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Splines: a new contribution to wavelet analysis

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Abstract

We present a new approach to the construction of biorthogonal wavelet transforms using polynomial splines. The construction is performed in a "lifting" manner and we use interpolatory, as well as local quasi-interpolatory and smoothing splines as predicting aggregates in this scheme. The transforms contain some scalar control parameters which enable their flexible tuning in either time or frequency domains. The transforms are implemented in a fast way. They demonstrated efficiency in application to image compression.

1 Introduction

Until recently, two methods have been used for the construction of wavelet schemes using splines. One is to construct orthogonal and semi-orthogonal wavelets in the spline spaces (Battle-Lemarié [2, 7], Chui-Wang [6], Unser-Aldroubi-Eden [12]). Another way was introduced by Cohen, Daubechies and Feauveau [3] who constructed symmetric compactly supported spline wavelets whose duals, remaining compactly supported and symmetric, do not belong to a spline space. However, since the introduction of the lifting scheme for the design of wavelet transforms [11], a new way was opened to use splines as a tool for devising a full discrete scheme of wavelet transforms. Namely, various splines can be employed as predicting aggregates in lifting constructions.

2 Lifting scheme of biorthogonal wavelet transform

The sequences $\{a(k)\}_{k=-\infty}^{\infty}$, which belong to the space l_1, we call the discrete-time signals. The $z-$transform of a signal $\{a(k)\}$ is defined as follows: $a(z) = \sum_{k=-\infty}^{\infty} z^{-k}\, a(k)$. Throughout the paper we assume that $z = e^{i\omega}$. We introduce a family of biorthogonal wavelet-type transforms that operate on the signal $\mathbf{x} = \{x(k)\}_{k=-\infty}^{\infty}$, which we construct through lifting steps.

The lifting scheme for the wavelet transform of a signal can be implemented in primal or dual modes. For brevity we consider only the primal mode.

Decomposition Generally, the primal lifting scheme for decomposition of signals consists of three steps: 1. Split. 2. Predict. 3. Update or lifting.

SPLIT - We split the array $\mathbf{x}$ into even and odd sub-arrays:

$$\mathbf{e}_1 = \{e_1(k) = x(2k)\}, \quad \mathbf{d}_1 = \{d_1(k) = x(2k+1)\}, \; k \in \mathbb{Z}.$$

PREDICT - We use the even array $\mathbf{e_1}$ to predict the odd array $\mathbf{d_1}$ and redefine the array $\mathbf{d_1}$ as the difference between the existing array and the predicted one. To be specific, we apply some filter with transfer function $zU(z)$ to the sequence $\mathbf{e_1}$ and predict the function $d_1(z^2)$ which is the z^2-transform of $\mathbf{d_1}$. The z^2-transform of the new $d-$array is defined as follows:

$$d_1^u(z^2) = d_1(z^2) - zU(z)e_1(z^2). \tag{2.1}$$

From now on the superscript u means an *update* operation of the array. Obviously, the prediction $zU(z)e_1(z^2)$ should approximate $d_1(z^2)$ well.

LIFTING - We update the even array using the new odd array:

$$e_1^u(z^2) = e_1(z^2) + \beta(z)z^{-1}\, d_1^u(z^2). \tag{2.2}$$

Generally, the goal of this step is to eliminate aliasing which appears while downsampling the original signal $\mathbf{x}$ into $\mathbf{e_1}$. Further on we will discuss how to achieve this effect by a proper choice of the filter β.

Reconstruction The reconstruction of the signal $\mathbf{x}$ from the arrays $\mathbf{e_1^u}$ and $\mathbf{d_1^u}$ is implemented in reverse order: 1. Undo Lifting. 2. Undo Predict. 3. Unsplit.

UNDO LIFTING - We restore the even array: $e_1(z^2) = e_1^u(z^2) - \beta(z)z^{-1}\, d_1^u(z^2)$.

UNDO PREDICT - We restore the odd array: $d_1(z^2) = d_1^u(z^2) + zU(z)e_1(z^2)$.

UNSPLIT - The last step represents the standard restoration of the signal from its even and odd components. In the $z-$domain this is $x(z) = e_1(z^2) + z^{-1}d_1(z^2)$.

The lifting scheme presented above, yields an efficient algorithm for the implementation of the forward and backward transform of $\mathbf{x} \longleftrightarrow \mathbf{e_1^u} \cup \mathbf{d_1^u}$. These operations can be interpreted as a transformation of the signal by a filter bank that possesses the perfect reconstruction properties and it is associated with the biorthogonal pairs of bases in the space of discrete-time signals. These basis signals are synthesis and analysis wavelets. Further steps of the transform are implemented in an iterative way by the same lifting operations.

3 Polynomial splines

We will construct polynomial splines of various kinds using the even subarray of a signal, calculate their values in the midpoints between nodes and use these values for prediction of the odd array. In this section we discuss some properties of such splines and derive the corresponding filters U.

3.1 $B-$splines

The central $B-$spline of first order on the grid $\{kh\}$ is defined as follows:

$$M_1^h(x) = \begin{cases} 1/h & \text{if } x \in [-h/2, h/2], \\ 0 & \text{elsewhere.} \end{cases}$$

The central $B-$spline of order p is the convolution $M_h^p(x) = M_h^{p-1}(x) * M_h^1(x) \quad p \geq 2$. Note that the $B-$spline of order p is supported at the interval $(-ph/2, ph/2)$. It is positive within its support and symmetric around zero. The nodes of $B-$splines of even orders are located at points $\{kh\}$ and of odd orders at points $\{h(k+1/2)\}$, $k \in \mathbb{Z}$. It is readily

verified that $hM_h^p(hx) = M^p(x)$, where $M^p(x) := M_1^p(x)$. Let

$$\mathbf{u}^p := \{hM_h^p(hk) = M^p(k)\}, \text{ and } \mathbf{w}^p := \{hM_h^p\left(h(k+1/2)\right) = M^p\left(k+1/2\right)\}, \; k \in \mathbb{Z}. \tag{3.1}$$

Due to the compact support of B–splines, these sequences are finite. We will use for our constructions only splines of odd orders $p = 2r-1$. In Table 1 we present the sequences for initial values r which are of practical importance.

k	-3	-2	-1	0	1	2	3
$\mathbf{u}^3 \times 8$	0	0	1	6	1	0	0
$\mathbf{u}^5 \times 384$	0	1	76	230	76	1	0
$\mathbf{w}^3 \times 2$	0	0	1	1	0	0	0
$\mathbf{w}^5 \times 24$	0	1	11	11	1	0	0

TAB. 1. Values of the sequences $\mathbf{u}^p$ and $\mathbf{w}^p$.

We need the z^2–transforms of the sequences $\mathbf{u}^p$ and $\mathbf{w}^p$:

$$u^p(z^2) := \sum_{k=-\infty}^{\infty} z^{-2k} u^p(k), \quad w^p(z^2) := \sum_{k=-\infty}^{\infty} z^{-2k} w^p(k).$$

These functions are Laurent polynomials, and are called the Euler-Frobenius polynomials [10].

Proposition 3.1. ([9]) *On the circle $z = e^{i\omega}$ the Laurent polynomials $u^p(z^2)$ are strictly positive. Their roots are all simple and negative. Each root ζ can be paired with a dual root θ such that $\zeta\,\theta = 1$. Thus, if $p = 2r+1$ is odd, then $u^p(z^2)$ can be represented as follows:*

$$u^p(z^2) = \prod_{n=1}^{r} \frac{1}{\gamma_n}(1+\gamma_n z^2)(1+\gamma_n z^{-2}), \quad 0 < \gamma_n < 1. \tag{3.2}$$

We denote

$$U_i^p(z) := z^{-1}\frac{w^p(z^2)}{u^p(z^2)}. \tag{3.3}$$

Proposition 3.2 *The rational functions $U_i^p(z)$ are real-valued and $U_i^p(-z) = -U_i^p(z)$. If $p = 2r+1$ is odd then*

$$1 - U_i^p(z) = \frac{(\alpha-2)^{r+1}\xi_r(\alpha)}{u^p(z^2)}, \quad 1 + U_i^p(z) = \frac{(-\alpha-2)^{r+1}\xi_r(-\alpha)}{u^p(z^2)}, \tag{3.4}$$

where $\alpha := z + z^{-1}$ and $\xi_r(\alpha)$ is a polynomial of degree $r-1$.

3.2 Interpolatory splines

The shifts of B–splines form a basis in the space $\mathbf{S}_h^p$ of splines of order p on the grid kh. Namely, any spline $S_h^p \in \mathbf{S}_h^p$ has the following representation:

$$S_h^p(x) = h\sum_l q(l)\, M_h^p(x - lh). \tag{3.5}$$

Let $\mathbf{q} := \{q(l)\}$, and $q(z^2)$ be the z^2-transform of $\mathbf{q}$. We introduce also the sequences $\mathbf{s}^p := h\{S_h^p(hk) = S_1^p(k)\}$ and $\mathbf{m}^p := \{S_h^p(h(k+1/2)) = S_1^p(k+1/2)\}$ of values of the spline on the grid points and on the midpoints. Let $s^p(z^2)$ and $m^p(z^2)$ be the corresponding z^2-transforms. We have

$$S_1^p(k) = \sum_l q(l)\, M_h^p(k-l), \quad \text{and} \quad S_1^p\left(k+\frac{1}{2}\right) = \sum_l q(l)\, M_h^p\left(k-l+\frac{1}{2}\right). \tag{3.6}$$

Respectively, $s^p(z^2) = q(z^2)u(z^2)$, and $m^p(z^2) = q(z^2)w(z^2)$.

From these formulae we can derive expression for the coefficients of a spline which interpolates a given sequence $\mathbf{e} := \{e(k)\}$ at grid points:

$$hS_h^p(hk) = e(k),\ k \in \mathbb{Z}, \Longleftrightarrow q(z^2)u^p(z) = e(z^2) \Longleftrightarrow q(z^2) = \frac{e(z^2)}{u^p(z^2)}. \tag{3.7}$$

The z^2-transform of the sequence $\mathbf{m}^p$ is:

$$m^p(z^2) = q(z^2)w^p(z^2) = zU_i^p(z)e(z^2). \tag{3.8}$$

Our further construction exploits the super-convergence property of the interpolatory splines of odd orders (even degrees).

Theorem 3.3. ([13]) *Let a function $f \in L^1(-\infty,\infty)$ have $p+1$ continuous derivatives and let $S_h^p \in \mathbf{S}_h^p$ interpolate f on the grid $\{kh\}$. Denote $\tilde{f}_k = f((k+1/2)h)$. Then in the case of odd $p = 2r+1$, the following asymptotic relation holds.*

$$S_h^p(h(k+1/2)) = \tilde{f}_k - h^{2r+2} f^{(2r+2)}(h(k+1/2))(2r+1)\frac{b_{2r+2}(0) - b_{2r+2}(\frac{1}{2})}{(2r+2)!} + o(h^{2r+2} f^{(2r+2)}), \tag{3.9}$$

where $b_s(x)$ is the Bernoulli polynomial of degree s.

Recall, that in general the interpolatory spline of order $2r+1$ approximates the function f with accuracy of h^{2r+1}. Therefore, we may claim that $\{(k+1/2)h\}$ are points of super-convergence of the spline S_h^p. Note, that the spline of order $2r+1$, which interpolates the values of a polynomial of degree $2r$, coincides with this polynomial. However, the spline of order $2r+1$ which interpolates the values of a polynomial of degree $2r+1$ on the grid $\{kh\}$ restores the values of this polynomial at the mid-points $\{(k+1/2)h\}$. This property will result in the vanishing moments property of the wavelets to be constructed later.

3.3 Quasi-interpolatory splines

We can see from (3.7) and (3.8) that in order to find values at the midpoints of the spline interpolating the signal $\mathbf{e}$, the signal has to be filtered with the filter whose transfer function is $zU_i^p(z)$. This filter has infinite impulse response (IIR). However, the property of super-convergence at the midpoints is not an exclusive attribute of the interpolatory splines. It is also inherent to the so called local quasi-interpolatory splines of odd orders, which can be constructed using finite impulse response (FIR) filtering.

Definition 3.4 *Let the function f have p continuous derivatives and $\mathbf{f} := \{f_k = f(hk)\}$, $k \in \mathbb{Z}$. The spline $S_h^p \in \mathbf{S}_h^p$ of order p given by (3.5) is said to be the local*

quasi-interpolatory spline if the array $\mathbf{q}$ *of its coefficients is derived by FIR filtering the array of samples* $\mathbf{f}$

$$\mathbf{q}(z^2) = \Gamma(z^2)\mathbf{f}(z^2), \tag{3.10}$$

where $\Gamma(z^2)$ *is a Laurent polynomial, and the difference* $|f(x) - S_h^p(x)| = O(f^{(p)}h^p)$. *If* f *is a polynomial of degree* $p-1$, *then the spline* $S_h^p(x) \equiv f(x)$.

If $\mathbf{w}^p$ is the sequence defined in (3.1) then the midpoint values $\mathbf{m}^p$ are produced by the following FIR filtering of the array of samples $\mathbf{f}$: $\mathbf{m}^p(z^2) = zU^p(z)\mathbf{f}(z^2)$, $U^p(z) := z^{-1}\Gamma(z^2)w^p(z^2)$. Explicit formulas for the construction of quasi-interpolatory splines as well as the estimations of the differences were established in [13]. In the present work we are interested in splines of odd orders $p = 2r+1$. There are many FIR filters which generate quasi-interpolatory splines but only one filter of minimal length $2r+1$ for each order $p = 2r+1$. Let $\lambda(z) := z^{-2} - 2 + z^2$.

Theorem 3.5 *A quasi-interpolatory spline of order* $p = 2r+1$ *can be produced by filtering (3.10) with filters* Γ *of length no less than* $2r+1$. *There exists a unique filter* Γ_m^r *of length* $2r+1$ *which produces the minimal quasi-interpolatory spline* $\tilde{S}_h^{2r+1}(x)$. *Its transfer function is:*

$$\Gamma_m^r(z^2) = 1 + \sum_{k=1}^{r} \beta_k^r \lambda^k(z), \quad \left(\frac{2\arcsin t/2}{t}\right)^{2r+1} = \sum_{k=0}^{\infty} (-1)^k \beta_k^r t^{2k}. \tag{3.11}$$

If the function f *has* $2r+3$ *derivatives then the following asymptotic relations hold for the midpoint values of the minimal quasi-interpolatory spline of odd order:*

$$\begin{aligned} &\tilde{S}_h^{2r+1}(h(k+1/2)) = f(h(k+1/2)) + h^{2r+2} f^{(2r+2)}(h(k+1/2))A^r + O(f^{(2r+3)}h^{2r+3}), \\ &A^r := \frac{(2r+1)b_{2r+2}(0)}{(2r+2)!} - \beta_{r+1}^r, \end{aligned} \tag{3.12}$$

where $b_s(x)$ *is the Bernoulli polynomial of degree* s.

This implies that the super-convergence property is similar to that of the interpolatory splines. The asymptotic representation (3.12) provides tools for custom design of predicting splines retaining or even enhancing the approximation accuracy of the minimal spline at the midpoints.

Proposition 3.6 *If the coefficients of the spline* $S_{h,\rho}^{2r+1} \in \mathbf{S}_h^{2r+1}$ *of order* $2r+1$ *are derived as in (3.10) using the filter* Γ_ρ^r *of length* $2r+3$, *with the transfer function* $\Gamma_\rho^r(z^2) = \Gamma_m^r(z^2) + \rho\lambda^{r+1}(z)$, *then the spline restores polynomials of degree* $2r+1$ *at the midpoints between nodes, for any real value* ρ. *However, if* $\rho = -A^r$ *then the spline restores polynomials of degree* $2r+3$.

If the parameter ρ is chosen such that $\rho = (-1)^r|\rho|$ then the spline $S_{h,\rho}^{2r+1}$ possesses the smoothing property [14].

3.4 Examples

3.4.1 *Quadratic splines*

Interpolatory spline Let $\alpha = z^{-1} + z$. Then

$$U_i^1(z) = \frac{4\alpha}{z^2 + 6 + z^{-2}}, \text{ and } 1 - U_i^1(z) = \frac{(\alpha - 2)^2}{z^{-2} + 6 + z^2},$$

Minimal spline The filters are

$$\begin{aligned} \Gamma_m^1(z^2) &= 1 - \frac{1}{8}\lambda(z),\ U_m^1(z) = \frac{-z^{-3} + 9z^{-1} + 9z - z^3}{16}, \\ \text{and } 1 - U_m^1(z) &= \frac{(\alpha - 2)^2(z^{-1} + 4 + z)}{16}. \end{aligned}$$

Extended spline

$$\Gamma_e^1(z) = \Gamma_m^1(z^2) + \frac{1}{64}\lambda^2(z),\ U_e^1(z) = \frac{3z^{-5} - 25z^{-3} + 150z^{-1} + 150z - 25z^3 + 3z^5}{256},$$

$$\text{and } 1 - U_e^1(z) = \frac{(\alpha - 2)^3(3z^{-2} + 18z^{-1} + 38 + 18z + 3z^2)}{256},$$

Remark 3.7 *In [5] Donoho presented a scheme where an odd sample is predicted by the value in the central point of the polynomial of odd degree which interpolates adjacent even samples. One can observe that our filter U_m^1 coincides with the filter derived by Donoho's scheme using the cubic interpolatory polynomial. The filter U_e^1 coincides with the filter derived using the interpolatory polynomial of fifth degree. On the other hand, the filter U_i^1 is closely related to the commonly used Butterworth filter [8]. Namely, in this case the filter transfer functions $\Phi_i^{1,l}(z) := (1+U_i^1(z))/2$, $\Phi_i^{1,h}(z) := (1-U_i^1(z))/2$ coincide with magnitude squared of the transfer functions of the discrete-time low-pass and high-pass half-band Butterworth filters of order 4, respectively.*

3.4.2 *Splines of fifth order (fourth degree)*

Interpolatory spline

$$U_i^2(z) = \frac{16(z^3 + 11z + 11z^{-1} + z^{-3})}{z^4 + 76z^2 + 230 + 76z^{-2} + z^{-4}},\ 1 - U_i^2(z) = \frac{(\alpha - 2)^3(\alpha - 10)}{z^4 + 76z^2 + 230 + 76z^{-2} + z^{-4}}.$$

Minimal spline The filter is

$$U_m^2(z) = \frac{47(z^{-7} + z^7) + 89(z^{-5} + z^5) - 2277(z^{-3} + z^3) + 15965\alpha}{27648}.$$

4 Wavelet transforms using spline filters

4.1 Choosing the filters for the *lifting* step

In the previous section we presented a family of filters U for the *predicting* step which were originated from splines of various types. But, as it is seen from (2.2), to accomplish the transform we have to define the filter β. There is a remarkable freedom in the choice of these filters. The only requirement needed to guarantee a perfect reconstruction property of the transform is that $\beta(-z) = \beta(z)$. In order to make synthesis and analysis filters

similar in their properties, we choose $\beta(z) = \breve{U}(z)/2$, where $\breve{U}$ means one of filters U presented above. In particular, $\breve{U}$ may coincide with the filter U which was used for the prediction.

We say that a wavelet ψ has m vanishing moments if the following relations hold: $\sum_{k\in\mathbb{Z}} k^s\psi(k) = 0, \quad s = 0, 1, \ldots, m-1.$

Proposition 4.1 *Suppose the filters $U(z)$ and $\beta(z) = \breve{U}(z)/2$ are used for the predicting and lifting steps, respectively. If $1 - U(z)$ contains the factor $(z - 2 + 1/z)^r$ then the high-frequency analysis wavelets $\tilde{\psi}^1$ have $2r$ vanishing moments. If, in addition $1-\breve{U}(z)$ contains the factor $(z-2+1/z)^p$ then the synthesis wavelet ψ^1_β has $2q$ vanishing moments, where $q = \min\{p, r\}$.*

4.2 Implementation of the transforms

Suppose, we have chosen the filter $\beta = \breve{U}/2$. The functions $zU(z)$ and $z\breve{U}(z)$ depend on z^2 and we write $F(z^2) := zU(z)$ and $\breve{F}(z^2) := z\breve{U}(z)$. Then the decomposition procedure is (see (2.1), (2.2)):

$$d^u_1(z) = d_1(z) - F(z)e_1(z), \quad e^u_1(z) = e_1(z) + \frac{1}{2z}\breve{F}(z)\, d^u_1(z). \tag{4.1}$$

Equation (4.1) means that in order to obtain the detail array $\mathbf{d}^u_1$, we must process the even array $\mathbf{e}_1$ with the filter F, with transfer function $F(z)$, and extract the filtered array from the odd array $\mathbf{d}_1$. In order to obtain the smoothed array $\mathbf{e}^u_1$, we must process the detail array $\mathbf{d}^u_1$ with the filter Φ that has the transfer function $\Phi(z) = z^{-1}\breve{F}(z)/2$ and add the filtered array to the even array $\mathbf{e}_1$. But the filter Φ differs from $\breve{F}_r/2$ only by one-sample delay and it operates similarly. Thus, both operations of the decomposition are, in principle, identical. For the reconstruction the same operation is conducted in reverse order.

Therefore, it is sufficient to outline the implementation of the filtering with the function $F(z)$.

Implementation of FIR filters originating from local splines is straightforward and, therefore we only make a few remarks on IIR filters originating from interpolatory splines. A detailed description can be found in [1]. Equations (3.2) and (3.3) imply that, while the interpolatory spline of order $2r+1$ is used, the transfer function $F(z) = P(z)/\prod_{n=1}^{r} \frac{1}{\gamma_n}(1+\gamma_n z)(1+\gamma_n z^{-1})$, where $P(z)$ is the Laurent polynomial. It means that the IIR filter F can be split into a cascade consisting of a FIR filter with the transfer function $P(z)$, r elementary causal recursive filters denoted by $\overrightarrow{R(n)}$, and r elementary anti-causal recursive filters, denoted by $\overleftarrow{R(n)}$. The causal and anti-causal filters operate as follows:

$$\mathbf{y} = \overrightarrow{R(n)}\mathbf{x} \Longleftrightarrow y(l) = x(l) + \gamma_n y(l-1), \quad \mathbf{y} = \overleftarrow{R(n)}\mathbf{x} \Longleftrightarrow y(l) = x(l) + \gamma_n y(l+1).$$

Example 4.2 (Example of recursive filter) We present IIR filters derived from the interpolatory splines of third order.

Let $\gamma_1^1 = 3 - 2\sqrt{2} \approx 0.172$. Then

$$F_i^1(z) = 4\gamma_1^1 \frac{1+z}{(1+\gamma_1^1 z)(1+\gamma_1^1 z^{-1})}.$$

The filter can be implemented with the following cascade:

$$x_0(k) = 4\gamma_1^1(x(k)+x(k+1)), \quad x_1(k) = x_0(k) - \gamma_1^1 x_1(k-1), \quad y(k) = x_1(k) - \gamma_1^1 y(k+1).$$

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