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Galerkin Method for Solving of Singular Integral Equation of Diffraction Problem*

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1 The statement of the diffraction problem

Let $P = \{x : 0 \leq x_1 \leq a, 0 \leq x_2 \leq b, 0 \leq x_3 \leq c\}$ be a resonator with perfectly conducting boundary. Let $Q$ be a three-dimensional body, located in $P$. $Q$ is characterized by tensor permittivity $\varepsilon$ and constant permeability $\mu_0$. We suppose that components of $\varepsilon$ are smooth functions in $Q$ and $\left(\frac{1}{\varepsilon_0} - I\right)$ is invertible in $Q$. $Q \cap \partial P = \emptyset$. Let $P \setminus Q$ be homogeneous and isotropic medium. Incident and diffraction fields depend on time variable as $e^{-iw}$.

We will find electromagnetic diffraction fields $E$ and $H$, satisfying Maxwell’s equations in $P \setminus \partial Q$:

$$\begin{align*}
\text{rot } \vec{H} &= -i\omega\varepsilon \vec{E} + \vec{J}_0 \\
\text{rot } \vec{E} &= i\omega\mu \vec{H} - \vec{J}_0. 
\end{align*}$$

The complete field should have continuous tangent components at $\partial Q$:

$$\left[\vec{n} \times \vec{E}^t\right]_{\partial Q} = \left[\vec{n} \times \vec{H}^t\right]_{\partial Q} = 0$$

and must satisfy the following boundary condition:

$$\vec{E}^t|_{\partial P} = 0.$$

2 Integro-differential equations for the diffraction problem

We will express the solution of the stated problem in terms of vector potentials $\vec{A}_E$ and $\vec{A}_H$ [4]:

$$\begin{align*}
\vec{A}_E &= \int_Q \vec{G}_E(x,y)\vec{J}_E(y)dy, \\
\vec{A}_H &= \int_Q \vec{G}_H(x,y)\vec{J}_H(y)dy, \\
\vec{E} &= i\omega\mu_0 \vec{A}_E - \frac{1}{i\omega\varepsilon_0} \text{grad div } \vec{A}_E - \text{rot } \vec{A}_H, \\
\vec{H} &= i\omega\varepsilon_0 \vec{A}_H - \frac{1}{i\omega\mu_0} \text{grad div } \vec{A}_H + \text{rot } \vec{A}_E.
\end{align*}$$

Here $\vec{J}_E = \vec{J}_0 + \vec{J}_E$, $\vec{J}_H = \vec{J}_0 + \vec{J}_H$, ($\vec{J}_0, \vec{J}_E, \vec{J}_H$ are polarization currents). $\vec{G}_E, \vec{G}_H$ are Green functions for Helmholtz equation, conforming to the arbitrary currents $\vec{J}_E, \vec{J}_H$.

$\vec{G}_E, \vec{G}_H$ are known [3] to have the form of diagonal tensors (the components of $\vec{G}_E$ are written out below):

$$\begin{align*}
G^{G}_E &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{2\pi n}{a}\text{sh}(\frac{n}{a}x_1)\text{sh}(\frac{n}{b}x_2)\text{cos}(\frac{m}{a}y_1)\text{cos}(\frac{m}{b}y_2) \left\{ \begin{array}{l}
\text{sh}y_3 \text{sh}y(c - y_3), x_3 < y_3 \\
\text{sh}y_3 \text{sh}y(c - x_3), x_3 > y_3
\end{array} \right.
\end{align*}$$

$$\begin{align*}
G^{G}_E &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{2\pi n}{a}\text{sh}(\frac{n}{a}x_1)\text{cos}(\frac{n}{b}x_2)\text{cos}(\frac{m}{a}y_1)\text{cos}(\frac{m}{b}y_2) \left\{ \begin{array}{l}
\text{sh}y_3 \text{sh}y(c - y_3), x_3 < y_3 \\
\text{sh}y_3 \text{sh}y(c - x_3), x_3 > y_3
\end{array} \right.
\end{align*}$$

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\text{ch}y_3 \text{ch}y(c - y_3), x_3 < y_3 \\
\text{ch}y_3 \text{ch}y(c - x_3), x_3 > y_3
\end{array} \right.
\end{align*}$$

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Here \( \gamma = \sqrt{\left(\varepsilon_n^m/\varepsilon_n^a\right)^2 + \left(\varepsilon_n^m/\varepsilon_n^b\right)^2 - k_0^2} \) (the proper branch for square root is chosen as in \([2], \S 2.3\), \(\varepsilon_0 = 1\) and \(\varepsilon_n = 2\) for \(n = 1, 2, 3, \ldots\).

We can obtain the following integro-differential equations (under the condition \(\mathbf{\hat{u}} = \mu_0 \mathbf{\hat{I}}\) in \(P\)):

\[
\begin{align*}
\mathbf{\hat{E}}(x) &= \mathbf{\hat{E}}(x) + k_0^2 \int \mathcal{G}_E \left[ \frac{\mathbf{\hat{E}}(y) - \mathbf{\hat{I}}}{\varepsilon_0} \right] \mathbf{\hat{E}}(y) dy + \nabla \cdot \nabla \mathbf{\hat{E}}(x) + \int \mathcal{G}_E \left[ \frac{\mathbf{\hat{E}}(y) - \mathbf{\hat{I}}}{\varepsilon_0} \right] \mathbf{\hat{E}}(y) dy ,
\end{align*}
\]

and we have

\[
\begin{align*}
\mathbf{\hat{H}}(x) &= \mathbf{\hat{H}}(x) - i\omega \varepsilon_0 \mathbf{\hat{I}} \int \mathcal{G}_E \left[ \frac{\mathbf{\hat{E}}(y) - \mathbf{\hat{I}}}{\varepsilon_0} \right] \mathbf{\hat{E}}(y) dy , x \in Q.
\end{align*}
\]

We can extract singularity of Green function \(\mathcal{G}\). Using Fourier transformation and interpolation polynomials we can obtain:

\[
\mathcal{G}_E(x, y) = \frac{1}{4\pi} \frac{e^{i|\mathbf{\hat{u}}|/\varepsilon_0}}{|x - y|} \cdot \mathbf{\hat{I}} + \text{diag}(g_1(x, y), g_2(x, y), g_3(x, y)),
\]

where \(g_k\) are smooth functions.

### 3 Galerkin method

Let us introduce the following auxiliary function

\[
\begin{align*}
\mathcal{G}(x, y) &= -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4}{\alpha y_1 b y_1} \sin \left(\frac{\pi n}{\alpha} x_1\right) \sin \left(\frac{\pi m}{b} x_2\right) \sin \left(\frac{\pi m}{b} y_1\right) \sin \left(\frac{\pi m}{b} y_2\right) \times
\end{align*}
\]

\[
\begin{align*}
&\times \left(\text{sh} y_3 \text{sh} (c - y_3), x_3 < y_3 \right)
&\times \left(\text{sh} y_3 \text{sh} (c - x_3), x_3 > y_3 \right).
\end{align*}
\]

The derivatives of \(\mathcal{G}\) are connected to the derivatives of \(G_k\) through the equalities:

\[
\frac{\partial G_k}{\partial x_i} = \frac{\partial \mathcal{G}}{\partial y_i} , i = 1, 2, 3.
\]

Before describing the method itself we should make some transformations of equation (5). Denoting \(\left(\frac{\mathbf{\hat{E}}(x) - \mathbf{\hat{I}}}{\varepsilon_0}\right)^{-1} = \mathcal{G}\) and \(\left(\frac{\mathbf{\hat{E}}(x) - \mathbf{\hat{I}}}{\varepsilon_0}\right) = \mathbf{\hat{J}}\) as \(\mathbf{\hat{J}}\) we obtain the following equation

\[
\begin{align*}
A \mathbf{\hat{J}} := \mathcal{G}(x, y) - k_0^2 \int \mathcal{G}_E \mathcal{J}(y) dy - \nabla \cdot \nabla \mathbf{\hat{J}} + \int \mathcal{G}_E \mathcal{J}(y) dy = \mathbf{\hat{E}}(x),
\end{align*}
\]

We can write vector equation (8) as a system of three scalar equations:

\[
\begin{align*}
\sum_{i=1}^{3} \xi_i J^i(x) - k_0^2 \int \mathcal{G}_E^i \mathcal{J}(y) dy - \nabla \cdot \nabla \mathbf{\hat{J}} + \int \mathcal{G}_E \mathcal{J}(y) dy = \mathbf{\hat{E}}(x),
\end{align*}
\]

We will determine the components of approximate solution \(\mathbf{\hat{J}}\) in the following way:

\[
\begin{align*}
J^1 = \sum_{k=1}^{N} a_k f^1_k(x),
J^2 = \sum_{k=1}^{N} b_k f^2_k(x),
J^3 = \sum_{k=1}^{N} c_k f^3_k(x),
\end{align*}
\]

where \(f^1_k\) are basis "hat"-functions dependent essentially on \(x^1\). The explicit form of \(f^1_k\) is given below.

Let \(Q\) be a parallelepiped: \(Q = \{x : a_k \leq x^1 \leq a_2, b_1 \leq x^2 \leq b_2, c_1 \leq x^3 \leq c_2\}, Q \subset P\). We will cover \(Q\) with smaller parallelepipeds

\[
\begin{align*}
\Pi_{k}^{x_k^i} &= \{x : x^1_k \leq x^1 \leq x^1_{k+1}, x^2 \leq x^2 \leq x^2_{k+1}, x^3 \leq x^3 \leq x^3_{m+1}\}
\end{align*}
\]

\[
\begin{align*}
x^1_k = a_1 + \frac{a_2 - a_1}{n} k, x^1_{k+1} = a_2 + \frac{a_2 - a_1}{n} , x^2_{k+1} = b_1 + \frac{b_2 - b_1}{n} k, x^3_{m+1} = c_1 + \frac{c_2 - c_1}{n} m;
\end{align*}
\]

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where \( k = 1, \ldots, n - 1; \ l, m = 0, 1, \ldots, \frac{n}{2} - 1 \).

Denoting \((x_k - x_{k-1})\) as \( h^2 \) we get the formulas for \( f_{klm}^1 \):

\[
f_{klm}^1 = \begin{cases} \frac{x_{k+1}^1 - x_k^1}{x_{k+1}^1 - x_k^1}, & \text{if } x^1 \in [x_{k-1}^1; x_k^1] \text{ and } x \in \Pi_{klm}^1 \\ \frac{x_k^1 - x_{k-1}^1}{x_k^1 - x_{k-1}^1}, & \text{if } x^1 \in [x_k^1; x_{k+1}^1] \text{ and } x \in \Pi_{klm}^1 \\ 0, & \text{if } x \notin \Pi_{klm}^1 \end{cases}
\]  

or

\[
f_{klm}^1 = \begin{cases} \frac{1}{h^2} |x^1 - x_k^1|, & \text{if } x \in \Pi_{klm}^1 \\ 0, & \text{if } x \notin \Pi_{klm}^1 \end{cases}
\]

Functions \( f_{klm}^2 \) and \( f_{klm}^3 \) should be determined by similar formulas. Since

\[
f_{klm}^2 |z^1 \in (x_{k-1}^1, x_k^1) = 0, \ f_{klm}^2 |z^2 \in (x_{k-1}^2, x_k^2) = 0, \ f_{klm}^3 |z^2 \in (x_{k-1}^2, x_k^2) = 0,
\]

every component of approximate vector solution vanishes at some side of \( Q \). However the constructed set of basis functions does satisfy the necessary approximation condition.

Introducing total enumeration for basis functions we get

\[
f_k^1, f_k^2, f_k^3; \ k = 1, \ldots, N,
\]

where \( N = \frac{1}{4} (n^1 - n^2) \).

It is convenient to represent the augmented matrix for determining unknown coefficients \( a_k, b_k, c_k \) in block form:

\[
\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}
\]

where columns \( B_k \) and matrices \( A_{kl} \) are determined by formulas:

\[
B_k^i = (\xi_k, f_k^i);
\]

\[
A_{kl}^{ij} = (\xi_k f_{j}^i, f_{l}^i) - \delta_{kl} k_0^2 \int_{Q} G_{E}^i(x,y)f_j^i(y)dy, f_l^i(x) - \left( \frac{\partial}{\partial x_k} \int_{Q} G_{E}^i(x,y)f_j^i(y)dy, f_l^i(x) \right),
\]

\[
(\xi_k f_{j}^i, f_{l}^i) - \delta_{kl} k_0^2 \int_{Q} G_{E}^i(x,y)f_j^i(y)dy, f_l^i(x) - \left( \frac{\partial}{\partial x_k} \int_{Q} G_{E}^i(x,y)f_j^i(y)dy, f_l^i(x) \right),
\]

\[
k = 1, 2, 3; \ i = 1, \ldots, N. (f,g) \text{ determines the scalar product in } L_2, (f,g) = \int_{Q} f(x)g(x)dx.
\]

Applying the formulas of integration by parts to both internal and external integrals and taking into account (7) and (14) we obtain:

\[
A_{kl}^{ij} = \int_{n_i} \xi_k f_{j}^i(x) f_{k}^i(x)dx - \delta_{kl} k_0^2 \int_{n_i} \int_{n_i} G_{E}^i(x,y)f_j^i(y)f_l^i(x)dydx - \int_{n_i} \int_{n_i} G(x,y) \frac{\partial}{\partial x_l} f_j^i(y) \frac{\partial}{\partial x_k} f_l^i(x)dydx.
\]

**References**

