_{1} - u_{j_{1}} - v_{k_{1}}$, and define

\[ \lambda_{i} = \lambda_{i_{1}} + a_{ij}^{1}, \quad i \in I \setminus \{i_{1}\}, \]
\[ u_{j} = u_{j_{1}} + a_{ij}^{1}, \quad j \in J \setminus \{j_{1}\}, \]
\[ v_{k} = v_{k_{1}} + a_{ij}^{1}, \quad k \in K \setminus \{k_{1}\}. \]

(3.8)

Then

\[ a_{ijk} = \lambda_{i} + u_{j} + v_{k}, \quad \forall (i,j,k) \in I \times J \times K. \]

(3.9)

Further, let $\chi$ be defined by

\[ \chi_{ijk} = \begin{cases} 1 & \text{if } i = j = k, \\ 0 & \text{otherwise}. \end{cases} \]
Then \( x \in P_I \), hence \( a x = a_0 \), or

\[
a_0 = \sum_{i \in I} x_i + \sum_{j \in J} \nu_j + \sum_{k \in K} \nu_k.
\]

**Corollary 3.4.** \( \dim P_I = n^3 - 3n + 2 \).

**Proof.** From Theorem 3.3, the smallest affine subspace containing \( P_I \) is the one defined by the system \( Ax = e \); the dimension of \( P_I \) is therefore the same as that of \( P \).

Next we turn to the constraints defining \( P \) and ask the question, which ones among these define facets of \( P_I \).

**Theorem 3.5.** Every inequality \( x_s \geq 0 \) for some \( s \in S \) defines a facet of \( P_I \).

**Proof.** The statement is true if and only if the polytope \( P_{I}^{S} = \{ x \in P_I \mid x_s = 0 \} \) has dimension \( \dim P_I - 1 = n^3 - 3n + 1 \). Clearly, \( \dim P_{I}^{S} \leq n^3 - 1 - r \), where \( r \) is the rank of the system \( A^S x = e \), and \( A^S \) is the matrix obtained from \( A \) by removing the column \( a^S \). The rank of \( A^S \) is easily seen to be the same as the rank of \( A \), i.e., \( r = 3n - 2 \). This is immediate in the case when \( a^S \) is not among those columns used in the proof of Lemma 3.1, and follows by symmetry for the other case. Hence the dimension of \( P_{I}^{S} \) is at most \( n^3 - 3n + 1 \). To show that this bound is actually attained, one can use the same argument as in the proof of Theorem 3.3 to show that any equation \( a x = a_0 \) (other than \( x_s = 0 \)) satisfied by every \( x \in P_{I}^{S} \) is a linear combination of the equations \( A^S x = e \). The argument goes through essentially unchanged.

As to the equations of the system \( Ax = e \), each of them defines an improper face of \( P_I \) (i.e., is satisfied by every \( x \in P_I \)).
We now turn to the inequalities (3.1) defined by the cliques of $G_A$.

Each clique of class 1 induces an inequality whose left hand side coefficient vector is one of the rows of $A$. Hence each such inequality is satisfied with equality by every $x \in P_1$, and therefore defines an improper face of $P_1$.

Next we consider the inequalities (3.1) induced by the cliques of class 2. Each clique in this class is defined relative to some index (triplet) $s \in S$, and has a node set of the form $\{s\} \cup T(s)$ (see Proposition 2.2). Let $C(s)$ denote the node set of the clique of class 2 defined relative to $s$.

Theorem 3.6. For $n \geq 3$, the inequality

\[(3.10) \quad \sum_{t \in C(s)} x_t \leq 1\]

defines a facet of $P_1$ for every $s \in S$.

Proof. Let $P_{C(s)}^c = \{x \in P_1 \mid \sum_{t \in C(s)} x_t = 1\}$ and let $n \geq 3$. We will show that $\dim P_{C(s)}^c \neq P_1$. Let $s = (i_0, j_0, k_0)$; then every triplet in $C(s) \setminus \{s\}$ contains two of the three elements of $s$. Hence any $x \in P_1$ whose support includes $(i_0, j', k'), (i', j_0, k'')$ and $(i'', j'', k_0)$, for any $i', i'' \neq i_0, j', j'' \neq j_0$ and $k', k'' \neq k_0$, satisfies $\sum_{t \in C(s)} x_t = 0 < 1$. Thus (3.9) does not define an improper face of $P_1$, i.e., $\dim P_{C(s)}^c \leq P_1 - 1$.

To show that this last inequality holds as equality, we prove that an arbitrary equation $ax = a_0$ satisfied by all $x \in P_{C(s)}^c$ is a linear combination of the equations of the system $Ax = e$ and

\[(3.11) \quad \sum_{t \in C(s)} x_t = 1.\]

Let $x \in C(s)$, $x \in (0, 1) \in R^{n^3}$, be such that $x_{i_1 j_1 k_1} x_{j_3 j_2 k_2} = 1$, where $(i_1, j_1, k_1), (i_3, j_2, k_2) \in C(s)$ and $n \geq 3$, there exist at least 3 disjoint triplets $(i, j, k)$ such that $x_{ijk} = 1$, and at most one of these can belong to
C(s); hence the vector \( x \in \mathbb{C}(s) \) postulated here exists. Note that \( x \) is of the same form as the vector \( x \) used in the proof of Theorem 3.3, and hence the same reasoning can be used to derive the relation

\[
\alpha_{i_1jk} - \alpha_{i_2jk} = \text{const., } \forall (j,k) \in \mathbb{J} \times \mathbb{K}; (i_1,j,k), (i_2,j,k) \not\in \mathbb{C}(s)
\]

for all \( i_1, i_2 \in \mathbb{I} \), analogous to (3.5). Using the notation (3.6), we then obtain

\[
\beta_{ijk} = \beta_{i_1j_1k_1} + \beta_{j_1k_1} + \beta_{i_2j_2k_2} + \beta_{i_2j_2k_2}, \quad \forall (i,j,k) \in \mathbb{S} \cap \mathbb{C}(s),
\]

and defining \( \gamma_i, \gamma_j, \gamma_k \), \( i \in \mathbb{I}, j \in \mathbb{J}, k \in \mathbb{K} \) as in (3.8) we get the relation

\[
(3.12) \quad \alpha_{ijk} = \gamma_i + \gamma_j + \gamma_k, \quad \forall (i,j,k) \in \mathbb{S} \setminus \mathbb{C}(s)
\]

analogous to (3.9).

Now let us define

\[
\tau_i = \pi_{i_0j_0k_0} - \lambda_i - \mu_j - \nu_k, \quad \forall i \in \mathbb{I}
\]

\[
(3.13) \quad \tau_j = \pi_{i_0j_0k_0} - \lambda_i - \mu_j - \nu_k, \quad \forall j \in \mathbb{J}
\]

\[
\tau_k = \pi_{i_0j_0k_0} - \lambda_i - \mu_j - \nu_k, \quad \forall k \in \mathbb{K}
\]

where \((i_0, j_0, k_0) = s\).

Since \( t \in \mathbb{C}(s) \) if and only if \( t \) is either of the form \((i_0,j_0,k_0)\), or \((i_0,j,k_0)\), or \((i_0,j_0,k)\), (3.13) defines \( \alpha_{ijk} \) for all \((i,j,k) \in \mathbb{C}(s)\). We will show that \( \tau_{i} = \tau_{j} = \tau_{k} \), \( \forall i \in \mathbb{I}, j \in \mathbb{J}, k \in \mathbb{K} \). From (3.13) it is easily seen that \( \tau_{i_0} = \tau_{j_0} = \tau_{k_0} \); hence, denoting \( \tau_{\mathbb{C}(s)} := \tau_{i_0} \) \( (= \tau_{j_0} = \tau_{k_0}) \), it is sufficient to show that \( \tau_{i} = \tau_{\mathbb{C}(s)} \), \( \forall i \in \mathbb{I} \); it then follows by symmetry that \( \tau_{j} = \tau_{\mathbb{C}(s)} \), \( \forall j \in \mathbb{J} \), \( \forall k \in \mathbb{K} \).
Let \( \bar{x}_I \in \mathbb{P}_n^C(s) \), \( \bar{x}_I(0,1)^n \), be such that \( \bar{x}_{i_0,j_0,k_0} = \bar{x}_{i,j,k} = 1 \), where \( (i',j',k') \notin C(s) \) is chosen arbitrarily; and let \( \bar{x}'_I(0,1)^n \) be defined by
\[
\bar{x}'_{i,j,k} = \begin{cases} 1, & \text{if } (i,j,k) \in C(s) \setminus \bar{C}(s), \\ 0, & \text{otherwise}. 
\end{cases}
\]
and \( \bar{x}'_{i,j,k} = \bar{x}_{i,j,k} \) for all other \((i,j,k)\). Then from \( \bar{x}_I \in \mathbb{P}_n^C(s) \) it follows that \( \bar{x}'_I \in \mathbb{P}_n^C(s) \), and from \( \alpha \bar{x} = \alpha \bar{x}'(=a_0) \) we obtain
\[
\alpha_{i,j,k} + a_{i,j,k} = a_{i,j,k} + a_{i,j,k}.
\]
Substituting for \( \alpha_{i,j,k} \) from (3.13) and \( a_{i,j,k} \) from (3.12) (since \( i_0,j_0,k_0 \), \( (i',j_0,k_0) \in C(s) \) and \( (i_0,j',k') \notin C(s) \)) we obtain
\[
\pi_{i,j,k} + \pi_{i,j,k} = \pi_{i,j,k} + \pi_{i,j,k}, \quad \text{or} \quad \pi_{i,j,k} = \pi_{i,j,k};
\]
and since \( i' \) was chosen arbitrarily (subject to the condition that \( (i',j',k') \notin C(s) \)), this proves that \( \pi_{i,j,k} = \pi_{i,j,k} = \pi_C(s) \), \( \forall i \in I \).
Furthermore, as mentioned above, \( \pi_{j,k} = \pi_C(s) \), \( \forall j \in J, k \in K \) follows by symmetry.

We have thus proved that
\[
\alpha_{i,j,k} = \begin{cases} \lambda_i + \mu_j + \nu_k + \pi_C(s), & \text{if } (i,j,k) \in C(s) \\ \lambda_i + \mu_j + \nu_k, & \text{if } (i,j,k) \in S \setminus C(s) 
\end{cases}
\]
Now let $\lambda \in (1,1)^n$ be defined by

$$
\lambda_{ijk} = \begin{cases} 
1 & \text{if } i = i_0 + \epsilon, j = j_0 + \epsilon, k = k_0 + \epsilon \pmod n \text{ for } \epsilon = 0, 1, \ldots, n-1 \cr 
0 & \text{otherwise} 
\end{cases}
$$

Then $\lambda \in P_I^C(s)$, hence $α\lambda = α_0$, i.e.,

$$
α_0 = \sum_{i \in I} λ_i + \sum_{j \in J} u_j + \sum_{k \in K} v_k + πC(s).
$$

This proves that the equation $α\lambda = α_0$ is a linear combination of the equations of the system defining $P_I^C(s)$; hence $\dim P_I^C(s) = P_I - 1$ and thus the inequality (3.10) defines a facet of $P_I$. ||

Finally, we turn to the inequalities (3.1) induced by cliques of class 3. Remember that each clique in this class is defined relative to an ordered pair $(s,t)$ of disjoint triplets, and has a node set of the form $[s, t_1, t_2, t_3]$, where each $t_i$, $i = 1, 2, 3$, contains one element of $s$ and two elements of $t$ (see Proposition 2.3). Let $C(s,t)$ denote the node set of the clique of class 3 defined relative to the ordered pair $(s,t)$.

**Theorem 3.7** For $n > 3$, the inequality

$$(3.14) \quad \sum_{u \in C(s,t)} x_u \leq 1$$

defines a facet of $P_I$ for all $s,t \in S$.

**Proof.** The proof of Theorem 3.6 can be used to show that $\dim P_I^C(s,t) \leq P_I - 1$, and to derive the relation

$$(3.15) \quad α_{ijk} = λ_i + u_j + v_k, \forall (i,j,k) \in S \setminus C(s,t).$$

analogous to (3.12).
Now let $s = (i_s, j_s, k_s), t = (i_t, j_t, k_t)$, and define

$$
\pi_s = \alpha_{i_s j_s k_s} - \lambda_{i_s} - \mu_{j_s} - \nu_{k_s}
$$

$$
\pi_{t_1} = \alpha_{i_t j_t k_t} - \lambda_{i_t} - \mu_{j_t} - \nu_{k_t}
$$

$$
(3.16)
$$

$$
\pi_{t_2} = \alpha_{i_t j_t k_t} - \lambda_{i_t} - \mu_{j_t} - \nu_{k_t}
$$

$$
\pi_{t_3} = \alpha_{i_t j_t k_t} - \lambda_{i_t} - \mu_{j_t} - \nu_{k_t}
$$

To show that $\pi_{t_i} = \pi_s$, $i = 1, 2, 3$, consider $\bar{x} \in (0, 1)^n$, $\bar{x} \in p_C(s, t)$ such that $\bar{x}_{i_s j_s k_s} = \bar{x}_{i_t j_t k_t} = 1$, and define $\bar{x}'$ by

$$
\bar{x}'_{i_s j_s k_s} = \bar{x}_{i_t j_t k_t} = 1,
$$

$$
\bar{x}'_{i_t j_t k_t} = \bar{x}_{i_s j_s k_s} = 1,
$$

and $\bar{x}'_{i_j k} = \bar{x}_{i_j k}$ for all other $(i, j, k)$. Then $\bar{x}' \in p_C(s, t)$ follows from $\bar{x} \in p_C(s, t)$, and $\alpha x = \alpha x'$ implies

$$
\alpha_{i_s j_s k_s} + \alpha_{i_t j_t k_t} = \alpha_{i_t j_t k_t} + \alpha_{i_s j_s k_s}.\tag{3.17}
$$

Since $(i_s, j_s, k_s), (i_t, j_t, k_t) \in C(s, t)$ and $(i_s, j_s, k_t), (i_t, j_s, k_s) \notin C(s, t)$, we have (from (3.15) and (3.16))

$$
\pi_s + \lambda_{i_s} + \mu_{j_s} + \nu_{k_s} + \lambda_{i_t} + \mu_{j_t} + \nu_{k_t} = \pi_{t_1} + \lambda_{i_t} + \mu_{j_t} + \nu_{k_t} + \lambda_{i_s} + \mu_{j_s} + \nu_{k_s}
$$

or $\pi_{t_1} = \pi_s$. By symmetry, $\pi_{t_2} = \pi_s = \pi_{t_3}$, and we are done. ||

**Theorem 3.8** The inequalities (3.1) induced by distinct cliques define distinct facets when $n \geq 3$.

To prove this theorem, we require the following auxiliary result:
Lemma 3.9  For \( n \geq 3 \), there is a feasible solution \( x \) with \( x_s = x_t = x_r = 1 \) and a pair of cliques with vertex sets \( C_1, C_2 \), such that 1. \( s \in C_1 \setminus C_2 \); 2. \( t \in C_2 \setminus C_1 \); 3. \( r \in S \setminus (C_2 \cup C_1) \); 4. if \( C_2 = C(t_0) \) then \( t = t_0 \) and \( t_0 \cap r = \emptyset \); and if \( C_2 = C(t,r_0) \) then \( r \neq r_0 \).

Proof. (i) Suppose \( C_1 = C(s_0) \), \( C_2 = C(t_0) \) where \( s_0 = (i_{s_0}, j_{s_0}, k_{s_0}) \), \( t_0 = (i_{t_0}, j_{t_0}, k_{t_0}) \). If \( t_0 \in C_1 \) then w.l.o.g. let \( i_{s_0} = i_{t_0} \) and \( j_{s_0} = j_{t_0} \), and choose \( s = (i_{s_0}, j_{s_0}, k_{s_0}) \) and \( t = (i_{t_0}, j_{t_0}, k_{t_0}) \) with \( i_{s_0} \neq i_{t_0} \), \( j_{s_0} \neq j_{t_0} \). If \( t_0 \notin C_1 \) then there is an \( s' \in C_1 \setminus C_2 \) such that \( s' \cap t_0 = \emptyset \). W.l.o.g. let \( s' = (i_{s_0}, j_{s_0}, k) \) where \( k \neq k_{t_0} \), and choose \( s = (i_{s_0}, j_{s_0}, k_{t_0}) \) and \( t = (i_{t_0}, j_{t_0}, k) \). Finally, select \( r \) such that \( r \cap s = r \cap t = \emptyset \). Then \( x \) satisfies conditions 1-4.

(ii) Suppose \( C_1 = C(s_0), \ C_2 = (t_0, r_0) \), where \( t_0 \notin C_1, \ s_0 \) and \( t_0 \) are as above and \( r_0 = (i_{r_0}, j_{r_0}, k_{r_0}) \). If \( r_0 \in C_1 \) then let \( s = r_0, \ t = t_0 \) and select any \( r \) such that \( r \cap s = r \cap t = \emptyset \). Suppose instead that \( r_0 \notin C_1 \). If \( t_0 \cap s_0 \neq \emptyset \) then there is a \( t' \in C_2 \) such that \( t' \cap s_0 = \emptyset \) and a corresponding \( r' \) such that \( C_2 = C(t', r') \); thus w.l.o.g. we can assume \( t_0 \cap s_0 = \emptyset \). Now, w.l.o.g., assume \( i_{r_0} \neq i_{s_0}, j_{r_0} \neq j_{s_0} \). Then there is an \( s' \in C_1 \) such that \( s' = (i_{s_0}, j_{s_0}, k) \) with \( k \neq k_{r_0} \), \( k \neq k_{t_0} \) (in particular, if \( k_{r_0} \neq k_{s_0} \) then we can take \( k = k_{s_0} \), otherwise there is at least one element of \( K \) which is not equal to \( k_{t_0} \) or to \( k_{s_0} \), and we can choose this element to be \( k \)). Choose \( s = (i_{s_0}, j_{s_0}, k_{r_0}), t = t_0 \) and \( r = (i_{r_0}, j_{r_0}, k_{r_0}) \). Then \( x \) satisfies conditions 1-4.

(iii) Suppose \( C_1 = C(s_0, w_0), \ C_2 = (t_0, r_0) \) where \( s_0, t_0 \) and \( r_0 \) are as above, \( s_0 \notin C_2, \ t_0 \notin C_1, \ s_0 \cap t_0 = \emptyset \) and \( w_0 = (i_{w_0}, j_{w_0}, k_{w_0}) \). Suppose first that \( w_0 \cap t_0 = \emptyset \). If \( w_0 \neq r_0 \), choose \( s = s_0, \ t = t_0, \ r = w_0 \). If \( w_0 = r_0 \) then \( C_1 \cap C_2 = \emptyset \) so we can write \( C_1 = C(s', w') \) with \( s' \neq s_0, \ w' \neq r_0 \). Then we choose \( s = s', \ t = t_0, \ r = w' \). Now suppose \( w_0 \cap t_0 \neq \emptyset \). It follows that
Define a new triplet $w'$ by selecting the element or elements of $w_0 \setminus t_0$ and taking the remaining element or elements from $t_0$. By definition $w \cap t_0 = \emptyset$.

If $w \cap s_0 = \emptyset$ and $w' \neq r_0$, choose $s = s_0$, $t = t_0$, $r = w'$.

If $w \cap s_0 = \emptyset$ and $w' = r_0$, then choose $s = t_0$, $t = s_0$, $r = w'$, exchanging the roles of $C_1$ and $C_2$.

If $w \cap s_0 \neq \emptyset$ then it follows from the definition of $w'$ that $r_0 \cap s_0 \neq \emptyset$ and that the common elements are in positions in which $w_0$ and $t_0$ share common elements. W.l.o.g. suppose $i_w = i_{r_0} = i_{s_0}$ and $i_w = i_{t_0}$. Then there is an $i \in I$ such that $i \neq i_w = i_{r_0} = i_{s_0}$ and $i \neq i_w = i_{t_0}$. The same holds for other elements of $w \cap s_0$, if any. Define the triple $w''$ from $w'$ by replacing $i_w = i_{r_0}$ with $i$ and doing the same for any other elements of $w \cap s_0$. Choose $s = s_0$, $t = t_0$, $r = w''$.

For each of the above cases, the resulting $x$ satisfies conditions 1-4. ||

**Proof of Theorem 3.8.** Let $C_1$ and $C_2$ be distinct cliques. Then $C_1 \setminus C_2 \neq \emptyset$ and $C_2 \setminus C_1 \neq \emptyset$. From Lemma 3.9, there exists a feasible solution $x$ such that

$$\sum_{s \in C_1} x_s = 1 \quad \text{and} \quad \sum_{t \in C_2} x_t = 1$$

We will show next how to modify that solution to produce a solution $x'$ with

$$\sum_{s \in C_1} x'_s = 1 \quad \text{and} \quad \sum_{t \in C_2} x'_t = 0$$
We construct $x'$ by exchanging one element of $t$ and the corresponding element of $r$, denoting the triplets thus created $t'$, $r'$. It follows from condition 4 of the lemma that for at least one of the three possible $t'$, $r'$ pairs, $t' \not\in C_2$ and $r' \not\in C_2$. Since $x_j' = x_j$ for all $j$ except $x_s' = x_t', x_r' = 1$ and $x_t' = x_r' = 0$, $x'$ is feasible and satisfies the inequality corresponding to $C_1$ with equality and the inequality corresponding to $C_2$ strictly. Since $x'$ lies on the facet corresponding to $C_1$ but not on the facet corresponding to $C_2$, it follows that $C_1$ and $C_2$ define distinct facets of $P_I$.

It is easy to see that each inequality $x_s > 0$ also defines a facet distinct from those defined by other trivial inequalities or by clique-inequalities.

4. Detecting Violated Clique-Facets

It is of great interest in terms of algorithm development to be able to determine, for an arbitrary noninteger solution to the LP-relaxation of an integer program, whether that solution violates a facet of the convex hull of integer solutions. One may solve the LP-relaxation, then identify a facet-defining inequality that cuts off the solution obtained and either add it to the constraint set of the LP, or take it into the objective function with a Lagrange multiplier. In general, for an NP-hard problem the facet-identification problem is also NP-hard, but for some subsets of the facets it may be possible to efficiently identify which, if any, members of the subsets are violated by an LP solution. Recent efforts to implement algorithms based on this strategy (and employing branch-and-bound techniques when a fractional solution is reached that does not violate any of the facets under consideration) have met with marked success [1], [8]. In this section we describe an efficient algorithm for detecting clique-facets violated by an
arbitrary \( x \in P \), i.e., an arbitrary solution to the LP-relaxation of AP3. Although the cardinality of the set of clique-facets is \( O(n^6) \), (namely, \( n^3 \) facets from cliques of class 2, and \( n^3(n-1)^3/4 \) from cliques of class 3), the proposed algorithm can be shown to have a worst-case running time of \( O(n^4) \). In terms of the number \( |S| \) of variables, this is \( O(|S|^{4/3}) \).

We first remark that given a noninteger \( x \in P \), it can be detected in \( O(n^4) \) steps whether any inequality induced by a clique of class 2 is violated. Indeed, each of the \( n^3 \) cliques of class 2 is associated with some \( s \in S \), and is induced by a node set of the form \( (s) \cup T(s) \), where \( T(s) \) is the set of those triplets that differ from the the triplet \( s \) in exactly one element. Since the cardinality of \( T(s) \) is \( 3(n-1) \), for each \( s \in S \) it requires \( O(n) \) steps to identify and add all \( x_{ijk} \) such that \( (i,j,k) \in C(s) \), in order to check whether the sum exceeds 1 (in which case the corresponding inequality is violated) or not. To execute this for all \( s \in S \) therefore requires \( O(n \times n^3) = O(n^4) \) steps.

For cliques of class 3 (whose number is \( O(n^6) \)) the complexity bound is not so straightforward. However, we will give an algorithm which performs that task too in \( O(n^4) \) steps. This is possible due to the following fact: Each clique of class 3 is of cardinality 4; therefore any \( x \in P \) that violates some inequality induced by a clique of class 3 must have at least one component of value \( \geq 1/4 \). On the other hand, we have

**Lemma 4.1** For any \( x \in P \) and any positive integer \( k \), the number of components with value \( \geq 1/k \) is \( \leq kn \).

*Proof.* The value of the linear program

\[
\text{(L) } \max \{ \langle x \rangle | x \in P \}
\]

is easily seen to be \( n \), since the vectors \( x \in R^{n^3} \) and \( u \in R^{3n} \), defined by

\[
x_s = 1/n^2, \quad \forall s \in S
\]
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and

\[ u_r = \frac{1}{3}, \forall r \in R, \]

are feasible solutions to (L) and its dual, respectively, with the common value of \( n \); hence they are optimal.

Now if \( x \) has more than \( kn \) components with value \( \geq \frac{1}{k} \), then \( ex > n \), a contradiction. [11]

**Theorem 4.2.** It can be determined in \( O(n^4) \) steps whether a given \( x \in P \) violates a facet defining inequality induced by a clique of class 3.

**Proof.** Let \( C(s,t) \) be the node set of a clique of class 3. Since \( |C(s,t)| = 4 \), if \( x \in P \) violates the facet-inequality corresponding to \( C(s,t) \), then from Lemma 4.1 \( x \) has at least one component \( \geq \frac{1}{4} \). Further, if \( C(s,t) = \{s,t_1,t_2,t_3\} \), from Proposition 2.4 there is no loss of generality in assuming that this happens for the component indexed by \( s \), i.e., that \( x_s \geq \frac{1}{4} \). Thus, instead of examining all ordered pairs \((s,t)\) such that \( a^s a^t = 0 \), we can restrict ourselves to examine those ordered pairs \((s,t)\) such that \( x_s \geq \frac{1}{4} \) and \( a^s a^t = 0 \).

Consider now the following algorithm.

1. Order \( S \) according to nonincreasing values of \( x_s \), \( s \in S \).

2. For each of the first \( 4n \) elements \( s = (i_s,j_s,k_s) \) of the ordered set \( S \) such that \( x_s > \frac{1}{4} \) and each of the \( (n-1)^3 \) triplets \( t = (i_t,j_t,k_t) \in S \) such that \( i_t \neq i_s \), \( j_t \neq j_s \) and \( k_t \neq k_s \), calculate the sum \( \varepsilon(s,t) = x_{i_s}x_{j_s}x_{k_s} + x_{i_t}x_{j_t}x_{k_t} + x_{i_s}x_{j_t}x_{k_t} \). If \( \varepsilon(s,t) > 1 \), stop: the inequality associated with \((s,t)\) is violated; otherwise continue.

Since the algorithm examines all pairs \((s,t)\) such that \( a^s a^t = 0 \) and \( x_s > \frac{1}{4} \), it either finds a pair whose corresponding facet inequality is violated by \( x \), or it stops with the conclusion that \( x \) satisfies all facet-inequalities induced by cliques of class 3. Step 1 is executed once and it
requires $O(n^3 \log n^3)$ operations. Step 2 is executed at most $4n(n-1)^3$ times, and each execution requires 3 additions. Hence, the overall complexity of the algorithm is $O(n^4)$. ||
References


Given three disjoint n-sets and the family of all weighted triplets that contain exactly one element of each set, the 3-index assignment (or 3-dimensional matching) problem asks for a minimum-weight subcollection of triplets that covers exactly (i.e., partitions) the union of the three sets. Unlike the common (2-index) assignment problem, the 3-index problem is NP-complete. In this paper we examine the facial structure of the 3-index assignment polytope (the convex hull of feasible solutions to the problem) with the aid of...
the intersection graph of the coefficient matrix of the problem's constraint set. In particular, we describe the cliques of the intersection graph as belonging to three distinct classes, and show that cliques in three of the three classes induce inequalities that define facets of our polytope. Furthermore, we given an \( O(n^3) \) procedure (note that the number of variables is \( n^3 \)) for finding a facet-defining clique-inequality violated by a given noninteger solution to the linear programming relaxation of the 3-index assignment problem, or showing that no such inequality exists.