Algorithms for ocean-bottom albedo determination from in-water natural-light measurements

Robert A. Leathers and Norman J. McCormick

A method for determining the ocean-bottom optical albedo $R_b$ from in-water upward and downward irradiance measurements at a shallow site is presented, tested, and compared with a more familiar approach that requires additional measurements at a nearby deep-water site. Also presented are two new algorithms for estimating $R_b$ from measurements of the downward irradiance and vertically upward radiance. All methods performed well in numerical situations at depths at which the influence of the bottom on the light field was significant. © 1999 Optical Society of America

OCIS codes: 010.4450, 030.5620, 100.3190.

1. Introduction

For shallow ocean waters, knowledge of the optical bottom albedo $R_b$ is necessary to model the underwater1,2 and above-water3 light field, to enhance underwater object detection or imaging, and to correct for bottom effects in the optical remote sensing of water depth5,6 or inherent optical properties (IOP’s).7,8 Measurements of $R_b$ can also help one identify the bottom-sediment composition,6 determine the distribution of benthic algal or coral communities,9 and detect objects embedded in the sea floor. Furthermore, values of $R_b$, defined as the upward irradiance emerging from the bottom divided by the downward irradiance into the bottom, can be used as an integral test for attempted measurements of the bottom bidirectional reflectance function.

Although the value of the irradiance ratio $R(z)$ equals $R_b$ at the bottom, it is not possible to measure $R(z)$ right at the bottom, and irradiance measurements just above the bottom are difficult to obtain because of instrument self-shadow. An estimate of $R_b$ can be made by extrapolating to the bottom measurements of $R(z)$ at several depths $z$ near the bottom. However, extrapolation is generally unreliable because profiles of $R(z)$ typically vary sharply with depth close to the bottom.10 In vitro $R_b$ measurements of small bottom samples can be obtained with the method in Ref. 5 or of larger bottom samples with a spectral radiometer. However, these processes are time-consuming. Also the in vitro value of $R_b$ is not necessarily equal to the in situ value, and it is not clear how representative these samples are of larger spatial regions of interest.

Because in situ estimates of $R_b$ from light measurements close to the bottom are subject to small-scale horizontal variability of the bottom, determining $R_b$ from measurements farther from the bottom may be preferable, thereby obtaining horizontally averaged values that are more appropriate for remote-sensing applications and one-dimensional radiative-transfer modeling. $R_b$ can be estimated6 from an $R(z)$ measurement just below the sea surface $R(0^-)$ together with simultaneous measurements in nearby deep water of $R(0^+)$ and the downward diffuse attenuation coefficient. Similarly, a qualitative algorithm has been proposed6 and tested11,12 for bottom characterization from remote radiance measurements at two wavelength bands simultaneously over both shallow and deep water. The disadvantage of both of these methods, however, is that they require a deep-water site nearby that has the same water composition, illumination, and surface conditions as the shallow-water site.

A new method of solving the inverse-radiative-transfer problem for determining $R_b$ is proposed in Section 2. Measurements of the upward and the downward irradiances at one wavelength and at least two mid-water-column depths at only one site are required. Also proposed are two related algorithms...
**Title:** Algorithms for ocean-bottom albedo determination from in-water natural-light measurements

**Authors:**

**Performing Organization:** University of Washington, Department of Mechanical Engineering, Box 352600, Seattle, WA, 98195-2600

**Abstract:**

Approved for public release; distribution unlimited

**Security Classification:**

- Report: unclassified
- Abstract: unclassified
- This Page: unclassified

**Number of Pages:** 7
for estimating $R_b$ from measurements of the vertically upward radiance and downward irradiance. These algorithms, like the previously developed ones, require knowledge of the measurement distances above the bottom. Results of specific numerical tests for all the $R_b$ estimation algorithms are presented in Section 3, and a discussion is given in Section 4.

2. Theory

A. Preliminaries

We are interested in the azimuthally averaged radiance $L(z, \mu)$ that satisfies the integro-differential radiative-transfer equation:

$$\left( \mu \frac{\partial}{\partial z} + c \right) L(z, \mu) = \int_{-1}^{1} \beta(z, \mu, \mu') L(z, \mu') d\mu',$$  \hspace{1cm} (1)

where $b$ and $c$ are the scattering and the beam attenuation coefficients, $\beta$ is the azimuthally integrated scattering phase function, and $\mu$ is the direction cosine with respect to the downward depth $z$. All the quantities in Eq. (1) implicitly depend on wavelength. The downward and upward irradiances are given by

$$E_d(z) = 2\pi \int_{0}^{1} \mu L(z, \mu) d\mu,$$

$$E_u(z) = 2\pi \int_{-1}^{0} \mu L(z, \mu) d\mu.$$  \hspace{1cm} (2)

The irradiance ratio is

$$R(z) = E_u(z)/E_d(z),$$  \hspace{1cm} (3)

and $R(z_b) = R_b$ for water depth $z_b$. The analogous radiance–irradiance ratio is

$$R^L(z) = \pi L_u(z)/E_d(z),$$  \hspace{1cm} (4)

where the vertically upward radiance $L_u(z) = L(z, -1)$. The factor $\pi$ is included in the definition of $R^L(z)$ so that $R^L(z_b) \approx R_b$. For a Lambertian bottom, $R^L(z_b) = R_b$.

With increasing depth in optically deep, spatially uniform, source-free waters, $R(z)$, $R^L(z)$, and the downward diffuse attenuation coefficient,

$$K_d(z) = -\frac{1}{E_d(z)} \frac{dE_d(z)}{dz},$$  \hspace{1cm} (5)

asymptotically approach values $R_\infty$, $R^L_\infty$, and $K_\infty$, respectively. These asymptotic values are IOP’s of the water that can be uniquely computed from $b$, $c$, and $\beta$.13

B. Irradiance-Ratio Approach

A well-known model for the irradiance ratio is\(^5,6\)

$$R(z) = R_{2nd}(z) + [R_b - R_{2nd}(z)] \exp[-2(z_b - z)K_{2nd}],$$  \hspace{1cm} (6)

where the water depth at the shallow site of interest is assumed to be known and the subscript $2nd$ denotes measurements taken at a nearby deep-water site characterized by the same illumination, sea-surface conditions, and water IOP’s as the site of interest. In the derivation of Eq. (6), $K$ is taken to be

$$K(z) = [K_d(z) + \kappa(z)]/2$$

averaged over depth in a nonspecified manner, where $\kappa$ is the coefficient of attenuation in the upward direction of upwelling photons\(^14,15\) (from both in-water scattering and the bottom); a qualitative and quantitative study of $\kappa$ is given in Refs. 5 and 15. In practice, however, the value of $K_{2nd}$ is typically approximated by the vertically averaged $K_d(z)$ at the deep-water site.\(^4\) A rearrangement of Eq. (6) gives the algorithm evaluated by Maritorena et al.\(^5\) for determining $R_b$ from irradiance measurements at two sites\(^9,16\):

$$R_b = R_{2nd}(z) + [R(z) - R_{2nd}(z)] \exp[2(z_b - z)K_{2nd}],$$  \hspace{1cm} (7)

Equation (7) is often written for $z = 0^*$ in hope of applying it to remote-sensing applications. However, it is valid for any depth $z$ and is more accurate at mid-water depths than at the surface. A difficulty with implementing Eq. (7) is that a deep-water (2nd) site may not be available that matches the water, surface, and illumination conditions of the shallow-water site.

An alternative shallow-water model was previously derived\(^17\) from the radiative-transfer equation with the eigenfunction expansion method. In this derivation the light field is approximated by the sum of two eigenmodes that decrease in magnitude with distance away from the surface and bottom. Expressed in the form of Eq. (7), this model is

$$R_b = R_\infty + [R(z) - R_\infty]\exp[2(z_b - z)K_\infty],$$  \hspace{1cm} (8)

which can be used to estimate $R_b$ from measurements at a single site provided that $z_b$ is known and $R_\infty$ and $K_\infty$ can be determined. Note that the asymptotic in-water irradiance ratio $R_\infty$ in Eq. (8) does not represent the same quantity denoted by that symbol in Refs. 4 and 9, where $R_\infty$ is our $R_{2nd}(0^-)$. Equations (7) and (8) were derived with entirely independent approaches. However, their final forms are very similar, and the two theoretically converge when $R_{2nd}$ and $K_{2nd}$ are measured at large depths in homogeneous, source-free waters, where $R_{2nd}(z) = R_\infty$ and $K_{2nd}(z) = K_\infty$.

Equation (8) provides a new interpretation of the attenuation coefficient in Eq. (7). The derivation of Eq. (7) suggests that $K \approx (K_d + \kappa)/2$, whereas the derivation of Eq. (8) suggests that $K \approx K_\infty$. Because $\kappa > K_d$,\(^4,15\) and therefore $K > K_d$, there is a question as to the appropriateness of taking $K$ to be the vertically averaged $K_d(z)$. Because typically $\kappa > K_d$, it follows that $K = [K_d(z) + \kappa(z)]_{2nd}/2 \approx K_{2nd}$. Therefore Eq. (7) may be best implemented by taking $K$ equal to the value of $K_d(z)$ deep in the euphotic zone, where $K_d(z) \approx K_\infty$ rather than as earlier pro-
posed.\textsuperscript{9,16} This hypothesis is further addressed in Section 3.

If $R_b$ and $K_o$ can be determined at the shallow site of interest, the method of Eq. (8) has the distinct advantage over that of Eq. (7) that measurements are required from only one site. One way to determine $R_b$ and $K_o$ is to calculate them with the procedure in Ref. 17 from measurements of $b$ and $c$ obtained, for example, from water samples\textsuperscript{18} or with a Wetlabs ed13 from local measurements of the IOP's or be measured.\textsuperscript{19} Alternatively, it is possible to estimate $R_b$ and $K_o$ from the same irradiance-profile measurements (at the shallow-water site) used to form $R(z)$ without any direct measurement of the water properties. The value of $R_b$ can be obtained with an equation derived\textsuperscript{17} from a shallow-water asymptotic approximation of the light field:

$$
\frac{1 - R_b}{1 + R_b} = \frac{[E_d(z) - E_u(z)]^2}{[E_d(z) + E_u(z)]^2}.
$$

(9)

To employ Eq. (9), one must subtract and add $E_d(z)$ and $E_u(z)$ at two depths, $z_1$ and $z_2$, square the results, and evaluate the differences between the two depths. Because these operations are susceptible to noise, it is important that the irradiance measurements be of high quality and their temporal variations be averaged out. The value of $K_o$ in Eq. (8) can be estimated as the maximum value attained by $K_d(z)$. Because the value of $K_d(z)$ is relatively insensitive to $R_b$,\textsuperscript{1} the value of $\max[K_d(z)]$ is typically approximately equal to $K_o$.

C. Radiance-Irradiance Ratio Approach

If $L_u(z)$ measurements are available rather than $E_u(z)$, $R_b$ can be estimated from $R^L(z)$ of Eq. (4) with a new model (derived in Appendix A) analogous to Eq. (8):

$$
R_b = R_o^{L} + [R^L(z) - R_o^{L} ] \exp[2(z_b - z)K_o].
$$

(10)

However, because an equation analogous to Eq. (9) has not been derived for $R_s^{L}$, implementation of Eq. (10) requires that the value of $R_s^{L}$ be calculated\textsuperscript{13} from local measurements of the IOP's or be measured in nearby deep water.

We found with numerical simulations that reasonable estimates of $R_b$ can alternatively be obtained from $R^L(z)$ with

$$
R_b = R_{2nd}^{L}(z) + [R^L(z) - R_{2nd}^{L}(z)] \exp[2(z_b - z)K_{2nd}],
$$

(11)

although we have no analytical justification for Eq. (11) other than its analogy to Eq. (7). As with Eq. (7) use of this equation requires measurements at a second (deep-water) site.

3. Numerical Tests

A. Methods

Numerical tests were performed to evaluate the accuracies of Eqs. (7)–(11) for determining $R_b$. Simulated $E_u(z)$, $E_d(z)$, and $L_u(z)$ values were generated at 0.25 optical depth spacing by using the discrete ordinates radiative-transfer code DISORT.\textsuperscript{20} The surface illumination was modeled as a combination of direct collimated sunlight and diffuse skylight. The water was defined to have locally homogenous optical properties, a relative index of refraction of 1.34 with respect to air, and scattering determined by the Petzold particle-scattering phase function.\textsuperscript{21} Spatially dependent internal sources, such as from fluorescence, Raman scattering, or bioluminescence, were neglected. A Lambertian bottom was assumed, which provides a good approximation to the more general, but usually poorly known, bidirectional reflectance function.\textsuperscript{1,2} Simulations were performed for various values of the single-scattering albedo ($\omega_0 = b/c$), $R_b$, percent direct sunlight, and water optical depth ($\tau_b = cz_b$). The values of $R_b$ were taken to be in the range $0 \leq R_b \leq 0.4$, which is consistent with observations in natural waters.\textsuperscript{1}

Equation (8) was applied to shallow-water simulations of $E_u(z)$ and $E_d(z)$ to determine bottom albedo estimates $\tilde{R}_b(z)$ as a function of the depth of the corresponding irradiance measurements. The values of $R_b$ and $K_o$ used in Eq. (8) were obtained in two different ways. First, $R_b$ was determined with Eq. (9) and $K_o$ was estimated from either $K_d(z) \approx K_d(z)$ or $K_o = \max[K_d(z)]$. Second, the IOP's of the water were assumed to be known from in situ measurements, and these IOP's were used to calculate $R_b$ and $K_o$. For comparison, $\tilde{R}_b(z)$ was also determined from $E_u(z)$ and $E_d(z)$ with Eq. (7) for combinations of shallow and deep-water simulations. Here $K$ was taken to be, alternatively, $K_d(z)$, $\overline{K_d(z)}$ [the vertical average of $K_d(z)$ between the surface and the depth of the shallow-water site], and $K_o$ [determined from a deep (asymptotic) value of $K_d(z)$].

The bottom albedo was also determined from simulations of $L_u(z)$ and $E_d(z)$. Equation (10) was used to calculate $\tilde{R}_b(z)$ from data at only the shallow-water site. The values of $R_o^{L}$ and $K_o$ were calculated from the known water optical properties. In addition $\tilde{R}_b(z)$ was determined from $L_u(z)$ and $E_d(z)$ with Eq. (11) for combinations of shallow- and deep-water sites. The value of $K_{2nd}$ was determined in the same manner as for the $E_u - E_d$ approach.

B. Results

For all the shallow-water simulations performed, estimates of $R_b$ obtained with Eqs. (8) and (9) approached the correct value in a nearly linear fashion within the bottom few optical depths of the water column. Extrapolation of $\tilde{R}_b(z)$ at two and one optical depths above the bottom consistently produced estimates of $R_b$ that were accurate to within $\pm 1\%$. In Table 1 example $\tilde{R}_b(z)$ is given at three, two, and one optical depths above the bottom as well as the linear extrapolation of the latter two to the bottom. For Table 1 the illumination conditions of the simulations were taken to be either overcast (100% diffuse) or sunny (75% direct sunlight from a zenith
angle of 30°, and in the solutions for $R_b$, the values of \( K_a \) in Eq. (8) were approximated by \( \text{max}[K_d(z)] \). At a given measurement depth, estimates generally improved with the increasing value of \( \omega_0 \). For example, for \( R_b = 0.2, \tau_b = 5, \) and sunny skies, the error at three optical depths above the bottom was 15% for \( \omega_0 = 0.7 \) but only 3.5% for \( \omega_0 = 0.9 \). The value of \( R_b \) had a small effect on the accuracy of its estimate. For large \( \omega_0 \) and small \( R_b, \) \( \tilde{R}_b(z) \) was greater than \( R_b \), whereas for small values of \( \omega_0 \) or for a combination of large \( \omega_0 \) and large \( R_b, \) \( \tilde{R}_b(z) \) was less than \( R_b \). Estimates were more accurate, but insignificantly so, with overcast conditions. Also, the depth of the water had little effect on the accuracy of \( \tilde{R}_b(z) \) at a given depth above the bottom. However, in practice, instrument noise will be more significant in relatively deep water than in shallow water.

In most cases, estimates of \( R_b \) with Eqs. (8) and (9) were more accurate if \( K_a \) in Eq. (8) was approximated by \( \text{max}[K_d(z)] \) than if it was replaced by \( K_d(z) \), because large values of \( K_a \) in Eq. (8) lead to large values of \( \tilde{R}_b \), and the estimates of \( R_b \) were typically less than the correct value. For cases in which the value of \( \omega_0 \) was high and the value of \( R_b \) small, the use of \( K_d(z) \) gave slightly, but insignificantly, better results than the use of \( \text{max}[K_d(z)] \).

Because Eq. (9) provides only an asymptotic approximation to \( R_a \), it was expected that estimates of \( R_b \) would improve if more accurate values of \( R_a \), calculated from the assumed known water IOP’s, were used in Eq. (8). However, the numerical tests showed the reverse to be true; errors in \( \tilde{R}_b(z) \) introduced by the approximation of Eq. (9) helped counteract errors in \( \tilde{R}_b(z) \) due to the assumptions inherent in Eq. (8). Although Eq. (8) with a calculated \( R_a \) performed similarly in the bottom half of the water column to Eq. (8) with \( R_a \) from Eq. (9), Eq. (8) with a calculated \( R_a \) performed poorly in the top half of the water column and even near the bottom slightly underperformed Eqs. (8) and (9).

For example, shown in Fig. 1 are the estimates of \( R_b \) as a function of the measurement optical depth obtained from simulated \( E_{\mu}(z) \) and \( E_d(z) \) with three different methods: from \( \tilde{R}_b(z) = R(z) \), from Eq. (8) with calculated \( R_a \) and \( K_a \), and from Eqs. (8) and (9) with \( K_b \) replaced by \( \text{max}[K_d(z)] \). In this simulation \( R_b = 0.2, \tau_b = 5, \) and \( \omega_0 = 0.8, \) and the sea-surface illumination was taken to be sunny (as defined above). Since \( R_b = R(z \rightarrow z_b) \), the first approach for determining \( R_b \) is the most straightforward. This approach gave a smooth profile of \( \tilde{R}_b(\tau) \) that monotonically approached \( R_b \) with increasing depth. However, this estimate is extremely inaccurate except very close to the bottom, with 41% error at only one optical depth above the bottom and 61% error at two optical depths above the bottom. This sharp increase in \( R(z) \) near the bottom is typical and makes extrapolation of \( R(z) \) from mid-water depths to the bottom impractical. At all depths off the bottom, far better estimates of \( R_b \) were obtained with Eq. (8) with \( R_a \) and \( K_a \) calculated from the known water IOP’s. The errors at two and one optical depths off the bottom were 11% and 5.8%, respectively. Even better estimates of \( R_b \) at all depths off the bottom, however, were obtained from Eq. (8) with \( R_a \) determined with Eq. (9) and \( K_a \) estimated by \( \text{max}[K_d(z)] \). This method gave an error at two and one optical depths off the bottom of 6.2% and 2.8%, respectively. Again, extrapolation from mid-depths gave an excellent estimate of \( R_b \).

Values of the bottom albedo from irradiance measurements at a combination of shallow- and deep-water sites with Eq. (7) were consistently underestimated. Therefore, since \( K_a \) was generally larger than \( K_d(z) \), results were more accurate when the value of \( K \) in Eq. (7) was determined from a deep value of \( K_d(z) \) in the deep-water site, where \( K_d(z) \approx K_a \), than when it was determined from either the deep-water \( K_d(z) \) or \( K_a \). For example, shown in Fig. 2 are \( \tilde{R}_b(\tau) \) calculated from Eq. (7) with \( K \) replaced by \( K_d(z) \) and by \( K_a \) [determined from \( K_d(\tau = 15) \)] for two sunny-sky simulations: (1) \( \omega_0 = 0.7, \tau_b \)
rather than the asymptotic and two-site Eu value of 

\[ K_d(z) \]

was large, but Eq. \ref{eq:asymptotic} performed well at all depths but still underperformed Eq. \ref{eq:one-site}. The simulations were for sunny conditions, \( \omega_0 = 0.9 \) and \( R_b = 0.1 \), and (right) for overcast conditions, \( \omega_0 = 0.9 \) and \( R_b = 0.2 \).

Shown in Fig. 3 are comparisons of \( \tilde{R}_b(\tau) \) obtained from \( E_d(z) \) and \( E_d(z) \) measurements at only the shallow-water site [Eqs. \ref{eq:shallow-water} and \ref{eq:deep-water}] and \( \tilde{R}_d(\tau) \) obtained from measurements at both shallow- and deep-water sites [Eq. \ref{eq:asymptotic}]. The two cases shown are sunny-sky simulations with \( \tau_b = 5 \) and (1) \( \omega_0 = 0.9 \) and \( R_b = 0.1 \) and (2) \( \omega_0 = 0.8 \) and \( R_b = 0.2 \). As these examples demonstrate, the two-site method was typically far more accurate than the one-site method near the surface, whereas the one-site algorithm usually outperformed the two-site method near the bottom.

Example \( \tilde{R}_b(\tau) \) obtained from \( L_d(z) \) and \( E_d(z) \) are shown in Fig. 4 for (1) \( \omega_0 = 0.9, R_b = 0.1, \) and sunny conditions and (2) \( \omega_0 = 0.9, R_b = 0.2, \) and overcast conditions. Because estimates of \( R_b \) with Eq. \ref{eq:two-site} were typically larger than the actual value, \( \tilde{R}_b(\tau) \) in Fig. 4 obtained with Eq. \ref{eq:two-site} was calculated with \( K_d(z) \) rather than the asymptotic \( K_d \) to make the estimated values as good as possible. In general the value of \( \omega_0 \) had a significant effect on the accuracy of both \( L_u - E_d \) methods, with the best estimates obtained when \( \omega_0 \) was large. The illumination conditions were also very important for the \( L_u - E_d \) methods, with the best results obtained for overcast conditions. The value of \( R_b \), on the other hand, had no significant effect on the accuracy of its estimate. In sunny conditions the one-site method of Eq. \ref{eq:one-site} and the two-site method of Eq. \ref{eq:two-site} performed similarly in the bottom one optical depth when \( \omega_0 \) was small and in the bottom three optical depths when \( \omega_0 \) was large, but Eq. \ref{eq:two-site} was considerably more accurate than Eq. \ref{eq:one-site} near the surface. For overcast conditions Eq. \ref{eq:one-site} performed well at all depths but still underperformed Eq. \ref{eq:two-site}.

4. Discussion

Several methods were evaluated here for determining \( R_b \) from common natural-light measurements. Each method returns accurate values of \( R_b \) if implemented close to the bottom. However, because it is difficult in practice to obtain light-field measurements close to the bottom, it is necessary to apply these algorithms at one or more optical depths off the bottom and, when possible, extrapolate the depth-dependent estimates to the bottom. Therefore it is desirable that the error of the method used be both small and linearly decreasing with depth. The methods were all found to be preferable to a straightforward extrapolation of \( R(z) = R_0 \) to the bottom, where \( R(z) = R_0 \), but differed in their accuracies when applied more than one or two optical depths away from the bottom.

Estimates of \( R_b \) can be obtained with Eq. \ref{eq:shallow-water} first by estimating \( R_b \) either with Eq. \ref{eq:deep-water} or by calculating it from known water IOP’s. This method does not require measurements at other wavelengths or at another site, and estimates of \( R_b \) from this method at one or two optical depths off the bottom were generally found to be more accurate than those obtained with Eq. \ref{eq:asymptotic}. Numerical simulations indicated that
the use of \( R_b \) from Eq. (9) produces better estimates of \( R_b \) than the use of the value of \( R_b \) computed from the IOP's, because fortuitously the error introduced by applying Eq. (8) where the light field is not well described by the assumed two-mode asymptotic model (see Appendix A) is mitigated by the deviation in the value of \( R_b \) predicted by Eq. (9) from its true value. Therefore, even if the water IOP's are known, employing Eq. (9) is preferable, provided the processed data have a relatively smooth \( \hat{R}_b(\tau) \) profile. Unfortunately this method is often inaccurate near the sea surface when the bottom signal is not strong.

If one wishes to estimate \( R_b \) from measurements close to the surface and a suitable deep-water site is available, Eq. (7) provides the most reliable method. However, it was found that if the deep-water site is vertically well mixed, Eq. (7) should be implemented by replacing \( K_{2nd} \) with \( K_{3rd} \), which can be directly measured deep in the euphotic zone of the deep-water site.

Estimates of \( R_b \) alternatively can be made from measurements of \( L_u(z) \) and \( E_d(z) \) with Eq. (10), provided that the bottom is approximately Lambertian. Equation (10) requires that \( R_b \) either be analytically computed from local measurements of the water IOP's or be measured at a nearby deep-water site. If a suitable second site is readily available, Eq. (11) should be used instead since it was found to be generally more accurate and reliable than Eq. (10). However, both \( L_u - E_d \) methods performed well when applied in the bottom half of the water column. The inaccuracy of Eq. (10) in sunny conditions makes it unsuitable for remote-sensing applications, and therefore Eq. (10) offers no advantage over the \( E_u - E_d \) method (which is the more natural approach to an in situ \( R_b \) estimation). Equation (11), on the other hand, shows some promise for remote-sensing applications for large \( \omega_0 \) or for any \( \omega_0 \) if \( \omega_0 \) is known.

Regardless of the method used, the determination of \( R_b \) is easiest when the bottom signal is strong. Thus \( R_b \) can be obtained most accurately when the water is shallow, the value of \( R_b \) is large, and the attenuation of the water is low (e.g., over tropical coral reefs or white sandy beaches). If the bottom composition is believed to be uniform over a large horizontal region, the determination of \( R_b \) should be made at the shallowest depth.

In practice the method to use for estimating \( R_b \) will be dictated by the instrumentation available and whether an appropriate deep-water site exists. Given the choice, however, the estimation of \( R_b \) should be made with measurements of \( E_d(z) \) rather than with \( L_u(z) \). The most informative approach would be to use all the methods discussed here and intercompare the results. If the estimates agree, they can be recorded with great confidence. On the other hand, the difference between estimated \( R_b \) values from \( R(z) \) and \( R^L(z) \) might serve as a crude measure of the degree to which the bottom behaves as a Lambertian surface. However, for purposes of modeling the light field, Mobley\textsuperscript{22} has demonstrated that measuring the magnitude of the effective bottom albedo is far more important than obtaining the detailed angular pattern of the bottom bidirectional reflectance function.

**Appendix A: Derivation of Eq. (10)**

If the optical properties of water are spatially uniform and there are no internal sources, when \( z \) is at least one optical depth away from a boundary the upward radiance and downward irradiance can be expressed as summations of eigenmodes\textsuperscript{13}:

\[
L_u(z) = L(z, -1) = \sum_{j=1}^{J} [C(v_j) \phi(v_j, -1) \exp(-cz/v_j) + C(-v_j) \phi(-v_j, -1) \exp(cz/v_j)],
\]

(A1)

\[
E_d(z) = \sum_{j=1}^{J} [C(v_j) \tilde{g}_1(v_j) \exp(-cz/v_j) + C(-v_j) \tilde{g}_1(-v_j) \exp(cz/v_j)],
\]

(A2)

where \( C(\pm v_j) \) are expansion coefficients, \( v_j \) are the eigenvalues of Eq. (1) corresponding to the eigenfunctions\textsuperscript{23} \( \phi(\pm v_j, \mu) \), and

\[
\tilde{g}_1(\pm v_1) = \int_{1}^{0} \phi(\pm v_j, \mu) P_j(\mu) \, d\mu.
\]

(A3)

for Legendre polynomial \( P_j(\mu) \). Far from the boundaries, \( E_d(z) \) and \( L_u(z) \) can be approximated by retaining only the asymptotic decreasing eigenmode corresponding to the largest eigenvalue \( v_1 \) i.e., \( C(v_j) = 0 \) for \( j > 1 \) and \( C(-v_j) = 0 \) for all \( j \), and since \( R^L(z) = \pi L_u(z)/E_d(z) \) the asymptotic value of \( R^L(z) \) is \( R^L = \pi\phi(v_1, -1)/\tilde{g}_1(v_1) \). At depths far from the surface but close to the topography bottom is present, \( E_d(z) \) and \( L_u(z) \) are better approximated by also including the eigenmode corresponding to the largest negative eigenvalue, \( -v_1 \), so that

\[
L_u(z) \approx C(v_1) \phi(v_1, -1) \exp(-cz/v_1) + C(-v_1) \phi(-v_1, -1) \exp(cz/v_1),
\]

(A4)

\[
E_d(z) \approx C(v_1) \tilde{g}_1(v_1) \exp(-cz/v_1) + C(-v_1) \tilde{g}_1(-v_1) \exp(cz/v_1).
\]

(A5)

After forming the ratio \( R^L(z) = \pi L_u(z)/E_d(z) \), dividing through by \( C(v_1) \tilde{g}_1(v_1) \exp(-cz/v_1) \), letting \( r = C(-v_1)/C(v_1) \), and recognizing that\textsuperscript{13} \( R^L = \phi(-v_1, 1)/\tilde{g}_1(v_1) \), we find that

\[
R^L(z) = \frac{R^L L + \pi r \phi(-v_1, -1) \exp(2cz/v_1) / \tilde{g}_1(v_1)}{1 + r R^L \exp(2cz/v_1) / \tilde{g}_1(v_1)}.
\]

(A6)

Subtraction of \( R^L \) from Eq. (A6) gives

\[
R^L(z) - R^L = \frac{r \exp(2cz/v_1)[\pi \phi(-v_1, -1) / \tilde{g}_1(v_1) - R^L L]}{1 + r R^L \exp(2cz/v_1)}.
\]
A rearrangement of Eq. (A7) is

$$ [R^l(z) - R^L] \exp(-2cz/v_1) [1 + rR_c \exp(2cz/v_1)] = r_0 \rho(-v_1, -1)/g_1(v_1) - R^L R_c. \quad (A8) $$

Since the right-hand side of Eq. (A8) is independent of $z$, the left-hand side at arbitrary depth $z$ equals that at the bottom. Therefore

$$ R^l(z) = R^L + [R^l(z_b) - R^L] \exp(-2c(z_b - z)/v_1) $$

$$ \times \left[ 1 + rR_c \exp(2cz_b/v_1) \right]. \quad (A9) $$

If $rR_c \exp(2cz_b/v_1) \ll 1$ or if $(z_b - z)$ is small, Eq. (A9) reduces to

$$ R^l(z) = R^L + [R^l(z_b) - R^L] \exp(-2c(z_b - z)/v_1), \quad (A10) $$

which is analogous to our equation for $R(z)$ derived in a similar manner.\(^{17}\) Rearrangement of Eq. (A10) gives Eq. (10).

This research was supported primarily by the U.S. Office of Naval Research, with additional support provided by NASA’s Earth System Science Office. The radiative-transfer numerical code was kindly provided by Knut Stamnes.

References