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On the $q$-Bernstein polynomials

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Abstract

We discuss here recent developments on the convergence of the $q$-Bernstein polynomials $B_n f$ which replaces the classical Bernstein polynomial with a one parameter family of polynomials. In addition, the convergence of iterates and iterated Boolean sum of $q$-Bernstein polynomial will be considered. Moreover a $q$–difference operator $D_q f$ defined by $D_q f = f[x, q x]$ is applied to $q$-Bernstein polynomials. This gives us some results which complement those concerning derivatives of Bernstein polynomials. It is shown that, with the parameter $0 < q \leq 1$, if $\Delta^k f \geq 0$ then $D_q^k B_n f \geq 0$. If $f$ is monotonic so is $D_q B_n f$. If $f$ is convex then $D_q^2 B_n f \geq 0$.

1 Introduction

First we begin by introducing some notations to be used. For any fixed real number $q > 0$, the $q$-integer $[k]$ is defined as

$$[k] = \begin{cases} \frac{(1-q^k)(1-q)}{1-q}, & q \neq 1, \\ k, & q = 1, \end{cases}$$

for all positive integer $k$. The term Gaussian coefficient is also used, since they were first studied by Gauss (see Andrews [1]).

Let $p(N, M, n)$ denote the number of partitions of a positive integer $n$ into at most $M$ parts, each less than or equal to $N$. Then the Gaussian polynomial, $G(N, M, n)$, appears as the generating function

$$G(N, M, n) = \left[ \begin{array}{c} N + M \\ M \end{array} \right] = \sum_{n \geq 0} p(N, M, n) q^n.$$ 

Note that $\left[ \begin{array}{c} n \\ k \end{array} \right]$ defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \begin{cases} \frac{[n]!}{[k]![n-k]!}, & n \geq k \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $[n]! = [n][n-1] \cdots [1]$ with $[0]! = 1$, is called Gaussian polynomial (or $q$-binomial coefficient) since it is a polynomial in $q$ with the degree $(n-k)k$. The $q$-binomial coeffi-
coefficients satisfy the recurrence relations,
\[
\begin{bmatrix} n+1 \\ k \end{bmatrix} = q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} + \begin{bmatrix} n \\ k \end{bmatrix}
\]
(1.1)

and
\[
\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n \\ k \end{bmatrix}.
\]
(1.2)

The following Euler identity can be verified using the recurrence relation (1.1) by induction that
\[
(1+x)(1+qx) \cdots (1+q^{k-1}x) = \sum_{r=0}^{k} q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} x^r.
\]
(1.3)

Phillips [8] introduced a generalization of Bernstein polynomials (q-Bernstein polynomials) in terms of q-integers
\[
B_n(f; x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x),
\]
(1.4)

where \( f_r = f \left( \frac{r}{n} \right) \) and an empty product denotes 1. When \( q = 1 \) the (1.4) reduces the classical Bernstein polynomials. The \( B_n(f; x) \) generalizes many properties of classical Bernstein polynomials. Firstly, generalized Bernstein polynomials satisfy the end point interpolation
\[
B_n(f; 0) = f(0), \quad B_n(f; 1) = f(1).
\]

Phillips [8] also states the generalization of well known forward difference form (see Davis [3]) of the classical Bernstein polynomials by the following theorem.

**Theorem 1.1** The generalized Bernstein polynomial, defined by (1.4), may be expressed in the q-difference form
\[
B_n(f; x) = \sum_{r=0}^{n} \Delta^r f_0 x^r
\]
(1.5)

where \( \Delta^r f_i = \Delta^{r-1} f_{i+1} - q^{-1} \Delta^{r-1} f_i \) for \( r \geq 1 \) and \( \Delta^0 f_i = f_i \).

It is easily verified by induction that q-differences satisfy
\[
\Delta^r f_i = \sum_{k=0}^{r} (-1)^k q^{k(k-1)/2} \begin{bmatrix} r \\ k \end{bmatrix} f_{r+i-k}.
\]
(1.6)

Using the q-difference form of the q-Bernstein polynomials (1.5), one may show that q-Bernstein polynomials reproduce linear functions, since \( B_n(1; x) = 1; \ B_n(x; x) = x \).

## 2 Convergence

In the discussion of the uniform convergence of the q-Bernstein operator, the Bohman-Korovkin Theorem (see Cheney [2]) is used as in the classical case. The Bohman-Korovkin Theorem states that for a linear monotone operator \( \mathcal{L}_n \), the convergence of
\( \mathcal{L}_n f \to f \) for \( f(x) = 1, x, x^2 \) is sufficient for the sequence of operators \( \mathcal{L}_n \) to have the uniform convergence property \( \mathcal{L}_n f \to f, \forall f \in C[0,1] \). Observe that the \( q \)-Bernstein operator is a monotone linear operator for \( 0 < q \leq 1 \). For a fixed value of \( q \) with \( 0 < q < 1 \)

\[
[n] \to \frac{1}{1-q} \quad \text{as} \quad n \to \infty.
\]

Notice that, since \( B_n(x^2; x) = x^2 + \frac{x(1-x)}{[n]} \), \( B_n(x^2; x) \) does not converge to \( x^2 \). Phillips [8] studies the uniform convergence of \( q \)-Bernstein polynomial.

**Theorem 2.1** Let \( q = q_n \) satisfy \( 0 < q_n < 1 \) and let \( q_n \to 1 \) as \( n \to \infty \). Then,

\[ B_n(f; x) \to f(x), \quad \forall f(x) \in C[0,1]. \]

The degree of \( q \)-Bernstein approximation to a bounded function on \([0, 1]\) may be described in terms of the modulus of continuity with the following theorem.

**Theorem 2.2** If \( f \) is bounded on \([0, 1]\) and \( B_n f \) denotes the generalized Bernstein operator associated with \( f \) defined by (1.4), then

\[ \|f - B_n f\|_{\infty} \leq \frac{3}{2} \omega(1/\lfloor n \rfloor^{1/2}). \]

An error estimate for the convergence of \( q \)-Bernstein polynomials is given in Phillips [8] by the Voronovskaya type theorem.

**Theorem 2.3** Let \( f \) be bounded on \([0, 1]\) and let \( x_0 \) be a point of \([0, 1]\) at which \( f''(x_0) \) exists. Further, let \( q = q_n \) satisfy \( 0 < q_n < 1 \) and let \( q_n \to 1 \) as \( n \to \infty \). Then the rate of convergence of the sequence of generalized Bernstein polynomials is governed by

\[ \lim_{n \to \infty} \lfloor n \rfloor (B_n(f; x_0) - f(x_0)) = \frac{1}{2} x_0(1 - x_0) f''(x_0). \]

It is well known that the classical Bernstein polynomials \( B_n f \) provide simultaneous approximation of the function and its derivatives. That is if \( f \in C^p[0,1] \), then

\[ \lim_{n \to \infty} B_n^{(p)}(f; x) = f^{(p)}(x) \]

uniformly on \([0, 1]\). It is worthwhile to examine if this property hold for \( q \)-Bernstein polynomials. Phillips [7] proved that the \( p^{th} \) derivative of \( q \)-Bernstein polynomials converges uniformly on \([0, 1]\) to the \( p^{th} \) derivative of \( f \) under some restrictions of the parameter \( q \). This property results from the generalization of the following theorem.

**Theorem 2.4** Let \( f \in C^1[0, 1] \) and let the sequence \( (q_n) \) be chosen so that the sequence \( (\epsilon_n) \) converges to zero from above faster than \((1/3^n)\), where

\[ \epsilon_n = \frac{n}{1 + q_n + q_n^2 + \cdots + q_n^{n-1}} - 1. \]

Then the sequence of derivatives of the generalized Bernstein polynomials, \( B_n f \), converges uniformly on \([0, 1]\) to \( f'(x) \).

Up to now the convergence of \( q \)-Bernstein polynomials is examined by taking a sequence \( q = q_n \) such that \( q_n \to 1 \) as \( n \to \infty \). In the recent developments, the convergence
of $q$-Bernstein polynomials is examined for fixed real $q$, $0 < q < 1$ and for $q \geq 1$. It is proved in Oruç and Tuncer [6] that for a fixed $q$, $0 < q < 1$, the uniform convergence holds if and only if $f$ is linear on the interval $[0,1]$. Moreover, if $q \geq 1$, $B_n f \to f$ as $n \to \infty$ if $f$ is a polynomial.

**Theorem 2.5** Let $q \geq 1$ be a fixed real number. Then, for any polynomial $p$, 

$$
\lim_{n \to \infty} B_n(p; x) = p(x).
$$

For any fixed integer $i$, the $q$-Bernstein polynomials of monomials (see Goodman et.al. [4]) can be written explicitly as

$$
B_n(x^i; x) = \sum_{j=0}^{i} \lambda_j [n]^{j-i} S_q(i,j) x^j,
$$

where

$$
\lambda_j = \prod_{r=0}^{j-1} \left( 1 - \frac{[r]}{[n]} \right),
$$

an empty product denotes 1, and

$$
S_q(i,j) = \frac{1}{[j]! q^{(j-1)/2}} \sum_{r=0}^{j} (-1)^r q^{r(j-1)/2} \binom{j}{r} [j-r]^i, \quad 0 \leq i \leq j,
$$

is the Stirling polynomial of second kind. Thus for any polynomial $p$ of degree $m$, one may write

$$
B_n(p; x) = a^T A x,
$$

where $a$ is the vector whose elements are the coefficients of $p$, $A$ is an $(m+1) \times (m+1)$ lower triangular matrix with the elements

$$
a_{i,j} = \begin{cases} 
\lambda_j [n]^{j-i} S_q(i,j), & 0 \leq j \leq i, \\
0, & i < j,
\end{cases}
$$

and $x$ is the vector whose elements form the standard basis for the space of polynomials $P_m$ of degree $m$.

**Lemma 2.1** Let $0 < q < 1$ be a fixed real number. Then

$$
\lim_{n \to \infty} B_n(p; x) = p(x)
$$

if and only if $p(x)$ is linear.

This lemma can be generalized for any function $f \in C[0,1]$.

**Theorem 2.6** Let $0 < q < 1$ be a fixed real number and $f \in C[0,1]$. Then

$$
\lim_{n \to \infty} B_n(f; x) = f(x)
$$

if and only if $f(x)$ is linear.
3 The iterates

The iterates of classical Bernstein polynomials were first studied by Kelisky and Rivlin [5]. The authors proved that iterates of Bernstein polynomials converge to linear end point interpolants on [0, 1]. Several generalization of the result due to Kelisky and Rivlin has been considered by many authors; see Sevy [9] and Wenz [10]. The recent result is the convergence of iterates of generalized Bernstein polynomials. It is proved in Oruç and Tuncer [6] that the $q$-Bernstein polynomials do preserve the convergence property of iterates of classical Bernstein polynomial. The iterates of generalized Bernstein polynomial are defined by

$$B_n^{M+1}(f; x) = B_n(B_n^{M}(f; x); x), \quad M = 1, 2, \ldots,$$

where $B_n^1(f; x) = B_n(f; x)$.

**Theorem 3.1** Let $q \geq 0$ be a fixed real number. Then

$$\lim_{M \to \infty} B_n^M(f; x) = f(0) + (f(1) - f(0))x.$$  \hspace{1cm} (3.2)

Let $A$ and $B$ be operators then the Boolean sum of $A$ and $B$ is defined to be

$$A \oplus B = A + B - A \circ B.$$  

We will be concerned with iterated Boolean sums of the generalized Bernstein polynomials in the form $B_n \oplus B_n \oplus \cdots \oplus B_n$ and will denote such an $M$-fold Boolean sum of the generalized Bernstein operators by $\oplus^M B_n$. Sevy [9] and Wenz [10] proved that the limit of iterated Boolean sums of Bernstein polynomials is the interpolation polynomial with respect to the nodes $(\frac{i}{n}, f(\frac{i}{n})) \; i = 0, \ldots, n$ as $M \to \infty$. The second theorem of this section will give a result for the convergence of iterates of Boolean sums of generalized Bernstein polynomials. It is proved in Oruç and Tuncer [6] that the iterates of Boolean sums of $q$-Bernstein polynomials converge to the interpolating polynomial at the nodes $(\frac{i}{n}, f(\frac{i}{n}))$.

**Theorem 3.2** The iterated Boolean sum of the $q$-Bernstein operator $\oplus^M B_n (f; x)$ associated with the function $f(x) \in C[0, 1]$ converges to the interpolating polynomial $L_n f$ of degree $n$ of $f(x)$ at the points $x_i = [i]/[n], \quad i = 0, 1, \ldots, n$.

4 A difference operator $D_q$ on generalized Bernstein polynomials

Given any function $f(x)$ and $q \in \mathbb{R}$ we define the operator $D_q$

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}.  \hspace{1cm} (4.1)$$

Thus $D_q f(x)$ is simply a divided difference, $D_q f(x) = f[x, qx]$. Note that, for a function $f$ and non-negative integer $k$

$$f[x, qx, \ldots, q^k x] = \frac{1}{[k]!} D_q^k f(x).$$
Theorem 4.1  For any integer $0 \leq k \leq n$,

$$\mathcal{D}_q^k B_n(f; x) = [n] \cdots [n-k+1] \sum_{r=0}^{n-k} \Delta^k f_r \left[ \begin{array}{c} n-k-r \\ r \end{array} \right] x^r \prod_{s=k}^{n-r-1} (1-q^s x).$$

Proof: Recall the $q$-difference form of generalized Bernstein polynomials (1.5) and apply the operator $\mathcal{D}_q$ to $B_n(f; x)$ repeatedly $k$ times to get,

$$\mathcal{D}_q^k B_n(f; x) = \sum_{r=0}^{n-k} \frac{[n]!}{[n-k-r]![r]!} \Delta^k f_r x^r. \quad (4.2)$$

It will be useful to express $\Delta^k+r$ in terms of $\Delta^k$. One may prove by induction on $m$ that, for $0 < m \leq n - k$ we may write

$$\Delta^{m+k} f_i = \sum_{t=0}^{m} (-1)^t q^{t(t+2k-1)/2} \left[ \begin{array}{c} m+t \\ t \end{array} \right] \Delta^k f_{m+t-t}. \quad (4.3)$$

Now applying the latter identity to (4.2) gives

$$\mathcal{D}_q^k B_n(f; x) = \sum_{r=0}^{n-k} \sum_{t=0}^{r} (-1)^t q^{t(t+2k-1)/2} \frac{[n]!}{[n-k-r]![r]!} \left[ \begin{array}{c} r \\ t \end{array} \right] \Delta^k f_{r-t} x^r. \quad (4.4)$$

Writing $m = r - t$

$$\frac{[n]!}{[n-k-m-t]![m+t]!} \left[ \begin{array}{c} m+t \\ t \end{array} \right] = \frac{[n]!}{[n-k-m]![m]!} \left[ \begin{array}{c} n-k-m \\ t \end{array} \right]$$

and putting (4.4) in (4.3) we obtain

$$\mathcal{D}_q^k B_n(f; x) = \sum_{m=0}^{n-k} \frac{[n]!}{[n-k-m]![m]!} \Delta^k f_m x^m \sum_{t=0}^{n-k-m} (-1)^t q^{t(t+2k-1)/2} \left[ \begin{array}{c} n-k-m \\ t \end{array} \right] x^t. \quad (4.5)$$

Now, it can be easily derived from generalized binomial expansion (1.3), on replacing $x$ by $q^k x$, that

$$\prod_{t=k}^{n-m-1} (1-q^t x) = \sum_{t=0}^{n-k-m} (-1)^t q^{t(t+2k-1)/2} \left[ \begin{array}{c} n-k-m \\ t \end{array} \right] x^t.$$

This completes the proof.

From Theorem 4.1 we see that, with $0 < q \leq 1$, if $\Delta^k f_r \geq 0$ for $0 \leq r \leq n - k$ then $\mathcal{D}_q^k B_n(f; x) \geq 0$. If $f$ is convex on $0 \leq x \leq 1$ then $\mathcal{D}_q^2 B_n(f; x) \geq 0$ for $0 < q \leq 1$. If $f$ is increasing then $\mathcal{D}_q B_n(f; x) \geq 0$, for $0 < q \leq 1$.

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