Planar Multiple-Valued Decision Diagrams

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Abstract

In VLSI, crossings occupy space and cause delay. Therefore, there is significant benefit to planar circuits. We propose the use of planar multiple-valued decision diagrams to produce planar multiple-valued circuits. Specifically, we show conditions on 1) threshold functions, 2) symmetric functions, and 3) monotone increasing functions that produce planar decision diagrams. Our results apply to binary functions, as well. For example, we show that all two-valued monotone increasing threshold functions of up to five variables have planar binary decision diagrams.

Index terms: binary decision diagram (BDD), dual function, threshold function, field programmable gate array (FPGA).

1 Introduction

Multiple-valued decision diagrams (MDDs) are multiple-valued extensions of binary decision diagrams (BDDs). MDDs are useful for designing multiple-valued logic networks; by replacing each node of an MDD with a multiple-valued multiplexer (MUX), we have a multiple-valued network for the function.

Fig. 1.1 shows a BDD for \( f = x_1 \overline{x}_2 x_3 x_4 \). This BDD has no crossing, which we denote as \( r \)-planar. Fig. 1.2 shows a BDD for the same function with a different ordering of the input variables. In this case, however, the BDD is not \( r \)-planar, since it has crossings. We say a function has an \( r \)-planar BDD if we can draw a planar BDD in a restricted form:

Definition 1.1 A BDD in which
1. a 1-edge emerges to the right of the node,
2. a 0-edge emerges to the left, and
3. the constant 1 node is to the left of the constant 0 node
is \( r \)-planar (restricted-planar) if it has no crossings.

\( r \)-planar graphs are special case of planar graphs. Fig. 1.3 shows a planar BDD that is isomorphic to the BDD in Fig. 1.2, which is not an \( r \)-planar BDD. Fig. 1.4 shows a network for \( f = x_1 x_2 \lor x_3 x_4 \). It corresponds to the BDD in Fig. 1.1, where each node in the BDD is replaced with a binary MUX. Note that this network has no crossings if we ignore the lines for the input variables. Fig. 1.5 is a network that corresponds to the BDD in Fig. 1.2. In this case, the network has crossings. When we implement networks in the form of LSIs, crossings are often expensive; they require additional channels and increase delay. Especially in the case of field programmable gate arrays (FPGAs) [2], crossings produce considerable delay. Since the delay of interconnections is the most important problem in...
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In VLSI, crossings occupy space and cause delay. Therefore, there is significant benefit to planar circuits. We propose the use of planar multiple-valued decision diagrams to produce planar multiple-valued circuits. Specifically, we show conditions on 1) threshold functions, 2 symmetric functions, and 3) monodiagrams. Our results apply to binary functions, as well. For example, we show that all two-valued monotone increasing threshold functions of up to five variables have planar binary decision diagrams.
FPGA design, networks without crossing are quite attractive. Also, in sub-micron LSIs, networks without crossings are desirable, since the delays in the interconnections and crossing are comparable to the delay for logic elements.

In this paper, we identify classes of logic functions whose MDDs and BDDs are r-planar. For these functions, we can easily derive logic networks whose layouts are relatively simple. Initially, we consider unrestricted MDDs and BDDs. Subsequently, we consider reduced ordered MDDs and BDDs that do not contain redundant nodes nor nodes representing the same function.

2 r-Planar MDDs

In this section, we define multiple-valued input two-valued output functions [11]. Then, we show some classes of functions having r-planar MDDs. These results will be used for the identification of functions having r-planar BDDs in Section 3. As for the definitions for BDDs and MDDs, refer to [1, 3, 7].

Definition 2.1 A multiple-valued input two-valued output function is

\[ g(x_1, x_2, \ldots, x_n) : \prod_{i=1}^{n} P_i \rightarrow B, \]

where \( x_i \) assumes a value in \( P_i = \{0, 1, \ldots, p_i - 1\} \) and \( B = \{0, 1\} \).

Definition 2.2 Let \( x_i \) be a variable taking values in \( P_i = \{0, 1, \ldots, p_i - 1\} \). Let \( S_i \) be a subset of \( P_i \). Then, \( x_i^{S_i} \) is a literal of \( S_i \), where \( x_i^{S_i} = 1 \) if \( x_i \in S_i \), and \( x_i^{S_i} = 0 \) otherwise. When \( S_i \) contains only one element \( a \in P_i \), \( x_i^{\{a\}} \) is written as \( x_i^a \).

Lemma 2.1 A multiple-valued input two-valued output function \( f(x_1, x_2, \ldots, x_n) \) can be represented by an expression

\[ f(x_1, x_2, \ldots, x_n) = \bigvee_{(S_1, S_2, \ldots, S_n)} x_1^{S_1} x_2^{S_2} \ldots x_n^{S_n}, \]

where \( \lor \) is OR and concatenation is AND.

Figure 1.4: An MUX network corresponding to Fig. 1.1.

Figure 1.5: An MUX network corresponding to Fig. 1.2.

A multiple-valued multiplexer (MUX), shown in Fig. 2.1, selects one terminal according to the value of \( z \), where \( z \in \{0, 1, \ldots, p - 1\} \). The function of the MUX is represented by

\[ f(z) = \bigvee_{\ell=0}^{n} f_{\ell} \lor z^{1} f_{1} \lor \cdots \lor z^{p-1} f_{p-1}. \]

Lemma 2.2 The tree network of MUXs shown in Fig. 2.2 realizes an arbitrary multiple-valued input two-valued output function.

Definition 2.3 Let \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) be vectors such that \( a_i, b_i \in \{0, 1, \ldots, p_i - 1\} \). We define a binary relation \( \preceq \) between vectors as follows: \( a \preceq b \) if \( a \) appears before \( b \) in lexicographical order.

For example, \((0, 0, 0) \preceq (0, 0, 1)\), and \((0, 1, 1) \preceq (1, 0, 0)\).

Definition 2.4 A function \( f(x) \) is l-monotonic (lexicographically monotonic) iff the following conditions hold: For vectors \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \), such that \( a_i, b_i \in \{0, 1, \ldots, p_i - 1\} \), \( a \preceq b \) implies \( f(a) \leq f(b) \), where the logic values are viewed as integers. \( f(X) \leq g(X) \) iff \( f(a) \leq g(a) \) for any \( a \).

Lemma 2.3 Suppose that a function \( f \) is l-monotonic. Let \( X_1 = (x_1, x_2, \ldots, x_t) \), and \( X_2 = (x_{t+1}, x_{t+2}, \ldots, x_n) \) be a partition of \( X = (x_1, x_2, \ldots, x_n) \). Then, \( f(a, X_2) \leq f(b, X_2) \) for any \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) such that \( a \preceq b \).
Definition 2.5 A complete MDD is an MDD that has a distinct node for every assignment of values to the variables. That is, no two nodes are merged.

Definition 2.6 Let f be a p-valued input two-valued output function. An MDD for f in which
1. an i-edge emerges to the right of an (i - 1)-edge, (1 <= i <= p - 1), and
2. the constant 1 node is to the left of the constant 0 node
is r-planar (restricted-planar) if it has no crossings.

Lemma 2.4 An l-monotonic function has an r-planar complete MDD.

Definition 2.7 A reduced ordered multiple-valued decision diagram (ROMDD) is an MDD where
1. two nodes are merged into one node if they represent the same function, and
2. a node η is removed if all the children of η represent the same function.

Lemma 2.5 An l-monotonic function has an r-planar ROMDD.

(Proof) Consider a complete MDD of function f, as shown in Fig. 2.2. Because f is l-monotonic, by Lemma 2.3, if a <= b then f(a, Xp) <= f(b, Xp). The functions represented by the nodes at the same level are totally ordered. In the lowest level, they are constant 0 or 1. From Lemma 2.4, the complete MDD for f is r-planar. Now, reduce the complete MDD into an ROMDD.

First, merge two nodes that represent same logic function. We show that the resulting MDD is also r-planar. Suppose that a, b, c, d, and e are nodes in the same level, where a <= b <= c <= d <= e. Also, suppose that b and d have the property,

f(b, X2) = f(d, X2).

(2.1)

Fig. 2.3(a) shows the situation. Because f is l-monotonic, we have

f(b, X2) <= f(c, X2) <= f(d, X2).

(2.2)

From (2.1) and (2.2), we have

f(b, X2) = f(c, X2) = f(d, X2).

This shows that the sub-tree for c also represents the same function as b and d. Thus, these three subtrees can be merged into one as shown in Fig. 2.3(b). Note that this operation does not introduce a crossing. It follows that merging two nodes that represent the same function preserves r-planarity. Also, it is clear that the reduction of redundant nodes preserves r-planarity. Hence, we have the lemma. (Q.E.D.)
Example 2.3 Consider the two-valued input function: \( f = x_1 \lor x_2 (x_3 \lor x_4) \). Note that \( f \) is a threshold function with the characteristic vector \((w_1, w_2, w_3, w_4 : T) = (5, 3, 1, 1 : 4)\). This vector satisfies the condition of Theorem 2.1. So, the function with the ordering \((x_1, x_2, x_3, x_4)\) has an r-planar ROBDD, as shown in Fig. 2.6(a). A different ordering \((x_4, x_1, x_3, x_2)\) produces a non r-planar ROBDD, as shown in Fig. 2.6(b). (End of Example)

Theorem 2.2 Suppose that a multiple-valued input two-valued output function \( f \) can be represented as

\[
f = X^A \cdot g \quad \text{or} \quad f = X^A \lor g,
\]

where \( X \) takes a value in \( P = \{0, 1, \ldots, p - 1\} \), \( A = \{a, a + 1, \ldots, p - 1\}, \) \( 1 \leq a \leq p - 1, \) and \( g \) does not depend on \( X. \) If \( g \) has an r-planar MDD, then \( f \) has an r-planar MDD.

(Proof) Fig. 2.7(a) and (b) show r-planar MDDs for \( f = X^A \cdot g \) and \( f = X^A \lor g, \) respectively. (Q.E.D.)

3 \ r-planar BDD

In this section, we consider the class of two-valued input two-valued output functions having r-planar ROBDDs. Here, for simplicity, function means two-valued input two-valued output function, unless otherwise noted.

Definition 3.1 A complete symmetric decision diagram (Fig. 3.1) is the decision diagram on variables \( x_1, x_2, \ldots, x_n \) that has \( n + 1 \) leaf nodes \( v_0, v_1, \ldots, v_n, \) such that \( v_i \) can be reached by only an assignment of values to \( X = (x_1, x_2, \ldots, x_n) \) whose weight (number of 1's) is \( i. \)

Fig. 3.2(a), (b) and (c) show the complete symmetric decision diagrams for \( n = 1, 2 \) and 3, respectively. Note that they are planar, and, in general, we have the following:

Lemma 3.1 A complete symmetric decision diagram has an r-planar ROBDD.

Definition 3.2 A voting function \( S_k(X) \) is a totally symmetric threshold function that can be represented as:

\[
S_k(X) = \begin{cases} 
1 & \text{if } ||X|| \geq k \\
0 & \text{otherwise},
\end{cases}
\]

where \( ||X|| \) represents the weight (number of 1's) in the inputs \( X. \)
Lemma 3.2 A voting function has an r-planar ROBDD.

(Proof) An ROBDD for an n variable voting function is derived from the complete symmetric decision diagram for n variables by assigning 0 to leaf nodes \( v_0 \) to \( v_i \), and 1 to leaf nodes \( \neg i+1 \) through \( \neg n \). Reduction operations (e.g. merging \( v_0 \) through \( \neg i \), and \( \neg i+1 \) through \( \neg n \)) preserves r-planarity. (Q.E.D.)

Example 3.1 Fig. 3.3 shows the construction described in the proof for \( n = 3 \). (End of Example)

Definition 3.3 Let \( X = (X_1, X_2, \ldots, X_r) \) be a partition of \( X = (x_1, x_2, \ldots, x_n) \). A function \( f \) is partially symmetric with respect to \( X_i (i = 1, 2, \ldots, r) \) if \( f \) is invariant under any permutation of the variables in \( X_i \).

Lemma 3.3 Let \( f \) be a partially symmetric function with respect to \( X_i \), where \( X_i \) contains \( n_i \) variables \( (i = 1, 2, \ldots, r) \). Then, \( f \) is represented by a multiple-valued input two-valued output function \( g(Y_1, Y_2, \ldots, Y_n) \), where \( Y_i \) takes one of \( n_i + 1 \) values representing the number of 1’s in \( X_i \).

Definition 3.4 The multiple-valued input two-valued output function \( g \) that corresponds to the partially symmetric function \( f \) in Lemma 3.3, is called a companion function of \( f \).

Theorem 3.1 A partially symmetric function has an r-planar ROBDD if the companion function has an r-planar ROMDD.

(Proof) Suppose that the r-planar MDD for the companion function \( g \) is given. By replacing each node of the MDD with a complete symmetric decision diagram, we can make a BDD for the partially symmetric function \( f \). By Lemma 3.1, the complete symmetric decision diagram is an r-planar BDD. Thus, the BDD for \( f \) is also r-planar.

Example 3.2 \( f = (x_1 \lor x_2)(x_3 x_4 \lor x_5 x_6) \) is partially symmetric with respect to \( X_1 = (x_1, x_2) \), \( X_2 = (x_3, x_4) \) and \( X_3 = (x_5, x_6) \). Let

\[
Y_i = 0 \quad \text{if} \quad X_i = (0, 0) \\
Y_i = 1 \quad \text{if} \quad X_i = (0, 1) \quad \text{or} \quad X_i = (1, 0), \quad \text{and} \\
Y_i = 2 \quad \text{if} \quad X_i = (1, 1).
\]

Then, the companion function \( g \) is represented by

\[
g(Y_1, Y_2, Y_3) = Y_1^{[1, 2]} \lor Y_2^{[2]} \lor Y_3^{[2]},
\]

By Theorem 2.2, we can see that \( g \) has an r-planar MDD. Fig. 3.4(a) shows the r-planar MDD for \( g \). By replacing each node with an r-planar BDD, we have an r-planar BDD for \( f \), as shown in Fig. 3.4(b). Note that \( f \) is not a threshold function. Also, note that companion functions can be generated iteratively. For example, (3.1) can be written as

\[
h(Y_1, Z_1) = Y_1^{[1, 2]} \lor Z_1^{[1, 2]},
\]

where

\[
Z_1^{[1, 2]} = Y_2^{[2]} \lor Y_3^{[2]}.
\]

In this way, companion functions can be constructed from other companion functions. (End of Example)

Lemma 3.4 A function \( f \) has an r-planar ROBDD iff \( f^d \) has an r-planar ROBDD, where \( f^d \) is the dual function of \( f \).

(Proof) Suppose that \( f \) has an r-planar ROBDD. In the BDD, for each node, interchange the 0-edge and 1-edge. Also, interchange the constant 0 and the constant 1. Then, the resulting ROBDD represents \( f^d \) and it is also r-planar. (Q.E.D.)
Figure 3.5: \(r\)-planar BDDs for voting functions.

Figure 3.6: Generation of \(r\)-planar BDDs for voting functions.

Lemma 3.5 Let \(S_k(X)\) be a voting function such that
\[
S_k(X) = \begin{cases} 
1 & \text{if } ||X|| \geq k \\
0 & \text{otherwise.}
\end{cases}
\]
There exists an \(r\)-planar ROBDD that produces \(S_0(X), S_1(X), \ldots, \text{and } S_n(X),\) simultaneously.

(Proof) For \(n = 1, 2, \text{and } 3,\) the voting functions are generated as shown in Fig. 3.5(a), (b), and (c), respectively. Assume that Fig. 3.6(a) is an \(r\)-planar ROBDD that generates all the voting functions of \(n\)-variables. Then, we can make an \(r\)-planar ROBDD that generates all the voting functions of \((n+1)\)-variables as shown in Fig. 3.6(b). (Q.E.D.)

Theorem 3.2 Suppose that \(X = (X_1, X_2)\) is a partition of variables \(X = (x_1, x_2, \ldots, x_n),\) where \(X_1 = (x_1, x_2, \ldots, x_k)\) and \(X_2 = (x_{k+1}, x_{k+2}, \ldots, x_n).\) Let \(\psi_i(X_1)\) be the symmetric function,
\[
\psi_i(X_1) = \begin{cases} 
1 & \text{if } ||X_1|| = i, \\
0 & \text{otherwise.}
\end{cases}
\]
Let \(S_k(X_2)\) be a voting function. If a function \(f\) can be represented as
\[
f(X_1, X_2) = \bigvee_{i=0}^{k} \psi_i(X_1) S_n(X_2),
\]
where \(S_n(X_2) \subseteq S_{n+1}(X_2),\) then \(f\) has an \(r\)-planar ROBDD.

(Proof) By Lemma 3.1, there is a planar BDD for \(\psi_i\) (a complete symmetric decision diagram). By Lemma 3.5, there is an \(r\)-planar BDD for \(S_k\) (a voting function). As shown in Fig. 3.7, consider the BDD where the upper block realizes \(\psi_i\)'s, and the lower block realizes \(S_k\)'s. By connecting appropriate terminals between two blocks, we have an \(r\)-planar BDD for the function \(f.\) (Q.E.D.)

Example 3.3 Consider the function \(f = (x_1 \oplus x_2) x_3 x_4 \lor x_1 x_2 (x_3 \lor x_4).\) f is partially symmetric with respect to \(X_1 = (x_1, x_2)\) and \(X_2 = (x_3, x_4).\) Note that \(f\) can be represented as \(f(X_1, X_2) = \psi_0(X_1) S_2(X_2) \lor \psi_1(X_1) S_1(X_2),\) where \(\psi_0(X_1) = x_1 x_2,\) \(\psi_1(X_1) = x_1 \oplus x_2,\) \(\psi_2(X_1) = x_1 x_2,\) \(S_0(X_2) = 0,\) \(S_1(X_2) = x_3 x_4,\) and \(S_2(X_2) = x_3 \lor x_4.\) Thus, by Theorem 3.2, \(f\) has an \(r\)-planar BDD.

(End of Example)

Corollary 3.1 A monotone increasing threshold function having at most two different weights has an \(r\)-planar BDD.

(Proof)
1) A monotone increasing threshold function \(f\) having only one weight is a voting function. Thus, by Lemma 3.2, \(f\) has an \(r\)-planar BDD.
Consider the threshold functions with characteristic vector \((\vec{w}, \vec{w}+1, \ldots, \vec{w}+t)\) for a function \(f\) has a characteristic vector \((w_1, w_2, \ldots, w_n : T)\), \(w_1 = 1\) and
\[
\sum_{r=1}^{n} w_r, \text{ and } \phi_i(X) \geq \phi_{i+1}(X).
\]
Then, both \(\psi_i(X) = \phi_i(X) \cdot \phi_{i+1}(X)\) (\(i = 1, 2, \ldots, t - 1\)) and \(\psi_t(X) = \phi_t(X)\) can be represented in an r-planar BDD.

Example 3.4 Consider the threshold functions with characteristic vector \((2,1,1:7)\). In this case,
\[
\begin{align*}
\phi_0(X) &= 1 \quad (T = 0) \\
\phi_1(X) &= x_1 \lor x_2 \lor x_3 \quad (T = 1) \\
\phi_2(X) &= x_1 \lor x_2 \lor x_3 \quad (T = 2) \\
\phi_3(X) &= x_1 x_2 x_3 \quad (T = 3) \\
\phi_4(X) &= x_1 x_2 x_3 \quad (T = 4).
\end{align*}
\]
Therefore,
\[
\begin{align*}
\psi_4(X) &= \phi_4(X) = x_1 x_2 x_3 \\
\psi_3(X) &= \phi_3(X) \cdot \phi_4(X) = x_1 (x_2 \lor x_3) \\
\psi_2(X) &= \phi_2(X) \cdot \phi_3(X) = x_1 x_2 x_3 \lor x_1 x_2 x_3 \\
\psi_1(X) &= \phi_1(X) \cdot \phi_2(X) = x_1 (x_2 \lor x_3) \\
\psi_0(X) &= \phi_0(X) \cdot \phi_1(X) = x_1 x_2 x_3.
\end{align*}
\]

Theorem 3.3 Suppose that \(X = (X_1, X_2)\) is a partition of variables \(X = (x_1, x_2, \ldots, x_n)\). If a function \(f\) can be represented as
\[
f(X_1, X_2) = \bigvee_{i=0}^{t} \psi_i(X_1) S_{a_i}(X_2),
\]
where \(S_{a_i}(X_2)\) is a symmetric function satisfying \(S_{a_i}(X_2) \leq S_{a_{i+1}}(X_2)\), and \(\psi_i (i = 1, 2, \ldots, t)\) is a function as defined in Lemma 3.6, then \(f\) has an r-planar BDD.

(Proof) We can prove this theorem in a similar way to Theorem 3.2. (Q.E.D.)

Corollary 3.2 Suppose that a monotone increasing threshold function \(f\) has a characteristic vector \((w_1, w_2, \ldots, w_k, w_{k+1}, \ldots, w_n : T)\), \(w_k = 1\),
\[
w_i \geq \sum_{j=i+1}^{k} w_j, \quad (i = 1, 2, \ldots, k - 1),
\]
and \(w_k = w_{k+1} = \cdots = w_n\). Then, \(f\) has an r-planar ROBDD.

(Proof) Note that \(f\) can be written in the form (3.3). Because \(f\) is monotone increasing, we can assume that \(S_{a_i}(X_2) \leq S_{a_{i+1}}(X_2)\). Thus, by Theorem 3.3, \(f\) has an r-planar ROBDD. (Q.E.D.)

Example 3.5 Consider the 5-variable function with the characteristic vector \((4,3,3,2,1:6)\). \(f\) is symmetric with respect to \(X_2 = (x_2, x_3)\). Also, the weights for \(X_1 = (x_1, x_4, x_5)\) satisfy the conditions of Lemma 3.6. Thus, \(f\) can be represented as
\[
f = \bigvee_{i=0}^{7} \psi_i(X_1) S_{a_i}(X_2).
\]
Fig. 3.10 shows the r-planar BDD for \(f\). The upper block generates \(\psi_1\), and the lower block generates \(S_{a_i}\). Note that each edge has a weight. In each path from the root node to the constant 1, the sum of the weights is greater than or equal to 6. On the other hand, in each path from the root node to the constant 0, the sum of the weights is less than 6. We can reduce the BDD without introducing crossings. (End of Example)
The converse operation of converting a monotone increasing function to a unate function, can be accomplished in the domain of BDDs, by interchanging 0 and 1 labels on all edges associated with some variable. This is the same as having 0-edges emerge to the right and 1-edges to the left. Thus, with minor modification, the results presented here can be made to apply to unate functions.

For a given monotone increasing function, in most cases, we can find an r-planar BDD among minimum BDDs (i.e., BDDs having the least number of nodes). However, some functions require additional nodes to make their BDDs r-planar. In the past, reduction of the number of nodes was the major subject in the optimization of BDDs. However, in implementing multilevel networks directly from the BDDs, the planarity of BDDs is also important, since crossing produces delay in LSIs. It is interesting to extend the theory for the decision diagrams with EXOR operators [13].

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