APPROXIMATIONS FOR CONVERTING
GEODETIC TO CARTESIAN
COORDINATES

Herbert R. Lotze

TECHNICAL REPORT NO. AFWL-TR-71-98

August 1971

AIR FORCE WEAPONS LABORATORY
Air Force Systems Command
Detachment 1
Holloman Air Force Base
New Mexico

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Approximations for converting geodetic to cartesian coordinates

Methods are investigated which apply to the conversion of latitude, longitude and height to cartesian coordinates in a plane tangent to the earth's surface in connection with onboard missile targeting. The criterion for the usefulness of the method is the error in the north and east coordinate with reference to Clarke's spheroid 1866. This error is determined as a function of vector length and azimuth for the spherical earth model referenced to the mean geodetic latitude at White Sands Missile Range utilizing a program prepared for a programmable electronic desk calculator (Marchant 1016 PR). An approximation of geocentric latitude is used in the program. It is explained how the error inherent to the spherical earth model can be reduced applying a certain correction. An expression for computing the reference coordinates in a plane tangent to the earth spheroid is derived which requires less numerical effort than the standard procedure.
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FOREWORD

This document has been prepared to describe work done under the Short Range Attack Missile (SRAM) project of AFSC, Weapon System 140A to support the flight test evaluation of B-52 missions and in particular to evaluate onboard targeting computations.

This technical report has been reviewed and is approved.

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ABSTRACT

Methods are investigated which apply to the conversion of latitude, longitude, and height to cartesian coordinates in a plane tangent to the earth's surface in connection with onboard missile targeting. The criterion for the usefulness of the method is the error in the north and east coordinate with reference to Clarke's spheroid 1866. This error is determined as a function of vector length and azimuth for the spherical earth model referenced to the mean geodetic latitude at White Sands Missile Range utilizing a program prepared for a programmable electronic desk calculator (Marchant 1016 PR). An approximation of geocentric latitude is used in the program. It is explained how the error inherent to the spherical earth model can be reduced applying a certain correction. An expression for computing the reference coordinates in a plane tangent to the earth spheroid is derived which requires less numerical effort than the standard procedure.
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SECTION I
INTRODUCTION

Approximations for Converting
Geodetic to Cartesian Coordinates

The purpose of a study related to the SRAM project was to simplify calculations for the post-flight evaluation of trajectory data and to reduce the number of numerical operations so that a programmable desk calculator could be used for the computations. Of primary interest was the process used to convert geodetic data of latitude, longitude and height to cartesian coordinates in a plane tangent to the surface of the earth. The Marchant Calculator, Model PR 1016, was available for these calculations and it was required that position fixes and target range computations performed during the flight of a B-52 aircraft be examined on the ground to obtain "quick-look" information revealing the accuracy of the onboard computations. This calculator had to be utilized to save the "turn-around" time spent in waiting for the results of a large electronic computer (CDC-3600).

To find an adequate procedure, conversion methods had to be compared with respect to their accuracy and a compromise made between the complexity of the calculations and the accuracy of the final results. Known approximations had to be examined and their limitations with respect to the maximum range between the fixed point or target and the aircraft had to be determined. The goal was to implement the calculator with a program capable of converting geodetic data to tangent plane data with an accuracy equivalent to a few feet compared to the solution of using a spheroidal earth model which was assumed to yield the exact solution. A specific question was then, up to what maximum range the spherical earth model would be adequate in this respect.

Target ranges in the White Sands test area barely exceed 100 nautical
miles; however, because of the significance of this subject, errors caused by some approximations were calculated for ranges up to 700 nautical miles. These ranges or distances between the aircraft and targets are calculated to determine the "range-to-go", which is derived from the output of the inertial platform sensors and the known target coordinates.

Since information about approximations for the required conversion process are scattered in the literature, the more important expressions related to this subject are reviewed and, in some cases, derived.
SECTION II
METHODS FOR THE CONVERSION PROCESS

It is standard practice to calculate position differences of an aircraft with respect to a ground target in cartesian coordinates. To relate the two positions to each other, it is necessary to convert the geodetic survey data of the target into cartesian coordinates of the platform system or vice versa. Normally a north/east orientation is used as a reference, and a deviation of the platform axes from this orientation is taken into account; results are presented with components in north and east direction counted positive which are called $R_N$ or $X$ and $R_E$ or $Y$ hereafter. Of less interest here is the Z component which may be computed but can also be found from the altitude measurement of the onboard radar altimeter combined with information of terrain altitude.

In the following, it is assumed, unless otherwise stated, that the point on the earth's surface is at a mean sea level. The models investigated with respect to their accuracy are:

Circular arcs in north/south and east/west direction and

Spherical earth with geocentric radius derived from the mean latitude of the two points.

The spheroidal earth model with major and minor axes specified for the basic ellipse serves as a reference to determine the error of the results derived both from circular arcs and spherical earth. It is noted that the spheroidal earth model is also an approximation, but it best approximates the actual earth shape.

a. Circular-arc Approximations

Circular arcs can be approximated using either geodetic or geocentric quantities for the earth radius and for latitude. Based on geodetic quantities the equations for the north range $R_N$ and for the east range $R_E$ in the local tangent plane are
\[ R_N = X = R_g \Delta \phi_g \quad (1a) \]
\[ R_E = Y = R_g \cos \phi_g \Delta \lambda \quad (1b) \]

where

- \( R_g \) = geodetic earth radius
- \( \Delta \phi_g = (\phi_g - \phi_{g1}) \) radians
- \( \Delta \lambda = (\lambda_2 - \lambda_1) \) radians
- \( \phi_g = (\phi_{g1} + \phi_{g2})/2 \)

\( \phi_{g1}, \phi_{g2} \) are the geodetic latitudes and \( \lambda_1, \lambda_2 \) the longitudes of point 1 and 2 respectively.

From standard text books (for instance, reference 7):

\[ R_g = \frac{R_{eq}}{1 - \varepsilon^2 \sin^2 \phi_g}^{-\frac{1}{2}} \]

where

- \( R_{eq} \) = equatorial radius of the earth
- \( \varepsilon \) = eccentricity of the earth ellipse

Equation (1b) is a valid approximation and can be used for quick estimates; (1a) should not be used because it is not a valid expression and leads to significant errors for values above 200 feet. In this context a few feet are considered as a tolerable error. Equation (1a) is mentioned here because it is occasionally used for quick-look estimates, with \( \Delta \phi_g \) and \( R_g \) being readily available. The error is shown for some typical values of \( R_N \) in the section on "Numerical Examples"; values of \( X(=R_N) \) computed from (1a) are designated as "approximation 1".

Selecting a certain earth model, the equatorial radius \( R_{eq} \) and the polar radius \( R_{pol} \) of the earth are given and can be used to compute the eccentricity \( \varepsilon \) which is defined by

\[ \varepsilon^2 = 1 - \left( \frac{R_{pol}}{R_{eq}} \right)^2 \quad (2) \]

A second approximation for \( X \) uses the mean geocentric earth radius \( R_C \) and geocentric latitudes \( \phi_C \):

\[ X = R_C \Delta \phi_C \quad (3a) \]

where

\[ \Delta \phi_C = \phi_{c2} - \phi_{c1} \]
(3a) is a valid approximation of $X$ and is adequate for a wider range compared to $X$ computed from (1a), provided the two points are on the same meridian.

$$Rc \cos \psi_c$$

**Geocentric and Geodetic Earth Radius**

Figure 1

The corresponding expression for the east/west component $R_E$

$$R_E = Y = Rc_2 \cos \psi_c \Delta \lambda$$

is equivalent to equation (1b). This can be seen from figure 1, since

$$Rc \cos \psi_c = Rg \cos \psi_g$$

Both (1b) and (3b) represent valid approximations of the east range $R_E$ for two points on the same parallel. The accuracy of the approximations is limited by:

$$\Delta Y = Rc_2 \cos \psi_c (\Delta \lambda - \sin \Delta \lambda)$$

The equation

$$Y = Rc_2 \cos \psi_c \sin \Delta \lambda$$

is exact for all earth models described here.

The computation of $X$ from (3a) and $Y$ from (3b) or (3c) requires the values of $R_c$ and $\psi_c$: they can be obtained from equations in standard textbooks. For instance, from reference 4:

$$R_c = \text{Req} \left[ \cos^2 \psi_c + (\text{Req}/\text{Rpol})^2 \sin^2 \psi_c \right]^{-\frac{1}{2}}$$

$$= \text{Req} \left[ 1 + \sin^2 \psi_c \left[ (\text{Req}/\text{Rpol})^2 - 1 \right] \right]^{-\frac{1}{2}}$$

(4a)

and using (2) we obtain

$$R_c = \text{Req} \left( 1 + \frac{\epsilon^2}{1 - \epsilon^2} \sin^2 \psi_c \right)^{-\frac{1}{2}}$$

(4b)
The geocentric latitude $\xi_c$ can be computed from

$$
\xi_c = \arctan\left(\frac{R_{pol}}{R_{eq}} \tan \xi_g\right) \quad (5a)
$$

Equation (5a) is exact and is found in reference 4. $\xi_c$ can also be obtained by applying the small angle approximation to $\tan(\xi_c - \xi_g)$ and then expanding in a power series. This is done in appendix A and the first two terms of the expansion are

$$
\xi_c - \xi_g = -0.19390737 \sin 2\xi_g - 0.00131249 \sin 2\xi_g \sin^2 \xi_g \quad (5b)
$$

where $\xi_c$ and $\xi_g$ are in degrees.

Two other forms of this series are given in references 2 and 3. One can derive the series of reference 3 by manipulating equation (5b) so that the second term in (5b) disappears for one particular value of $\xi_g$. This is described in appendix A. Choosing 33° for this particular value of $\xi_g$ yields:

$$
\xi_c - \xi_g = -0.19429670 \sin 2\xi_g + 0.00038933 (1 - 3.371184 \sin^2 \xi_g) \sin 2\xi_g \quad (5c)
$$

(Corresponding to equation (A7) in appendix A).

If one neglects the second term of (5c) one finds an improved approximation of $(\xi_c - \xi_g)$ compared to a one-term-only solution of (5b). This improvement is obtained of course for the specific value of $\xi_g$ and to some degree also for values of $\xi_g$ in the neighborhood of the specific value. Since the mean geodetic latitude at WSMR was 33° or close to 33° for those target ranges, which were evaluated, this number was considered adequate as basis for the one-term approximation.

$$
\xi_c - \xi_g = -0.19429670 \sin 2\xi_g \quad (5d)
$$

which follows from (5c).

The values of $X$ and $Y$ from equations (3a) and (3b) can now be computed provided the geodetic latitudes $\xi_g$ are given. Examples are discussed in the section 'Numerical Examples'. Equation (5d) was used to compute $\xi_c$ in equation (3a) and in (3b) and the approximations for $X$ and $Y$ found in this way are referred to as 'approximation 2'. The errors caused by the one-term approximation are plotted in figure 2 for 30° and 33°.

In practice, another procedure is more frequently used to estimate the length
of an arc of a meridian or a parallel, and this estimate in turn is used for approximating north or east components in the local tangent plane. This is the method of using precalculated scaled factors of distance in feet per angular degrees or per arc minute. These scale factors can be derived from a spheroidal model of the earth and computed as functions of geodetic latitude. In figure 3 and tables 1 and 2, these factors are presented both for arcs in latitude and longitude based on Clarke's spheroid of 1866. They were found by converting a one-arc-minute difference in latitude or longitude respectively to tangent plane coordinates. As a result, the following expressions can be used for a geodetic latitude of \( 33^\circ \) to estimate the north/south (X) and east/west (Y) coordinates in a tangent plane system.

\[
\begin{align*}
X &= 6064.1 \Delta \phi_g \\
Y &= 6093.2 \Delta \lambda \cos 33^\circ
\end{align*}
\]

where \( \Delta \phi_g \) and \( \Delta \lambda \) are in minutes of arc, or

\[
\begin{align*}
X &= 363840 \Delta \phi_g \\
Y &= 365592 \Delta \lambda \cos 33^\circ
\end{align*}
\]

where \( \Delta \phi_g \) and \( \Delta \lambda \) are in degrees.

It is emphasized here that equations (3a), (3b), (6a), (6b), (6c) and (6d) are valid approximations only if the two points of interest are located on the same meridian, or on the same parallel, respectively. The error caused by not being on the same meridian or parallel depends on the value of the angular difference. This limitation is related to the subject of the next section.

b. Spherical Earth Approximation

A better approximation is obtained by using a spherical earth model with a radius \( R_e \) equal to the geocentric radius which is computed as the mean of the geocentric radii of the two particular points of the surface of the earth. The expressions for the north and east components of the distance between two points in the local tangent plane are then derived from figure 4 as follows:
Figure 4. Spherical Earth Model
<table>
<thead>
<tr>
<th>Degree</th>
<th>Feet per Degree of Geodetic Latitude</th>
<th>Feet per Arc Minute of Geodetic Latitude</th>
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<tr>
<td>0</td>
<td>362,760</td>
<td>6046.0</td>
</tr>
<tr>
<td>10</td>
<td>362,874</td>
<td>6047.9</td>
</tr>
<tr>
<td>20</td>
<td>363,192</td>
<td>6053.2</td>
</tr>
<tr>
<td>30</td>
<td>363,684</td>
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<tr>
<td>35</td>
<td>363,948</td>
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</tr>
<tr>
<td>40</td>
<td>364,290</td>
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<td>60</td>
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<td>70</td>
<td>366,036</td>
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<td>6106.0</td>
</tr>
<tr>
<td>90</td>
<td>366,474</td>
<td>6107.9</td>
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Table II
FEET PER DEGREE AND PER ARC MINUTE OF LONGITUDE
FOR CLARKE'S 1866 SPHEROID

<table>
<thead>
<tr>
<th>Degree</th>
<th>Feet per Degree of Longitude</th>
<th>Feet per Arc Minute of Longitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>365,225</td>
<td>6087.1</td>
</tr>
<tr>
<td>10</td>
<td>359,713</td>
<td>5995.2</td>
</tr>
<tr>
<td>20</td>
<td>343,334</td>
<td>5722.3</td>
</tr>
<tr>
<td>30</td>
<td>316,562</td>
<td>5276.0</td>
</tr>
<tr>
<td>33</td>
<td>306,611</td>
<td>5110.2</td>
</tr>
<tr>
<td>40</td>
<td>280,170</td>
<td>4669.5</td>
</tr>
<tr>
<td>50</td>
<td>235,230</td>
<td>3920.5</td>
</tr>
<tr>
<td>60</td>
<td>183,078</td>
<td>3051.3</td>
</tr>
<tr>
<td>70</td>
<td>125,289</td>
<td>2088.1</td>
</tr>
<tr>
<td>80</td>
<td>63,650</td>
<td>1060.6</td>
</tr>
<tr>
<td>90</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
It can be seen in this figure that $R_N$ is the base and 

$$(R_C + h_2)\sin \theta_2 - (R_C + h_2)\cos \theta_2 \cos \Delta \tan \theta_1$$

is the hypotenuse of a right triangle with the angle $\theta_1$ between base and hypotenuse. Note that the angles $\theta_1$ and $\theta_2$ represent geocentric latitudes and $R_c$ is the mean geocentric radius.

Therefore

$$X = R_N \cos \theta_2 \left[ (R_C + h_2)\sin \theta_2 - (R_C + h_2)\tan \theta_2 \cos \theta_2 \cos \Delta \right]$$

where

$$\Delta = \lambda_2 - \lambda_1$$

or

$$R_N = (R_C + h_2)(\cos \theta_2 \sin \theta_2 - \sin \theta_1 \cos \theta_2 \cos \Delta \lambda) \quad (7a)$$

Setting $\Delta = 0$, under the assumption that the two points are on the same meridian, one obtains as can be expected

$$R_N = (R_C + h_2)\sin (\theta_2 - \theta_1) \quad (7b)$$

Only for small angular differences ($\theta_2 - \theta_1$) and $h_2 = 0$ does this equation yield approximately the same results as equation (3a) which indicates the limitation of the circular-arc approximation. The east component $R_E$ is

$$Y = R_E = (R_C + h_2)\cos \theta_2 \sin \Delta \lambda \quad (7c)$$

Where $R_C$ is the geocentric radius of point 2.

Equation (7c) yields an approximation for $Y$ with errors depending on the value of $h_2$.

The exact value of $Y$ is found from

$$Y = R_E = (R_C \cos \theta_2 + h_2 \cos \theta_2 \sin \Delta \lambda) \quad (7d)$$

It is noted here that the expression for the north component (7a) is an approximation because of the use of a spherical earth model to represent an elliptical earth; but (7d) is an exact expression for the calculation of the east component of the range. It is noteworthy also that (7d) uses the geocentric radius at point 2.

$R_{C2}$, instead of the mean geocentric radius $R_c$ which is used in (7a).

In many practical cases, equation (7a) is accurate enough and its simplicity makes computation possible on a small programmable electronic desk calculator. An example of the program which was prepared for the Marchant 1016 PR is listed in appendix B. The program consists of four parts each of which is recorded.
on a separate magnetic tape. These tapes are read into the core memory sequentially and require four quantities as manual input:

\[ \$g_1 = \text{Geodetic latitude (in degrees) of point 1} \]
\[ \$g_2 = \text{Geodetic latitude (in degrees) of point 2} \]
\[ \Delta \lambda = \lambda_2 - \lambda_1 \text{ (in degrees)} \]
\[ h_2 = \text{Height of point 2 in feet (multiplied by a scale factor of } 10^{-8}) \]

Typical values of X and Y were calculated from equations (7a) and (7c) or (7d) respectively and the results are discussed in the section "Numerical Examples". When the approximated geocentric latitudes \( \xi_c \) from equation (5d) are used to compute X and Y the approximations are referred to as "approximation 3"; when the exact value of \( \xi_c \) from equation (5a) is inserted into equations (7a), (7c) or (7d) the values of X and Y are referred to as "approximation 4". With this arrangement the approximations are easy to distinguish. Those designated with higher numbers should produce more accurate results with respect to the earth spheroid.

A simple relationship (7e) can be used to approximate the Z component of the vector connecting point 1 and 2:

\[
Z = \left[ (R_{c2} + h_2)^2 - X^2 - Y^2 \right]^{\frac{1}{2}} - (R_{c1} + h_1)
\]

Note that (7e) is not an exact equation since the heights \( h_1 \) and \( h_2 \) do not form a straight line with \( R_{c1} \) and \( R_{c2} \) respectively; however, this approximation is accurate enough for the applications considered here. For higher accuracy requirements the Z component also referred to as \( R_Z \), should be derived from the spheroidal earth model.

c. Spheroidal Earth Model

The most accurate estimate of position differences is obtained by applying a spheroidal or ellipsoidal model of the earth in the coordinate transformation process. Frequently used models are Clarke's spheroid of 1866 and Hayford's.
spheroid of 1910, also called the international spheroid. Improved earth model parameters were recently (1966) derived by the Smithsonian Astrophysical Observatory (SAO) from earth satellite data. The model is referred to later in the text as SAO spheroid. The equatorial and polar radius for these three models are as follows:

<table>
<thead>
<tr>
<th></th>
<th>$R_{eq}$ (ft)</th>
<th>$R_{pol}$ (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clarke's spheroid</td>
<td>20,925,832</td>
<td>20,854,892</td>
</tr>
<tr>
<td>Hayford's spheroid</td>
<td>20,926,470</td>
<td>20,856,010</td>
</tr>
<tr>
<td>SAO spheroid</td>
<td>20,925,738</td>
<td>20,855,776</td>
</tr>
</tbody>
</table>

Derived from $R_{pol}$ and $R_{eq}$ are by definition the ellipticity $E = 1 - R_{pol}/R_{eq}$ and the eccentricity $e = [1 - R_{pol}/R_{eq}]^{1/2}$, which are listed below together with $e^2$ and the frequently used term $\frac{e^2}{(1-e^2)}$.

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$e$</th>
<th>$e^2$</th>
<th>$e^2/(1-e^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clarke's spheroid</td>
<td>1/295</td>
<td>0.082271770</td>
<td>0.0067686441</td>
<td>0.0068147708</td>
</tr>
<tr>
<td>Hayford's spheroid</td>
<td>1/297</td>
<td>0.08199218</td>
<td>0.0067227183</td>
<td>0.0067681701</td>
</tr>
<tr>
<td>SAO spheroid</td>
<td>1/298.25</td>
<td>0.08182018</td>
<td>0.0066945419</td>
<td>0.0067396608</td>
</tr>
</tbody>
</table>

Geodetic survey data of targets, impact locations and radar sites, etc., at WSMR are in general based on Clarke's spheroid of 1866 and the latter is used in connection with numerical examples discussed later. There are two ways to compute the north and east components of the range between two points from given geodetic data. First, the "classical" or conventional procedure consisting of the conversion from geodetic to earth-centered coordinates, followed by a translation and rotation of the coordinates. Second, relatively simple explicit formulae can be derived for $X$, $Y$ and $Z$, also called $R_N$, $R_E$ and $R_Z$ in the text. The classical method is described as follows:

The given geodetic coordinates of point one and two are converted to earth-centered cartesian system (ecs) which is a left-handed system. The coordinates of point two are translated to point one as new origin which yields:
The other procedure, mentioned above, for converting geodetic data to local tangent plane coordinates is described next. It utilizes in a straightforward manner the geometry of the ellipse representing the earth and of the cartesian coordinates connecting the two points of interest. Similarly as for the spherical earth model in figure 4, point 2 and straight line connections through point 2 are shown in figure 5 projected into the vertical plane through the meridian on which point 1 is located. In the derivation of the expression for \( R_N \), the two equations of the vertical and horizontal cartesian coordinates \( R_V \) and \( R_H \) are used as follows:

\[
\begin{align*}
R_V &= R_g(1 - e^2)\sin \phi g \\ 
R_H &= R_g \cos \phi g 
\end{align*}
\]

Equation (9b) can be derived from figure 5 readily and equation (9a) is found by inserting (9b) into the expression for \( \tan \phi_c \):

\[
\tan \phi_c = \frac{R_V}{R_H} = \frac{b^2}{a^2} \tan \phi_g
\]

which yields

\[
R_V = \frac{b^2}{a^2} R_g \sin \phi_g
\]

The expressions for \( R_N \) and \( R_Z \) can be found from figure 5 which shows that \( R_N \) is the base and

\[
H = (R_g + h_g) \sin \phi_2 - e^2 (R_g \sin \phi_2 - R_1 \sin \phi_1) \\
- (R_g + h_g) \cos \phi_2 \cos \Delta \tan \phi_1
\]

\[ (9d) \]
Figure 5. Spheroid Geometry
is the hypotenuse of a right triangle with \( \theta_1 \) being the angle between base and hypotenuse. Therefore,

\[
R_N = (R_0 + h_0)(\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \cos\lambda) - c^2(R_0 \sin\theta_2 - R_1 \sin\theta_1) \cos\theta_1
\]

As can be expected, this formula is similar to the expression for \( R_N \) which was derived from spherical earth (7c) using the mean geocentric radius. Since (9e) is based on the elliptical earth model utilizing the geodetic radii \( R_1 \) and \( R_0 \) and the geodetic latitude \( \theta_1 \) and \( \theta_2 \), an additional term must be applied. This term, the second part of (9e), accounts for the distance between the centers of the radii \( c^2(R_0 \sin\theta_2 - R_1 \sin\theta_1) \).

The expression for \( R_Z \) is found as

\[
R_Z = R_1 + h_1 - \left[ (R_0 + h_0) \cos\theta_2 \cos\lambda / \cos\theta_1 + RN \tan\theta_2 \right]
\]

The equation for \( R_E \) (9g) is equivalent to equation (7c) with the geodetic radius \( R_0 \) and geodetic latitude \( \theta_2 \) used instead of the corresponding geocentric data:

\[
R_E = (R_0 + h_0) \cos\theta_2 \sin\lambda
\]

To compare the two procedures described above for the computation of the LTP components, one may count the numerical operations (products, divisions, additions, subtractions and squaring) and the subroutine entries. Under the assumption that in either case, as is usual, the geodetic coordinates of the two points are given, it is found in this way that the second procedure requires

- 32 operations including 7 trigonometric functions
- and 2 square roots to compute \( R_N \) and \( R_E \);

and in addition
- 8 operations to compute \( R_Z \).

The former procedure requires a minimum of

- 48 operations including 8 trigonometric functions
- and 2 square roots to compute \( R_N \) and \( R_E \);

and in addition
- 7 operations to compute \( R_Z \).

Local Tangent Plane
A double precision version of the former program was used as reference for investigating the accuracy of the two programs. It was found that the second program, the new version, yields slightly more accurate results because fewer operations are involved. The difference was more pronounced for the Z component than for the X component, and more for the X component than for the Y component. Results for RZ obtained from the new version agree to 9 decimal places with the double precision results, whereas the older procedure leads to an agreement of 8 places.

By manipulating the matrix product in equation (C4) of appendix C representing the classical procedure it can be shown that (C4) leads to equations identical with (9e), (9f) and (9g).
SECTION III
NUMERICAL EXAMPLES

Accuracy limitations of the circular arc and of the spherical earth approximations become evident from numerical examples discussed in this section. The errors $\Delta R_N$ in $R_N$ are presented in figures 6 through 10 which are made by the 4 approximations defined as follows:

- Approximation 1: circular arcs $R_g \Delta \theta_g$
- Approximation 2: circular arcs $R_C \Delta \theta_c$
- Approximation 3: spherical earth using (5d) for $\theta_c$
- Approximation 4: spherical earth using (5a) for $\theta_c$

Those values of $R_N$ served as a reference for all presented errors $\Delta R_N$ which were computed from equation (9e) based on Clarke's spheroid of 1866 for various ranges and azimuth angles. These reference values are listed in tables 3 and 4 as functions of the range $R_N$ (or $R_E$) and of the azimuth.

The input for the computation is assumed to be given in geodetic coordinates, latitude $\phi$ and longitude $\lambda$ in degrees and height $h$ in feet above mean sea level. For applications considered in this context, it is adequate to carry six decimal places for $\phi$ and $\lambda$ and to use a rounded integer number for $h$ considering a one-foot accuracy as well within accuracy requirements.

Some details of the computation which apply to approximations 3 and 4 are clarified first. Neither version computes the mean geocentric radius $R_C$ as the mean of the two geocentric radii $R_{C1}$ and $R_{C2}$ which would appear necessary from a theoretical standpoint; instead $R_C$ is derived from the mean geocentric latitude $\bar{\phi}_C$ as

$$R_C = R_e q \left( 1 + \frac{e^2}{1-e^2} \sin^2 \bar{\phi}_C \right)^{-\frac{1}{2}}$$  \hspace{1cm} (10a)
Figure 9
Approximation 3 and 4
Azimuth 89° to 91°
### Table III

**R_N AND R_E FOR CLARKE'S SPHEROID 1866 AS FUNCTION OF RANGE AND AZIMUTH**

Azimuth: 0°, 36°, 136°

| R_N Nautical Miles | Azimuth: 0° | | Azimuth: 36° | | Azimuth: 136° |
|-------------------|---|---|---|---|
| Δρ (°) | R_N (ft) | Δρ (°) | R_N (ft) | Δρ (°) | R_N (ft) |
| 60    | 1.0 | 363825 | 1.0 | 363837 | -1.0 | 361744 |
| 0     | 0   | 0     | 0.185 | 361245 | 0.185 | 365344 |
| 100   | 1.666 | 606085 | 1.666 | 612588 | -1.666 | 600221 |
| 0     | 0   | 0     | 0.975 | 599684 | 0.975 | 621642 |
| 150   | 2.5 | 909322 | 2.5 | 915571 | -2.5 | 995867 |
| 0     | 0 | 0     | 0.975 | 897777 | 0.975 | 922844 |
| 200   | 3.32 | 1207300 | 3.32 | 1228462 | -3.32 | 1183256 |
| 0     | 0 | 0     | 0.94 | 1183569 | 0.94 | 1229189 |
| 400   | 6.64 | 2410558 | 6.64 | 2459249 | -6.64 | 2369093 |
| 0     | 0 | 0     | 0.684 | 2310904 | 0.684 | 2490640 |
| 500   | 8.33 | 3020212 | 8.33 | 3129414 | -8.33 | 2856144 |
| 0     | 0 | 0     | 0.671 | 2942274 | 0.671 | 3145028 |

* Approximate values for nautical miles.

**Exact values for Δρ and Δλ**

\[
\Delta \rho = \rho_{2} - \rho_{1} \\
\Delta \lambda = \lambda_{2} - \lambda_{1}
\]
Table IV

R_N AND R_E FOR CLARKE’S SPHEROID S866 AS FUNCTION OF RANGE AND AZIMUTH

Azimuth: 89°, 90°, 91°

<table>
<thead>
<tr>
<th>R_E *</th>
<th>Azimuth 89°</th>
<th>Azimuth 90°</th>
<th>Azimuth 91°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nautical Miles</td>
<td>Δφ'</td>
<td>R_N(ft)</td>
<td>Δφ'</td>
</tr>
<tr>
<td>60</td>
<td>0.00075</td>
<td>8400</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1.19682</td>
<td>364750</td>
<td>1.19</td>
</tr>
<tr>
<td>100</td>
<td>0.02917</td>
<td>36340</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1.98303</td>
<td>607798</td>
<td>1.98</td>
</tr>
<tr>
<td>130</td>
<td>0.04375</td>
<td>28799</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2.97455</td>
<td>971396</td>
<td>2.97</td>
</tr>
<tr>
<td>200</td>
<td>0.05834</td>
<td>44135</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3.96600</td>
<td>1244667</td>
<td>3.97</td>
</tr>
<tr>
<td>400</td>
<td>0.15669</td>
<td>333794</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>7.93222</td>
<td>2422718</td>
<td>7.93</td>
</tr>
<tr>
<td>500</td>
<td>0.14586</td>
<td>0.95078</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3.15215</td>
<td>3022454</td>
<td>3.15</td>
</tr>
</tbody>
</table>

* Approximate values for nautical miles

Exact values for Δφ and Δλ:

Δφ = ϕ₂ - ϕ₁  
Δλ = λ₂ - λ₁
This simplification produces an error which is insignificant compared to other errors, but it substantially reduces the numerical effort. The geocentric earth radius at point 2, $R_{c2}$, is used to compute the east/west component $R_E$ from

$$R_E = R_{c2} \cos \phi_2 \sin \alpha$$  \hspace{1cm} (10b)

This expression is exact and values of $R_E$ computed from (10b) are identical with those derived from (9g) if the exact value of $\phi_2$ is used. But even for approximated geocentric angles $\phi_2$ the value of $R_E$ computed from (10b) agreed for all examples very closely (the maximum difference found was 3.4 feet) with the value of $R_E$ resulting from approximation 4. $R_E$ values are therefore omitted from further discussion.

To make various results comparable, the assumption was made in all cases that the center of the range between the aircraft and the ground target was at 31° geodetic latitude. Basically, two parameters have been varied in the numerical examples. These are, first, the length of the range vector for a given constant azimuth and, second, the azimuth for a given constant range vector.

The azimuth $A_1$ is the angle between north and the range vector, counted positive in clockwise direction. It is computed as an approximation from angular data $(\Delta \lambda, \Delta \phi_g)$ instead of from the cartesian coordinates.

$$A_1 = \arctan \left( \frac{306611 \Delta \lambda}{363848 \Delta \phi_g} \right)$$

Six different azimuth angles are considered to demonstrate the general behavior of $R_N$ as function of the range vector length. They are 0°, 45°, 90°, 93°, 96° and 99°. For these angles, $R_N$ was computed from equation (9c) as function of the range; the results are listed in Tables 3 and 4.

As can be expected, $R_N$ grows nonlinearly with the length of one of the vector coordinates which is listed in nautical miles. The sign of $R_N$ is positive or negative (same as the sign of $\Delta \phi$) and stays so, independent of the vector length. An exception to this rule is found for an azimuth slightly larger than 90° as can be seen at an angle of 91°. (Table 4). This unusual situation can be explained as follows:
The sign of \( R_N \) depends on this special case or where point 2 is located with respect to a borderline which is defined by

\[
\cos \theta_2 \sin \lambda_2 = \cos \theta_1 \sin \lambda_1 \cos \Delta
\]

This equation is found from (7a) by setting \( R_N = 0 \).

If this equation is satisfied, \( R_N \) vanishes; if it is not satisfied, the location of point 2 north or south determines the sign. For a 90° azimuth therefore, \( R_N \) is a relatively small positive or negative quantity, depending on whether point 2 is west or east of point 1, respectively. Increasing the vector length and, as a result, also increasing \( R_E \) for the case of a 91° azimuth, point 2 changes its position from one side to the other side of this borderline and \( R_N \) changes its sign accordingly. This situation has some effect on an application discussed in the next section.

Considering the accuracy of the circular arc approximation first, it is found from Figure 6 that approximation 1 is not adequate for ranges \( R_N \) exceeding 0.6 nautical miles and approximation 2 is restricted to ranges \( R_N \) up to 33 nautical miles provided, of course, that the two points are located on the same meridian; for larger values of \( R_N \), an error of at least 10 feet is made. If the two points are not on the same meridian, approximations 3 and 4 can be used over a comparatively wide range to estimate \( R_N \) with tolerable errors. This is demonstrated in Figures 7 through 10.

Inspecting these figures, we find that as a general trend for both approximations of \( R_N \) (approximations 3 and 4) the error \( \Delta R_N \) is negative and its absolute value grows as a nonlinear function with the vector length. Its value varies between 20 feet and 16 feet at 200 nautical miles depending on the azimuth angle and reaches more than 600 feet at 300 nautical miles. In the neighborhood of 90°, the two approximations 3 and 4 yield almost identical values; the differences are negligible for practical purposes between 89° and 91°. Of particular interest is the fact that for the azimuth of 45° the "poorer" approximation (3), representing the spherical earth model, leads to a value of \( R_N \) more closely resembling the-
value corresponding to the spheroidal earth. This, of course, is an effect of the bias error in the geocentric latitudes derived from the truncated equation mentioned earlier.

More information on the behavior of approximations 3 and 4 is found from figure 10; it shows for a constant vector length of about 150 nautical miles the errors $R_N$ as functions of the azimuth angle. The maximum is found at 90°. From this diagram, it is also evident that within some interval between 10° and 90°, approximation 3 yields more accurate values of $R_N$ than approximation 4. This includes the case of 45° mentioned earlier.

It is noted that all results of $\Delta R_N$ shown in figures 6 through 10 were obtained for an altitude of the target $h_0$ equal to zero. An experiment showed that the value of $h_0$ has no significant effect on $\Delta R_N$. Two values of $h_0$ were considered: $h_0 = 0$ and $h_0 = 4000$ feet MSL, which is approximately the elevation of the basin of White Sands Missile Range. The effect was 2 feet or less when $h_0$ was changed from 0 to 4000 feet. As can be seen in equation (9e), $h_0$ does have an effect on $R_N$ itself, whereas the height of the airplane $h_1$ does not.
SECTION IV

IMPROVEMENTS OF THE SPHERICAL EARTH APPROXIMATIONS

An improvement of the approximations based on a spherical earth model
(approximations 3 and 4 in the previous sections) is desirable in the sense that \( R_N \),
derived from a slightly modified model, approaches more closely the value of \( R_N \)
which is based on a spheroidal earth model. This is desired over a certain in-
terval of \( R_N \) values. There may be various ways of achieving this improvement.
As it has been mentioned earlier, approximation 3 may yield better results in
this respect than approximation 4; the improvement is caused in this case by a
bias error affecting geocentric latitudes. One way which appears practical and
efficient or account of some experiments is described in this section.

Considering equation (7a), and rewriting it as

\[
R_N = F_1 F_2
\]

where

\[
F_1 = R_c + h
\]

\[
F_2 = \cos \phi_1 \sin \Delta - \cos \phi_2 \sin \phi_2 \cos \Delta
\]

it is evident that a systematic or bias type error either in \( F_1 \) or \( F_2 \) or in both may
affect the value of \( R_N \). Introducing a bias in a controlled way may lead to the im-
provement of the approximation. This has been done by adding a correction term
\( \Delta R \) to \( F_1 \) using the following equations:

\[
F_1 = (R_c + h) + \Delta R
\]

where

\[
\Delta R = \frac{\Delta R_{No}}{R_{No}} \frac{R_c}{R_{No}} \frac{\Delta \phi}{\Delta \nu_c}
\]

\[
\Delta \phi (R_{No}) \quad \text{is the error of position latitude between point 1 and 2.}
\]

\[
\Delta \nu_c \quad \text{is the specified} \Delta \phi \quad \text{for which} \Delta R = 0 \quad \text{by adjustment.}
\]

\[
R_c \quad \text{mean geocentric earth radius.}
\]

\[
\Delta R_{No} \quad \text{the error to be adjusted or compensated.}
\]

\[
R_{No} \quad \text{the north component of the total range for which} \Delta R_{No} \quad \text{is}
\]

\text{to be adjusted.}
The following numerical example explains the procedure for finding the value of \( \Delta R_C \). The intention is to make \( \Delta R_N \) zero at a range \( R_N \) of 200 nautical miles for an azimuth angle of 135°. The corresponding, uncompensated error \( \Delta R_N \) for approximation 3 is found from figure 8b as

\[
\Delta R_N = -135 \text{ feet}
\]

To compensate this error, \( R_C \) must be reduced by a proportionate value.

\[
\Delta R_C = -135 \frac{R_C}{R_N} \text{ feet}
\]

With \( R_C = 20,904,910 \) feet from equation (4b) for \( \theta_g = 33^\circ \) and with \( R_N = 1,183,256 \) feet from table 3 for 200 nautical miles (\( \theta_N = 135^\circ \)), we obtain:

\[
\Delta R_C = -2,367 \text{ feet}
\]

For values of \( R_N \) other than 200 nautical miles

\[
\Delta R_C = \Delta R_C \frac{\Delta \theta(R_N)}{\Delta \theta_0}
\]

For instance, for \( R_N = 100 \) nautical miles, using the value of \( \Delta \theta = 1.666^\circ \) from table 3 and \( \Delta \theta_0 = 3.32 \) for 200 nautical miles, we find

\[
\Delta R_C = -2367 \frac{1.666}{3.32} = -1187 \text{ feet}
\]

In the same way, additional correction factors can be determined between \( R_N = 6 \) and \( R_N = 500 \) nautical miles.

For later reference (12b) is rewritten as

\[
\Delta R_C = C_1 \Delta \theta(R_N)
\]

where \( C_1 \) is a constant which depends on the azimuth and the range for which \( \Delta R_N \) is compensated. For one particular vector length, the error \( \Delta R_N \) is compensated and for values above and below it, \( \Delta R_N \) is reduced to some extent. Numerical values of \( \Delta R_C \) were used in connection with the Markhart Calculator Program described in appendix B to compute improved approximations of \( R_N \) and the results are referred to in this report as 'approximation 5'. Results obtained by this error compensation are shown in figures 11, 12a, 12b, 14, and 14'. For \( C_1 \)

These figures also contain the 'unreduced error' \( \Delta R_N \) as obtained from approximation 4.
azimuth (figure 11), the error adjustment is of practical value only at large values of \( R_N \) (more than 300 nautical miles), simply because approximation 4 leads to comparatively small errors \( \Delta R_N \) at lower \( R_N \) values. For azimuth angles other than 0°, a significant improvement is found if \( R_N \) (or \( R_E \)) exceeds 100 nautical miles.

A similar reduction of \( \Delta R_N \) as for 45° and 135° (figures 12a and 12b) was obtained for 67.5° azimuth also by applying equation (13). This equation can be used in general for reducing the error except near 90° azimuth. For 90° itself, a very simple and effective error correction was achieved by adding a constant amount of

\[
\Delta R_C = 98925 \text{ feet}
\]

to the mean geocentric earth radius \( R_C \). The result is shown in figure 14. It is noted here that the value of this constant and of those listed below depend on the mean geodetic latitude which in the case considered here was always 33°.

The special situation for azimuth angles slightly bigger or smaller than 90° requires a slightly different adjustment. Considering first the case of an azimuth slightly bigger than 90° (e.g. 91°), the value of \( \Delta R_C \) was computed from

\[
\Delta R_C = C_1 \frac{\Delta \psi}{\Delta \psi - C_0}
\]

and \( \Delta R_C \) is used again to adjust the factor \( F_1 \) in (12a) as

\[
F_1 = R_0 + R_E + \Delta R_C
\]

The resulting reduced error \( \Delta R_N \) is shown in figure 13. In a small region which is close to 186 nautical miles, equation (14) cannot be used because the denominator becomes zero. To circumvent this difficulty, another set of coefficients \( C_0 \) and \( C_1 \) must be chosen.

In the other case, if the azimuth is slightly below 90° (e.g. 89°), a different equation is used to compute \( \Delta R_C \). A second degree approximation was found adequate for the error adjustment:

\[
\Delta R_C = C_1 \Delta \psi + C_2 \Delta \psi^2
\]

* A better solution is probably to use two sets of coefficients for equation (13) corresponding to the positive and negative values of \( R_N \).
For the examples shown in figures 11, 12a, 12b and 13, the following coefficients have been used in equations (13), (14) and (15):

<table>
<thead>
<tr>
<th>Azimuth (degrees)</th>
<th>6</th>
<th>45</th>
<th>135</th>
<th>89</th>
<th>91</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C_2</td>
<td>C_1</td>
<td>C_1</td>
<td>C_1</td>
<td>C_0</td>
</tr>
<tr>
<td></td>
<td>90.030 feet/degree</td>
<td>399.24 feet/degree</td>
<td>713.08 feet/degree</td>
<td>1,219,609 feet/degree</td>
<td>5,691,402 feet/degree</td>
</tr>
</tbody>
</table>

For an azimuth of 67.5°, the coefficient C_1 was 713.08 foot/degree.
SECTION V

CONCLUSIONS

The study of methods and of numerical expressions for the conversion of geodetic data to local tangent plane coordinates leads to the following conclusions:

1. The equations for computing the north and east components $R_N$ and $R_E$, based on a spherical earth model can be mechanized on the programmable Marchant desk calculator 1016 PR and stored on a minimum of three magnetic tapes with a storage capacity of 100 bits each. The program uses an approximation for computing the geocentric latitude.

2. Depending on the azimuth of the local tangent plane vector, errors of up to 37 feet in $R_N$ are caused by this program for a 100 nautical mile distance in north (or east) direction using the spheroidal earth model as reference. This error grows rapidly with increasing length of the vector. The error in $R_E$ is negligible.

3. The error in $R_N$ as mentioned can be reduced by using precalculated correction factors which depend on the azimuth of the local tangent plane vector. Further investigation is needed to determine the optimum correction factors as function of azimuth.

4. The effect of the height of the target point (point 2) in the interval from 0 to 4,000 feet (MSL) is found insignificant with respect to the error $\Delta R_N$. Discrepancies in the results did not exceed 2 feet for all ranges investigated.

5. A simplified equation is derived to determine $R_N$, $R_E$ and $R_Z$ based on the spheroidal earth model; the numerical effort for solving it is reduced compared to the effort associated with the standard procedure.

6. Because of the reduction of the numerical effort, the results and simplifications presented are applicable for quick look postflight evaluation and to some extent for real time data processing.
SECTION V:

ACKNOWLEDGEMENTS

Capt J. D. Hopkins, SRAM Project Officer at Detachment 1, Air Force Weapons Laboratory, during the time this work was done, recommended preparing this report based on a memorandum on the same subject submitted by the author; he also gave advice and guidance to the latter in discussions during the report writing period. The support given by Mr. D. H. Liston was valuable; his suggestions for modifying and rearranging the text and the tables and for omitting some details helped to enhance the clarity of the report.
APPENDIX A

COMPUTATION OF GEOCENTRIC LATITUDE

Based on equation (5a) in the basic report, a power series is derived to approximate $\Phi_c$ and a truncated version of this series can be used to compute $\Phi$ for angles in the neighborhood of a given value of $\Phi_g$. To find the power series, equations (5a) and (2) are inserted into

$$\tan(\Phi_g - \Phi_c) = \frac{\tan\Phi_g - \tan\Phi_c}{1 + \tan\Phi_c \tan\Phi_g}$$

which yields, after manipulation of trigonometric relationships,

$$\tan(\Phi_g - \Phi_c) = \frac{\epsilon^2 \sin 2\Phi_g}{2 \left( 1 - \epsilon^2 \sin^2 \Phi_g \right)} \quad (A1)$$

Assuming that $\Phi_g - \Phi_c$ is a small quantity and applying the binomial expansion, one finds the difference $\Phi_g - \Phi_c$ in degrees as

$$\Phi_g - \Phi_c \approx \frac{180}{\pi} \frac{\epsilon^2}{2} \sin 2\Phi_g (1 + \epsilon^2 \sin^2 \Phi_g + \epsilon^4 \sin^4 \Phi_g + \ldots)$$

Solving for $\Phi_c$ and carrying only one term yields

$$\Phi_c = \Phi_g - \frac{180}{\pi} \left( \frac{\epsilon^2}{2} + \frac{\epsilon^4}{2} \sin^2 \Phi_g \right) \sin 2\Phi_g \quad (A2)$$

One may rewrite (A2) so that the term containing $\sin^2 \Phi_g$ disappears for one particular value of $\Phi_g$, and that the equation for $\Phi_c$ is of the form

$$\Phi_c = \Phi_g - K_1 \sin 2\Phi_g + K_2 (1 - K_3 \sin^2 \Phi_g) \sin 2\Phi_g \ldots \quad (A3)$$

To determine the value of the coefficients $K_1$, $K_2$, and $K_3$, one may then proceed as follows: assuming a certain special value for $\Phi_g$ called $\Phi_g$, is chosen, one substitutes this value into equation (A2) and finds

$$K_1 = \frac{180}{\pi} \left( \frac{\epsilon^2}{2} + \frac{\epsilon^4}{2} \sin^2 \Theta_g, \Phi_g \right)$$
The two following equations, derived from (A2) and (A3), are then compared to determine the coefficients $K_0$ and $K_3$

$$\xi_c \approx \xi_g - (K_1 - K_2) \sin 2\xi_g - K_0 \sin^2 \xi_g \sin 2\xi_g$$
$$\xi_c \approx \xi_g - \frac{180}{\pi} \left( \frac{\pi^2}{2} + \frac{\pi^4}{8} \sin^2 \xi_g \right) \sin 2\xi_g$$

To satisfy both equations, it is found that

$$K_0 = \frac{180 \pi^2}{2} \sin^2 \xi_g, s$$
$$K_3 = 1/ \sin^2 \xi_g, b$$

Inserting the numerical value for $\sin^2 \xi_g$, for $\xi_g = 30^\circ$ equation (A2) is changed to be

$$\xi_c \approx \xi_g - \frac{180}{\pi} \left( \frac{\pi^2}{2} + \frac{\pi^4}{8} \right) \sin 2\xi_g + \frac{180}{\pi} \frac{\pi^2}{8} (1 - 4 \sin^3 \xi_g) \sin 2\xi_g$$

(A4)

In a similar way for $\xi_g = 33^\circ$

$$\xi_c \approx \xi_g - \frac{180}{\pi} \left( \frac{\pi^2}{2} + \frac{\pi^4}{8} 0.24663168 \right) \sin 2\xi_g$$
$$+ \frac{180}{\pi} \frac{\pi^2}{8} 0.29663168 (1 - 3.371848 r^2 \xi_g)$$

(A5)

Applying a value for $\xi_g$ corresponding to Clarke's spheroid 1866, finally for $\xi_g = 30^\circ$

$$\xi_c \approx \xi_g - 0.19423549 - 0.003281 \frac{1}{2} (1 - 4 \sin^3 \xi_g) \sin 2\xi_g$$

(A6)

and for $\xi_g = 33^\circ$

$$\xi_c \approx \xi_g - 0.19429670 - 0.0038933 \frac{1}{2} (1 - 3.371848 \sin^2 \xi_g) \sin 2\xi_g$$

(A7)

Using equations (A6) and (A7), the geodetic latitude can be calculated accurately enough for many purposes in the neighborhood of 30 or 33 geodetic latitude.

Equation (A6) is found in reference 3 with rounded-off coefficients. If only the terms use: $\xi_c$, for $\xi_g, s$

$$\xi_c - \xi_g \approx 0.19423549 \sin 2\xi_g$$

(A8)

and for $\xi_g, b - 33$

$$\xi_c - \xi_g \approx 0.19429670 \sin 2\xi_g$$

(A9)

a still relatively small error is made. It is depicted in figure 2. Equation (A9) is used in the March 1970 NML computer program described in appendix B.
APPENDIX B

COMPUTER PROGRAM FOR THE MARCHANT 1016 PR CALCULATOR

Purpose of the program was to compute from given geodetic data (longitude, latitude and altitude) the tangent plane coordinates $R_N$ and $R_E$ of point B with respect to the origin at point A based on a spherical earth model. A minimum of memory space of the calculator had to be used without significant loss of accuracy. The program consists of 4 sections each recorded on a magnetic tape, read into the calculator from the IOTA tape unit.

Tape 1:

This part of the program computes the geocentric latitudes $\phi$ for point 1 and point 2 using the approximation:

$$\phi_c = \phi - 0.1942967 \sin 2\phi$$

where

$$\phi = \text{geodetic latitude in degrees}$$

and it computes the mean geocentric latitude $\overline{\phi}$.

Input:

$\phi_1$, $\phi_2$, (in degrees) are loaded manually into the key card register

Output:

$\phi_1$ in W1, $\phi_2$ in W2, $\overline{\phi}$ (1/4 $\phi_1$ + $\phi_2$) in W4

All three angles in degrees.

Code:

$ W2 M X .03490 8504 X W M T W $ 
$ M N X W 6 R 1 4 S 1 0 S W A B O 1 6 X 2 2 T R X -1 T 4 X .1942967 -W2 R -S M $ 
$ W1 R BO 2 1 T 2 W1 M K T M K BX 0 0 $ 

This expression is derived in appendix A.

** Merchant symbols are used for the code except for the negative sign in the code list above. N stands for negative sign. Merchant uses $\phi$ for $\phi_1$ and negative sign.
Controls set before input values are read into the keyboard:

10 digits, clear all registers, round-off switch on.

Explanations:

$\hat{\theta}_g, 1$ is inserted in the keyboard before the "run" key is operated and the value of $\hat{\theta}_g, 1$ is printed immediately before the actual computation starts. Insertions of $\hat{\theta}_g, 2$ follows when the program comes to the first stop.

Tape 2:

This part of the program computes the cosines of $\theta_c, 1$, $\theta_c, 2$, $\theta$ and $\Delta \lambda$.

Input:

$\hat{\theta}_c, 1$ in W1, $\hat{\theta}_c, 2$ in W2, $\overline{\theta}_c$ in W3 (all in degrees), $\Delta \lambda$ in degrees manually into the keyboard register at the first stop.

Output:

$\cos \hat{\theta}_c, 1$ in W1, $\cos \hat{\theta}_c, 2$ in W2, $\cos \overline{\theta}_c$ in W3, $\cos \Delta \lambda$ in W4

Code:

X..01743329252 X : W6 M W5 1 M N X
W6 R 1 S 1 S W5 A BO 47 BX 23 T R +
W1 R BO 76 W2 R BO 84 W3 R BO 92 T W4 M
K T M W2 BX 00 T M W3 BX 00 T M K # BX 01

Controls set before running section 2 of the program:

10 digits, round-off switch on, select $N$.

Tape 3:

This part computes the mean geocentric earth radius $R_e$ and the north coordinate $R_y$ of point 2 in a tangent plane with point 1 as the origin based on the equations (10-1) and (10-2).

\[
R = \frac{1}{1 - \cos^2 \theta_c, 1} 
\]

(10-1)

10 digits are shown above an error of 2 feet, which occurs if the function \( \cos \theta_c, 1 \) is computed with 10 digits only, for \( \theta_c, 1 < 0.01 \) degree.
where
\[
A = \frac{R_{eq}}{\left[\epsilon^2/(1-\epsilon^2)\right]^{\frac{1}{2}}}
\]
\[
B = \frac{1-\epsilon^2}{\epsilon^2}
\]

with
\[
\epsilon^2 = 0.0067686441
\]

and
\[
P_{eq} = 20,925,832
\]
\[
A = 2.5348793
\]
\[
B = 147.7461
\]

Note that \(R_c\), \(A\), \(R_{eq}\), and \(R_N\) are all scaled down by a factor of \(10^4\). The formulation of (B1) reduces the memory space requirement to a minimum.

\[
R_N = (R_c + h_0) [\sin \beta \cos \gamma - \sin \gamma \cos \beta] \quad \text{(B2)}
\]

Input:
- \(\cos \beta\) in \(W_1\), \(\cos \gamma\) in \(W_2\), \(\cos \alpha\) in \(W_3\), \(\cos \delta\) in \(W_4\), \(h_0\) into keyboard at the first stop.

Printed Output:
- \(R_N\) in feet.

Code:
\[
\begin{align*}
W3 & \cdot X & -147.7461 & \cdot T \cdot 2.5348793 \\
+ T & A & X & A & W2 & R & X & -1 & T & / & X & W & 1 & R & \cdot & M & T & X \\
+ & -1 & T & / & X & W2 & R & X & W4 & R & - & W & 1 & R & \cdot & T & X & W & 3 & R & \cdot & K \\
\end{align*}
\]

Controls set before running the programs:
- 16 digits, round-off switch on.

Tape 4

To compute the fast component \(R_{eq}\) first the calculation of the generator angles is repeated using tape 1 and tape 2 programs. Then tape 4 is read into the calculator to compute

\[
R_{eq} = \frac{1}{(B - \cos^2 \gamma)^{\frac{1}{2}}}
\]
where as before:

\[ A = \frac{R^q}{\left[ \frac{1}{e^2/(1-e^2)} \right]^2} \]

\[ B = \frac{1-e^2}{e^2} + 1 \]

and

\[ R_E = (R_c, a + h) \cos \phi_2 \sin \Delta \lambda \]

Input:

\( \cos \phi_2, a \) in W2, \( \cos \Delta \lambda \) in W4, \( h \) into keyboard at the first stop.

Printed Output:

\( R_E \) in feet

Code:

\[ W2 R X = -147.7401 + T/ + 2.5348783 + T \]

\[ W3 A K # A W4 R X = -1 + T/ X W3 X W2 R = \# K \]

Controls for running the program: 10 digits, round-off switch on.

The codes listed above are not necessarily the shortest possible codes.

Remarks on Program Operation With Three Tapes Instead of Four.

With very little extra effort, tape 4 can be eliminated and both components, \( R_N \) and \( R_E \), can be computed using only 3 tapes. Tape 3 has enough unused space to carry the program for computing \( R_E \). After \( R_N \) has been computed tapes 1 and 2 have to be rerun to generate \( \cos \phi_2, a \) unless the value of this quantity was saved at the end of the program 2 run by printing the contents of W2. Before running tape 3 again, \( \cos \phi_2, a \) must be transferred into W3 (and also kept in W2). The code of the additional, second part of tape 3 for computing \( R_E \) is listed below:

\[ W4 R X = -1 + T/ X W3 R X W2 R = \# K \]

The stop code K at the end of the first part of the tape 3 program (see page 47) must be replaced by the first letter of the second part, which is W. If this operation is performed with only 3 tapes, then the result of \( R_E \) printed at the end of the first path and the result of \( R_N \) printed at the end of the second path are disregarded.
APPENDIX C
TRANSLATING AND ROTATING GEOCENTRIC
COORDINATES TO A LOCAL TANGENT PLANE

This appendix derives formulas which are used in a standard computer
program for coordinate transformation under the option for translating geodetic
to local tangent plane coordinates. The pertinent FORTRAN statements are
listed in appendix D. First, the expressions for earth centered (ecs) coordi-
nates (equation 8)*

\[
\begin{align*}
X_{ecs} &= R_c \cos \phi \cos \lambda \\
Y_{ecs} &= R_c \cos \phi \sin \lambda \\
Z_{ecs} &= R_c \sin \phi
\end{align*}
\]

are manipulated by inserting the equation for the geocentric earth radius \( R_c \)
(equation 4b)* and replacing in the expression for \( R_c \) the term \( e^2/(1-e^2) \) by

\[(a^2-b^2)b^2\]

where

\[
\begin{align*}
a &= \text{equatorial radius} \\
b &= \text{polar radius}
\end{align*}
\]

We find

\[
X_{ecs} = a(1 + \frac{a^2}{b^2} \tan^2 \phi_c)^{-\frac{1}{2}} \cos \lambda
\]

and using (5a)

\[
X_{ecs} = a^2(a^2 + b^2 \tan^2 \phi_g)^{-\frac{1}{2}} \cos \lambda
\]

(C2a)

In a similar way we find

\[
\begin{align*}
Y_{ecs} &= a^2(a^2 + b^2 \tan^2 \phi_g)^{-\frac{1}{2}} \sin \lambda \\
Z_{ecs} &= b^2(a^2 + b^2 \tan^2 \phi_g)^{-\frac{1}{2}} \tan \phi_g
\end{align*}
\]

(C2b)

(C2c)

Note that in the equations (C2), the coordinates are functions of the geo-
detic latitude which is more readily available than the geocentric latitude \( \phi_c \).
in equations (C1).

* equation in the basic report
If the tilt angle \( h \), which was assumed as zero so far, has a finite value, we obtain:

\[
X_{\text{ecs}} = \left[ a^2 (a^2 + b^2 \tan^2 \frac{h}{2})^{\frac{1}{2}} + h \cos \frac{h}{2} \right] \cos \lambda
\]

\[
Y_{\text{ecs}} = \left[ a^2 (a^2 + b^2 \tan^2 \frac{h}{2})^{\frac{1}{2}} - h \cos \frac{h}{2} \right] \sin \lambda
\]

\[
Z_{\text{ecs}} = b^2 (a^2 + b^2 \tan^2 \frac{h}{2})^{\frac{1}{2}} \tan \frac{h}{2} + h \sin \frac{h}{2}
\]

The following steps are then performed: First, the ecs coordinates are computed for point I using (C3) and the new coordinates serve as coordinates of the origin of the new system. The ecs coordinates of point 2 are computed and then translated to the new origin:

\[
X_D = X_{\text{ecs},2} - X_{\text{ecs},1}
\]

\[
Y_D = Y_{\text{ecs},2} - Y_{\text{ecs},1}
\]

\[
Z_D = Z_{\text{ecs},2} - Z_{\text{ecs},1}
\]

The coordinates \( X_D, Y_D, Z_D \) are rotated in three steps. First, \( X_D \) and \( Y_D \) are rotated by an amount of \((\pm \lambda_1)\) degrees around the vertical \( Z_D \) axis so that the \( X \) axis is in the vertical plane through the meridian of point 1. The new coordinates are called \( X, Y, Z \). Second, \( Y \) and \( Z \) are rotated by \( 180^\circ \) around \( X \). Finally \( X \) and \( Z \) are rotated in north direction by an amount of \((90 + \phi)\) degrees around the \( Y \) axis which points into east direction. As a result, \( X \) points north and \( Z \) upwards along the local vertical at point 1. The 3 rotations are expressed in the following equation:

\[
\begin{bmatrix}
R_N \\
R_E \\
R_Z
\end{bmatrix} = \begin{bmatrix}
\sin \phi & 0 & -\cos \phi \\
0 & 1 & 0 \\
\cos \phi & 0 & -\sin \phi
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \lambda & -\sin \lambda \\
0 & \sin \lambda & \cos \lambda
\end{bmatrix} \begin{bmatrix}
X_D \\
-Y_D \\
Z_D
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sin \phi \cos \lambda & \sin \phi \sin \lambda & \cos \phi \\
-\sin \lambda & -\cos \lambda & 0 \\
\cos \phi \cos \lambda & -\cos \phi \sin \lambda & \sin \phi
\end{bmatrix} \begin{bmatrix}
X_D \\
-Y_D \\
Z_D
\end{bmatrix} = \begin{bmatrix}
\cos \frac{h}{2} \\
\sin \frac{h}{2} \\
0
\end{bmatrix}
\]
The final result represents a left-handed system with X replaced by $R_N$, Y by $R_E$ and Z by $R_Z$.

Note: The component $Y_D$ in west direction in (C4) has a negative sign to obtain positive values along the $Y_{EC}$ axis from negative longitudes. In the corresponding FORTRAN program in appendix D the second column of the $3 \times 3$ matrix is negative instead of $Y_D$:

$$
\begin{bmatrix}
R_N \\
R_E \\
R_Z
\end{bmatrix} =
\begin{bmatrix}
-sin\theta \cos\lambda & -sin\theta \sin\lambda & \cos\theta \\
-sin\lambda & \cos\lambda & 0 \\
\cos\theta \cos\lambda & \cos\theta \sin\lambda & \sin\theta
\end{bmatrix}
\begin{bmatrix}
X_D \\
Y_D \\
Z_D
\end{bmatrix}
$$
APPENDIX D

FORTRAN STATEMENTS OF CONVERSION PROGRAMS

1. Conventional Program

The FORTRAN statements listed here are an excerpt of a program for coordinate transformation which was coded for a variety of applications by D. Dickinson and D. Walter at AFMDc, HAFB in 1969. Symbols have been changed to aid the understanding.

Before execution of the program at the computer (CDC 3600/3800), one card is read to insert the origin of the LTP system with coordinates:

- Geodetic Latitude $\phi_1$ - PH1 in degrees
- Longitude $\lambda_1$ - LAM1 in degrees
- Altitude $h_1$ - H1 in feet

The next card contains the geodetic coordinates of the point 2:

- Geodetic Latitude $\phi_2$ - PH2 in degrees
- Longitude $\lambda_2$ - LAM2 in degrees
- Altitude $h_2$ - H2 in feet

Input data are based on Clarke's spheroid 1866.

Data (DTR 0.743292819), (REQ 2.0925832),
(RPOL 20854.92)

Statement
Number

1. TYPE REAL LAM1, LAM2
2. PHI PH DTR $ LAM1, LAM2 * DTR
3. PH2 PH2 DTR $ LAM2, LAM2 * DTR
4. SL2 = SINF(LAM2) $ CL2 $ COSF(PH2)
5. SP2 SINF(PH2) $ CP2 $ COSF(PH2) $ TP2 - SP2/CP2
Statt- mnt

Number

6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25

\[ A1 = \sqrt{\left( R\hat{\varphi}_L \cdot TP2 \right)^2 + \left( R\hat{\varphi}_L \right)^2 + \text{REQ}^2 + 2} \]

\[ R = \frac{\text{REQ}^2 + 2}{A1} + H2 \cdot \text{SP2} \]

\[ XECS2 = R \cdot CL2 \]

\[ YECS2 = -R \cdot SL2 \]

\[ ZECS2 = \frac{R\hat{\varphi}_L \cdot 2 \cdot TP2}{A1} + H2 \cdot \text{SP2} \]

\[ A21 = -\sin(LAM1) \]

\[ A22 = -\cos(LAM1) \]

\[ A13 = \cos(\text{PH1}) \]

\[ A33 = \sin(\text{PH1}) \]

\[ A11 = A22 \cdot A33 \]

\[ A12 = -A21 \cdot A33 \]

\[ A31 = -A13 \cdot A22 \]

\[ A32 = A13 \cdot A21 \]

\[ A01 = \sqrt{\left( R\hat{\varphi}_L \cdot A33 / A13 \right)^2 + \text{REQ}^2 + 2} \]

\[ R0 = \frac{\text{REQ}^2 + 2}{A01} + H1 \cdot A13 \]

\[ XECS1 = -R0 \cdot A22 \]

\[ YECS1 = \frac{1}{R0} \cdot A21 \]

\[ ZECS1 = \frac{R\hat{\varphi}_L \cdot 2 \cdot A33 / (A13 \cdot A01)}{A13} \]

\[ DX = XECS2 - XECS1 \]

\[ DY = YECS2 - YECS1 \]

\[ DZ = ZECS2 - ZECS1 \]

\[ RN = A11 \cdot DX + A12 \cdot DY + A13 \cdot DZ \]

\[ RE = A21 \cdot DX + A22 \cdot DY \]

\[ RZ = A31 \cdot DX + A32 \cdot DY + A33 \cdot DZ \]

2. New Program:

DATA (DTR = 0.017453292519), (REQ = 20925842)

(E2 = 0.006768644065)
1 TYPE REAL LAM1, LAM2

2 PHI1 = PHI1 * DTR $ PHI2 = PHI2 * DTR

3 DL = (LAM1 - LAM2) * DTR

4 S1 = SINF(PHI1) $ C1 = COSF(PHI1) $ T1 = S1/C1

5 S2 = SINF(PHI2) $ C2 = COSF(PHI2)

6 SDL = SINF(DL) $ CDL = COSF(CDL)

7 R1 = REQ/SQRTF(1 - E2*S1^2)

8 R2 = REQ/SQRTF(1 - E2*S2^2)

9 RN = (R2 + H2) * (C1*S2 - S1*C2*CDL) - C2*(R2*S2 - R1*S1)*C1

10 RE = (R2 + H2)*C2*SDL

11 RZ = R1 + H1 - (R2 + H2)*C2*CDL/C1 - RN*T1
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