A Performance and Plume Comparison of Xenon and Krypton Propellant on the SPT-100

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The use of krypton as an alternative to xenon for Hall thruster propellant is an interesting option for satellite system designers due to its lower cost. However, this cost-savings comes at the expense of thrust efficiency. Reduction in efficiency can be caused by energy losses from Joule heating, radiation, and the ionization process as well as degradation of plume quality from an increase in velocity distribution spread (most often from an increase in multiply charged ion populations) and geometric beam divergence. In order to quantify this performance reduction for the case of the flight model SPT-100 HET (1.35 kW), performance measurements were made using an inverted pendulum thrust stand. The plume was also characterized by a Faraday probe and RPA measurements to examine how plume qualities change with operating condition. Krypton operating conditions were tested over a large range of operating powers from 800 W to 3.9 kW. Analysis of how performance is impacted by propellant and operating condition is presented. A simple mission analysis was done based on the performance measurements to evaluate the practicality of krypton propellant for an SPT-100 subsystem using krypton propellant for north-south station keeping (NSSK) for a typical communications spacecraft in geosynchronous orbit.
A Performance and Plume Comparison of Xenon and Krypton Propellant on the SPT-100

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Introduction

• Motivation
  – Kr less expensive than Xe (~ 10× by vol., ~ 6× by mass)
  – Kr may increase $I_{sp}$ (25% greater velocity for a given discharge voltage due to smaller atomic mass)
    • Beneficial for station-keeping missions
  – Simple integration into existing Xe propellant management system

• Study Objectives
  – Characterize differences between Xe and Kr performance and plume characteristics for a flight qualified thruster over a wide range of operating conditions
  – Study how individual thruster efficiency components change with propellant and operating conditions
  – Determine if Kr is a viable alternative propellant for the flight model SPT-100 for NSSK purposes

• Methodology
  – Measure thrust for a wide range of operating conditions
    • Kr test matrix: 800 W - 3.9 kW
    • Smaller set of Xe test cases for comparison
    • Evaluate how measured performance translates into firing time and propellant mass required for typical NSSK duties for a GEO COMM satellite
  – Measure plume characteristics with electrostatic probes
    • Calculate beam efficiency, current utilization efficiency, and voltage utilization efficiency from probe data
Testing Facilities

Chamber 1, AFRL Edwards AFB
- 2.4 m dia., 4.1 m length, stainless steel
- Two cryogenic pumps
- 1.2 m dia., LN2 baffled (70 K), 2 stage He (15 K)
- SPT-100 operation background pressure
  - Xe: $1.1 \times 10^{-5}$ Torr (56.4 sccm flow rate)
  - Kr: $1.0 \times 10^{-5}$ Torr (65.4 sccm flow rate)

Inverted Pendulum Thrust Stand
- Null-Type Design
  - Electromagnetic force generated by PID feedback system counteracts displacement
  - Restoring force linear proportional to voltage applied to null coil
  - In-situ calibration performed with weights loaded and unloaded by mechanical pulley system
  - Design by Haag at NASA GRC

Probe Translation System
- Two stage combination
  - Rotary stage for angular movement
  - Linear stage for radial movement
- Stepper motor driven
  - $r$ limit: 100 cm
  - $\theta$ limit: $\pm 90^\circ$
SPT-100 Hall Effect Thruster

Thruster Characteristics

- Anode Power: 1350 W
- Conventional 5 magnetic core design
  - One inner, four outer connected in series
  - Magnetic circuit current supplied by anode current
- Acceleration channel: 100 mm outer dia., 69 mm inner dia., 28 mm depth
- 2 lanthanum hexaboride (LaB$_6$) cathodes
- Space Systems/Loral flight heritage
  - 7 spacecraft with SPT-100 propulsion subsystems in orbit
  - 13 years of cumulative on-orbit experience
  - Single thruster has accumulated over 6 years of near-daily operation
  - 11 more spacecraft under construction

<table>
<thead>
<tr>
<th>Thrust</th>
<th>83 mN</th>
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<tr>
<td>Thrust Efficiency</td>
<td>50%</td>
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<tr>
<td>$I_{sp}$</td>
<td>1600 s</td>
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NASA life and performance testing:


Kr and Kr/Xe mixture performance testing:
### Operating Conditions Tested

#### Krypton

<table>
<thead>
<tr>
<th>Anode Potential</th>
<th>Power Values (W)</th>
<th>-20%</th>
<th>-10%</th>
<th>Nominal</th>
<th>+10%</th>
<th>+20%</th>
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<tr>
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<td></td>
<td>813</td>
<td>950</td>
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<td>1209</td>
<td>1339</td>
<td>1476</td>
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<td>915</td>
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<tr>
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<td><strong>1356</strong></td>
<td>1516</td>
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<tr>
<td>+10% (332 V)</td>
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<td>1121</td>
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<td>1670</td>
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<td>1242</td>
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#### Xenon

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<th>-20%</th>
<th>-10%</th>
<th>Nominal</th>
<th>+10%</th>
<th>+20%</th>
<th>+30%</th>
<th>+40%</th>
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<tr>
<td>-10% (272 V)</td>
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<td>1785</td>
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</table>

**Anode Potential:**
-30% (211 V)
-20% (242 V)
-10% (272 V)
Nominal (302 V)
+10% (332 V)
+20% (363 V)
+30% (393 V)

**Propellant Flow Rate:**
-30% (3.88 mg/s)
-20% (4.44 mg/s)
-10% (4.99 mg/s)
Nominal (5.54 mg/s)
+10% (6.10 mg/s)
+20% (6.65 mg/s)
+30% (7.21 mg/s)

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+20% (363 V)
+30% (393 V)
Performance Results: Thrust

-20% Flow Rate (3.27 mg/s)
-10% Flow Rate (3.68 mg/s)
Nominal Flow Rate (4.09 mg/s)
+10% Flow Rate (4.50 mg/s)
+20% Flow Rate (4.90 mg/s)
+30% Flow Rate (5.31 mg/s)
+40% Flow Rate (5.72 mg/s)

Nominal Condition (300 V, 1350 W)
- Xe: 82.1 mN
- Kr: 62.9 mN

Thrust increases linearly with discharge potential and propellant flow rate
Thrust gains level out around 510 – 540 V
Performance Results: Specific Impulse

\[ I_{sp} = \frac{1}{g_e} \sqrt{2 \frac{q V_a}{m}} \]

Krypton Data
- -20% Flow Rate (3.27 mg/s)
- -10% Flow Rate (3.68 mg/s)
- Nominal Flow Rate (4.09 mg/s)
- +10% Flow Rate (4.50 mg/s)
- +20% Flow Rate (4.90 mg/s)
- +30% Flow Rate (5.31 mg/s)
- +40% Flow Rate (5.72 mg/s)

Xenon Data
- Nominal Flow Rate (5.54 mg/s)
- Nominal Discharge Potential (302 V)

• Nominal Condition (1350 W)
  - Xe: 1511 s
  - Kr: 1568 s

• Performance Gap Between Actual and Ideal \( I_{sp} \)
  - Xe: 520 s
  - Kr: 1000 s

• Higher Kr \( I_{sp} \) not realized due to inferior propellant utilization fraction \( \frac{m_i}{m_a} \) and higher momentum divergence half-angle
Performance Results: Thrust Efficiency

- Krypton Data
  - -20% Flow Rate (3.27 mg/s)
  - -10% Flow Rate (3.68 mg/s)
  - Nominal Flow Rate (4.09 mg/s)
  - +10% Flow Rate (4.50 mg/s)
  - +20% Flow Rate (4.90 mg/s)
  - +30% Flow Rate (5.31 mg/s)
  - +40% Flow Rate (5.72 mg/s)

- Xenon Data
  - Nominal Flow Rate (5.54 mg/s)
  - Nominal Discharge Potential (302 V)

• Nominal Condition (300 V, 1350 W)
  - Xe: 45%
  - Kr: 36%

• Kr efficiency improves with higher operating powers
Krypton Mission Example: NSSK for GEO COMM S/C

Can Krypton Propellant Satisfy Mission Needs?

Mission Parameters

- *Initial spacecraft mass*: 3760 kg
- *Mission lifetime*: 15 years
- *Thruster cant angle*: 40 deg. (directional cosine loss)
- *Quantity of SPT-100 thrusters*: 2
- *Delta-V required*: 
  \( (51 \text{ m/s/year}) \times 15 \text{ years} = 765 \text{ m/s} \)
Krypton Mission Example: NSSK for GEO COMM S/C

Initial spacecraft mass: 3760 kg
Mission lifetime: 15 years
Thruster cant angle: 40 deg.
Quantity of SPT-100 thrusters: 2
Delta-V required: 765 m/s

Nominal Xe Operating Condition
Vd = 302 V, Id = 4.49 A, P = 1356 W
T = 82 mN, Isp = 1500 s
1.82 MN-s, 6160 hr

\[ M_p = M_0 - M_0 \exp \left( \frac{-\Delta V}{g_e I_{sp} \cos \theta} \right) \]

Krypton Data
- -20% Flow Rate (3.27 mg/s)
- -10% Flow Rate (3.68 mg/s)
- Nominal Flow Rate (4.09 mg/s)
- +10% Flow Rate (4.50 mg/s)
- +20% Flow Rate (4.90 mg/s)
- +30% Flow Rate (5.31 mg/s)
- +40% Flow Rate (5.72 mg/s)

Contour Lines:
Thruster Power for Kr Operation

Xenon Data
- Nominal Flow Rate (5.54 mg/s)
- Nominal Discharge Potential (302 V)
• Kr appears capable of satisfying mission requirements (despite reduced performance)
  – Higher power required
  – Longer firing time required
  – Erosion rates for Kr unknown

• 390 V, nominal flow rate operating point vs. nominal Xe condition
  – 54 kg of total propellant mass saved
  – 400 additional Watts of power per thruster
  – 400 additional hours of firing time per thruster
# Krypton Test Matrix

<table>
<thead>
<tr>
<th>Discharge Potential</th>
<th>Propellant Flow Rate</th>
<th>Faraday Probe</th>
<th>RPA</th>
</tr>
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<tbody>
<tr>
<td></td>
<td><strong>Nominal</strong></td>
<td>(4.09 mg/s)</td>
<td>(302 V)</td>
</tr>
<tr>
<td></td>
<td>+20%</td>
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<td></td>
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<td>(mg/s)</td>
<td>(483 V)</td>
</tr>
<tr>
<td></td>
<td>+80%</td>
<td>(mg/s)</td>
<td>(544 V)</td>
</tr>
</tbody>
</table>

- Faraday Probe: $r = 50$ cm to 100 cm, 5 cm increments
- $\theta = -90$ to 90 deg, 1 deg. increments

- RPA: $r = 100$ cm
- $\theta = -90$ to 90 deg, 5 deg. increments
Faraday Probe Results: Xe

**$V_d$ Variations**

$r = 100 \text{ cm}$

**Flow Rate Variations**

$r = 100 \text{ cm}$
Faraday Probe Results: Kr

**V<sub>d</sub> Variations**

Discharge Potential (V)
-20% (242)
nominal (302)
+20% (363)
+40% (423)
+60% (483)
+80% (544)

**Flow Rate Variations**

Propellant Flow Rate (mg/s)
- Nominal (4.09)
+20% (4.90)
+40% (5.72)

**Graphs**

- **V<sub>d</sub> Variations**
  - Charge Flux (A/m<sup>2</sup>)
  - θ (deg.)
  - r = 100 cm

- **Propellant Flow Rate**
  - Charge Flux (A/m<sup>2</sup>)
  - θ (deg.)
  - r = 100 cm
Faraday Results: Xe/Kr Comparison

Nominal Conditions

- Xe, r = 100 cm
- Kr, r = 100 cm
- Xe, r = 50 cm
- Kr, r = 50 cm

Krypton Data
- Nominal Flow Rate (4.09 mg/s)
- +20% Flow Rate (4.90 mg/s)
- +40% Flow Rate (5.72 mg/s)

Xenon Data
- Nominal Flow Rate (5.54 mg/s)
- Nominal Discharge Potential (302 V)
RPA Results: Xe/Kr, 0 deg.

**Xe**

- **Nominal Flow Rate**

  ![Xe Nominal Flow Rate Graph](image1)

- **Nominal Discharge Potential**

  ![Xe Nominal Discharge Potential Graph](image2)

**Kr**

- **Nominal Flow Rate**

  ![Kr Nominal Flow Rate Graph](image3)

- **Nominal Discharge Potential**

  ![Kr Nominal Discharge Potential Graph](image4)
RPA Results: Xe/Kr, angular position

\[ \theta = 0 \text{ deg.} \quad \theta = 15 \text{ deg.} \quad \theta = 30 \text{ deg.} \quad \theta = 45 \text{ deg.} \]

\[ \theta = 60 \text{ deg.} \quad \theta = 75 \text{ deg.} \quad \theta = 90 \text{ deg.} \]

EDF most probable energy 6 V less for Kr
Probe Derived Efficiency
Components: Faraday Probe

Total Beam Current

\[ I_b \approx \pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} j(\theta) \sin \theta \, d\theta \]

Current Utilization Efficiency

\[ \frac{I_b}{I_d} \]

Beam Efficiency (divergence loss factor)

\[ \Psi_B = <\cos \theta>_{mv}^2 \]

\[ \approx (\frac{I_{axial}}{I_b})^2 \approx \left( \frac{\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} j(\theta) \cos \theta \sin \theta \, d\theta}{\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} j(\theta) \sin \theta \, d\theta} \right)^2 \]

Krypton Data
- ▲ Nominal Flow Rate (4.09 mg/s)
- ▲ +20% Flow Rate (4.90 mg/s)
- ▲ +40% Flow Rate (5.72 mg/s)

Xenon Data
- ▲ Nominal Flow Rate (5.54 mg/s)
- ▲ Nominal Discharge Potential (302 V)
Probe Derived Efficiency Components: RPA – Voltage Utilization Efficiency

EDF mean method

\[
\left( \frac{\Delta V}{V_d} \right) = \left( \frac{<V(\theta)>_m}{V_d} \right) \approx \frac{1}{V_d} \left( \frac{\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} j(\theta) \bar{V}(\theta) |\sin \theta| d\theta}{\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} j(\theta) |\sin \theta| d\theta} \right)
\]

Krypton Data
- ▲ Nominal Flow Rate (4.09 mg/s)
- ● +20% Flow Rate (4.90 mg/s)
- ● +40% Flow Rate (5.72 mg/s)

Xenon Data
- ▲ Nominal Flow Rate (5.54 mg/s)
- ◁ Nominal Discharge Potential (302 V)

EDF > 100 eV/q method

EDF most probable energy method

Kr EDF mean values

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Conclusions

• Performance Study
  – Thrust Efficiency
    • Kr is 9% less efficient at nominal operating power (36% vs. 45%)
    • Efficiency improves with increasing power
  – Specific Impulse
    • Kr not significantly higher than Xe
    • Due to lower propellant utilization fraction and higher plume divergence
  – On-orbit use
    • May be feasible for NSSK mission requirements despite lower performance
    • Life tests with erosion study would necessary to validate thruster life time for inherent higher energy throughput required
    • Potential benefit is lower propellant cost, but not mass savings (due to similar $I_{sp}$)

• Plume Study
  – Faraday Probe
    • Beam focusing improves with higher discharge potential (both Xe and Kr)
    • Kr has lower beam efficiency than Xe (by 0.06 for nominal condition)
    • Similar current utilization efficiency for Xe and Kr
  – RPA
    • Similar EDF shape for Xe and Kr (but higher low energy peak for Xe at oblique angles)
    • Most probable energy lower for Kr (6 V at nominal condition)
    • Kr has lower voltage utilization efficiency
Back Up Slides

Back Up Slides
Krypton Mission Example: NSSK for GEO COMM S/C

• Examine propulsion system requirements for each operating condition tested
  – Total propellant mass
  – Total firing time

• Calculate propellant mass with rocket equation:

\[ M_p = M_0 - M_0 \exp \left( \frac{-\Delta V}{g_e I_{sp} \cos \theta} \right) \]

• Calculate total firing time with propellant mass and operating condition flow rate:

\[ t = \frac{M_p}{\dot{m}_t} \]
Performance Evaluation Techniques

\[ \eta_a = \left( \frac{\Delta V}{V_d} \right) \left( \frac{I_b}{I_d} \right) \Phi_P \Psi_B \]

Anode Efficiency
Voltage Util. Efficiency
Current Util. Efficiency
Energy Efficiency
Propellant Util. Eff.
Beam Eff.

Losses:
- Joule heating
- Radiation from jet
- Ionization losses

Loss: Non-uniform VDF throughout plume
Loss: Plume Divergence
Calculation requires velocity (or charge species fractions)
Geometric characteristic of beam flux

Mathematically separated from jet vector properties

\[ \eta_a = \frac{P_{\text{thrust}}}{P_d} = \frac{T^2}{2mP_d} \]

Faraday Probe

• Measures ion charge flux
• Molybdenum electrodes
• Guard ring ensures planar sheath over collection disk
• Biased to -30 V (relative to chamber ground) to ensure ion saturation
• Effective current collection area assumed to include a portion of the gap surface area

\[
A_{\text{eff}} = A_c + A_{\text{gap}} \left( \frac{d_c}{d_g} \right)
\]

(both electrodes have same thickness)
Traces from $r = 30$ to $r = 100$ cm

Linearly extrapolated to $r = 0$ cm
Propellant Utilization Fraction

- Ion charge species fractions unknown
- Data suggests \( \left( \frac{m_i}{m_a} \right) \) significantly lower for Kr
- Increasing flow rate for Kr seems to improve propellant utilization
Retarding Potential Analyzer (RPA)

- Energy filtered Faraday probe
- Four grid design
- I-V traces fit with smoothing spline
- Differentiated to calculate Energy-per-charge Distribution Function (EDF)

Graphs and diagrams illustrating the device's design and performance metrics.
Methodology: Thrust

\[ T = \dot{m} <\bar{v}>_m <\cos(\theta)>_{mv} \]

\[ = \pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{m}(\theta)\bar{v}(\theta)\cos(\theta)|\sin(\theta)| \, d\theta \]

\[ \bar{v} = \frac{\int f(v)v \, dv}{\int f(v) \, dv} \]

\[ <\bar{v}>_m = \frac{\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{m}(\theta)\bar{v}(\theta)|\sin(\theta)| \, d\theta}{\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{m}(\theta)|\sin(\theta)| \, d\theta} \]

\[ <\cos(\theta)>_{mv} = \frac{\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{m}(\theta)\bar{v}(\theta)\cos(\theta)|\sin(\theta)| \, d\theta}{\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{m}(\theta)\bar{v}(\theta)|\sin(\theta)| \, d\theta} \]
Methodology: Anode Efficiency

\[
\eta_a = \frac{P_{\text{thrust}}}{P_{\text{input}}} = \frac{1}{2} \frac{T^2}{\dot{m}} = \frac{1}{2} \frac{\dot{m} < \cos(\theta) >_{mv} < \bar{v} >_m^2}{P_d}
\]

\[
= \frac{1}{2} \frac{\dot{m} < \cos(\theta) >_{mv} < \bar{v} >_m^2}{P_d} \cdot \frac{< \bar{v} >_m^2}{< \bar{v} >_m^2}
\]

\[
= \frac{1}{2} \frac{\dot{m} < \bar{v} >_m^2}{P_d} \cdot < \cos(\theta) >_{mv} \cdot \frac{< \bar{v} >_m^2}{< \bar{v} >_m^2}
\]

\[
= \frac{I_b \Delta V}{I_d V_d} \cdot \Psi_B \cdot \Phi_P
\]

\[
< \bar{v} >_m = \frac{\pi r^2 \int_{-\pi/2}^{\pi/2} \dot{m}(\theta)\bar{v}^2(\theta)\sin(\theta) d\theta}{\pi r^2 \int_{-\pi/2}^{\pi/2} \dot{m}(\theta)\sin(\theta) d\theta}
\]

\[
= \eta_E \cdot \Psi_B \cdot \Phi_P
\]
Charge Flux Weighting Approximations

\[ j(\theta) \approx \frac{I(\theta)}{A_{\text{probe}}} = \dot{m}_i(\theta) \frac{N_A Q(\theta)}{m A_{\text{probe}}} = C Q(\theta) \dot{m}_i(\theta) \]

• To approximate momentum weighting (beam efficiency):

\[ \dot{m}(\theta) \bar{v}(\theta) \propto j(\theta) \]

Must assume \( Q(\theta) \) AND \( \bar{v}(\theta) \) are constant

• To approximate mass weighting (voltage util. efficiency):

\[ \dot{m}(\theta) \propto j(\theta) \]

Only need to assume \( Q(\theta) \) is constant