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AUTHORITY
AFFDL ltr, 13 Jul 1979
A FINITE ELEMENT MACH BOX PROCEDURE FOR FLUTTER PREDICTION OF PANELS IN THREE-DIMENSIONAL SUPersonic unsteady Potential FLOW

DECEMBER 1976

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FINAL REPORT FOR PERIOD JUNE 1976 - DECEMBER 1976

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This technical report has been reviewed and is approved for publication.

James J. Olsen
Project Engineer

FOR THE COMMANDER

Howard L. Farmer, Colonel, USAF
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A finite element formulation and solution procedure are developed for flutter predictions of rectangular panels with one surface exposed to three-dimensional supersonic unsteady potential flow. The effect of in-plane force is included. Each element is divided into several Mach boxes. The aerodynamic influence coefficients between each pair of sending and receiving boxes are computed by the method of Gaussian quadrature. The aerodynamic matrix is based on the numerically computed velocity potentials for all boxes.
This development is particularly useful in the low supersonic range for panels with chord-span ratio less than about one, while the piston theory does not give satisfactory solution. Examples are demonstrated by using the 16 d.o.f. rectangular plate elements. Results for flutter boundaries for the unstressed panels agree well with an alternative Galerkin's modal solution. The examples demonstrate that flutter boundaries are dominated by higher modes for panels with higher chord-span ratio. They also demonstrate that the dominating flutter boundaries abruptly change modes as the Mach number is varied. The beneficial effect of in-plane tension is demonstrated.
FOREWORD

This report was prepared by Dr. T. Y. Yang, Visiting Scientist in the Optimization Group, Analysis and Optimization Branch, Structural Mechanics Division, Air Force Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio. This work was performed under Project Nr. 2307, "Research in Flight Vehicle Dynamics," Task Nr. 230705, "Basic Research in Structures and Dynamics."

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Flutter boundaries for clamped panels with $l/w = 1$, $M = 1.5$, and various tension parameters $F$

Flutter boundaries for clamped panels with $l/w = 1$, $M = 1.52$, and various tension parameters $F$

Flutter boundaries for clamped panels with $l/w = 1$, $M = 1.54$, and various tension parameters $F$

Flutter boundaries for clamped panels with $l/w = 1$, $M = 1.6$, and various tension parameters $F$
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SYMBOLS

a, b = length and width of the rectangular plate finite element as defined in Fig. 1.

[A] = aerodynamic matrix of the panel system

B_x, B_{xs} = number of boxes in the stream and cross-stream directions, respectively

D = Eh^3/12(1 - \gamma^2), bending rigidity with \gamma being Poisson's ratio

F = N_x/\xi^2 w^2 c h, Initial in-plane tension parameter

f_j(x,y) = shape function associated with degree of freedom "j"

g = structural damping coefficient

h = panel thickness

i, j = subscripts indicate d.o.f. number

[K] = stiffness matrix of the panel system

k_x, k_c = \omega z/V, \omega c/V, respectively, reduced frequency

z = length of panel in stream direction

[M] = mass matrix of the panel system

M = Mach number

m, n = receiving box index numbers in the stream and cross-stream directions, respectively

[N] = incremental stiffness matrix of the panel system

N_x, N_y, N_{xy} = panel in-plane line forces

r, s = (m - \lambda), (n - \nu), respectively

u, v = transformed variables of integration based on c/2 as reference length, \nu c/2 = x_m - c, \nu/2 = y_n - n

V = speed of undisturbed airstream
LIST OF SYMBOLS (Continued)

\[ w = \text{width of panel in cross-stream direction} \]

\[ W_j = \text{downwash velocity at panel surface for d.o.f. "j"} \]

\[ x, y = \text{panel coordinates as defined in Fig. 1} \]

\[ \bar{x}, \bar{y} = \frac{x}{i}, \frac{y}{w}, \text{respectively} \]

\[ \chi_{m}, \gamma_{n} = \text{values of } \frac{x}{c} \text{ and } \frac{y}{c} \text{ at center of box } m, n \]

\[ \alpha_{\phi} = \text{aerodynamic influence coefficient relating the velocity potential at a box to unit downwash on another box (Eq. 12)} \]

\[ \beta = (M^2 - 1)^{\frac{1}{2}} \]

\[ c = \frac{w}{B_{xs}}, \text{ box width} \]

\[ \xi, \eta = \text{values of } \frac{x}{c} \text{ and } \frac{y}{c} \text{ at any point in the sending box} \]

\[ \nu = \text{panel-air mass density ratio, } \frac{\sigma h}{\rho k} \]

\[ \lambda, \nu = \text{sending box index number in the stream and cross-stream directions, respectively} \]

\[ \rho = \text{density of undisturbed airstream} \]

\[ \sigma = \text{density of panel} \]

\[ \tau = \text{length-width ratio of box as defined in Fig. 1} \]

\[ \phi_i(m, n) = \text{velocity potential at center of box } m, n \text{ for d.o.f. "j"} \]

\[ \tilde{\phi}(m, n) = \text{velocity potential at center of box } m, n \text{ due to unit downwash over box } \lambda, \nu \]

\[ \omega = \text{frequency of flutter motion} \]

\[ \omega_1 = \text{first natural frequency of the panel} \]

\[ \Omega = (\omega_1/\omega)^2 (1 + ig) \]

\[ \tilde{\Omega} = M^2 k_\epsilon/\beta^2 \]
SECTION I
INTRODUCTION

Ever since the earliest days of manned flight, panel flutter has been known as one of the most important problems in the design of aircraft, missiles, launched vehicles, and spacecraft. Extensive progress in wind tunnel tests and theoretical studies has been achieved. The basic theories and an account of the developments on panel flutter can be found in, among other books, a recent text by Dowell (Reference 1). A list of keyed bibliography and collection of some significant survey papers and original papers were prepared by Garrick (Reference 2).

The theoretical solution for a panel flutter problem usually requires an accurate aerodynamic and structural theory to formulate a set of complex eigenvalue equations of motion interacted between the panel and the flow. One of the most common theoretical methods is the modal method where the aerodynamic pressure and the inertial and elastic forces of the panel are obtained by assuming the displacements as composed of a number of natural modes and generalized coordinates. The natural frequencies and corresponding normal mode shapes are obtained either theoretically or experimentally, or both. Since the finite element method is powerful and practical in the free vibration analysis of panels with arbitrary geometrical and boundary conditions, it is commonly used in obtaining natural frequencies and mode shapes in the modal method.

As an alternative approach, the finite element workers have formulated the matrix of aerodynamic pressure by using the
displacement functions as composed of the nodal degrees of freedom and shape functions rather than the generalized coordinates and natural mode shapes. Such approach can directly solve for the flutter frequencies and corresponding normal mode shapes without having to seek the natural frequencies and modes and choose the number of modes before the eigensolution, and also without having to compute the flutter mode shapes after the eigensolution. Such approach permits expression of the equations of motion in an elegant and straightforward form. Such approach also permits generality in panel configurations and boundary conditions, and allows for flexibility to accurately include physical effect such as in-plane forces.

The finite element method was first extended to the panel flutter problems by Olson (Reference 3). He formulated the aerodynamic matrix explicitly for an infinite plate element. Olson (Reference 4) later formulated the aerodynamic matrices for two rectangular (12 and 16 d.o.f.) and an 18 d.o.f. triangular plate elements. Simultaneously, but independently, Appa and Somashekar (Reference 5) formulated the aerodynamic matrix for a 12 d.o.f. rectangular plate element. Appa, Somashekar and Shah (Reference 6) later extended their work by accounting for skew panels and yawed flow by means of coordinate transformation. Sander, Don, and Geradin (Reference 7) employed the CQ conforming quadrilateral plate finite element for flutter analysis of rectangular panels with yawed flow and in-plane stresses.
In all the above finite element works (References 3 - 7), Lighthill's linearized piston theory was employed. The Mach numbers considered were above approximately 1.6. Recently, Yang (Reference 8) developed a finite element procedure using the exact linearized two-dimensional theory (strip theory) for unsteady supersonic flow to formulate an infinite plate finite element by means of numerical integration. The flutter speed considered thus could be in the lower supersonic range. Such formulation cannot, however, be adequately applied to the more general case of rectangular panels with finite aspect ratios.

In this report, the three-dimensional supersonic unsteady potential flow theory is employed to formulate the plate finite elements so that the flutter problems of the finite panels in low supersonic range can be treated.

The aerodynamic matrix is derived by using the principle of virtual work. The aerodynamic forces or velocity potentials that produce the work are obtained for each d.o.f. by the Mach box method. Each finite element is divided into several boxes. The aerodynamic influence coefficients for each pair of sending and receiving boxes are evaluated, for each d.o.f., by the method of Gaussian quadrature. The velocity potential at each receiving box is obtained, for each d.o.f., by summation of the product of corresponding downwash and influence coefficients for all sending boxes. It should be noted that a similar box method was used by Cunningham (Reference 9) in conjunction with a Galerkin's modal method for panel flutter analysis. Such 3-D aerodynamic theory was also employed by Dowell and Voss (Reference 10) in a theoretical and experimental correlation study of panel flutter.
The 16 d.o.f. conforming rectangular plate element (Reference 11) was used for example demonstrations. Flutter boundaries were found for clamped rectangular panels with various aspect ratios. The thickness ratios required to prevent flutter of an aluminum panel at sea level were plotted for Mach numbers ranging from 1.05 to 3. Results found are in favorable agreement with Cunninhgam's solution (Reference 9).

The initial in-plane tensile stresses were then included in finding the flutter boundaries for a clamped aluminum panel. The thickness ratios required to prevent flutter of the panel at 25,000 feet altitude were obtained for Mach numbers ranging from 1.05 to 2.0 for various values of tension. It was found that the dominating flutter mode changed abruptly as Mach number was varied.
SECTION II
FORMULATION OF EQUATIONS OF MOTION

The free vibration equations of motion for a finite element panel subjected to the effect of stiffness, in-plane force, inertia, and aerodynamic pressure may be written as

\[ [K](q) + [N](q) + [M](\ddot{q}) + [A](q) = 0; \]

where \([K]\), \([N]\), \([M]\), and \([A]\) are, respectively, the stiffness, incremental stiffness, mass, and aerodynamic matrices assembled for the whole finite element system. The vector \([q]\) contains the nodal degrees of freedom for the whole panel system. In this section, only the system aerodynamic matrix is formulated.

1. PRINCIPLE OF VIRTUAL WORK

For a system of plate finite elements, the deflection and aerodynamic pressure may be written by separating the time and space variables as,

\[ \ddot{z}(x,y,t) = z(x,y)e^{i\omega t} \]
\[ \ddot{p}(x,y,t) = p(x,y)e^{i\omega t} \]

(2)

where the coordinates are defined in Fig. 1.

The strain energy for the panel system is equal to the work produced by the aerodynamic pressure

\[ U = \frac{1}{2} \iint z(x,y)p(x,y) \, dx \, dy \]

(3)
Figure 1. Panel divided into finite elements (solid lines) and boxes (dash lines) with coordinates, dimensions, and each cone.

\[ x_m - \xi = \beta(y_n - \eta) \]

---

Figure 2. Three types of integrals for computing the aerodynamic influence coefficients.
The deflection function for the total finite element system may be assumed as

\[ z(x,y) = \sum_{j=1}^{N} f_j(x,y)q_j = \{f\}^T\{q\} \quad (4) \]

where \( f_j(x,y) \) represents the assembled shape function corresponding to the nodal d.o.f. "q\(_j\)" and \( N \) is the number of degrees of freedom for the entire panel system. Usually, \( q_j \) represents deflection, slopes, twist, and curvatures of the plate at the nodal point \( j \).

Similarly, the aerodynamic pressure may be assumed as

\[ p(x,y) = \sum_{j=1}^{N} P_j(x,y)q_j = \{P\}^T\{q\} \quad (5) \]

where \( P_j(x,y) \) is defined in Eq. (8) as the pressure function associated with the shape function \( f_j(x,y) \) and d.o.f. "q\(_j\)."

Substituting Eqs. (4) and (5) into (3), the strain energy expression becomes

\[ U = \frac{1}{2} \{q\}^T[A]\{q\} \quad (6) \]

with

\[ [A] = \iint (f)(P)^T \, dx \, dy \quad (7) \]

Using the principle of virtual work, it can be interpreted that Eq. (7) yields the aerodynamic matrix.

2. AERODYNAMIC MATRIX AND AERODYNAMIC PRESSURE

The aerodynamic perturbation pressure is obtained from the linearized three-dimensional supersonic unsteady potential flow theory. When associated
with the d.o.f. "q_j" and the shape function \( f_j(x,y) \), the perturbation pressure may be expressed in terms of the velocity potential through the relation (see, for example, Ref. 12),

\[
p_j(x,y) = -\frac{\rho V}{2} \left( \frac{\partial \phi_j}{\partial x} + i \frac{\omega}{V} \phi_j \right)
\]

where \( \phi_j \) is the velocity potential associated with d.o.f. "q_j".

The coefficient for the \( i \text{th} \) row and \( j \text{th} \) column of the aerodynamic matrix \([A]\) is obtained by substituting Eq. (8) into Eq. (7).

\[
A_{ij} = -\frac{\rho V}{2} \int \int f_i(x,y) \left( \frac{\partial \phi_j}{\partial x} + i \frac{\omega}{V} \phi_j \right) d(x)d(y)
\]

Performing integration by parts and assuming restrained trailing edge, Eq. (9) becomes

\[
A_{ij} = \frac{\rho V}{2} \int \int \phi_j \left( \frac{\partial f_i}{\partial x} - i \frac{\omega}{V} f_i \right) d(x)d(y)
\]

where the quantity within the parentheses is the complex conjugate of the downwash ratio for unit d.o.f. "q_i" and unit panel length. The coefficient \( A_{ij} \) is evaluated numerically through the following simple summation

\[
A_{ij} = \frac{\rho V}{2} \sum \phi_j \left( \frac{\partial f_i}{\partial x} - i \frac{\omega}{V} f_i \right) \text{(Box Area)}
\]

where the summation is to be made for every box and the quantities related to \( \phi_j \) and \( f_i \) are evaluated numerically at the center of each box. The method for numerically evaluating the velocity potential at the center of each box is given as follows.
3. VELOCITY POTENTIAL AND AERODYNAMIC INFLUENCE COEFFICIENT

The method of Mach box as employed by Cunningham in Reference 9 is used here to derive the aerodynamic influence coefficients between each pair of boxes and the resulting velocity potentials at each box.

In this method, each finite element is divided into several equal-size boxes as shown in Fig. 1. The numbers of boxes in the stream and cross-stream directions are defined as \( B_x \) and \( B_{xs} \), respectively. The width and length of each box are defined as \( c \) and \( c_e \), respectively, where \( c = w/B_{xs} \) and \( e = r_{xs}/w_{ox} \). The boxes are numbered in sequence from the origin with \( m \) (or \( \lambda \)) and \( n \) (or \( v \)) as the box number in the stream and the cross-stream directions, respectively. The numbers \((m,n)\) and \((\lambda,v)\) refer to the receiving and sending boxes, respectively.

The boxes are assumed as sufficiently small so that the downwash over any sending box is considered as uniformly distributed at any instant, and the resulting perturbation pressure at the center of each receiving box represents the average of the pressure distributed over the box.

The velocity potential at the center of a receiving box \((m,n)\) due to a uniformly distributed but otherwise unspecified downwash \( w(\lambda,v) \) over the sending box \((\lambda,v)\) can be expressed, for harmonic motion, as

\[
\Phi(m,n) = c w(\lambda,v) \Phi(r,s)
\]

where the relative locations in stream and cross-stream directions, respectively, between the two boxes are defined as \( r = m - \lambda \) and \( s = n - v \). The aerodynamic influence coefficient from Reference 9 is,
\[ a_{\phi}(r,s) = \frac{1}{2\pi} \int_{u_1}^{u_2} e^{-i(\phi'/\phi)}u \left\{ \cos^{-1} \frac{v_1}{u} - \cos^{-1} \frac{v_2}{u} + \int_{v_1}^{v_2} \frac{\cos \frac{\pi}{2M} \left( \frac{1}{u^2 - v^2} \right)^b}{(u^2 - v^2)^2} \, dv \right\} \, du \]  

where all the parameters are defined in the list of Symbols.

In Eq. (13), the surface integration limits \( u_1, u_2, v_1, \) and \( v_2 \) result in only three different forms \( I_{G1}, I_{G2}, \) and \( I_{G3} \) as shown in Fig. 2.

The first form is for any portion of a sending box \((r, \alpha, \beta)\) cut by both sides of the Mach cone so that \( v_2 = -v_1 = u \) and \( s = 0. \)

\[ I_{G1} = \frac{1}{2} \int_{u_1}^{u_2} e^{-i(\phi'/\phi)}u J_0 \left( \frac{u}{2M} \right) \, du \]  

where \( J_0 \) is the Bessel function of the first kind of order zero.

The second form is for portions of a box cut by one side of the Mach cone so that the limits \( v_2 = u \) and \( v_1 = 2s - 1, 1 \geq 1. \)

\[ I_{G2} = \frac{1}{2\pi} \int_{u_1}^{u_2} e^{-i(\phi'/\phi)}u \left\{ \cos^{-1} \frac{v_1}{u} + \int_{v_1}^{u} \frac{\cos \frac{\pi}{2M} \left( \frac{1}{u^2 - v^2} \right)^b}{(u^2 - v^2)^2} \, dv \right\} \, du \]  

The third form is for boxes that are completely within the Mach cone and also for portions of boxes ahead of the point where the Mach line cuts the side of the box.

\[ I_{G3} = \frac{1}{2\pi} \int_{u_1}^{u_2} e^{-i(\phi'/\phi)}u \left\{ \cos^{-1} \frac{v_1}{u} - \cos^{-1} \frac{v_2}{u} + \int_{v_1}^{v_2} \frac{\cos \frac{\pi}{2M} \left( \frac{1}{u^2 - v^2} \right)^b}{(u^2 - v^2)^2} \, dv \right\} \, du \]  

The complete \( a_{\phi}(r,s) \) for any one sending box \((r, \alpha, \beta)\) consists of \( I_{G1}, I_{G2}, \) or \( I_{G3}, \) or a combination of \( I_{G1} \) and \( I_{G2}, \) or of \( I_{G2} \) and \( I_{G3}, \) or of \( I_{G1} \) and \( I_{G3}. \) The types of integrals and limits of integration for computing \( a_{\phi}(r,s) \) for all possible relative locations of box \((r, \alpha, \beta)\) and the Mach cone from \((r, \alpha, \beta)\) are given in Reference 9.
The evaluation of the above three integrals is carried out through the method of Gaussian quadrature. In the subsequent numerical examples, three Gaussian points are used in both x and y directions for computing $I_{G3}$. Five Gaussian points in both x and y directions are used for computing $I_{G1}$ and $I_{G2}$.

Once all possible values of the aerodynamic influence coefficient $c(r,s)$ are obtained for a certain shape function $f_j(x,y)$, the total velocity potential at the center of a receiving box $(m,n)$ for the downwash associated with the same shape function is a weighted sum of the $\phi_j(m,n)$ defined in Eq. (12)

$$\phi_j(m,n) = \omega \sum_{\lambda, \nu} \frac{W_j(\lambda, \nu)}{V} \alpha(r,s) \quad (17)$$

The summation is extended over all the sending boxes. The downwash ratios $w_j(\lambda, \nu)/V$ for a unit d.o.f. "$q_j$" are the total time derivatives of the shape function $f_j(x,y)e^{i\omega t}$

$$\frac{w_j}{V} = \frac{1}{k}(\frac{\partial f_j}{\partial x} + i \frac{\omega x}{V} f_j) \quad (18)$$

Once the velocity potentials are obtained for each box for each shape function, they can readily be substituted into Eq. (11) for computing the aerodynamic matrix for the entire panel system.
In the present finite element flutter formulation, the parameters are so grouped and nondimensionalized that the solution is general enough to include every parameter. Assuming harmonic motion with natural frequency $\omega$, the equations of motion (1) are rewritten in an element form as

$$\left\{ \Omega \left[ c_1[k] + Fc_2[n] \right] - \left[ c_3[m] - c_4[A] \right] \right\} \{q\} = \{0\}$$

(19)

where $\Omega = \omega_1^2/\omega^2(1 + ig)$ is the flutter eigenvalue parameter, and

$$c_1 = \frac{\mu(\xi/a)^5 h}{\omega_1^2 \sigma h^4} ; \quad c_2 = \mu(\xi/a)^3 \frac{1}{\omega_1^2 h^4} ;$$

$$c_3 = \mu(\xi/a) ; \quad c_4 = \left( \frac{\xi}{a} \right)^3 \left( \frac{E}{\chi} \right) \left( \frac{1}{k} \right)^2 \left( \frac{\xi}{h} \right)^2$$

(20)

The element matrix terms are obtained as

$$k_{ij} = \int_0^1 \int_0^1 \left[ \frac{\partial^2 f_i}{\partial x^2} \frac{\partial^2 f_j}{\partial x^2} + (\frac{a}{b})^4 \frac{\partial^2 f_i}{\partial y^2} \frac{\partial^2 f_j}{\partial y^2} \right] \frac{\partial^2 f_i}{\partial x^2} \frac{\partial^2 f_j}{\partial y^2}$$

$$+ \nu(\frac{a}{b})^2 \frac{\partial^2 f_i}{\partial x^2} \frac{\partial^2 f_j}{\partial y^2} + 2(1 - \nu)(\frac{a}{b})^2 \frac{\partial^2 f_i}{\partial x \partial y} \frac{\partial^2 f_j}{\partial y \partial x} d\tilde{x} d\tilde{y}$$

(21)

$$n_{ij} = \int_0^1 \int_0^1 \left[ \frac{\partial f_i}{\partial x} \frac{\partial f_j}{\partial x} + \left( \frac{x}{a} \right) \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial y} + \left( \frac{x}{b} \right) \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial y} + \left( \frac{x}{a} \right) \frac{\partial f_i}{\partial y} \frac{\partial f_j}{\partial x} \right] d\tilde{x} d\tilde{y}$$

(22)

$$m_{ij} = \int_0^1 \int_0^1 f_i f_j d\tilde{x} d\tilde{y}$$

(23)

$$A_{ij} = \sum \frac{\phi_j}{V \epsilon} \left( \frac{\partial f_i}{\partial x} - i \omega \frac{\partial f_i}{\partial y} \right)$$

(24)
The shape functions \( f_i(x, y) \) are usually cubic or higher order functions of nondimensional coordinate variables \( x \) and \( y \). Their differentiations with respect to \( x \) and \( y \) are performed analytically. The subscript "i" indicates the degree of freedom number with which the shape function is associated. Eqs. (19-24) are suitable for any plate finite element so long as it is a displacement model with assumed shape functions. According to the state-of-the-art of the finite element development, the stiffness matrix \([k]\), mass matrix \([m]\), and incremental stiffness matrix \([n]\) have been formulated analytically and explicitly for almost every plate and shell finite element. Only the aerodynamic matrix \([A]\) remains to be formulated and it is to be obtained by numerical integration here.

Eq. (19) constitutes an eigenvalue problem. The flutter solution is obtained by first assuming a value of reduced frequency \( k_\infty \), then varying the air-panel mass ratio \( 1/\mu \) incrementally and solving for the corresponding eigenvalues \( \omega \) or \((\omega/\omega)^2(1 + ig)\). When the structural damping coefficient \( g \) changes its value from negative to positive, the panel goes from the stable region to unstable region, and vice versa. The values of \( k_\infty \) and \( 1/\mu \) that correspond to zero \( g \) value define the flutter boundary and the corresponding mode shape defines the flutter mode. The complex eigenvalue problem is solved by using a subroutine in EISPACK provided by Argonne National Laboratory.

Before performing analysis for each class of problems, a convergence study by varying the meshes of Mach boxes and finite elements must be made in order to find the suitable meshes needed for obtaining converged results. Such meshes depend on the panel geometry, boundary conditions, flow speed, and flutter modes. However, due to the enormous computations needed for obtaining massive data in this study, not the finest meshes are used. Most of the present results are compared with those obtained by the modal method (Reference 9) and good results are obtained.
SECTION IV
RESULTS

The 16 d.o.f. conforming rectangular plate finite elements were employed to demonstrate the present formulation and procedure. The examples chosen were rectangular panels, stressed as well as unstressed, with clamped edges. For the unstressed panels, a sophisticated analytical solution by Cunningham (Reference 9) using the box method (400 boxes) and Galerkin's modal approach (6 to 16 modes) was available for comparison.

In all examples studied here, the flutter mode shapes were assumed as symmetrical about the center chord-line of the panel. Thus, only half of each panel was modeled by finite elements. In all cases, two elements in the cross stream direction were used for half of the panel. The number of elements in the stream direction varied from 4 to 10 dependent on the dominating flutter mode shapes. Each element was divided into 4 by 2 boxes in the stream and cross stream directions, respectively.

It is important to note that symmetry does not exist for the mesh of boxes unless the Mach cone apex is located at the center chord-line of the panel. The boxes on the other (disregarded) half of the panel can, however, still be accounted for since their deflection shapes are known by symmetry.

Case 1  Clamped Panels with Various Chord-span Ratios at $M = 1.3$

The rectangular panels with all edges clamped and chord-span ratios $(t/w)$ equal to 0, 1/4, 1/2, 1, 2, and 4 were studied for $M = 1.3$. 

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The resulting flutter boundaries are presented as plots of each dominating mode with the air-panel mass ratio $1/\mu$ as the vertical coordinate and the stiffness parameter $\omega \cdot \ell/\rho$ as the horizontal coordinate. The values of reduced frequency $k\gamma$ or $\omega\kappa/V$ are marked along each curve. The result for the infinite panel with $\ell/\rho = 0$ are in exact agreement with those obtained by Cunningham, therefore, they are not presented here.

Fig. 3 shows that for the wide panel with $\ell/\rho = 1/4$, the first mode flutter boundary crosses the second mode flutter boundary. The critical flutter boundaries are thus dominated by the first mode in the high mass ratio range and by the second mode in the low mass ratio range. A slight amount of structural damping was also included ($\zeta = 0.01$). The results agree well with those obtained by Cunningham (Reference 9).

The first flutter mode shape (real part) corresponding to point A ($k\gamma = 0.635$) in Fig. 3 is shown in Fig. 4. It is seen that, due to the effect of the flow, the mode is not symmetrical about the cross stream centerline. The maximum deflection occurs at a point behind the center of the panel. The second flutter mode shape (real part) for the center chord-line of the panel corresponding to point B ($k\gamma = 1.59$) in Fig. 3 is shown in Fig. 5. The effect of the flow that pushes down the front part of the panel is seen.

Fig. 6 shows that, as the panel width is reduced ($\ell/\rho = 1/2$), the first mode boundary shifts to the left and the second mode boundary becomes the critical flutter boundary. Again, the results agree well with Cunningham's solution.
Cunningham, (Ref. 9)

First Mode, g=0

Second Mode, g=0

First Mode, g=0.01

Second Mode, g=0.01

Figure 3 Flutter boundaries for clamped panel with \( l/w = 1/4 \) and \( M = 1.3 \)
Figure 4. The first mode shape (real part) for panel corresponding to point A in Fig. 3

Figure 5. The second mode shape (real part) for the center line of panel corresponding to point B in Fig. 3
Figure 6 Flutter boundaries for clamped panel with $l/w = 1/2$ and $M = 1.3$
The first and second flutter mode shapes (real part) corresponding to points A and B in Fig. 6 are plotted in Figs. 7 and 8, respectively. They are quite similar to those shown in Figs. 4 and 5.

Fig. 9 shows that, as the panel becomes square, the third and fourth mode boundaries emerge and the third mode boundary becomes the critical flutter boundary. The results are in good agreement with Cunningham's solution. The results for the first mode boundary obtained by Houbolt using the piston theory are also shown in the figure.

The third and fourth flutter mode shapes (real part) corresponding to points A and B in Fig. 9 are shown in Figs. 10 and 11, respectively. The effect of the flow that tends to blow flat the front part of the panel is seen.

Fig. 12 shows that, for a long panel with $\ell/w = 2$, the fifth mode flutter boundary becomes dominant in the low mass ratio range and the first mode boundary dominates the high mass ratio range. It also shows that Houbolt's solution for first mode and this solution are very close. There are, however, some discrepancies between this solution and Cunningham's solution for the first mode boundary. This may be due to the fact that the stiffness coupling effect between assumed beam natural modes was uniformly neglected by Cunningham. Such coupling effect can become significant as the chord-span ratio increases.

It should be noted that, due to the limited number of elements and boxes used, the present results are not the most accurate that this approach can produce.
Figure 7 The first mode shape (real part) for panel corresponding to point A in Fig. 6.

Figure 8 The second mode shape (real part) for the center line of panel corresponding to point B in Fig. 6.
Figure 9: Flutter boundaries for clamped panel with $\ell/w = 1$ and $M = 1.3$.
Figure 10  The third mode shape (real part) for the center line of panel corresponding to point A in Fig. 9

Figure 11  The fourth mode shape (real part) for the center line of panel corresponding to point B in Fig. 9
Figure 12: Flutter boundaries for clamped panel with \( \ell/w = 2 \) and \( M = 1.3 \)
The first and third flutter mode shapes (real part) corresponding to points A and B in Fig. 12 are shown in Figs. 13 and 14, respectively. The difference between a symmetrical in vacuo first mode and the first flutter mode with front part blown flat is clearly shown in Fig. 13.

Fig. 15 shows that, for a very long panel with \( \nu/w = 4 \), the tenth mode flutter boundary dominates the lower mass ratio range while the first mode boundary dominates the upper mass ratio range. The discrepancy between the present solution and Cunningham's solution may be attributed to the same reasons as explained for Fig. 6. Houbolt's solution for first mode boundary is also shown.

The second and tenth flutter mode shapes (real part) corresponding to points A and B in Fig. 15 are plotted in Figs. 16 and 17, respectively.

**Case 2. Clamped Panel with \( \nu/w = 2 \) at Various Mach Numbers**

A clamped panel with \( \nu/w = 2 \) and with one surface exposed to air stream with various Mach numbers was studied. One of the main purposes was to establish a curve for the thickness ratios required to prevent flutter of panel at various air speeds and at sea level.

The first mode flutter boundaries for panel with \( \nu/w = 2 \) and \( M = 1.05, 1.1, 1.4, 1.5, 2.0, \) and \( 3.0 \) are shown in Figs. 18, 19, and 20, respectively. Corresponding to each curve, a dashed parabola is shown. Each parabola is plotted for the equation \( xy = C \) or \( (\nu/\sin)(w \nu/V) = C \). The constant \( C \) is dependent upon
Figure 13 The first mode shape (real part) for panel corresponding to point A in Fig. 12.

Figure 14 The fifth mode shape (real part) for the center line of panel corresponding to point B in Fig. 17.
Figure 15 Flutter boundaries for clamped panel with $l/w = 4$ and $M = 1.3$. 

**Graph Details:**
- **Houbolt (Ref 13)**
- **Cunningham (Ref. 9)**
- **8 elements or 20 elements for half panel**

**Key Parameters:**
- $k_2 = 2.335A$
- $\omega, l/v$

**Points and Modes:**
- **Second Mode:** $\omega = 7.15$
- **Tenth Mode:** $\omega = 7.2178$
- **Tenth Mode (20 elements):** $\omega = 7.236$
Figure 16 The second mode shape (real part) for panel corresponding to point A in Fig. 15.

Figure 17 The tenth mode shape (real part) for the center line of panel corresponding to point B in Fig. 15.
Figure 18 Flutter boundaries for clamped panel with $l/w = 2$ and $M = 1.05, 1.10$. 
Figure 19  Flutter boundaries for clamped panel with $\ell/\nu = 2$ and $M = 1.40, 1.50$. 

M = 1.5 (8 elements) 
M = 1.4 (8 elements)
Figure 20  Flutter boundaries for clamped panel with $l/w = 2$ and $M = 2.0$ and $3.0$. 

STABLE
the densities of the aluminum and the air at sea level, the dimensions of the panel, the air speed, and the bending rigidity of the panel. The intersecting point between a pair of dashed and solid curves gives the thickness ratio $h/\lambda$ required to prevent flutter of the aluminum panel at sea level. For all Mach numbers considered here, the first mode flutter boundaries are critical for the dashed curves for the sea level altitude.

The results are shown in Fig. 21. The present results are slightly on the unconservative side as compared to the results of Cunningham.

Case 3  Pinned-Edge Panels with Various Chord-Span Ratios at $M = 1.3$

The panels with simply-supported edges and various chord length-span width ratios were studied. The flow speed considered was $M = 1.3$. The dominating flutter boundaries for panels with $\ell/w = 0.25, 0.5, 1.0, \text{ and } 2.0$ are shown in Figs. 22, 24, 26 and 28, respectively. They are all in good agreement with those found by Cunningham (Reference 9).

Fig. 22 shows that the first mode flutter boundary is the critical boundary for panels with $\ell/w = 0.25$. The mode shape (real part) is shown in Fig. 23.

Fig. 24 shows that the second mode flutter boundary is the critical boundary for panels with $\ell/w = 0.5$. The first and second flutter mode shapes (real part) are shown in Fig. 25.

Fig. 26 shows that the third mode flutter boundary is the critical boundary for panel with $\ell/w = 1.0$. The first mode boundary found by Hedgepeth (Reference 11) using the piston theory is also shown in the figure. The corresponding first and third flutter mode shapes (real part) are shown in Fig. 27.
Figure 21  Thickness required to prevent flutter for clamped aluminum panel with \( \frac{q}{\pi} = 2 \) at sea level.
Figure 22  Flutter boundaries for pinned-edge panel with $l/w = 1/4$ and $M = 1.3$. 

Cunningham (Ref. 9)

8 elements for half panel

First Mode
$q = 0.01$

$k_b = 0.55$

$\omega / \sqrt{V}$

Stiffness Parameter
Figure 23: The First Mode Shape (Real Part) for Panel Corresponding to Point A in Fig. 22.
Figure 24  Flutter boundaries for pinned-edge panel with $l/w = 1/2$ and $M = 1.3$. 
Figure 25 The First Mode Shape (Real Part) for Panel Corresponding to Point A in Fig. 24 and the Second Mode Shape (Real Part) for the Center Line of Panel Corresponding to Point B in Fig. 24.
Figure 26  Flutter boundaries for pinned-edge panel with $k/w = 1$ and $M = 1.3$. 
Figure 27 The First Mode Shape (Real Part) for Panel Corresponding to Point A in Fig. 26 and the Third Mode Shape (Real Part) for the Center Line of Panel Corresponding to Point B in Fig. 26.
Fig. 28 shows that the fifth mode flutter boundary is the critical boundary for panels with \( \frac{\ell}{w} = 2.0 \). The results by Hedgepeth and Cunningham are seen to be in good agreement with the present results. The corresponding first and fifth flutter mode shapes (real part) are shown in Fig. 29.

**Case 4  Clamped Square Panels with Various Tension Parameters and Mach Numbers**

One of the advantages of the finite element method is that the effect of in-plane forces can be included in a direct and accurate fashion. To demonstrate this, a square clamped panel was chosen and four different tension parameters \( F = 0.01, 0.1, 0.5, \) and \( 1 \) were considered. The flutter boundaries were obtained for various Mach numbers.

Fig. 30 shows the dominating first mode flutter boundaries and the hyperbola for the square, aluminum panel at 25,000 feet above sea level and at \( M = 1.1 \). Four values of tension parameters were included.

The mode shapes (real part) corresponding to points A (\( F = 0.01 \)) and point B (\( F = 0.1 \)) of the flutter boundaries in Fig. 30 are plotted in Figs. 31 and 32. The mode shapes for the case \( F = 0.5 \) and \( 1.0 \) are similar to those shown in Figs. 31 and 32. They are thus not shown here.

Fig. 33 shows the first and the third mode flutter boundaries for \( M = 1.2 \) and various tension parameters. For aluminum panels at 25,000 feet above sea level (dashed parabola), the third mode boundaries are the critical flutter boundaries. The first flutter mode shapes (real part) corresponding to points A and B in Fig. 33 are presented in Figs. 34 and 35, respectively.
Figure 28  Flutter boundaries for pinned-edge panel with $l/u = 2$ and $M = 1.3$. 

Hedgepeth (Ref. 14)
Cunningham (Ref. 9)
8 elements or 12 elements for half panel

First Mode (8 elements)
Fifth Mode (12 elements)
At Point A

At Point B

Figure 29: The First Mode Shape (Real Part) for Panel Corresponding to Point A in Fig. 28 and the Fifth Mode Shape (Real Part) for the Center Line of Panel Corresponding to Point B in Fig. 28.
Figure 30. Flutter boundaries for clamped panels of \( l/w = 1.0 \) with \( M = 1.1 \) and various tension parameters \( F \).
Figure 31 The First Mode Shape (Real Part) for Panel Corresponding to Point A in Fig. 30.

Figure 32 The First Mode Shape (Real Part) for Panel Corresponding to Point B in Fig. 30.
Figure 33 Flutter boundaries for clamped panels of $Q/\omega=1.0$ with $M=1.2$ and various tension parameters $P\psi$. 

**First Mode**

- $F=1.0$
- $F=0.5$
- $F=0.1$

**Third Mode**

- $D_{k,b}=0.628$
- $A=0.628$
- $B=1.5$
- $C=1.2$
- $D=1.08$
- $E=0.68$

Aluminum Panel at 25,000 ft
Figure 34 The First Mode Shape (Real Part) for Panel Corresponding to Point A in Fig. 33.

Figure 35 The First Mode Shape (Real Part) for Panel Corresponding to Point B in Fig. 33.
The third flutter mode shapes (real part) for the center chord lines corresponding to points C, D, E, and F are shown in Fig. 36.

Fig. 37 shows the first mode flutter boundaries for $M = 1.3$ and the third mode flutter boundaries for $M = 1.3, 1.32, 1.35$ for various values of tension. The flutter boundaries are dominated by the third mode. Since the flutter mode shapes are similar to those shown in the previous figures, they are not presented here.

Fig. 38 shows the flutter boundaries for $M = 1.4, 1.45, 1.48$ and various tension parameters. The third mode flutter boundaries shift to the right as the Mach number increases. It is interesting to see that the top portions of the third mode flutter boundaries begin to bend down to the left as the Mach number increases.

Fig. 39 shows that for $M = 1.5$ and various values of tension, the third mode flutter boundaries continue to bend down rapidly with a slight increase in Mach number.

Figs. 40 and 41 show that for $M = 1.52$ and 1.54 and various values of tension, the third mode flutter boundaries bend down substantially to be small loops. They only dominate the lower range of the mass ratio.

Fig. 42 shows that for $M = 1.6$, all the third mode flutter boundaries disappear and the first mode flutter boundaries once again become the critical boundaries. Such a phenomenon is also the case for $M = 2.0$ as shown in Fig. 43.

Also shown in Figs. 37-43 are the dashed parabolae for an aluminum panel at 25,000 feet above sea level.
Figure 36  The Third Mode Shape (Real Part) for the Center Line of Panel with various tension parameter Corresponding to Point C, D, E, and F respectively in Fig. 35.
Figure 37 Flutter boundaries for clamped panels of $l/w=1.0$ with $K=1.3$, $M=1.32$, $K=1.35$ and various values of tension parameters $F$. 

STIFFNESS PARAMETER $\omega l/v$ vs. STIFFNESS RATIO $1/\mu$ 

- $A$--$A$ Third Mode for $M=1.32$ 
- $B$--$B$ Third Mode for $K=1.35$ 
- $O$--$O$ First Mode for $M=1.30$ 
- $O$--$O$ Third Mode for $M=1.30$ 
- $F=0.01$ 
- $F=0.1$ 
- $F=0.5$ 
- $k_g=0.628$ 
- $k_g=2.0$ 
- Aluminum panel at 25,000 ft.

4E
Figure 38: Flutter boundaries for clamped panels with $l/w=1.0$ with $M=1.4$, $M=1.45$, $M=1.48$ and various values of tension parameter $F$.
Figure 39: Flutter boundaries for clamped panels of $\ell/y = 1.0$ with $K = 1.5$ and various values of tension parameters $F$.
Figure 40. Flutter boundaries for clamped panels of $l/w = 1.0$ with $M = 1.52$ and various values of tension parameters $F$. 

STIFFNESS PARAMETER $\frac{\omega l}{V}$
Figure 41 Flutter boundaries for clamped panels of $l/w=1.0$ with $M=1.54$ and various values of tension parameters $F$.
Figure 42 Flutter boundaries for clamped panels of $L/w=1.0$ with $M=1.6$ and various values of tension parameters $F$. 

Stable

Mass Ratio $1/\mu$

STIFFNESS PARAMETER $\omega q/V$

Aluminum panel at 25,000 ft.
Figure 43 Flutter boundaries for clamped panel of $l/w=1.0$ with $M=2.0$ and various values of tension parameters $F$. 

Aluminum panel at 25,000 ft.
By collecting all the intersecting points between the dashed parabolas and the critical flutter boundaries in Figs. 30, 33, and 37-43, the results for thickness ratios required to prevent flutter of the panel are shown in Fig. 44 for various values of Mach numbers and tension.

In Fig. 44, the critical flutter boundaries were dominated by the third mode in the Mach region between approximately 1.2 and 1.5. For other lower and higher Mach regions, the first mode flutter boundaries dominate. The sharp drops in the curves are due to the abrupt changes in critical flutter modes.

The beneficial effect of introducing in-plane tensions to reduce the required panel thickness to avoid flutter is clearly demonstrated. Fig. 44 should be of value to panel flutter analysts and designers.
Figure 44 Thickness ratio required to prevent flutter for various values of tension parameter $F$ for a square clamped aluminum panel at 25,000 ft. above sea level.
A basic finite element procedure for panel flutter analysis has been developed and performed with examples. The following concluding remarks may be made.

(1) The three-dimensional supersonic unsteady potential flow theory was employed. This theory allows the treatment of panels with finite aspect ratio. It is particularly advantageous at low supersonic range ($1 < M < \sqrt{2}$) for panels with chord-span ratio less than one, when the piston theory does not give satisfactory results.

(2) The finite element method offers a set of elegant and straightforward eigenvalue equations. It can be used directly to solve for flutter frequencies and mode shapes without requiring the natural vibration frequencies and mode shapes before the eigensolution, and also without requiring the computation of mode shapes after the eigensolution.

(3) If the modal method is used in panel flutter analysis, one could use the natural frequencies and modes found by the finite element method. Such procedure is, however, basically different from the present finite element method.

(4) The formulations here (Lqs. 19-24) are general. They are readily applicable to any plate finite element displacement model whose shape functions are known.
(5) The present results agree well with Galerkin's modal solution by Cunningham.

(6) The critical flutter boundaries for a square clamped panel change modes abruptly as the Mach number is varied.

(7) Fig. 44 clearly demonstrates the beneficial effect of in-plane tensions.

(8) The present development may provide a basic finite element procedure for panel flutter analysis. The present results may provide useful data to the flutter analysts and designers.

(9) A logical next step appears to be the development of a method using triangular plate finite elements combined with triangular Mach boxes. Such development will be of great value to the flutter predictions of wing panels with arbitrary configurations. It is suggested that the triangular element developed by Bell (Reference 15) be used.
REFERENCES


APPENDIX
THE COMPUTER PROGRAM

The description and listing of the computer program are provided in this appendix. The flow chart is given on pages 69 and 70. The procedure for input is given on pages 62, 67, and 68. The use of the program is demonstrated by considering an example of a square clamped panel with $k_x = 0.5$, $M = 1.2$, and $F = 0.1$. Both input and output data are provided.

The stiffness, mass, and incremental stiffness matrices are based on the known explicit coefficients. These coefficients are read in as input data given on pages 63-66. The program uses the subroutine EISPACK for solving the general complex eigenvalue equations.
INPUT DATA

1. Control Card (2F5.2, 713, 4F7.4)

Columns 1-5  Panel chordwise length (XL)
6-10  Panel spanwise width (YL)
11-13  Number of elements in chordwise direction (NX)
14-16  Half number of elements in spanwise direction (NY)
17-19  Number of boxes in streamwise (or chordwise) direction for one element (IBX)
20-22  Number of boxes in cross-streamwise (or spanwise) direction for one element (IBY)
23-25  Total number of degrees of freedom for the panel (NXD)
26-28  Number of nodes in streamwise direction (NDX)
29-31  Number of nodes in cross-streamwise direction (NDY)
32-38  Structural damping coefficient (SG)
39-45  Mach number (MACH)
46-52  Starting value for the mass ratio range (MUO)
53-59  Increment for the values of mass ratios (XINCR)

2. Element matrix cards (8I4, 2I2, 2I4, 2I6)

Read in the data given on pages 63-66.
<table>
<thead>
<tr>
<th>TABLE I</th>
<th>DATA INFORMATION FOR GQ(I,J), QM(I,J) AND QH(I,J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GQ: I-J ELEMENT DIFFUSION COEFFICIENTS</td>
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<tr>
<td>QM: I-J ELEMENT MOC DATA</td>
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<tr>
<td>QH: I-J ELEMENT INCREMENTAL DIFFUSION COEFFICIENTS</td>
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<td>Number of Card Numbers: 1-24</td>
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<table>
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<th>GQ(I,J) DATA</th>
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<th>GQ(I,J) DATA</th>
<th>QM(I,J) DATA</th>
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Note: The table continues with similar entries for other locations and card numbers.
TABLE I
DATA INFORMATION FOR QQ(I,J), GQ(I,J) AND GQ(I,J), continued...
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</tbody>
</table>

**TABLE I**  **DATA INFORMATION FOR QQ(I,J), QM(I,J) and QM(I,J), continued ...**
3. Element Cards with the boundary condition code at each node (2213).

Columns 1-3  The sequence number for the element
4-6  The starting matrix row position for the degrees of freedom at nodal point I  (ID(1))
7-9  The starting matrix row position for the degrees of freedom at nodal point J  (ID(2))
10-12 The starting matrix row position for the degrees of freedom at nodal point K  (ID(3))
13-15 The starting matrix row position for the degrees of freedom at nodal point L  (ID(4))
16-18 Boundary condition code for deflection Z at nodal point I  (w(1))
19-21 Boundary condition code for $Z_x$ at nodal point I  (w(2))
22-24 Boundary condition code for $Z_y$ at nodal point I  (w(3))
25-27 Boundary condition code for $Z_{xy}$ at nodal point I  (w(4))
28-30 Boundary condition code for Z at nodal point J  (w(5))
31-33 Boundary condition code for $Z_x$ at nodal point J  (w(6))
34-36 Boundary condition code for $Z_y$ at nodal point J  (w(7))
37-39 Boundary condition code for $Z_{xy}$ at nodal point J  (w(8))
40-42 Boundary condition code for Z at nodal point K  (w(9))
43-45 Boundary condition code for $Z_x$ at nodal point K  (w(10))
46-48 Boundary condition code for $Z_y$ at nodal point K  (w(11))
49-51 Boundary condition code for $Z_{xy}$ at nodal point K  (w(12))
52-54 Boundary condition code for Z at nodal point L \( (w(13)) \)

55-57 Boundary condition code for \( Z_{x} \) at nodal point L \( (w(14)) \)

58-60 Boundary condition code for \( Z_{y} \) at nodal point L \( (w(15)) \)

61-63 Boundary condition code for \( Z_{xy} \) at nodal point L \( (w(16)) \)

The 'ID' array at I, J, K and L is assigned for each element for a specific mesh of the elements of the panel. The permutation for I, J, K and L is clockwise and I is corresponding to the origin of each local element as shown in Fig. 2. If ID(I) = 0, all the degrees of freedom at node I are zero. Same situation is applied for other nodes for the element.

The 'w' array defines the boundary condition codes for the degrees of freedom at each node. If w(I) = 0, the associated degree of freedom is zero and if w(I) \neq 0, the associated degree of freedom is taken into account.
Input 'Control Card' and Generating Parameters

Subroutine CHECK

Generating Aerodynamic Influence Coefficients

Input Data Deck of 'The Element Matrix Cards'

Generating Element Stiffness, Mass, and Incremental Stiffness Matrices

Input 'Element Cards with the boundary condition code at each node'

Assembling the Stiffness, Mass and Incremental Stiffness Matrices

Program Flow Chart
Subroutine ACRO

Computing Velocity Potentials for various d.o.f.

Generating Master Aerodynamic Matrices

Formulating the Eigenvalue Program

Set values for Mass Ratios

List Eigenvalues and Eigenvectors

STOP

Program Flow Chart continued ..............
PROGRAM LISTING AND SAMPLE PROBLEM

PROGRAM FLUTTER (INPUT=OUTPUT,THRES=INPUT,THRES=OUTPUT)
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF
CALL flux cc(1,18)=1:10=RT+CF

Panel flutter for clamped panel

The D.O.F. # for each nodal point: on the element

NOD#-X, Y, Z, IX, IY, IZ

Panel parameters

Panel: 0.32

The first frequency parameter for clamped panel

K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70
K=0.45, 70

Reference parameter

K=0.5
K=0.5
K=0.5
K=0.5
K=0.5
K=0.5
K=0.5
K=0.5
K=0.5
K=0.5
K=0.5
K=0.5

Initial in-plane parameter

FLUX
F=0.1
PRINT 103, F

FLOW PARAMETERS
PRINT 103, Mach
BETA=SOFT(Mach, **2, -1.)
OMEGAO<(Mach**2-KL**(2L))/BETA**2)
MACH=OMEGAO**2-MACH
USE=OMEGAO**2-MACH
C=0.1**2/3**14.159

GENERATING AERODYNAMIC INFLUENCE COEFFICIENTS

PHASE=P1
SIPH=2*MACH
DO 101 I=1, RMAX
DO 101 J=1, RMAX
GRCP(I, J)=0.
101 CONTINUE

GENERATING ELEMENT STIFFNESS MATRIX, ELEMENT INCREMENTAL STIFFNESS

STIFFNESS MATRIX AND ELEMENT MASS MATRIX

DO 103 I=1, L6
DO 103 J=1, L6
PMAX(L6)=JS(L6)+12**1+TV*(JM(L6)+L6+1)+JR(L6)+L6+1)
Q0=J=1, S4**2*J=1, S6**2*J=1, S8**2*PD=J=1, S10**2*PO=J=1, S12**2
J=1, S2**2*J=1, S4**2*J=1, S6**2*J=1, S8**2*PD=J=1, S10**2*PO=J=1, S12**2

ASSEMBLY ELEMENT STIFFNESS MATRIX, ELEMENT INCREMENTAL STIFFNESS

MATRIX AND ELEMENT MASS MATRIX

72
CALL ASSEMB (NM+NY+NDY)

GENERATING VELOCITY POTENTIAL FUNCTION

CALL RLAG (NXY+NDY+1+IV+V+VS+LS+VS+SB+DA+JG+NDY+NY)

GENERATING MASTER AERODYNAMIC MATRIX

DO 120 I=1,NDY
   DO 122 J=1,NDY
      MUL=4
      IF (JY.EQ.NDY) NDY=2
      DO 121 K=1,NDY
         I=II+1+NDY
         II=(IY-1)*4+NDY
         IF (IY.EQ.NDY) II=(IY-1)*4+NDY+(JY-1)*2+NDY
      DO 120 JY=1+NDY
      DO 122 JY=1+NDY
      NDY=2
      IF (JY.EQ.NDY) NDY=2
      DO 126 NDY=1+NDY
      D=JY*(JY-1)*4+NDY*(JY-1)*4+NDY
      IF (JY.EQ.NDY) D=JY*(JY-1)*4+NDY*(JY-1)*2+NDY
      G0=0.
      DO 118 NL=1+100
         H=NL-1
         IF (HL.LT.0) GO TO 118
         IF (HL.GT.1) GO TO 119
         DO 117 K=1,HL+1
            H2=H2-1
            IF (H2.LT.0) GO TO 117
            IF (H2.GT.1) GO TO 119
            IF (HL.EQ.0) GO TO 105
            IF (H2.EQ.0) GO TO 106
            IPL=H2*(NDY)
            GO TO 105
         105
         IF (H2.EQ.0) GO TO 107
         IPL=H2*(NDY)
         GO TO 106
         106
         IPL=H2*(NDY)
         GO TO 103
         107
         IPL=H2*(NDY)
         GO TO 100
         108
         DO 115 IK=1,IK
            DO 115 I=1,IK
            ML=(NL-2-L)*IRY+NS
            NR=IRY-L*M+M
            XFLG=FLG*(N-1)+*4+REV*2
            XH=FLG*(N-1)+*4+REV*2
            CALL SHARE (FLG+IF+II+MDY+EA)
            IF (JY.EQ.NDY) GO TO 110
            U=4*(JY-1)*4+NDY*(JY-1)*2+NDY
            IF (JY.EQ.NDY) GO TO 110
            109
         109
            IF (JY.EQ.NDY) GO TO 110
            GO TO 111
            110
            VR=0.
            73
GO TO 114

IF (X-J) I=10, 114, 113

J2=J+2*(NDK+1)+((CN+NDK-2)-J2)*4*(NDX+1)

GO TO 114

112

V=VJ+JMM+VJ +JMM

GO TO 114

113

V=VJ+JMM

GO TO 114

114

C-=C+FV*DFA*(KL*V*2)*(SIGMA*(EPIS**2))

GO TO 114

115

CONTINUE

116

CONTINUE

117

CONTINUE

118

CONTINUE

119

AK(J,J) = 0

120

CONTINUE

121

CONTINUE

122

CONTINUE

123

CONTINUE

FINDING EIGENVALUES AND EIGENVECTORS FOR VARIOUS RATIOS OF AIR MASS TO PANEL MASS

V=AV*2

TO 124 I=1, 160

TO 124 J=1, 160

124

(CM*J)=0, 0.01 + (CM*J)+ F*X*CM*J)

CALL GIMP2 (C1H, C2, C3, V, 16, 16, 16, 16)

IF (X-J) I=10, 125, 124

DF=1 + 1.00 + FACTOR

DO 125 J=1, 160

DO 125 J=1, 160

125 CONTINUE

DO 126 I=1, 160

126 CONTINUE

CONTINUE

DO 128 I=1, 160

74
DO 138 J=1,NLN
  IC=0.
  B=0.
  DO 125 I=1,NLH
    W(I,J)=K(I,J)*((MK(I,J)-MK(J,I))
          +E(I,J)*G(K(I,J)))
    W(I,J)=G(K(I,J))*((JG(J,I))
          +E(I,J)*G(K(J,I))))
  CONTINUE
  C(I,J)=HNLH(I,J)*ES
127 CONTINUE
CALL COUTG (NLH+IC+CS+IND+HS)
DO 125 I=1,NLH
  C(I,HNLH(I,J)*0.0)
  IC=IC+1
  CONTINUE
  CONTINUE
  CONTINUE
  CONTINUE
129 PRINT 142, IC(NLH)+G+SP+1A
PRED-NP
130 CONTINUE
GO TO 132
131 PRINT 145
132 STOP
CALL COUTG (NLH+IC+CS+IND+HS)
133 FORMAT (2NH1, PANEL FLUTTER FOR CLAMPED PANEL)
134 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
135 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
136 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
137 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
138 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
139 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
140 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
141 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
142 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
143 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
144 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
145 FORMAT (2H1, PANEL FLUTTER FOR CLAMPED PANEL)
  END
  END
SUBROUTINE COUTG (NLH+IC+CS+IND+HS)
  CONTINUE
CALL GIPS (1K+AI+MI+2, YG2)
GO (*K)= YG3+YG2

112 CONTINUE
RETURN
113 IF (*E+FO+0) GO TO 120
IF (*DVEC+GT, (BI+0)+LETAD) GO TO 114
GO TO 116
114 IF (*E+FO+0) GO TO 114
CALL GIPA (1K+AI+MI+4, YG)
GO (*K)= YG
115 CONTINUE
RETURN
116 IF (*E+FO+0, (BI+0)+LETAD) GO TO 118
DO 117 E=1,2
CALL GIPA (1K+0.1, (E+34+YG3)
GO (*K.+2, *YG)
117 CONTINUE
RETURN
118 DO 119 E=1,2
CALL GIPA (1K+AI+1.1+YG)
CALL GIPA (1K+0.1, (E+34+YG3)
GO (*K.+2, *YG)
119 CONTINUE
RETURN
120 IF (*E+FO, (BI+0)+LETAD) GO TO 122
DO 121 E=1,2
CALL GIPA (1K+0.1, (E+34+YG)
GO (*K)= YG
121 CONTINUE
RETURN
122 DO 123 E=1,2
CALL GIPA (1K+0.1, (E+34+YG)
CALL GIPA (1K+0.1, (E+34+YG3)
GO (*K.+2, *YG)
123 CONTINUE
RETURN

END
FUNCTION FY(I-K, DELTA, YL, YU)
RETURN

CEND

SUBROUTINE GIP (IK, YL, YH, XU, YG)
RETURN

CEND

FUNCTION FY(I-K, DELTA, YL, YU)
RETURN

CEND
SUBROUTINE RESJ (N+1,BJ,D+IER)
  B J=0
  IF (D) 101+102

101 IEP=1
  RETURN
102 IF (D) 103+104
103 IEP=2
  RETURN
104 IF (D<15.) 105+106
105 HTEST0-0.+X.*X.*X.*X.*X.
  GO TO 107
106 HTEST10+.123+456+789+100
  GO TO 107
107 IF (H-HTEST) 109+108
108 IEP=4
  PLO=H

109 HLP=0
  HH=1
  EPREV=0

C CONFIDE STARTING VALUE OF M

C IF (H-5.) 110+111+111

110 MP=100.
  GO TO 112
111 MP=1.1*10.4
  GO TO 112
112 M0=1.1*10.4+1.1*10.4+1.1*10.4
  MP=MP+M0

C SET UPPER LIMIT OF M

C MP=HTEST
  GO TO 121

C SET FMH,F(M-1)

C FMH=1.0*2A
  FMH=0
  IF (M-1) 114+113+114
113 JT=1
  GO TO 115
114 JT=1
  GO TO 115

C MP=M2
  GO TO 118+117
115 M2=1.1+2
  GO TO 118+117
116 M2=1.1+M2
  MP=M2
  IF (M-1) 117+116+117
117 JT=JT+2
  IF (M-1) 116+117
118 MP=FM/10.4+FM*
  GO TO 117
119 JT=JT+2
  IF (M-1) 116+117
120 IEP=IEP+100.
  IF (IIEP<0) 122
121 EPREV=BJ
  IER=5

C 122 RETURN

END
SUB(GO)
INC SWAP (FA,CA;IK1;Y)BA
K 10
F108=1,132*(2*3)-5,*(2*2)
K 20
F108=2*(2-1);*(2*2)
K 30
F108=30,*(2*2)-0,8
K 40
F108=3,*(2*2)-4.8;1.
K 50
F108=3,*(2*2)-2.8
K 60
IF (IK,EO,0) GO TO 104
K 70
IF (IK,EO,0) GO TO 104
K 80
IF (IK,EO,1,DP,IK,EO,3,DP,IK,EO,5,DP,IK,EO,7) GO TO 101
K 90
IF (IK,EO,1,DP,IK,EO,3,DP,IK,EO,5,DP,IK,EO,7) GO TO 101
K 100
IF (IK,EO,3,DP,IK,EO,4,DP,IK,EO,6,DP,IK,EO,9) GO TO 102
K 110
IF (IK,EO,10,DP,IK,EO,12,DP,IK,EO,14,DP,IK,EO,16) GO TO 104
K 120
101 n=DF1(0)
K 130
CC=DF1(0)
K 140
GO TO 105
K 150
102n=DF2(0)
K 160
CC=DF2(0)
K 170
GO TO 105
K 180
103 n=n+1,DF1(0)
K 190
CC=DF1(0)
K 200
GO TO 105
K 210
104 n=n+1,DF2(0)
K 220
CC=DF2(0)
K 230
GO TO 105
K 240
105 IF (IK,EO,1,DP,IK,EO,2,DP,IK,EO,3,DP,IK,EO,14) BE=DF1(0)
K 250
IF (IK,EO,1,DP,IK,EO,2,DP,IK,EO,3,DP,IK,EO,14) BE=DF1(0)
K 260
IF (IK,EO,3,DP,IK,EO,4,DP,IK,EO,5,DP,IK,EO,16) BE=DF2(0)*BA
K 270
IF (IK,EO,3,DP,IK,EO,4,DP,IK,EO,5,DP,IK,EO,16) BE=DF2(0)*BA
K 280
IF (IK,EO,7,DP,IK,EO,8,DP,IK,EO,11,DP,IK,EO,12) BE=DF4(0)*EA
K 290
IF (IK,EO,7,DP,IK,EO,8,DP,IK,EO,11,DP,IK,EO,12) BE=DF4(0)*EA
K 300
PARA=0,
K 310
DFA=0,
K 320
RETURN
K 330
C
K 340
END
K 350
Panel flutter for clamped panel

Length-Width Ratio = 1.000
Chordwise Element Length = 1.000
Spanwise Element Width = 1.000
Box Width = .500
Ratio of Box Length to Box Width = .500
Reduced Frequency = .500
Initial In-Plane Force Parameter = .100
Mach Number = 1.20
Structural Damping Coefficient = 0.000

Density Ratio Mu = .0150

Eigenvalue = 5.097E-01 - 1.442E-02
G = -.08829
Frequency Ratio = .71404
Stiffness Parameter = .95702

Eigenvector
6.265E-01 -4.834E-03 9.239E-01 -1.192E-02

Eigenvalue = 1.3142E-01 5.670E-04
G = .00471
Frequency Ratio = .36282
Stiffness Parameter = .18185

Eigenvector
-1.403E-01 -1.275E-02 1.371E-01 -5.065E-03 -1.947E-03 -1.156E-02 -1.163E-02 -8.304E-03 8.969E-03 4.734E-03 4.734E-03 2.770E-02 7.026E-02 -1.607E-02 -6.516E-03 -2.874E-03 3.533E-01 -1.614E-01 6.681E-03 -1.874E-03 5.811E-01 -1.938E-02 6.342E-02 -1.848E-02 1.330E-01 -2.320E-02 1.783E-02 -1.401E-02
6.440E-02 -1.430E-02 1.409E-02 -2.781E-02 -5.560E-02 -7.186E-02

Eigenvalue = 7.189E-02 -7.137E-04
G = -.00993
Frequency Ratio = .28287
Stiffness Parameter = .13463

Eigenvector
1.423E-02 -1.939E-02 -5.211E-02 -6.549E-02 -3.202E-02 4.104E-02

Eigenvalue = 5.163E-02 -1.921E-03
G = .03722
Frequency Ratio = .22872
Stiffness Parameter = .11362

Eigenvector
-1.555E-01 1.681E-01 -7.064E-01 -1.437E-01 -5.323E-02 -1.365E-01
<p>| EIGENVALUE | 4.9998E-02 -8.7809E-03 | G | = | 0.0500 | FREQUENCY RATIO | = | 0.22165 | STIFFNESS PARAMETER | = | 0.11053 |
| DENSITY RATIO Mu | = | 0.0250 |
| EIGENVALUE | 5.3596E-01 -2.1646E-02 | G | = | 0.0402 | FREQUENCY RATIO | = | 0.73017 | STIFFNESS PARAMETER | = | 0.36508 |
| EIGENVALUE | 1.3881E-02 1.0493E-03 | G | = | 0.00756 | FREQUENCY RATIO | = | 0.37258 | STIFFNESS PARAMETER | = | 0.19039 |
| EIGENVALUE | 7.6589E-02 -2.5325E-03 | G | = | 0.03307 | FREQUENCY RATIO | = | 0.27679 | STIFFNESS PARAMETER | = | 0.13839 |
| EIGENVALUE | 6.7926E-02 -2.2525E-03 | G | = | 0.04169 | FREQUENCY RATIO | = | 0.25092 | STIFFNESS PARAMETER | = | 0.12546 |
| EIGENVALUE | 5.6059E-02 -5.0903E-03 | G | = | 0.02701 | FREQUENCY RATIO | = | 0.23701 | STIFFNESS PARAMETER | = | 0.11851 |</p>
<table>
<thead>
<tr>
<th>EIGENVALUE</th>
<th>5.4934E+01</th>
<th>-2.7489E+00</th>
<th>G</th>
<th>-0.0502</th>
<th>FREQUENCY RATIO</th>
<th>0.7407</th>
<th>STIFFNESS PARAMETER</th>
<th>0.37036</th>
</tr>
</thead>
</table>

EIGENVECTOR

| 1.0984E+01 | 1.8092E+00 | 6.6266E+00 | 2.0149E+01 | 1.6305E+01 | 6.5795E+00 | 2.5098E+01 | 1.5959E+01 | 6.7754E+00 |

| 9.3652E-01 | 1.0572E+00 | 1.5869E+00 | 1.6305E+01 | 2.5098E+01 | 6.7754E+00 |

| 3.1113E-01 | 1.1201E-01 | 2.7232E-01 | 1.1620E+00 | 2.5098E+01 | 6.7754E+00 |

| 1.4746E-01 | 1.4478E-02 | 2.2752E-02 | 1.8680E+00 | 2.5098E+01 | 6.7754E+00 |

| 1.4669E+01 | 9.6772E+00 | 2.6603E+00 | 2.6603E+00 | 2.6603E+00 | 2.6603E+00 |

| 1.4505E+00 | 1.4505E+00 | 1.4505E+00 | 1.4505E+00 | 1.4505E+00 | 1.4505E+00 |

| 8.6747E-02 | 5.5556E-03 | 5.5556E-03 | 5.5556E-03 | 5.5556E-03 | 5.5556E-03 |

| 5.3556E+01 | 1.5849E+01 | 1.5849E+01 | 1.5849E+01 | 1.5849E+01 | 1.5849E+01 |

| 2.6532E+02 | 7.1758E+02 | 6.2072E+02 | 7.1758E+02 | 6.2072E+02 | 7.1758E+02 |

| 3.1902E+02 | 1.5849E+01 | 1.5849E+01 | 1.5849E+01 | 1.5849E+01 | 1.5849E+01 |

| 2.8556E+02 | 1.5574E+02 | 1.5574E+02 | 1.5574E+02 | 1.5574E+02 | 1.5574E+02 |

| 1.9762E+02 | 1.9762E+02 | 1.9762E+02 | 1.9762E+02 | 1.9762E+02 | 1.9762E+02 |

| 7.5929E+00 | 5.3278E+00 | 5.3278E+00 | 5.3278E+00 | 5.3278E+00 | 5.3278E+00 |

| 2.7854E+01 | 2.7854E+01 | 2.7854E+01 | 2.7854E+01 | 2.7854E+01 | 2.7854E+01 |

| 1.5120E+01 | 1.5120E+01 | 1.5120E+01 | 1.5120E+01 | 1.5120E+01 | 1.5120E+01 |

| 1.2373E+01 | 1.2373E+01 | 1.2373E+01 | 1.2373E+01 | 1.2373E+01 | 1.2373E+01 |

| 1.2573E+01 | 1.2573E+01 | 1.2573E+01 | 1.2573E+01 | 1.2573E+01 | 1.2573E+01 |

| 6.3250E+00 | 6.3250E+00 | 6.3250E+00 | 6.3250E+00 | 6.3250E+00 | 6.3250E+00 |


ENGINENUM
1.3046E+01 1.8880E-03 1.6686E-01 6.2526E-04 1.9415E-01 1.2649E-03 2.5913E-01 4.3655E-03 2.9742E-01 4.0038E-03
9.7547E-02 1.0878E-02 4.0435E-01 2.8635E-03 4.7902E-02 8.8934E-02 2.8428E-02 1.2451E-02 2.1578E-01 1.4214E-03
3.1155E-01 2.0067E-02 3.2327E-01 1.1676E-02 2.5000E-01 0.0 3.5000E-01 3.3692E-03 5.1078E-01 1.0192E-02
1.4746E-01 1.4458E-02 4.2758E-02 1.8053E-02 2.0634E-01 6.7559E-03

EIGENVALUE = 1.4669E-01 9.6772E-04 G = -0.0502 FREQUENCY RATIO = 0.3630 STIFFNESS PARAMETER = 0.1915

EIGENVALUE = -1.0041E-01 -4.3428E-03 -1.3739E-03 4.1174E-03 -1.0478E-01 -6.1935E-02 7.0736E-02 -8.0361E-02 1.7017E-02 1.2596E-02
5.6165E-01 1.6593E-02 2.4345E-01 5.6405E-02 9.6973E-01 9.1512E-03 7.0393E-01 3.1745E-02 2.5038E-01 7.6116E-02
1.1958E+00 1.9572E-01 -5.1618E-03 -7.6897E-02 -1.3948E-01 -2.6349E-02 -1.2664E-01 -2.5844E-02 2.4304E-01 -1.4767E-02
1.6755E-00 7.0603E-02 1.4262E+00 1.2668E-01 2.6697E-01 6.9861E-02

EIGENVALUE = 9.6747E-02 -5.5556E-03 G = -0.0645 FREQUENCY RATIO = 0.2947 STIFFNESS PARAMETER = 0.1473

-2.8428E-02 -1.7433E-02 8.4513E-02 -1.9366E-02 1.8959E-01 -2.9293E-02 1.0245E-01 -8.6143E-04 2.8111E-01 1.4797E-02
4.7256E-01 -9.3511E-02 2.6135E-01 6.1333E-03 3.9259E-02 2.1324E-03 2.8703E-02 -2.6701E-03 6.8876E-02 -2.6023E-03
1.6305E-01 4.3374E-02 5.1328E-01 -2.7740E-02 3.5496E-02 2.2751E-02

EIGENVALUE = 7.2016E-02 3.5303E-00 G = -0.0504 FREQUENCY RATIO = 0.2884 STIFFNESS PARAMETER = 0.1342

EIGENVALUE = 2.0728E-01 7.7936E-03 2.8470E-01 2.1582E-02 -0.9936E-02 2.5615E-02 -2.5625E-02 9.9027E-01 2.6039E-01 1.4959E-01
1.4916E+00 -5.3363E-01 2.7257E-02 -4.7564E-02 -5.4301E-02 2.0130E-02 -1.6269E-01 -1.6269E-02 4.5778E-02 -1.5712E-01
-2.3002E-02 -2.3002E-02 1.5300E+00 -2.7257E-01 2.5746E+00 4.9414E-04

EIGENVALUE = 6.1903E-02 -6.2573E-03 G = -0.1018 FREQUENCY RATIO = 0.2491 STIFFNESS PARAMETER = 0.1245

EIGENVALUE = -3.2134E-03 9.0454E-02 9.1000E-03 1.5390E-02 1.3183E-02 1.8258E-02 -2.3026E-02 2.0130E-02 -1.9915E-02 1.0792E-01
-1.5436E+00 4.2154E-01 4.1955E+00 1.2237E+00 1.8821E+00 -7.2547E-02 2.9426E-02 3.9408E-02 8.2037E-02 2.5396E-02
-6.1753E-01 2.5301E-02 1.3832E+00 6.2108E+00 1.0966E+00 1.5277E+00

DENSITY RATIO MU = 0.0500

EIGENVALUE = 5.4346E-01 -3.1875E-02 G = -0.0544 FREQUENCY RATIO = 0.7452 STIFFNESS PARAMETER = 0.3726

DENSITY MU = 0.0450

EIGENVALUE = 1.2737E-01 2.3692E-03 1.9538E-01 1.5673E-03 1.9611E-01 -1.5419E-02 2.7637E-01 3.4722E-03 3.2057E-01 3.7246E-03
1.9438E-01 1.2543E-02 4.4732E-01 1.1853E-02 8.4123E-02 1.2417E-02 2.8924E-01 1.5052E-02 2.3116E-01 8.5591E-03
1.3425E-01 1.3105E-02 -2.5102E-02 1.2017E-02 1.3309E-02 2.5000E-02 0.0 1.6756E+00 -2.9107E-01 3.1509E-01
2.0738E-01 1.7157E-02 5.0198E-01 -2.2108E-02 -4.20157E-01 6.2768E-03
| EIGENVALUE | $4.52993E-02$ | $2.4715$ | $0.67832$ | $2.39916$ |
| EIGENVECTOR | $9.49792E-02$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ |
| $1.17894E-01$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ |
| $5.14393E-02$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ |
| $2.27335E-02$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ |
| $1.06935E-02$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ | $8.24827E-02$ |

**DECEMBER RATIO MU**

- 0.0500

| EIGENVALUE | $4.53230E-01$ | $1.12030E-01$ | $1.24715$ | $0.67832$ | $0.39916$ |
| EIGENVECTOR | $1.13344E-01$ | $4.51439E-02$ | $7.87673E-02$ | $2.4715$ | $0.67832$ |
| $1.72743E-01$ | $1.83935E-01$ | $1.83935E-01$ | $1.83935E-01$ | $1.83935E-01$ |

**EIGENVALUE**

- $4.52993E-02$
- $2.4715$
- $0.67832$
- $0.39916$

**FREQUENCY RATIO**

- $9.49792E-02$
- $8.24827E-02$
- $8.24827E-02$
- $8.24827E-02$

**STIFFNESS PARAMETER**

- $1.17894E-01$
- $8.24827E-02$
- $8.24827E-02$
- $8.24827E-02$

**DENSITY RATIO MU**

- 0.0500

**EIGENVALUE**

- $4.53230E-01$
- $1.12030E-01$

**FREQUENCY RATIO**

- $1.13344E-01$
- $4.51439E-02$
- $7.87673E-02$
- $2.4715$

**STIFFNESS PARAMETER**

- $1.72743E-01$
- $1.83935E-01$
- $1.83935E-01$
- $1.83935E-01$

**DENSITY RATIO MU**

- 0.0500