Reliability-based Design Optimization with Confidence Level for Non-Gaussian Distributions Using Bootstrap Method

Yoojung Noh, K. K. Choi, Ikjin Lee, David Gorsich, David Lamb

University of Iowa @ Iowa City
Office of Sponsored Programs
2 Gilmore hall
Iowa City, IA 52242 -1316

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

Reliability-based design optimization, input statistical model, confidence level, non-Gaussian distribution, bootstrap method

For reliability-based design optimization (RBDO), generating an input statistical model with confidence level has been recently proposed to offset inaccurate estimation of the input statistical model with Gaussian distributions. For this, the confidence intervals for the mean and standard deviation are calculated using Gaussian distributions of the input random variables. However, if the input random variables are non-Gaussian, use of Gaussian distributions of the input variables will provide inaccurate confidence intervals, and thus, yield an undesirable confidence level of the reliability-based optimum design meeting the target reliability. In this paper, an RBDO method using a bootstrap method is presented.
ABSTRACT

For reliability-based design optimization (RBDO), generating an input statistical model with confidence level has been recently proposed to offset inaccurate estimation of the input statistical model with Gaussian distributions. For this, the confidence intervals for the mean and standard deviation are calculated using Gaussian distributions of the input random variables. However, if the input random variables are non-Gaussian, use of Gaussian distributions of the input variables will provide inaccurate confidence intervals, and thus, yield an undesirable confidence level of the reliability-based optimum design meeting the target reliability. In this paper, an RBDO method using a bootstrap method, which accurately calculates the confidence intervals for the input parameters for non-Gaussian distributions, is proposed to obtain a desirable confidence level of the output performance for non-Gaussian distributions. The proposed method is examined by testing a numerical example and M1A1 Abrams tank roadarm problem.
Continuation for Block 13

ARO Report Number  56025.35-NS
Reliability-based Design Optimization with Confi...

Block 13: Supplementary Note
© 2011 . Published in ASME Journal of Mechanical Design, Vol. Ed. 0 (2011), (Ed. ). DoD Components reserve a royalty-free, nonexclusive and irrevocable right to reproduce, publish, or otherwise use the work for Federal purposes, and to authorize others to do so (DODGARS §32.36). The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

Approved for public release; distribution is unlimited.
Reliability-Based Design Optimization With Confidence Level for Non-Gaussian Distributions Using Bootstrap Method

For reliability-based design optimization (RBDO), generating an input statistical model with confidence level has been recently proposed to offset inaccurate estimation of the input statistical model with Gaussian distributions. This is because for a gradient-based or the most probable point (MPP)-based RBDO, the MPP is located on the β−contour (see Sec. 2.1 for definition), and thus, the size of β−contour, which can be controlled by the standard deviations of the input random variables, determines how reliable the optimum design is. However, for many input random variables, such as load- and material properties, and manufacturing geometric variability, only limited data are available due to excessive testing costs. If an input statistical model is obtained from insufficient data, it could yield an unreliable optimum design. To deal with the input statistical uncertainties, the possibility-based design optimization (PBDO) [1,2], the interval-based method [3–5], and the Bayesian reliability-based design optimization (RBDO) have been recently proposed [6,7]. However, the PBDO and interval-based method do not consider data size to properly quantify the uncertainties, and the Bayesian RBDO method assumes that input distributions are known, which is not common in real applications. In this paper, to offset inaccurate estimation of the input model, generating an input model with a confidence level has been recently proposed by using adjusted standard deviations and a correlation coefficient [8].

The adjusted standard deviation and correlation coefficient proposed in Ref. 8 are obtained from the confidence intervals for the input distribution parameters such as the mean, standard deviation, and correlation coefficient. The confidence intervals for the mean and standard deviation are usually calculated using Gaussian distributions of the input variables [9]. If the input variables have marginal Gaussian distributions, the confidence intervals for the mean and standard deviation can be explicitly and exactly calculated [8].

Keywords: reliability-based design optimization, input statistical model, confidence level, non-Gaussian distribution, bootstrap method

1 Introduction

Obtaining an accurate input statistical model, which includes marginal distributions and a joint distribution of input random variables, is crucial to obtain an accurate reliability-based optimum design. However, for many input random variables, such as loading, material properties, and manufacturing geometric variability, only limited data are available due to expensive testing costs. If an input statistical model is obtained from insufficient data, it could yield an unreliable optimum design. To deal with the input statistical uncertainties, the possibility-based design optimization (PBDO) [1,2], the interval-based method [3–5], and the Bayesian reliability-based design optimization (RBDO) have been recently proposed [6,7]. However, the PBDO and interval-based method do not consider data size to properly quantify the uncertainties, and the Bayesian RBDO method assumes that input distributions are known, which is not common in real applications. In this paper, to offset inaccurate estimation of the input model, generating an input model with a confidence level has been recently proposed by using adjusted standard deviations and a correlation coefficient [8].

The adjusted standard deviation and correlation coefficient proposed in Ref. 8 are obtained from the confidence intervals for the input distribution parameters such as the mean, standard deviation, and correlation coefficient. The confidence intervals for the mean and standard deviation are usually calculated using Gaussian distributions of the input variables [9]. If the input variables have marginal Gaussian distributions, the confidence intervals for the mean and standard deviation can be explicitly and exactly calculated [8].

1Corresponding author
Contributed by the Design Automation Committee of ASME for publication in the Journal of Mechanical Design. Manuscript received February 13, 2011; final manuscript received June 28, 2011; published online September 7, 2011. Assoc. Editor: Zisisimos P. Mourelatos.
design will be. A mathematical example and an M1A1 Abrams tank roadarm problem with non-Gaussian correlated variables are used to illustrate how the input model with a target confidence level using the bootstrap method provides a more desirable output confidence level compared with the one without using the proposed method for correlated input distributions. The proposed method is focused on correlated random variables only. Uncorrelated but dependent variables are beyond the scope of the paper.

2 Confidence Level of Input Model

When an input model is estimated from given limited data, we want to know how reliable the RBDO result obtained from the estimated input model is. Even though the target probability of failure is 2.275%, the probability evaluated at the RBDO optimum using the estimated input model could be much larger than the target probability of failure due to inaccurate estimation of the input model. However, it is difficult to predict an accurate confidence level of the output performance of the RBDO optimum because the confidence level of the output performance depends on problems even though the same input model is used. Thus, the confidence level of the input model needs to be first estimated before stepping into estimation of the confidence level of the output performance. Even though the confidence level of the input model is not necessarily equivalent to the output confidence level, if a conservative measure for estimating the input confidence level, i.e., a \( \beta_t \)-contour is used, then it can be assured that the confidence level of the output performance is at least larger than the confidence level of the input model. In this paper, the confidence level of the input model is defined as the probability that the \( \beta_t \)-contour obtained using the estimated input model covers the \( \beta_t \)-contour obtained using the true input model. Since a larger \( \beta_t \)-contour assures the RBDO optimum design to satisfy the target reliability, the confidence level of the input model will provide a confidence level that the RBDO optimum design satisfies the target reliability. This measure of input confidence level, \( \beta_t \)-contour, is explained in Sec. 2.1 in detail.

In Sec. 2.2, the adjusted parameters obtained from the confidence intervals for the input parameters are introduced.

2.1 Measure of Input Confidence Level

The RBDO formulation is defined to

\[
\text{min. Cost}(d) \\
\text{s.t.} \quad P(G_j(X) > 0) \leq P_{k, j}^{tu}, \quad j = 1, \ldots, nc \\
d = \mu(X), \quad d_L \leq d \leq d_U, \quad d \in R^{nd} \text{ and } X \in R^n
\]

where \( X \) is the vector of random variables; \( d \) is the vector of design variables, which is the mean values of random variables; \( G_j(X) \) represents the \( j \)-th constraint function; \( P_{k, j}^{tu} \) is the given target probability of failure for the \( j \)-th constraint; and \( nc, ndv, \) and \( n \) are the number of probabilistic constraints, design variables, and random variables, respectively. To satisfy the probabilistic constraint in Eq. (1), the constraint function evaluated at the MPP (\( X^* \)), \( G_j(X^*) \) should be less than zero, where the MPP is obtained by solving the inverse reliability analysis as

\[
\text{maximize} \quad g_j(u) \\
\text{subject to} \quad ||u|| = \beta_t
\]

where \( g_j(u) \) is the \( j \)-th constraint function in the standard normal U-space, i.e., \( g_j(u) \equiv G_j(x(u)) = G_j(x) \) and \( \beta_t \) is the target reliability index such that \( P_{k, j}^{tu} = \Phi(-\beta_t) \) for the \( j \)-th constraint. Once the hyper-sphere in Eq. (2), \( ||u|| = \beta_t \), is transformed from the U-space to the X-space using the Rosenblatt transformation [15], it is called as \( \beta_t \)-contour, and the MPP search is carried out on the \( \beta_t \)-contour for the inverse reliability analysis in Eq. (2).

The Rosenblatt transformation uses a conditional joint distribution of input random variables, which is obtained from a joint distribution. However, since it is difficult to obtain a joint distribution directly from limited data, a copula function, which consists of marginal distributions and correlation parameters, is used to model the joint distribution as

\[
F_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = C(F_{X_1}(x_1), \ldots, F_{X_n}(x_n) | \theta)
\]

where \( F_{X_i, X_j}(x_i, \ldots, x_n) \) is the joint cumulative distribution function (CDF) of \( X_1, \ldots, X_n \), and \( F_{X_i}(x_i) \) is the marginal CDF of input random variables \( X_i \) for \( i = 1, \ldots, n \). \( C \) is a copula function of the marginal CDFs with the given matrix of correlation parameters \( \theta \) between \( X_1, \ldots, X_n \). Since the correlation parameters are different for different copula types, a common correlation measure, the Kendall’s tau [17] is used. The sample version of the Kendall’s tau is calculated as

\[
t = \frac{c - d}{c + d} = \frac{2(c - d)}{ns(2n - 1)}
\]

where \( c \) and \( d \) are the numbers of concordant and discordant pairs, respectively, and \( c + d = \frac{ns}{2} \) where \( ns \) is the number of samples. A pair of two variable data sets \((X_1, Y_1)(X_2, Y_2) \) is called concordant if \( (X_1 - X_2)(Y_1 - Y_2) > 0 \), and called discordant if \( (X_1 - X_2)(Y_1 - Y_2) < 0 \). Once the Kendall’s tau is obtained from samples using Eq. (4), the correlation parameter \( \theta \) of two random variables can be implicitly obtained using the following equation:

\[
\tau = 4 \int \int_C \frac{c(u,v; \theta)dCd(\tau, \theta) - 1}{C(u,v; \theta)dCd(\tau, \theta)}
\]

where \( u = F_{X_1}(x_1) \) and \( v = F_{X_2}(x_2) \) are the marginal CDFs of \( X_1 \) and \( X_2 \), respectively, \( dCd(\tau, \theta) = \frac{\partial^2 C(u,v; \theta)}{\partial u \partial v}dudv \), and the unit square \( I^2 \) is the product \( I \times I = [0, 1] \times [0, 1] \) of the domain of the two marginal CDFs \( u \) and \( v \). For some copulas, there exist explicit formulas between \( \theta \) and \( \tau \), which are given in Ref. [18].

Using the copula function, the Rosenblatt transformation can be written as

\[
u_i = \Phi^{-1}[F_{X_i}(x_i| x_1, x_2, \ldots, x_{i-1})] = \Phi^{-1}\left[\frac{\partial^2 C(z_1, \ldots, z_{i-1}, z_i; \theta)}{\partial z_1 \partial z_{i-1}} f_{X_i}(x_i) f_{X_1}(x_1, \ldots, x_{i-1})\right]
\]

where \( F_{X_i}(x_i| x_1, x_2, \ldots, x_{i-1}) \) is the conditional distribution function of \( X_i \) given \( X_1, \ldots, X_{i-1} \), \( z_i = F_{X_i}(x_i) \), and \( f_{X_i}(x_i) \) and \( f_{X_1}(x_1, \ldots, x_{i-1}) \) are the marginal and joint probability density function (PDF) for \( i = 1, \ldots, n \). Inserting Eq. (6) into the second equation of Eq. (2), the hyper-sphere in the U-space can be expressed as the \( \beta_t \)-contour. Most copula applications consider bivariate data because only few copula families have \( n \)-dimensional generalization. In engineering applications, such as the strain-based fatigue analysis, the fatigue strength coefficient and exponent are correlated and the fatigue ductility coefficient and exponent are correlated. However, they are known not to be cross correlated [19,20]. There are other types of problems such that the random variables are pair-wise correlated [21,22]. Thus, in this paper, only the bivariate copulas are considered.

The \( \beta_t \)-contour acts as a safety barrier that locates the optimum design point away from the constraint boundary with the target probability of failure. Therefore, if the \( \beta_t \)-contour is large enough for the optimum design point to be away from the constraint boundary with the target probability of failure, it means that the obtained optimum design satisfies the target reliability. Consider two \( \beta_t \)-contours obtained from an estimated input model (dotted line) and the true input model (solid line) shown in

091001-2 / Vol. 133, SEPTEMBER 2011
Transactions of the ASME
Adding the ratio of portion to the change in the sample mean. Thus, it is proposed that
Since the coefficient of variation (COV), size of the parameter is.

dence level is, the bigger the confidence interval for the estimated parameter falls within the confidence interval, the larger the confi-
these parameters, which is likely to include unknown population

shown in Eqs.(2) and (6). The estimated input parameters, such as the sample mean, standard deviations and correlation coeffi-
cient is calculated is provided. Then, the confidence level of the
description of how the confidence interval for the correlation coef-
cient are explained. Since accurate estimation of the upper bound of the
mean and standard deviation determine the location and variabili-
ty of the distributions regardless of distribution types, the standard deviations for non-Gaussian distributions can be used to enlarge
the \\beta_r\text{-contour for RBDO. Accordingly, once the mean and standard deviation are calculated from the given data, the input param-
ters can be calculated using the explicit functions, which are presented for various marginal distributions in Ref. 18.

The adjusted parameters in Eqs. (7) and (8) for Gaussian input random variables are calculated explicitly and exactly [8]. How-
ever, if the input random variables are not Gaussian, the assump-
tion that input random variables have Gaussian distribution may yield inaccurate confidence intervals, which lead to unreliable op-
timum designs. Thus, the bootstrap method is proposed to calcu-
late the confidence intervals for the distribution parameters for
non-Gaussian distributions in this paper.

3 Estimation of Input Statistical Model With Confidence Level

For obtaining RBDO result, the input model, i.e., the marginal distributions and joint distribution, needs to be estimated from the given data. In Secs. 3.1 and 3.2, identification and quantification of the marginal and joint distributions using bootstrap methods are explained. Since accurate estimation of the upper bound of the standard deviation is important to have a desirable input confidence level, various bootstrap methods for accurate estimation of the confidence interval for the standard deviation are discussed and tested using simulation studies in Sec. 3.3. In Sec. 3.4, a description of how the confidence interval for the correlation coeffi-
cient is calculated is provided. Then, the confidence level of the input model generated using the adjusted parameters, which are obtained from confidence intervals for input parameters in Eqs. (7) and (8), will be tested for various input distributions in Sec. 3.5.

3.1 Identification of Input Model. To identify the joint or marginal distribution types from the given data, a Bayesian method is used to select one candidate distribution that best describes the given data among candidate marginal or joint distri-

butions. The input model can be identified by a one-step procedure, which directly tests all candidate marginal and joint distributions simultaneously, or by a two-step procedure, which first identifies marginal distributions and then a copula [18,23–26]. The two-step procedure is more efficient and accurate than the one-step procedure [27]. For example, if seven candidate
marginal distributions of $X_1$ and $X_2$ and nine candidate copulas are used to identify a joint distribution, the one-step procedure requires to test $7 \times 7 \times 9 = 441$ cases, whereas the two-step procedure requires to test $7 + 7 + 9 = 23$ cases. It is more challenging to identify a correct joint distribution from 441 candidates compared to 23 candidates, so the two-step procedure is preferred. Based on the measure of identification, the weight-based method [18] and Markov chain Monte Carlo (MCMC)-based method [28] can be used. The weight-based method calculates the weights of all candidates by integrating the likelihood functions of the candidates, and then selects one candidate with the highest weight. The MCMC-based method identifies a correct distribution among candidates using a criterion such as a deviance information criterion. However, the MCMC-based method uses random samples of the posterior distribution, which causes randomness of identification results. Thus, in this paper, a two-step weight-based Bayesian method is used to identify the input model. More detailed information on the two-step weight-based method is presented in Ref. 18.

3.2 Quantification of Input Model Using Bootstrap Method. If the input random variables have a Gaussian distribution, the confidence intervals for the mean and standard deviation can be explicitly and exactly obtained. However, if not, there are no explicit functions for calculation of the confidence intervals for the mean and standard deviation for a non-Gaussian distribution, so a bootstrap method needs to be introduced to obtain accurate confidence intervals for the mean and standard deviation for non-Gaussian distributions.

Because the confidence interval for the mean obtained assuming input variables have a Gaussian distribution can be accurately estimated even for most non-Gaussian distributions [29,30], the bootstrap method is tested to calculate the confidence interval for the standard deviation only. However, for non-Gaussian distributions with extremely high skewness such as highly skewed extreme distribution, the bootstrap method needs to be used to calculate the confidence interval for the mean as well as the standard deviation.

If the input variable follow a Gaussian distribution with the mean $\mu$ and standard deviation $\sigma$, the lower and upper bounds of the confidence interval for the mean ($\mu^L$ and $\mu^U$) can be obtained as [8,9]

$$\mu^L = \bar{\mu} - t_{\alpha/2,\text{ns}-1} \frac{\tilde{\sigma}}{\sqrt{\text{ns}}} \quad \text{and} \quad \mu^U = \bar{\mu} + t_{\alpha/2,\text{ns}-1} \frac{\tilde{\sigma}}{\sqrt{\text{ns}}}$$

(9)

where $\text{ns}$ is the number of samples, $\bar{\mu}$ and $\tilde{\sigma}$ are the sample mean and sample standard deviation, respectively, and $t_{\alpha/2,\text{ns}-1}$ is the value of the student’s $t$-distribution with $(\text{ns}-1)$ degrees of freedom at two-sided confidence level, $100 \times (1-\alpha/2)$, $\alpha$ indicates a confidence level, which is the difference that the true mean is not within the confidence interval in Eq. (9). For example, for the given 95% confidence level, $\alpha = 0.05$.

Using a similar procedure of calculating the confidence interval for the mean, the lower and upper bounds of the confidence interval for the standard deviation of Gaussian distribution, $\tilde{\sigma}^L$ and $\tilde{\sigma}^U$, respectively, are calculated as [8,9]

$$\tilde{\sigma}^L = \frac{(\text{ns} - 1) \hat{\sigma}^2}{c_{\alpha/2,\text{ns}-1}} \quad \text{and} \quad \tilde{\sigma}^U = \frac{(\text{ns} - 1) \hat{\sigma}^2}{c_{1-\alpha/2,\text{ns}-1}}$$

(10)

where $c_{\alpha/2,\text{ns}-1}$ and $c_{1-\alpha/2,\text{ns}-1}$ are the critical values of the chi-square distribution evaluated at two-sided confidence level $100 \times (\alpha/2)$ and $100 \times (1-\alpha/2)$ with $(\text{ns}-1)$ degrees of freedom, respectively.

For non-Gaussian distributions, the assumption that input variables have Gaussian distributions may yield inaccurate estimation of the confidence interval for the standard deviation. Thus, a bootstrap method, which does not require any assumption on the distribution types of input variables, needs to be used. To calculate the confidence interval for an estimated standard deviation $\tilde{\sigma}$, the bootstrap method constructs a distribution of the standard deviation using the frequency distribution of $\tilde{\sigma}$ obtained from randomly generated bootstrap samples based on the given data.

The first step is to construct an empirical distribution $F_{\text{ns}}(x)$ or a parametric distribution $F(x)$ from the given samples, $x = [x_1, x_2, \ldots, x_{\text{ns}}]$. If a random sample of size $\text{ns}$ with replacement is drawn from the empirical distribution $F_{\text{ns}}(x)$, then this is called a nonparametric approach. If the sample is drawn from the specified model $F(x)$ determined from the given samples, this is called a parametric approach. The empirical distribution is obtained as

$$F_{\text{ns}}(x) = \frac{1}{\text{ns}} \sum_{i=1}^{\text{ns}} I[x_i \leq x]$$

(11)

where $I[A]$ is the indicator function of event $A$, that is, $I[A] = 1$ if the event $A$ occurs, otherwise, $I[A] = 0$.

In the second step, bootstrap samples are generated from an empirical or parametric distribution or parametric distribution. The third step is to calculate $\tilde{\sigma}$ from the resample, drawn from either an empirical or parametric distribution, yielding $\tilde{\sigma}^{*1}$. In the fourth step, the second and third steps are repeated $B$ times (e.g., $B = 1000$). Then, the fifth step is to construct a probability distribution from $\tilde{\sigma}^{*1}, \tilde{\sigma}^{*2}, \ldots, \tilde{\sigma}^{*B}$. This distribution is the bootstrap sampling distribution $G^{(\tilde{\sigma}^{*})}$ of $\tilde{\sigma}$, which is used to calculate the confidence interval for $\tilde{\sigma}$. To obtain the bootstrap sampling distribution of $\tilde{\sigma}$, five bootstrap methods such as the normal approximation, percentile, bias corrected, percentile-$t$, or bias corrected accelerated methods can be used and their performances are tested in this paper. Table 1 summarizes how to calculate the confidence interval for the standard deviation using the bootstrap method.

3.2.1 Normal Approximation Method. The normal approximation method assumes that the distribution of estimated standard deviation is a Gaussian distribution. Using the assumption, the confidence interval for the standard deviation is obtained as [8,31]

$$\tilde{\sigma} - z_{\alpha/2} \hat{\sigma} \leq \tilde{\sigma} \leq \tilde{\sigma} + z_{\alpha/2} \hat{\sigma}$$

(12)

where $\hat{\sigma}^{2} = \frac{\sum_{i=1}^{B} (\tilde{\sigma}^{*i} - \tilde{\sigma})^2}{(B-1)}$, $\tilde{\sigma} = \frac{\sum_{i=1}^{B} \tilde{\sigma}^{*i}/B$, and $z_{\alpha/2}$ are the values of standard Gaussian distribution CDF at $\alpha/2$.

3.2.2 Percentile Method. The percentile method calculates the confidence interval for the parameter based on the bootstrap sampling distribution $G^{(\tilde{\sigma}^{*})}$ approximating the population distribution $G(\tilde{\sigma})$. The basic idea of this method is that the confidence interval for $(1-\alpha)\%$ level includes all the values of $\tilde{\sigma}^{*}$ between the $(\alpha/2 \times 100)^{th}$ and $(1-\alpha/2 \times 100)^{th}$ percentiles of $G^{(\tilde{\sigma}^{*})}$. The sorting vector of $\tilde{\sigma}^{*}$ is obtained from each bootstrap sample for $bs = 1, \ldots, B$ and the values of $\tilde{\sigma}^{*}_{bs}$ evaluated at the $(\alpha/2 \times 100)^{th}$

<table>
<thead>
<tr>
<th>Table 1 Bootstrap procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bootstrap procedures</strong></td>
</tr>
<tr>
<td><strong>Step 1</strong> From given samples $x = [x_1, x_2, \ldots, x_{\text{ns}}]$, construct an empirical distribution $F_{\text{ns}}(x)$ or parametric distribution $F(x)$.</td>
</tr>
<tr>
<td><strong>Step 2</strong> Generate bootstrap samples $x' = [x'_1, x'<em>2, \ldots, x'</em>{\text{ns}}]$ from the constructed distribution in Step 1.</td>
</tr>
<tr>
<td><strong>Step 3</strong> Calculate a statistic of interest $\tilde{\theta}$ from bootstrap samples, yielding $\tilde{\theta}_{bs}, bs = 1, \ldots, B$</td>
</tr>
<tr>
<td><strong>Step 4</strong> Repeat Step 2 and 3 $B$ times (e.g., $B = 1000$).</td>
</tr>
<tr>
<td><strong>Step 5</strong> Construct a probability distribution $G^{(\tilde{\theta}^{<em>})}$ from $\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_B$, and then calculate confidence interval for estimated parameter, $\tilde{\theta}$ using $G^{(\tilde{\theta}^{</em>})}$.</td>
</tr>
</tbody>
</table>
and \((1 - z/2) \times 100\)th percentiles of \(\tilde{G}^*(\hat{\sigma}^*)\) are used as the lower and upper bounds of \(\hat{\sigma}\), respectively,

\[
\tilde{\sigma}^*_{z/2} < \hat{\sigma} < \tilde{\sigma}^*_1 - z/2
\]

Since the percentile method does not assume that the bootstrap sampling distribution follows a Gaussian distribution such as the normal approximation method, it allows \(\tilde{G}^*(\hat{\sigma}^*)\) conforming to any shape that the data follow. For this reason, it is the most widely used bootstrap technique among applied statisticians [32]. However, when the number of samples is small, \(\tilde{G}^*(\hat{\sigma}^*)\) might be a biased estimator of \(\tilde{G}^*(\hat{\sigma})\), i.e., \(\hat{\sigma}^*\) is a biased estimator of \(\hat{\sigma}\). In that case, the percentile method can be inaccurate.

### 3.2.3 BC Method

The BC method corrects the bias term by introducing an adjusted parameter \(z_0\). Suppose that there exist some monotonic transformations between \(\hat{\sigma}^*\) and \(\hat{\sigma}\), say, \(\hat{\sigma} = \phi(\hat{\sigma}^*)\). Instead of assuming that \(\hat{\sigma}^* - \hat{\sigma}\) is centered on zero, the BC method assumes that \(\phi(\hat{\sigma}^*) - \phi(\hat{\sigma}) + z_0 = z\) follows a standard Gaussian distribution. Since \(\hat{\sigma}^*\) and \(\hat{\sigma}\) are monotonic functions, it holds that \(\Pr(\hat{\sigma}^* \leq \hat{\sigma}) = \Pr(\hat{\sigma} \leq z_0) = \Phi(\hat{\sigma}_0)\), where \(\Phi()\) is the standard Gaussian CDF. Accordingly, \(z_0\) is calculated using \(\Phi^{-1}\)

\[
z_0 = \Phi^{-1}(\Pr(\hat{\sigma}^* \leq \hat{\sigma}))
\]

where \(z_0\) is a biasing constant that compensates the bias between \(\hat{\sigma}^*\) and \(\hat{\sigma}\), and \(\Pr(\hat{\sigma}^* \leq \hat{\sigma}) = \Phi(z_0)\). Since \(\tilde{G}^*(\hat{\sigma}^*)\) is invariant to the transformation, the transformation does not need to be known. Using \(z_0\), the confidence interval for \(\hat{\sigma}\) is obtained as

\[
\tilde{\sigma}_{\Phi(2z_0+z_0/2)} < \hat{\sigma} < \tilde{\sigma}_{\Phi(2z_0+z_0/2)}
\]

where \(\tilde{\sigma}_{\Phi(2z_0+z_0/2)}\) is the value of \(\hat{\sigma}\) evaluated at the \(\Phi(2z_0 + z_0/2) \times 100\) percentile and \(\tilde{\sigma}_{\Phi(2z_0 + z_0/2)}\) is the value of \(\hat{\sigma}\) evaluated at the \(\Phi(2z_0 + z_0/2) \times 100\) percentile. The BC method corrects the bias term, but it still requires the parametric assumption that there exist monotonic transformations between \(\hat{\sigma}^*\) and \(\hat{\sigma}\).

### 3.2.4 BCa Method

The BCa method generalizes the BC method in a way that the BC method only corrects the bias, whereas the BCa method corrects both the bias and the skewness. The BCa method assumes that, for certain monotonic transformations \(\phi\) and \(\phi\), the bias constant \(z_0\) and acceleration constant A result in Ref. [11]

\[
\frac{(\hat{\phi} - \phi)}{\sigma_\phi} \sim N(-z_0\sigma_{\hat{\phi}}, \sigma^2) , \sigma_\phi = 1 + Ah
\]

where \(\sigma_{\hat{\phi}}\) is the constant standard error of \(\hat{\phi}\). The acceleration A is defined as

\[
A = \sum_{i=1}^{n} \left\{ \frac{1}{n(n-1)} \left[ \hat{\sigma} - \hat{\sigma}(i) \right]^3 \right\}^{3/2}
\]

where \(\hat{\sigma}(i)\) is the estimated parameter of \(X_{(j)} = (x_1, \ldots, x_{n-1}, x_i, \ldots, x_n)\) without the ith point \(x_i\) and \(\hat{\sigma} = \sum_{i=1}^{n} \hat{\sigma}(i)/n\). Using Eq. (16), the BCa confidence interval is defined as

\[
\tilde{\sigma}_{\Phi(2z_0+z_0/2)} < \hat{\sigma} < \tilde{\sigma}_{\Phi(2z_0+z_0/2)}
\]

where \(\tilde{\sigma}_{\Phi(2z_0+z_0/2)}\) is the estimated parameter from the resampled data and \(\tilde{\sigma}_{\Phi(2z_0+z_0/2)}\) is the estimated parameter from Ref. [11].

### 3.2.5 Percentile-t Method

The percentile-t method uses the distribution of a standardized estimator to calculate the confidence interval. The percentile-t interval is expected to be accurate to the extent that standardizing depends less on the bootstrap sampling estimator, \(\hat{\sigma^*}\), than the percentile method. The standardized parameter \(t_0\) can be defined as [32]

\[
t_0 = \frac{(\hat{\sigma}_0 - \hat{\sigma})}{\tilde{\sigma}_0} 
\]

where \(\hat{\sigma}_0\) is the estimated parameter from the resampled data and \(\tilde{\sigma}_0\) is the estimated parameter from the original data, \(x = (x_1, x_2, \ldots, x_n)\) and \(\tilde{\sigma}_0\) is the standard deviation of \(\hat{\sigma}\) obtained from a double bootstrap, which is another level of resampling. That is, the double bootstrap sample \(x'_{(1)} = (x_{1}, x_{2}, \ldots, x_{n})\) for \(d = 1, \ldots, D\) is resampled from the bootstrap samples, where \(x_{bs} = (x_{1bs}, x_{2bs}, \ldots, x_{nbs})\) for \(bs = 1, \ldots, B\). Thus, the percentile-t method requires a large number \((D \times B)\) of bootstrap samples. Using the double bootstrap samples, \(\tilde{\sigma}_0\) is obtained as

\[
\tilde{\sigma}_0 = \left( \sum_{d=1}^{D} \sum_{bs=1}^{B} \left[ \frac{x_{bs} - \tilde{x}_{bs}}{\tilde{\sigma}_0} \right]^2 \right) / (D - 1)
\]

where \(\hat{\sigma^*}\) is the interval that the BCa method highly depends on the acceleration A, if it is not accurate, the BCa is also inaccurate.

### 3.3 Tests of Bootstrap Methods

To test and compare the performance of the five bootstrap methods, a lognormal distribution is considered as the true marginal distribution with \(\mu = 5.0\) and \(\sigma = 5.0\), which means a relatively large COV. For calculation of confidence interval for the standard deviation, an empirical distribution (nonparametric approach) or a parametric distribution (parametric approach) needs to be first constructed as explained in Sec. 3.2. For the parametric approach, a distribution type and its parameters need to be determined from samples, which are randomly generated from the assumed true marginal distribution (i.e., lognormal distribution). The distribution type is identified from the generated samples over 1000 data sets with different sample sizes of \(ns = 30, 100, \) and \(300\) using the two-step weight-based Bayesian method where seven candidate distributions are the Gaussian, Weibull, gamma, lognormal, Gumbel, extreme, and extreme type II distributions.

The distribution type is differently identified for different data sets. In this testing, the true lognormal distribution is identified 650 times out of the 1000 data sets from the seven candidates for \(ns = 30\). Weibull and gamma distributions are identified 113 and 210 times, respectively, because they have the most close distribution shapes to the lognormal distribution with \(\mu = 5\) and \(\sigma = 5\) among the seven candidates. For \(ns = 100\) and \(300\), the correct distribution is identified 875 and 980 times out of the 1000 data samples.
Table 2 Obtained confidence levels (% of confidence interval for standard deviation)

<table>
<thead>
<tr>
<th>ns</th>
<th>Approach</th>
<th>Normal approximation</th>
<th>Percentile</th>
<th>BC</th>
<th>BCa</th>
<th>Percentile-t</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>Nonpar.</td>
<td>64.9</td>
<td>61.3</td>
<td>64.8</td>
<td>70.4</td>
<td>83.7</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>74.9</td>
<td>79.0</td>
<td>85.0</td>
<td>88.9</td>
<td>89.6</td>
</tr>
<tr>
<td>100</td>
<td>Nonpar.</td>
<td>89.3</td>
<td>95.2</td>
<td>88.3</td>
<td>92.4</td>
<td>88.3</td>
</tr>
<tr>
<td></td>
<td>Par. (True)</td>
<td>76.0</td>
<td>75.1</td>
<td>78.1</td>
<td>85.1</td>
<td>88.8</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>87.2</td>
<td>87.5</td>
<td>87.4</td>
<td>89.9</td>
<td>92.0</td>
</tr>
<tr>
<td>300</td>
<td>Nonpar.</td>
<td>95.2</td>
<td>97.7</td>
<td>86.9</td>
<td>89.9</td>
<td>87.1</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>82.3</td>
<td>82.7</td>
<td>84.5</td>
<td>87.9</td>
<td>91.3</td>
</tr>
<tr>
<td></td>
<td>Par. (True)</td>
<td>95.9</td>
<td>97.5</td>
<td>86.8</td>
<td>88.4</td>
<td>85.7</td>
</tr>
<tr>
<td></td>
<td>Par. (True)</td>
<td>96.7</td>
<td>99.0</td>
<td>86.3</td>
<td>88.2</td>
<td>85.3</td>
</tr>
</tbody>
</table>

Note: Method using Gaussian distribution of input variable yields 65.8% for ns = 30, 66.0% for ns = 100, and 65.0% for ns = 300.

Table 3 Mean values of upper bound of confidence interval for standard deviation

<table>
<thead>
<tr>
<th>ns</th>
<th>Approach</th>
<th>Normal approximation</th>
<th>Percentile</th>
<th>BC</th>
<th>BCa</th>
<th>Percentile-t</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>Nonpar.</td>
<td>6.900</td>
<td>6.458</td>
<td>6.803</td>
<td>7.379</td>
<td>20.01</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>7.830</td>
<td>8.396</td>
<td>12.72</td>
<td>15.22</td>
<td>16.62</td>
</tr>
<tr>
<td>100</td>
<td>Nonpar.</td>
<td>8.742</td>
<td>9.998</td>
<td>12.57</td>
<td>15.83</td>
<td>16.04</td>
</tr>
<tr>
<td></td>
<td>Par. (True)</td>
<td>6.882</td>
<td>6.367</td>
<td>6.604</td>
<td>7.144</td>
<td>10.78</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>7.167</td>
<td>7.596</td>
<td>9.204</td>
<td>10.43</td>
<td>10.83</td>
</tr>
<tr>
<td>300</td>
<td>Nonpar.</td>
<td>7.367</td>
<td>7.978</td>
<td>9.342</td>
<td>10.60</td>
<td>10.77</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>6.416</td>
<td>6.763</td>
<td>7.324</td>
<td>7.877</td>
<td>7.642</td>
</tr>
<tr>
<td></td>
<td>Par. (True)</td>
<td>6.491</td>
<td>6.824</td>
<td>8.034</td>
<td>7.422</td>
<td>7.984</td>
</tr>
</tbody>
</table>

Table 4 Standard deviations of upper bound of confidence interval for standard deviation

<table>
<thead>
<tr>
<th>ns</th>
<th>Approach</th>
<th>Normal approximation</th>
<th>Percentile</th>
<th>BC</th>
<th>BCa</th>
<th>Percentile-t</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>Nonpar.</td>
<td>3.629</td>
<td>3.130</td>
<td>3.491</td>
<td>3.990</td>
<td>32.78</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>3.873</td>
<td>4.316</td>
<td>11.62</td>
<td>12.96</td>
<td>20.66</td>
</tr>
<tr>
<td>100</td>
<td>Nonpar.</td>
<td>3.587</td>
<td>3.958</td>
<td>10.18</td>
<td>11.81</td>
<td>21.21</td>
</tr>
<tr>
<td></td>
<td>Par. (True)</td>
<td>2.272</td>
<td>2.132</td>
<td>2.365</td>
<td>2.813</td>
<td>11.71</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>1.977</td>
<td>2.019</td>
<td>4.725</td>
<td>5.373</td>
<td>12.34</td>
</tr>
<tr>
<td>300</td>
<td>Nonpar.</td>
<td>1.219</td>
<td>1.200</td>
<td>1.308</td>
<td>1.594</td>
<td>3.930</td>
</tr>
<tr>
<td></td>
<td>Par. (Iden.)</td>
<td>0.971</td>
<td>0.873</td>
<td>2.722</td>
<td>2.978</td>
<td>4.692</td>
</tr>
<tr>
<td></td>
<td>Par. (True)</td>
<td>0.929</td>
<td>0.815</td>
<td>2.598</td>
<td>2.946</td>
<td>4.695</td>
</tr>
</tbody>
</table>

Note: Method using Gaussian distribution of input variable yields 2.484 for ns = 30, 1.488 for ns = 100, and 0.834 for ns = 300.

Figure 2 shows histograms of the upper bounds of the confidence interval for the standard deviation using the parametric percentile bootstrap method with identified CDF. The estimated upper bounds of the standard deviations mostly centered at the true standard deviation (5.0) even for ns = 30. As the number of samples increases, a large amount of the upper bounds of standard deviation tends to be very close to the true standard deviation with a small variation.

As shown in the example, for the lognormal distribution with a large COV such as 1.0, the performances of the BC, BCa, and percentile-t methods are not as good as the percentile method. This is because non-Gaussian distribution with a large COV has a large variation of bias corrected terms, which may yield an over conservative confidence interval. Likewise, the double bootstrap samples in percentile-t method depend more on the estimated distribution than other methods, but the estimated parameters are not accurately estimated for the non-Gaussian distribution with a large COV, which leads to inaccurate estimation of confidence interval for the standard deviation. Even though the bootstrap methods do not achieve the target confidence level for a small number of samples, as the number of samples increases, the obtained confidence levels tend to converge to the target confidence level while the method assuming Gaussian distribution of input variable does not.

When the input variable has a Gaussian distribution, the method using the Gaussian distribution of the input variable needs to be
used because it has an exact formulation of calculating the confidence interval for the standard deviation. Even in that case, the bootstrap method can be used, but it might not be as accurate as the method using the Gaussian distribution of input variable because of the randomness of the bootstrap samples. The bootstrap method can be applied to any types of distribution, and the test results for various types of distributions are presented in Ref. 30.

The accuracy of the confidence interval for the standard deviation could be improved by including higher moments such as the skewness or fourth moment [12]. However, the higher moments could yield quite inaccurate confidence interval especially for small number of samples because accuracy of the estimated confidence interval highly depends on the accuracy of the estimated moments from given samples. Even though the parametric percentile method does not achieve the target confidence level for small number of samples because accuracy of the estimated confidence interval for the standard deviation and the correlation parameter for the confidence level of 100 is known that as the number of samples increases, the length of confidence interval for the standard deviation and the correlation parameter for the confidence level of 100 converges to the target confidence level.

3.4 Confidence Interval for Correlation Coefficient. It is known that as the number of samples $ns$ goes to infinity, the sample correlation parameter follows a Gaussian distribution as [33]

$$
\hat{\theta} \sim N\left(\theta, \frac{1}{ns} \left\{ 4w \frac{d^2 g^{-1}(\hat{\theta})}{d\hat{\theta}^2} \right\} \right)
$$

(22)

where

$$
\theta = g^{-1}(\tau), \quad w = \sqrt{\frac{4}{\pi}} \sum_{i=1}^{ns} (w_i + \bar{w} - 2\bar{w})^2,
$$

$$
w_i = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{ns} t_i, \quad \bar{w} = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{ns} \bar{t}_j, \quad \bar{w} = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{ns} w_j.
$$

If $x_1 < x_2$ or $x_2 < x_1$, then $I_{ij} = 1$, otherwise, $I_{ij} = 0$. Thus, the confidence interval for the correlation parameter for the confidence level of 100 $\times$ $(1 - z)$ is obtained as

$$
\Pr\left[ \hat{\theta} - h \leq \theta \leq \hat{\theta} + h \right] = 1 - z
$$

(23)

where $z_{\alpha/2}$ is the CDF value of the Gaussian distribution evaluated at $\alpha/2$ and $h = z_{\alpha/2} \sqrt{\frac{4w}{\pi ns}} \frac{d^2 g^{-1}(\hat{\theta})}{d\hat{\theta}^2}$.

Using the lower and upper bounds of the confidence interval for the correlation parameter $\theta$, the upper and lower bounds of the confidence interval for the correlation coefficient $\tau$ are calculated from $\tau = g(\theta)$ using Eq. (5) or the explicit functions in Ref. 18. To verify Eq. (22), the confidence levels of the 97.5% confidence interval for the correlation coefficient are tested by randomly generating 1000 data sets with $ns = 30, 100, 300$ from four representative copulas with different correlation coefficient of $\tau = 0.2, 0.5, 0.8$. The confidence interval for the correlation parameter is accurately estimated regardless of copula function types as shown in Table 5. Thus, the bootstrap method is not necessary to obtain the confidence interval for the correlation parameter in this paper.

Once the input model with the adjusted parameters using the bootstrap method is obtained, the $\beta_i$-contour can be obtained as explained in Sec. 2.1. To check how the $\beta_i$-contour obtained using the adjusted parameters covers the $\beta_i$-contour of the true input model, the confidence level of the input model is measured by counting enough number of data sets that the $\beta_i$-contour obtained from the adjusted parameters fully covers the true $\beta_i$-contour. The numerical test results are shown in the Sec. 3.5.

3.5 Confidence Levels of Input Model. Let $X_1$ have a log-normal distribution with $\mu_{X_1} = 5.0$ and $\sigma_{X_1} = 5.0$ (COV=1.0); $X_2$ have a Gaussian distribution with $\mu_{X_2} = 5.0$ and $\sigma_{X_2} = 1.0$ (COV=0.2). Let $X_1$ and $X_2$ are correlated with the Frank copula. From the true input model, a different number of samples, $ns = 30, 100, 300$, are randomly generated for a sufficient number of trials, 300. Using the generated samples, the marginal distributions and copulas are identified and their parameters are quantified. If the marginal Gaussian distributions are identified, Eq. (10) is used to calculate the confidence interval for the standard deviation. If not, the parametric percentile bootstrap method is used. Then, the input confidence level is assessed by calculating the probability that the obtained $\beta_i$-contour covers the true $\beta_i$-contour.

Table 5 Confidence level (%) of confidence interval for correlation coefficient using Eq. (22)

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$ns$</th>
<th>Clayton</th>
<th>Gumbel</th>
<th>Frank</th>
<th>Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>30</td>
<td>95.7</td>
<td>98.7</td>
<td>96.6</td>
<td>94.2</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>96.9</td>
<td>96.6</td>
<td>97.3</td>
<td>96.8</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>97.2</td>
<td>97.7</td>
<td>97.3</td>
<td>97.3</td>
</tr>
<tr>
<td>0.5</td>
<td>30</td>
<td>95.3</td>
<td>95.6</td>
<td>96.0</td>
<td>99.0</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>97.2</td>
<td>97.2</td>
<td>96.8</td>
<td>96.6</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>97.3</td>
<td>97.8</td>
<td>97.6</td>
<td>97.2</td>
</tr>
<tr>
<td>0.8</td>
<td>30</td>
<td>96.1</td>
<td>96.1</td>
<td>97.2</td>
<td>92.5</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>97.3</td>
<td>97.5</td>
<td>96.6</td>
<td>96.2</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>97.8</td>
<td>97.8</td>
<td>97.1</td>
<td>97.8</td>
</tr>
</tbody>
</table>
contour over 300 trials. For this test, the target confidence level of 97.5% is used.

Table 6 shows the obtained input confidence levels using the identified marginal distributions and copulas. Since the bootstrap method more accurately calculates the upper bound of the confidence interval for the standard deviation for a non-Gaussian distribution, it yields a more accurate confidence level than the method using the Gaussian distribution for calculation of the upper bound. Likewise, when the true marginal distributions types are used, the input confidence levels using the bootstrap method are more accurate than the method using the Gaussian distribution for calculation of the upper bound as shown in Table 7. When the correct distribution types are used, the performance of the bootstrap method is even more improved as shown in Tables 6 and 7.

To test the confidence levels for various input models, four input models with distinct \( \beta \)-contour shapes shown in Fig. 3 are tested. The statistical information on four input models is presented in Table 8.

Randomly generating 100 data sets of \( ns = 30, 100, \) and 300 from the four input models, the distribution types are identified using the Bayesian method, and the adjusted parameters are estimated using the parametric percentile bootstrap method. As shown in Table 9, even though the input confidence levels for four input models do not reach the target confidence level, 97.5%, they tend to converge to the target as the number of samples increases.

It is possible that the true joint distribution is not one of candidates in the Bayesian method. If so, the obtained input confidence level may not converge to the target confidence level even for infinite number of samples. However, the Bayesian method selects the one that best describes the given data among candidates. Thus, even though the selected distribution may not be the true one, it will yield a similar confidence level as long as the selected distribution has similar distribution shape with the true one.

If the number of samples is very limited, i.e., less than 20, the identification of marginal distributions and copulas could be meaningless. The research on how the confidence levels are incorporated in the problems with very limited data will be a future research.

### 4 Numerical Examples

In this section, a mathematical example and an M1A1 tank roadarm with correlated non-Gaussian input variables are used to demonstrate how the parametric percentile bootstrap method yields more reliable design than the method assuming a Gaussian distribution of input variable. To carry out RBDO, the MPP-based dimension reduction method (DRM) [34] is used for more accurate calculation of the probability of failure than the first-order reliability method (FORM).

#### 4.1 Mathematical Example

Let \( X_1 \) and \( X_2 \) have lognormal and Gaussian distributions, \( X_1 \sim LN(3, 1.5^2) \) and \( X_2 \sim N(3, 0.3^2) \), respectively, which are correlated with the Frank copula and \( \tau = 0.7 \). From the true input model, 100 data sets with \( ns = 30, 100, \) and 300 are randomly generated, and the marginal distribution, the copula type, and their parameters are determined from each data set.

For the given data, the adjusted standard deviation and correlation coefficients are obtained using Eqs. (7) and (8), respectively. For the adjusted standard deviation for the non-Gaussian distribution, the parametric percentile bootstrap method is used to calculate the confidence interval for the standard deviation. In the RBDO formulation, the input model with the adjusted parameters is used to estimate the probabilistic constraint for RBDO.

For the comparison study, one model with the estimated parameters and another model with the adjusted parameters obtained using the parametric percentile bootstrap method are tested. The two input models are used to estimate the probabilistic constraints of RBDO. The output confidence levels are assessed by counting the number that the probabilities of failure evaluated at optimum designs obtained from 100 data sets are smaller than the target probability of failure, 2.275%.

The RBDO problem is formulated to

\[
\begin{align*}
\text{minimize} & \quad \text{cost}(d) = d_1 + d_2 \\
\text{subject to} & \quad P(G_j(X) > 0) \leq P_{F_j}^{\text{tar}} (= 2.275\%), \quad j = 1, 2, 3 \\
& \quad d = \mu(X), \quad 0 \leq d_1, d_2 \leq 10 \\
& \quad G_1(X) = 1 - (0.9010X_1 - 0.4339X_2 + 1.5)^2 \\
& \quad \times (0.4339X_1 + 0.9010X_2 + 2)/20 \\
& \quad G_2(X) = 1 - (X_1 + X_2 - 2.8)^2 / 30 - (X_1 - X_2 - 12)^2 / 120 \\
& \quad G_3(X) = 1 - 200 / (2.5(0.9010X_1 - 0.4339X_2 - 3)^2 \\
& \quad + 8(0.4339X_1 + 0.9010X_2 + 5))
\end{align*}
\]

where three constraints are shown in Fig. 4.

Table 10 shows the minimum, median, and maximum values of the probabilities of failure \( P_{F_i} \), and \( P_{F_j} \) for two active constraints \( G_1 \) and \( G_2 \) evaluated at the optimum designs using the Monte Carlo simulation (MCS), and the output confidence levels of optimum designs, which are obtained from 100 data sets. Figures 5 and 6 show box plots of the statistical test results in Table 10. As
shown in Table 10, when the input model with the estimated parameters is used for $n_s = 30, 49$ and $30$ out of $100$ data sets for $G_1$ and $G_2$, respectively, failed to achieve the target probability of failure. Thus, the confidence levels of the output performance are significantly lower than the target confidence level $97.5\%$ as shown in Table 10. This phenomenon does not change significantly even when the number of samples in each data set increases. As shown in Fig. 5, as the number of sample data increases, boxes whose edges indicate the $25$th and $75$th percentiles become smaller, which means that the input estimation becomes accurate. However, even with $300$ samples, the output confidence level for $G_1$ is still $52\%$ as shown in Table 10, which is still far less than the target confidence level of $97.5\%$. In this example, the reason that the confidence level for $G_2$ is relatively close to the target is because there exists reliability analysis error due to the nonlinearity of $G_2$ function and highly nonlinear transformation. Even if the MPP-based DRM is used to reduce the reliability analysis error, the probability of failure for $G_2$ at the optimum design obtained using the true input model is $1.922\%$, which is less than the target. Thus, as the number of samples increases, probabilities of failure at optimum designs obtained using the estimated parameters tend to converge to $1.922\%$ instead of $2.275\%$.

On the other hand, when the input model with the adjusted parameters obtained using the parametric percentile bootstrap method is used, the median values of $P_{G_1}$ and $P_{G_2}$ are smaller than $2.275\%$, which yields more confidence than the case of the estimated parameter. Accordingly, the obtained output confidence levels using the adjusted parameters is much closer to the target confidence level, $97.5\%$ as shown in Table 6. As the number of samples increases, the output confidence levels using the parametric percentile bootstrap method are getting much closer to the target confidence level as shown in Fig. 6, whereas the input model with the estimated parameters does not provide the output confidence level near the target confidence level as shown in Table 5. Also, as the number of samples increases, the average cost value at the optimum design decreases, which means that we can obtain a better optimum design if we can spend more on accurate input estimation.

### Table 8 Statistical information for four cases

<table>
<thead>
<tr>
<th>Case</th>
<th>Marginal distribution</th>
<th>Copula</th>
<th>Kendall’s tau</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>Weibull (COV = 1.0)</td>
<td>Gumbel</td>
<td>A12</td>
</tr>
<tr>
<td>Case 2</td>
<td>Lognormal (COV = 0.5)</td>
<td>Gaussian</td>
<td>Gumbel</td>
</tr>
<tr>
<td>Case 3</td>
<td>Gaussian (COV = 0.5)</td>
<td>Weibull</td>
<td>Frank</td>
</tr>
<tr>
<td>Case 4</td>
<td>Extreme-II (COV = 0.2)</td>
<td>Lognormal</td>
<td>Clayton</td>
</tr>
</tbody>
</table>

4.2 M1A1 Abrams Tank Roadarm. The roadarm in the M1A1 tank is modeled using 1572 eight-node isoparametric finite elements (SOLID45) and four beam elements (BEAM44) of a commercial program, ANSYS [35], as shown in Fig. 7. The material of the roadarm is S4340 steel with Young’s modulus $E = 3.0 \times 10^5$ psi and Poisson’s ratio $\nu = 0.3$. The durability analysis of the roadarm is carried out to obtain the fatigue life contour using the Durability and Reliability Analysis Workspace (DRAW) code [36,37]. The fatigue lives at the 13 critical nodes are selected for the design constraints of the RBDO. Detailed information on the location of 13 critical nodes and design parameterization of the roadarm can be found in Ref. 34.

Table 11 shows the assumed statistical information of random variables and parameters for RBDO of the roadarm. Since the true statistical information on S4340 steel except its nominal values is not available, it is assumed in the paper using the study of SAE 950X. First, it is assumed that Frank copula ($\tau = -0.683$) for $\sigma_1$ and $b$, and Gaussian copula ($\tau = -0.906$) for $\varepsilon_1$ and $c$, respectively, are the true copulas. As the two copulas well describe the experimental data of SAE 950X [19] as shown in Fig. 8, it seems to be reasonable to select these two copulas to model the joint CDFs of the four correlated random parameters of S4340 steel. Furthermore, the marginal distribution types of S4340 steel are assumed to be the same as those of SAE 950X.

Second, once the copula and marginal distribution types are obtained, the mean and standard deviation of S4340 need to be determined. The mean values of four fatigue material properties of S4340 are obtained from a materials standard book, but the standard deviations are unknown. Therefore, the standard deviations are assumed using COV of SAE 950X. The coefficient of variation of SAE 950X is $115\%$ for $\varepsilon_1$ and $25\%$ for other material properties [19]. Since S4340 steel is a stronger material than SAE 950X, in this paper, it is assumed that the COV of S4340 is $50\%$ for $\varepsilon_1$ and $25\%$ for other material properties to estimate the standard deviation as shown in Table 11.
Assuming that S4340 steel has the true statistical information as shown in Table 11, 30 paired data are randomly generated from the assumed true statistical information, and two input statistical models with estimated and adjusted parameters, respectively, are computed using the generated 30 data and will be used to carry out RBDO. Table 12 shows the estimated and adjusted parameters, and the target confidence level is specified as 97.5% in this roadarm example.

The RBDO for the roadarm is formulated to

<table>
<thead>
<tr>
<th>RBDO with</th>
<th>ns</th>
<th>Minimum</th>
<th>Median</th>
<th>Maximum</th>
<th>CL</th>
<th>Minimum</th>
<th>Median</th>
<th>Maximum</th>
<th>CL</th>
<th>Average cost at optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated parameter</td>
<td>30</td>
<td>0.128</td>
<td>2.149</td>
<td>22.003</td>
<td>51</td>
<td>0.158</td>
<td>1.515</td>
<td>11.982</td>
<td>70</td>
<td>5.376</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.369</td>
<td>2.329</td>
<td>8.759</td>
<td>48</td>
<td>0.262</td>
<td>1.556</td>
<td>5.010</td>
<td>84</td>
<td>5.422</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>0.805</td>
<td>2.246</td>
<td>4.689</td>
<td>52</td>
<td>0.842</td>
<td>1.709</td>
<td>3.081</td>
<td>90</td>
<td>5.434</td>
</tr>
<tr>
<td>Adjusted parameter</td>
<td>30</td>
<td>0.000</td>
<td>0.273</td>
<td>4.162</td>
<td>93</td>
<td>0.004</td>
<td>0.466</td>
<td>8.082</td>
<td>92</td>
<td>6.374</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.003</td>
<td>0.551</td>
<td>4.687</td>
<td>95</td>
<td>0.122</td>
<td>1.018</td>
<td>4.788</td>
<td>93</td>
<td>6.032</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>0.329</td>
<td>0.954</td>
<td>2.847</td>
<td>96</td>
<td>0.582</td>
<td>1.540</td>
<td>2.611</td>
<td>97</td>
<td>5.792</td>
</tr>
</tbody>
</table>

*Indicates the confidence level.

Fig. 5 Box plot for estimated parameters

Fig. 6 Box plot for adjusted parameters

Fig. 7 Finite element model of roadarm
minimize $\text{cost}(\mathbf{d})$

subject to $P(G_j(X) \geq 0) \leq P_{T_{tar}}^{f}, \ j = 1 \sim 13$

$d = \mu(X), \ \mathbf{d}^l \leq \mathbf{d} \leq \mathbf{d}^u, \ \mathbf{d} \in R^n, P_{T_{tar}}^{f} = 2.275\%$

$G_j(X) = 1 - \frac{L(X)}{L_t}, \ j = 1 \sim 13$

(25)

cost($\mathbf{d}$) : Weight of Roadarm
$L(X)$ : Crack Initiation Fatigue Life,
$L_t$ : Crack Initiation Target Fatigue Life (= 5years)

Table 13 shows a comparison of RBDO results for various input models where the assumed true input model is the one in Table 11. First, the RBDO results are compared for the independent and correlated input fatigue material properties. As shown in the table, when the correlation between material properties is considered, the optimized weight of the roadarm is significantly reduced from 592.22 to 514.02 for the same target reliability. This is because the material properties are highly and negatively correlated as shown in Fig. 8. Thus, it is very important to correctly model the correlation between material properties to carry out the RBDO.

Second, when the estimated input model is used, the underestimated standard deviations (see Tables 11 and 12) yield a smaller optimum cost than the optimum cost obtained using the true input model (509.44 versus 514.02). On the other hand, when the input model with the adjusted parameters is used, the obtained optimum cost is higher than the optimum cost obtained from the true input model (531.64 versus 514.02). Since the MCS cannot be used for the benchmark test for this problem due to computational cost, it is difficult to determine which input model yields reliable optimum designs by comparing the optimum costs. Thus, at each optimum design, fatigue analysis is carried out using the true input.

Table 11 Random variables and fatigue material properties

<table>
<thead>
<tr>
<th>Random variables</th>
<th>Lower bound $\mathbf{d}^l$</th>
<th>Initial design $\mathbf{d}^0$</th>
<th>Upper bound $\mathbf{d}^u$</th>
<th>Standard deviation</th>
<th>Distribution Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>1.3500</td>
<td>1.7500</td>
<td>2.1500</td>
<td>0.0875</td>
<td>Gaussian</td>
</tr>
<tr>
<td>$d_2$</td>
<td>2.6496</td>
<td>3.2496</td>
<td>3.7496</td>
<td>0.1625</td>
<td>Gaussian</td>
</tr>
<tr>
<td>$d_3$</td>
<td>1.3500</td>
<td>1.7500</td>
<td>2.1500</td>
<td>0.0875</td>
<td>Gaussian</td>
</tr>
<tr>
<td>$d_4$</td>
<td>2.5703</td>
<td>3.1703</td>
<td>3.6703</td>
<td>0.1585</td>
<td>Gaussian</td>
</tr>
<tr>
<td>$d_5$</td>
<td>1.3565</td>
<td>1.7563</td>
<td>2.1563</td>
<td>0.0878</td>
<td>Gaussian</td>
</tr>
<tr>
<td>$d_6$</td>
<td>2.4377</td>
<td>3.0377</td>
<td>3.5377</td>
<td>0.1519</td>
<td>Gaussian</td>
</tr>
<tr>
<td>$d_7$</td>
<td>1.3517</td>
<td>1.7517</td>
<td>2.1517</td>
<td>0.0876</td>
<td>Gaussian</td>
</tr>
<tr>
<td>$d_8$</td>
<td>2.5085</td>
<td>2.9085</td>
<td>3.4085</td>
<td>0.1454</td>
<td>Gaussian</td>
</tr>
</tbody>
</table>

Fatigue material properties

<table>
<thead>
<tr>
<th>Nondesign uncertainties</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Distribution Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fatigue strength coefficient, $\sigma_f^j$</td>
<td>177000</td>
<td>44250</td>
<td>Lognormal</td>
</tr>
<tr>
<td>Fatigue strength exponent, $b^j$</td>
<td>-0.0730</td>
<td>0.018</td>
<td>Gaussian</td>
</tr>
<tr>
<td>Fatigue ductility coefficient, $\varepsilon_f^j$</td>
<td>0.4100</td>
<td>0.205</td>
<td>Lognormal</td>
</tr>
<tr>
<td>Fatigue ductility exponent, $c^j$</td>
<td>-0.6000</td>
<td>0.150</td>
<td>Gaussian</td>
</tr>
</tbody>
</table>

Table 12 Estimated and adjusted parameters

<table>
<thead>
<tr>
<th>$\sigma_f^j$</th>
<th>$b$</th>
<th>$\varepsilon_f^j$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\sigma}_f^j$</td>
<td>176738</td>
<td>-0.073</td>
<td>0.395</td>
</tr>
<tr>
<td>$\hat{b}$</td>
<td>35141</td>
<td>0.015</td>
<td>0.143</td>
</tr>
<tr>
<td>$\hat{\varepsilon_f}^j$</td>
<td>45356</td>
<td>0.020</td>
<td>0.223</td>
</tr>
<tr>
<td>$\hat{c}$</td>
<td>-0.701</td>
<td>-0.596</td>
<td>-0.921</td>
</tr>
</tbody>
</table>

Copula

Frank

Gaussian
model in Table 11. At the optimum design obtained using the estimated input model, the lowest fatigue life is 1.5 years, which occurs at the MPP of $G_1$. This fatigue life is much less than the target (5 years). On the other hand, at the optimum design obtained using the adjusted parameters, the lowest fatigue life is 6.16 years, which occurs at the MPP of $G_4$, which is larger than the target fatigue life. Accordingly, the input model with the adjusted standard deviation and correlation coefficient is indeed necessary to obtain a reliable optimum design.

5 Conclusions

In many engineering applications, only limited test data are available for input variables, and thus, the input statistical model obtained from the insufficient data could yield an unreliable optimum design. Thus, the RBDO with confidence level is proposed to offset the inaccurate estimation of the input model by using the adjusted standard deviation and correlation coefficient. The adjusted standard deviation is obtained from the confidence intervals for the standard deviation, mean, and correlation coefficient. If the input variables have a Gaussian distribution, the method using Gaussian distribution of input variables is explicit and exact. If not, it yields an inaccurate estimation of the confidence interval for the standard deviation. Thus, in this paper, the bootstrap method is used to calculate the confidence interval for the standard deviation, and thus, the adjusted standard deviation. The input model with the adjusted parameters obtained from the bootstrap method is used to input and output confidence levels for the non-Gaussian distributions. Numerical test shows that the percentile method has the most desirable performance out of five candidate bootstrap methods for the parametric approach. Numerical examples also show that the input model using the parametric percentile bootstrap method yields more reliable design than the one using other methods for the non-Gaussian distributions.

Acknowledgment

Research is primarily supported by the Automotive Research Center, which is sponsored by the U.S. Army TARDEC and ARO Project W911NF-09-1-0250. This research was also partially supported by the World Class University Program through the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education, Science and Technology (Grant Number R32-2008-000-10161-0 in 2009). These supports are greatly appreciated.

References