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THE INVERSE SOURCE PROBLEM FOR MAXWELL'S EQUATIONS FINAL REPORT

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14. ABSTRACT
The inverse source problem for Maxwell's equations is considered. We show that the problem of finding a volume current density from surface measurements does not have a unique solution, and we characterize the non-uniqueness. We also show that if further information is available the inverse source problem may have a unique solution. The method is useful for the quantitative determination of interior brain currents from surface electroencephalographic measurements. The application is to prosthesis control.

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THE INVERSE SOURCE PROBLEM FOR MAXWELL'S EQUATIONS

R. ALBANESE* AND P.B. MONK†

Abstract. The inverse source problem for Maxwell's equations is considered. We show that the problem of finding a volume current density from surface measurements does not have a unique solution, and characterize the non-uniqueness. We also show that if further a priori information is available, the inverse source problem may have a unique solution (in particular for surface currents or dipole sources).

1. Introduction. The goal of this paper is to investigate the inverse source problem for the general Maxwell system. This problem arises in medical applications where measurements are taken of the electric and magnetic surface currents on the surface of the human head, and it is desired to infer from these currents the source currents in the brain that produced the measured fields. It is hoped that such measurements could be used to diagnose abnormalities in the brain and also to allow the control of prosthetic limbs.

From the point of view of mathematical modeling, we shall first make the simplifying assumption that the measurements can be made on a surface containing the entire head. Later we shall discuss how the theory developed in this ideal case might be extended to the more realistic problem where measurements are made on a portion of the skull. Thus we assume that there is a bounded smooth domain $\Omega \subset \mathbb{R}^3$ (connected with connected complement) such that the known electromagnetic parameters ϵ, σ and μ (all assumed real) have the following properties:

1. In $\mathbb{R}^3 \setminus \overline{\Omega}$ (the air) the parameters have the values $\epsilon = \epsilon_0 > 0$ and $\mu = \mu_0 > 0$ where ϵ_0 and μ_0 are the permitivitty and permeability of free space respectively. In addition the conductivity $\sigma = 0$.
2. In Ω (the head) there are constants ϵ_{min} and ϵ_{max} such that
$$0 < \epsilon_{min} \leq \epsilon(\boldsymbol{x}) \leq \epsilon_{max} \text{ for all } \boldsymbol{x} \in \Omega.$$
In addition there are a constant σ_{max} such that
$$0 \leq \sigma(\boldsymbol{x}) \leq \sigma_{max}, \text{ for all } \boldsymbol{x} \in \Omega.$$
In view of the biological application it is also reasonable to assume that $\mu = \mu_0$ in all Ω and we shall make this assumption in the rest of the paper.
3. The coefficients ϵ and σ are assumed to be piecewise C^1 functions and any surfaces of discontinuity are assumed to be smooth.

The latter assumption is made so that we can apply the unique continuation results of [21] (in [19] it is commented that the continuation property extends to piecewise C^1 functions under even more general conditions on the interface between regions where the coefficients are smooth).

Given a current density $\boldsymbol{J}$ supported in Ω, the resulting electromagnetic field described by the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ satisfies the following Maxwell system where $\omega > 0$ is the frequency of the time harmonic field:

$$-\mathrm{i}\omega\epsilon\boldsymbol{E} + \sigma\boldsymbol{E} - \nabla \times \boldsymbol{H} = -\boldsymbol{J} \text{ in } \mathbb{R}^3, \tag{1.1}$$

$$-\mathrm{i}\omega\mu\boldsymbol{H} + \nabla \times \boldsymbol{E} = 0 \text{ in } \mathbb{R}^3. \tag{1.2}$$

In order to uniquely determine the fields, we must also impose the Silver-Müller radiation condition at infinity:

$$\lim_{|\boldsymbol{x}|\to\infty} (\boldsymbol{H} \times \boldsymbol{x} - |\boldsymbol{x}|\boldsymbol{E}) = 0 \tag{1.3}$$

uniformly in $\hat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$. Under suitable conditions on the smoothness of the source $\boldsymbol{J}$ (for example $\boldsymbol{J} \in L^2(\mathbb{R}^3)$ having compact support), the forward problem of finding $\boldsymbol{E}$ and $\boldsymbol{H}$ for a known current source $\boldsymbol{J}$ is well posed (see for example [19]).

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Having defined the above notation, we can now discuss the inverse source problem. Let Γ denote the surface of Ω. Let $\boldsymbol{n}$ denote the unit outward normal to $\Gamma = \partial\Omega$. For the inverse source problem we assume that ϵ, σ and μ are known and in addition that the surface electric current $\boldsymbol{n} \times \boldsymbol{E}$ and the surface magnetic current $\boldsymbol{n} \times \boldsymbol{H}$ are known on the entire surface Γ. From this data we would like to find the corresponding unknown source current density $\boldsymbol{J}$. As we shall see this problem is severely ill-posed.

The inverse source problem has been exhaustively studied in the literature both from the point of view of applied biomedical engineering and also as a mathematical problem (see for example [3, 22, 15, 7, 14, 9, 6, 12] where we have emphasized mathematical references of relevance to this paper). It is well-known that volume sources cannot be uniquely determined from surface measurements due to the existence of non-radiating sources [3, 18, 17, 8]. One goal of this paper is to extend the results of [3] and characterize the non-radiating sources when the background media is non-constant as is found in the head. This is done using weak solutions and variational methods.

With the exception of the work of Ammari, Bao and Flemming [1] and He and Romanov [13] most theoretical work on biological applications has focused on the static (or quasi-static) case since it is clear that the frequency of the radiation is very low (perhaps 100Hz). Even in [1] the numerical scheme suggested is based on a static model. However, since the goal of the inverse source problem is to monitor dynamic neuronal events (an action potential has a rise time on the order of 0.5 milliseconds [23]) it may be that the displacement current is not negligible. This has already been pointed out in studies of source problems related to monitoring neurons in the arm [16]. In particular there are two important dimensionless parameters that must be considered when assessing whether the displacement current may be neglected [11] namely

$$\alpha = \sqrt{\omega\mu\sigma}L \text{ and } \beta = \sqrt{\mu\epsilon}\omega L$$

where L is a representative length for the conductor. Of course neglecting the displacement current results in an eddy current model - the first step to a static model

If we choose $\omega = 100Hz$ then, using the Gabriel database [10], we find that for grey matter the relative permitivity is $\epsilon_r = 3.9 \times 10^6$ and $\sigma = 0.089\text{Sm}^{-1}$. Taking $\epsilon_0 = 8.85 \times 10^{-12}\text{Fm}^{-1}$, $\mu_0 = 4\pi \times 10^{-7}\text{Hm}^{-1}$ and taking as a representative length $L = 0.01\text{m}$ (roughly the radius of the head) we obtain

$$\alpha = 3.3 \times 10^{-5} \text{ and } \beta = 6.6 \times 10^{-6}.$$

While both these terms are small compared to unity, they are of the same order of magnitude and thus we cannot a priori neglect the displacement current (this is also the conclusion of the computational studies in [16]). It therefore seems prudent to consider the full Maxwell system. The remainder of the paper is devoted to uniqueness questions for the inverse source problem for the Maxwell system. Our work is strongly motivated by [12] in which uniqueness questions are examined for the corresponding static problem.

The plan of this paper is as follows. In the next section (Section 2) we address the question of uniqueness for the inverse problem assuming that the unknown sources are distributed throughout the domain Ω. We show that such sources cannot be determined from the given data (without further a priori information) and characterize the non-uniqueness. This tells us the limits on the inverse source problem in the sense that, even with perfect data, the volume current density cannot be reconstructed. If the source currents are a priori known to have smaller support than the entire domain Ω, it is sometimes possible to prove that the data uniquely determines the source. We consider two cases found in applications. In Section 3 we suppose that it is known a priori that the source current is given by a delta function that is supported on the surface of a smooth subdomain in Ω. In the case of a surface current we show that the current density is uniquely determined by the data. In the second case, in Section 4, we consider the case when the source current is due to finitely many dipole sources. In this case we prove that the data uniquely determines the number, position and strength of the dipoles. Finally, in Section 5, we show how the variational characterization can be extended to allow for measurements only on a portion of the boundary Γ and draw some conclusions in Section 6.

The uniqueness theory in this paper does not address the stability of the inverse problem to data perturbations. We also do not address the question of what algorithm could be used in each of the three cases. However the variational characterization of the current source that we shall derive can be used as part of an inversion scheme.

In general vector quantities are denoted by boldface type (e.g. $\boldsymbol{E} = (E_1, E_2, E_3)^T$ as already used above). Much of our theory makes use of the standard energy space for solutions of Maxwell's equations

$$H(\mathrm{curl};\Omega) = \left\{\boldsymbol{u} \in L^2(\Omega) \mid \nabla \times \boldsymbol{u} \in L^2(\Omega)\right\}.$$

2. Non uniqueness of volume currents. Our goal in this section is to derive a variational equation relating the unknown source $\boldsymbol{J}$ to the data on Γ. This derivation is motivated by the work of Ammari, Bao and Fleming [1] who derived a similar expression in their study of methods for identifying point sources. We then use this variational equation to study the uniqueness question when $\boldsymbol{J}$ is a distributed source in Ω.

Suppose $\boldsymbol{J} \in L^2(\Omega)$ has support in Ω and that $\epsilon, \sigma \in L^\infty(\mathbb{R}^3)$ satisfy the assumptions noted in the previous section. Then we can reformulate (1.1)–(1.3) on Ω alone using a suitable Dirichlet-to-Neumann (DtN) map T on Γ. In order to define the map T we need the trace space $H^{-1/2}(\mathrm{Div};\Gamma)$ defined by

$$H^{-1/2}(\mathrm{Div};\Gamma) = \left\{\boldsymbol{u} \in H^{-1/2}(\Gamma) \mid \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma \text{ and } \nabla_\Gamma \cdot \boldsymbol{u} \in H^{-1/2}(\Gamma)\right\}$$

where $\nabla_\Gamma\cdot$ is the surface divergence. Note that since we assumed that the surfaces in this problem are smooth, classical definitions of the surface gradient, curl and divergence may be used [4].

Using this space, the map T can be defined as follows. For $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div};\Gamma)$ we define $T\boldsymbol{\lambda} = \boldsymbol{n}\times\boldsymbol{v}$ where $(\boldsymbol{u},\boldsymbol{v}) \in H_{loc}(\mathrm{curl};\mathbb{R}^3\setminus\overline{\Omega})^2$ satisfy

$$\begin{aligned} -\mathrm{i}\omega\epsilon_0\boldsymbol{u} - \nabla\times\boldsymbol{v} &= 0 \text{ in } \mathbb{R}^3\setminus\overline{\Omega},\\ -\mathrm{i}\omega\mu_0\boldsymbol{v} + \nabla\times\boldsymbol{u} &= 0 \text{ in } \mathbb{R}^3\setminus\overline{\Omega},\\ \boldsymbol{n}\times\boldsymbol{u} &= \boldsymbol{\lambda} \text{ on } \Gamma, \end{aligned}$$

together with the Silver-Müller radiation condition (1.3). It follows from the solvability of the exterior scattering problem in $H_{\mathrm{loc}}(\mathrm{curl};\mathbb{R}^3\setminus\overline{\Omega})$ that $T : H^{-1/2}(\mathrm{Div};\Gamma) \to H^{-1/2}(\mathrm{Div};\Gamma)$. For an introduction to vector trace spaces and the properties of T see [20, 19, 2].

Later, we shall also need to use the adjoint operator $T^* : H^{-1/2}(\mathrm{Curl};\Gamma) \to H^{-1/2}(\mathrm{Curl};\Gamma)$ where

$$H^{-1/2}(\mathrm{Curl};\Gamma) = \left\{\boldsymbol{u} \in H^{-1/2}(\Gamma) \mid \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma \text{ and } \nabla_\Gamma \times \boldsymbol{u} \in H^{-1/2}(\Gamma)\right\}$$

and where $\nabla_\Gamma\times\boldsymbol{u}$ is the surface curl of $\boldsymbol{u}$. Of course $H^{-1/2}(\mathrm{Div};\Gamma)$ and $H^{-1/2}(\mathrm{Curl};\Gamma)$ are dual spaces so that T and T^* are related by

$$\int_\Gamma T\boldsymbol{\lambda}\cdot\overline{\boldsymbol{\eta}}\,dA = \int_\Gamma \boldsymbol{\lambda}\cdot\overline{T^*\boldsymbol{\eta}}\,dA \quad \forall\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div};\gamma) \text{ and } \boldsymbol{\eta} \in H^{-1/2}(\mathrm{Curl};\Gamma).$$

The map T^* is also related to a boundary value problem. Given $\boldsymbol{\eta} \in H^{-1/2}(\mathrm{Curl};\Gamma)$ define $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ to satisfy the exterior scattering problem

$$\begin{aligned} -\mathrm{i}\omega\epsilon\boldsymbol{\phi} - \nabla\times\boldsymbol{\psi} &= 0 \text{ in } \mathbb{R}^3\setminus\overline{\Omega},\\ -\mathrm{i}\omega\mu\boldsymbol{\psi} + \nabla\times\boldsymbol{\phi} &= 0 \text{ in } \mathbb{R}^3\setminus\overline{\Omega},\\ (\boldsymbol{n}\times\boldsymbol{\phi})\times\boldsymbol{n} &= \boldsymbol{\eta} \text{ on } \Gamma,\\ \lim_{|\boldsymbol{x}|\to\infty}(\boldsymbol{\psi}\times\boldsymbol{x} + |\boldsymbol{x}|\boldsymbol{\phi}) &= 0. \end{aligned}$$

This is just an exterior problem for Maxwell's equations (recognizing that the boundary condition on Γ may be rewritten as $\boldsymbol{\phi}\times\boldsymbol{n} = \boldsymbol{\eta}\times\boldsymbol{n}$) where the radiation condition has the opposite sign to that in (1.3). Standard techniques show that this problem has a unique solution for any $\boldsymbol{\eta} \in H^{-1/2}(\mathrm{Curl};\Gamma)$ (see for example [19]).

Using integration by parts and the equations for the fields $(\boldsymbol{u},\boldsymbol{v})$ and $(\boldsymbol{\phi},\boldsymbol{\psi})$ in a ball B_R of radius R containing Ω we have (recalling that $\boldsymbol{n}$ is the outward normal to Γ and hence inwards to B_R) that

$$-\int_\Gamma T(\boldsymbol{n}\times\boldsymbol{u})\cdot\overline{\boldsymbol{\eta}}\,dA = \int_{B_R\setminus\overline{\Omega}} \nabla\times\boldsymbol{v}\cdot\overline{\boldsymbol{\phi}} - \boldsymbol{v}\cdot\nabla\times\overline{\boldsymbol{\phi}}\,dV - \int_{\partial B_R} \boldsymbol{n}\times\boldsymbol{v}\cdot\overline{\boldsymbol{\phi}}\,dA$$

$$
\begin{aligned}
&= \int_{B_R\setminus\overline{\Omega}} -\mathrm{i}\omega\epsilon \boldsymbol{u}\cdot\overline{\boldsymbol{\phi}} + \nabla\times\boldsymbol{u}\cdot\overline{\boldsymbol{\psi}}\,dV - \int_{\partial B_R} \boldsymbol{n}\times\boldsymbol{v}\cdot\overline{\boldsymbol{\phi}}\,dA \\
&= \int_{B_R\setminus\overline{\Omega}} -\mathrm{i}\omega\epsilon \boldsymbol{u}\cdot\overline{\boldsymbol{\phi}} + \boldsymbol{u}\cdot\nabla\times\overline{\boldsymbol{\psi}}\,dV - \int_{\Gamma} \boldsymbol{n}\times\boldsymbol{u}\cdot\overline{\boldsymbol{\psi}}\,dA + \int_{\partial B_R} \boldsymbol{n}\times\boldsymbol{u}\cdot\overline{\boldsymbol{\psi}} - \boldsymbol{n}\times\boldsymbol{v}\cdot\overline{\boldsymbol{\phi}}\,dA \\
&= -\int_{\Gamma} \boldsymbol{n}\times\boldsymbol{u}\cdot\overline{\boldsymbol{\psi}}\,dA + \int_{\partial B_R} \boldsymbol{n}\times\boldsymbol{u}\cdot\overline{\boldsymbol{\psi}} - \boldsymbol{n}\times\boldsymbol{v}\cdot\overline{\boldsymbol{\phi}}\,dA.
\end{aligned}
$$

But we may write

$$
\int_{\partial B_R} \boldsymbol{n}\times\boldsymbol{u}\cdot\overline{\boldsymbol{\psi}} - \boldsymbol{n}\times\boldsymbol{v}\cdot\overline{\boldsymbol{\phi}}\,dA = \int_{\partial B_R} (\boldsymbol{v}\times\boldsymbol{n}-\boldsymbol{u})\cdot\overline{\boldsymbol{\phi}} + \boldsymbol{u}\cdot(\overline{\boldsymbol{\psi}\times\boldsymbol{n}+\boldsymbol{\phi}})\,dA.
$$

Using the fact that $|\boldsymbol{u}| = O(1/|\boldsymbol{x}|)$, $|\boldsymbol{\phi}| = O(1/|\boldsymbol{x}|)$, $|\boldsymbol{v}\times\boldsymbol{n}-\boldsymbol{u}| = O(1/|\boldsymbol{x}|^2)$ and $|\boldsymbol{\psi}\times\boldsymbol{n}+\boldsymbol{\phi}| = O(1/|\boldsymbol{x}|^2)$ for $|\boldsymbol{x}| = R$ large (see [4]), we see that the integral on ∂B_R vanishes in the limit $R\to\infty$ and we have shown that $T^*\boldsymbol{\eta} = (\boldsymbol{n}\times\boldsymbol{\psi})\times\boldsymbol{n}$ on Γ.

Using the DtN map T, we can reformulate (1.1)-(1.3) as the problem of finding $(\boldsymbol{E},\boldsymbol{H}) \in H(\mathrm{curl};\Omega)\times H(\mathrm{curl};\Omega)$ such that

$$
-\mathrm{i}\omega\epsilon\boldsymbol{E} + \sigma\boldsymbol{E} - \nabla\times\boldsymbol{H} = -\boldsymbol{J} \text{ in } \Omega, \tag{2.1}
$$

$$
-\mathrm{i}\omega\mu\boldsymbol{H} + \nabla\times\boldsymbol{E} = 0 \text{ in } \Omega, \tag{2.2}
$$

$$
\boldsymbol{H}\times\boldsymbol{n} = T(\boldsymbol{E}\times\boldsymbol{n}) \text{ on } \Gamma. \tag{2.3}
$$

It is now convenient to derive a vector wave equation for $\boldsymbol{E}$ alone by using (2.2) to write $\boldsymbol{H} = 1/(\mathrm{i}\omega\mu)\nabla\times\boldsymbol{E}$ and eliminating $\boldsymbol{H}$ from (2.1) to obtain

$$
\nabla\times\mu^{-1}\nabla\times\boldsymbol{E} - (\omega^2\epsilon + \mathrm{i}\sigma\omega)\boldsymbol{E} = \mathrm{i}\omega\boldsymbol{J} \text{ in } \Omega. \tag{2.4}
$$

Multiplying by the complex conjugate of a smooth test vector function $\boldsymbol{\xi}$ and integrating by parts (using the expression for $\boldsymbol{E}$ just derived) we obtain the variational equation

$$
\int_\Omega \mu^{-1}\nabla\times\boldsymbol{E}\cdot\nabla\times\overline{\boldsymbol{\xi}} - (\omega^2\epsilon + \mathrm{i}\sigma)\boldsymbol{E}\cdot\overline{\boldsymbol{\xi}}\,dV + \int_\Gamma \mathrm{i}\omega\boldsymbol{n}\times\boldsymbol{H}\cdot\overline{\boldsymbol{\xi}}\,dA = \int_\Omega \mathrm{i}\omega\boldsymbol{J}\cdot\overline{\boldsymbol{\xi}}\,dV. \tag{2.5}
$$

Replacing the magnetic surface current using the DtN map we obtain the problem of finding $\boldsymbol{E}\in H(\mathrm{curl};\Omega)$ such that

$$
\int_\Omega \mu^{-1}\nabla\times\boldsymbol{E}\cdot\nabla\times\overline{\boldsymbol{\xi}} - (\omega^2\epsilon + \mathrm{i}\sigma\omega)\boldsymbol{E}\cdot\overline{\boldsymbol{\xi}}\,dV + \int_\Gamma \mathrm{i}\omega T(\boldsymbol{n}\times\boldsymbol{E})\cdot\overline{\boldsymbol{\xi}}\,dA = \int_\Omega \mathrm{i}\omega\boldsymbol{J}\cdot\overline{\boldsymbol{\xi}}\,dV \tag{2.6}
$$

for all $\boldsymbol{\xi}\in H(\mathrm{curl};\Omega)$. This variational formulation can then be used to prove that, for a given $\boldsymbol{J}\in L^2(\Omega)$, there exists a unique electromagnetic field $(\boldsymbol{E},\boldsymbol{H})$ satisfying (2.6) [19].

We now return to the inverse problem and derive an identity relating the source current $\boldsymbol{J}$ to the data. The approach is closely connected to the characterization of non-radiating sources in [17] but we now allow a more general scattering problem for which the Maxwell operator is not self adjoint. Using (2.5) and recalling that $\boldsymbol{\xi}$ is assumed to be a smooth vector function, we integrate by parts one more time to obtain the indentity

$$
\begin{aligned}
\mathrm{i}\omega\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{\xi}}\,dV = &\int_\Omega \boldsymbol{E}\cdot(\overline{\nabla\times\mu^{-1}\nabla\times\boldsymbol{\xi} - (\omega^2\epsilon - \mathrm{i}\sigma\omega)\boldsymbol{\xi}})\,dV \\
&+ \int_\Gamma \boldsymbol{n}\times\boldsymbol{E}\cdot\mu^{-1}\nabla\times\overline{\boldsymbol{\xi}} + \mathrm{i}\omega\boldsymbol{n}\times\boldsymbol{H}\cdot\overline{\boldsymbol{\xi}}\,dA.
\end{aligned}
$$

Now we choose $\boldsymbol{\xi}\in H(\mathrm{curl},\Omega)$ to be such that

$$
\int_\Omega \mu^{-1}\nabla\times\boldsymbol{\xi}\cdot\nabla\times\boldsymbol{\psi} - (\omega^2\epsilon - \mathrm{i}\sigma\omega)\boldsymbol{\xi}\cdot\boldsymbol{\psi}\,dV = 0 \tag{2.7}
$$

for all smooth vector functions ψ of compact support in Ω. Note in particular that this implies, in the sense of distributions, that $\boldsymbol{\xi}$ is just a weak solution of the adjoint Maxwell system

$$\nabla \times \mu^{-1} \nabla \times \boldsymbol{\xi} - (\omega^2 \epsilon - \mathrm{i}\sigma\omega)\boldsymbol{\xi} = 0 \text{ in } \Omega.$$

The fact that $\boldsymbol{\xi} \in L^2(\Omega)$ then implies that $\nabla \times \mu^{-1} \nabla \times \boldsymbol{\xi} \in L^2(\Omega)$ and hence that $\boldsymbol{n} \times \mu^{-1} \nabla \times \boldsymbol{\xi} \in H^{-1/2}(\mathrm{Div}; \Gamma)$ is well defined. Making this choice of $\boldsymbol{\xi}$ we see that (2.7) becomes

$$\mathrm{i}\omega \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{\xi}}\, dV = \int_\Gamma \left(\boldsymbol{n} \times \boldsymbol{E} \cdot \mu^{-1} \nabla \times \overline{\boldsymbol{\xi}} + \mathrm{i}\omega \boldsymbol{n} \times \boldsymbol{H} \cdot \overline{\boldsymbol{\xi}} \right) dA. \tag{2.8}$$

for all $\boldsymbol{\xi} \in H(\mathrm{curl}; \Omega)$ satisfying (2.7).

Suppose that we have an exact knowledge of $\boldsymbol{n} \times \boldsymbol{E}$ and $\boldsymbol{n} \times \boldsymbol{H}$ on Γ (and of ϵ and σ in Ω). Then equation (2.8) provides a link between the known data and the unknown source $\boldsymbol{J}$. Let us define the set of solutions of the adjoint problem to be

$$\mathcal{H}(\Omega) = \{\boldsymbol{u} \in H(\mathrm{curl}; \Omega) \mid \boldsymbol{u} \text{ satisfies (2.7)}\}.$$

Then we define $H(\Omega)$ to be the closure of $\mathcal{H}(\Omega)$ in the $L^2(\Omega)$ norm. Using $H(\Omega)$ we can write the L^2-orthogonal decomposition

$$L^2(\Omega) = H(\Omega) \oplus H(\Omega)^\perp.$$

We note that $H(\Omega)^\perp$ is an infinite dimensional subspace of $L^2(\Omega)$ as the following lemma shows.

LEMMA 2.1. *Suppose $\boldsymbol{\chi} \in (C_0^\infty(\Omega))^3$. Then if*

$$\boldsymbol{\phi} = \nabla \times \mu^{-1} \nabla \times \boldsymbol{\chi} - (\omega^2 \epsilon - \mathrm{i}\omega\sigma)\boldsymbol{\chi}$$

we have $\boldsymbol{\phi} \in H(\Omega)^\perp$.

Remark: An interesting question is whether the above lemma characterizes all of $H(\Omega)^\perp$.

Proof. Suppose $\boldsymbol{u} \in \mathcal{H}(\Omega)$ then, integrating by parts,

$$\begin{aligned} \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{\phi}}\, dV &= \int_\Omega \boldsymbol{u} \cdot \left(\overline{\nabla \times \mu^{-1} \nabla \times \boldsymbol{\chi} - (\omega^2 \epsilon - \mathrm{i}\sigma\omega)\boldsymbol{\chi}} \right) dV \\ &= \int_\Omega \mu^{-1} \nabla \times \boldsymbol{u} \cdot \overline{\nabla \times \boldsymbol{\chi}} - (\omega^2 \epsilon + \mathrm{i}\sigma\omega) \boldsymbol{u} \cdot \overline{\boldsymbol{\chi}}\, dV = 0, \end{aligned}$$

using (2.7). By a density argument this holds for all functions in $H(\Omega)$. □

Recall that $\boldsymbol{J} \in L^2(\Omega)$. From Lemma 2.1 we see that $H(\Omega)$ is a proper subspace of $L^2(\Omega)$ and using (2.8) we see that only the component of $\boldsymbol{J}$ in $H(\Omega)$ can be determined from the data by using equation (2.8). It is possible that some other equation could be derived to determine the component of $\boldsymbol{J}$ in $H(\Omega)^\perp$ from the data, but the following theorem rules this out.

THEOREM 2.2. *Suppose $\boldsymbol{J} \in L^2(\Omega)$. Then $\boldsymbol{J} = \boldsymbol{J}_H + \boldsymbol{J}_{H^\perp}$ where $\boldsymbol{J}_H \in H(\Omega)$ and $\boldsymbol{J}_{H^\perp} \in H(\Omega)^\perp$. The inverse source problem uniquely determines $\boldsymbol{J}_H$ via equation (2.8). The component $\boldsymbol{J}_{H^\perp}$ does not produce surface currents on Γ and hence cannot be identified.*

Remark: The functions in $H(\Omega)^\perp$ are termed *non radiating sources* [3]. Computing the component of the source in $H(\Omega)$ corresponds to computing a "minumum energy" solution [18].

Proof. Given the discussion preceding this theorem, the only part of the statement of the theorem that needs to be proved is that $\boldsymbol{J}_{H^\perp}$ produces no surface currents on Γ. To simplify notation, suppose now that $\boldsymbol{J} \in H(\Omega)^\perp$. Then from (2.8) we have that

$$\int_\Gamma \boldsymbol{n} \times \boldsymbol{E} \cdot \mu^{-1} \nabla \times \overline{\boldsymbol{\xi}} + \mathrm{i}\omega \boldsymbol{n} \times \boldsymbol{H} \cdot \overline{\boldsymbol{\xi}}\, dA = 0$$

for all $\boldsymbol{\xi} \in \mathcal{H}(\Omega)$. Using the Dirichlet to Neumann map T

$$\int_\Gamma \boldsymbol{n} \times \boldsymbol{E} \cdot \mu^{-1} \nabla \times \overline{\boldsymbol{\xi}} + \mathrm{i}\omega T(\boldsymbol{n} \times \boldsymbol{E}) \cdot \overline{\boldsymbol{\xi}}\, dA = 0$$

so using the adjoint operator $T^* : H^{-1/2}(\mathrm{Curl};\Gamma) \to H^{-1/2}(\mathrm{Curl};\Gamma)$,

$$\int_\Gamma \boldsymbol{n} \times \boldsymbol{E} \cdot (\overline{\mu^{-1}\nabla \times \boldsymbol{\xi} - \mathrm{i}\omega T^*(\boldsymbol{\xi}_T)})\, dA = 0 \tag{2.9}$$

where $\boldsymbol{\xi}_T = (\boldsymbol{n} \times \boldsymbol{\xi}) \times \boldsymbol{n}$ on Γ. We now choose $\boldsymbol{\xi} \in H(\mathrm{curl};\Omega)$ to satisfy (2.7) together with the boundary condition

$$\overline{\mu^{-1}(\nabla \times \boldsymbol{\xi})_T + \mathrm{i}\omega T^*(\boldsymbol{\xi}_T)} = \boldsymbol{\eta} \text{ on } \Gamma$$

where $\boldsymbol{\eta} \in H^{-1/2}(\mathrm{Curl};\Gamma)$. More precisely, $\boldsymbol{\xi} \in H(\mathrm{curl};\Omega)$ satisfies the variational problem

$$\int_\Omega \mu^{-1}\nabla \times \boldsymbol{\xi} \cdot \nabla \times \boldsymbol{\phi} - (\omega^2\epsilon - \mathrm{i}\omega\sigma)\boldsymbol{\xi} \cdot \boldsymbol{\phi}\, dV + \int_\Gamma (\overline{\boldsymbol{\eta}} - \mathrm{i}\omega T^*(\boldsymbol{\xi}_T)) \cdot \boldsymbol{\phi} \times \boldsymbol{n}\, dA = 0$$

for all $\boldsymbol{\phi} \in H(\mathrm{curl};\Omega)$. We have already shown at the start of this section that T^* is characterized by solving an exterior boundary value problem for Maxwell's equations using the radiation condition with a sign change. Thus the above variational problem is equivalent to solving the following strong form of the boundary value problem of finding $\boldsymbol{\xi} \in H_{\mathrm{loc}}(\mathrm{curl};\mathbb{R}^3)$ such that

$$\nabla \times \mu^{-1}\nabla \times \boldsymbol{\xi} - (\omega^2\epsilon - \mathrm{i}\sigma\omega)\boldsymbol{\xi} = 0 \text{ in } \mathbb{R}^3 \setminus \Gamma, \tag{2.10}$$

$$[\![\boldsymbol{\xi}]\!]_T = \text{ on } \Gamma, \tag{2.11}$$

$$[\![\mu^{-1}\nabla \times \boldsymbol{\xi}]\!]_T = -\overline{\boldsymbol{\eta}} \text{ on } \Gamma, \tag{2.12}$$

$$\mu^{-1}(\nabla \times \boldsymbol{\xi}) \times \boldsymbol{x} + \mathrm{i}\omega|\boldsymbol{x}|\boldsymbol{\xi} \to 0 \text{ as } |\boldsymbol{x}| \to \infty, \tag{2.13}$$

where, for a smooth function $\boldsymbol{u}$ the jump $[\![\boldsymbol{u}]\!]_T$ on Γ is given, for $\boldsymbol{x} \in \Gamma$, by

$$[\![\boldsymbol{u}]\!]_T(\boldsymbol{x}) = \lim_{h\to 0} ((\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{u}(\boldsymbol{x} - h\boldsymbol{n}(\boldsymbol{x}))) \times \boldsymbol{n}(\boldsymbol{x}) - (\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{u}(\boldsymbol{x} + h\boldsymbol{n}(\boldsymbol{x}))) \times \boldsymbol{n}(\boldsymbol{x})))$$

where we recall that $\boldsymbol{n}$ is the unit outward normal to Ω.

Standard analysis of uniqueness and existence for the Maxwell system can now be applied to the above problem to show the existence of a unique solution $\boldsymbol{\xi}$ to (2.10)-(2.13) for any $\boldsymbol{\eta} \in H^{-1/2}(\mathrm{Curl};\Omega)$. Hence (2.9) shows that

$$\int_\Gamma \boldsymbol{n} \times \boldsymbol{E} \cdot \overline{\boldsymbol{\eta}}\, dA = 0 \quad \text{for all } \boldsymbol{\eta} \in H^{-1/2}(\mathrm{Curl};\Omega)$$

so $\boldsymbol{n} \times \boldsymbol{E} = 0$ (and hence $\boldsymbol{H} \times \boldsymbol{n} = T(\boldsymbol{E} \times \boldsymbol{n}) = 0$ on Γ). Thus a current in $H(\Omega)^\perp$ produces no measurable signal. □

The preceding result shows that only a certain component of a volume current $\boldsymbol{J}$ can be determined directly without further a priori information. Furthermore since the measurable component is a solution of the homogeneous adjoint Maxwell solution, it's support will be all of Ω (unless there is no measurable field!). Hence the support of the source also cannot be determined without further a priori information.

3. Uniqueness of surface currents. Next we consider the inverse source problem in a case where extra a priori information is assumed to be available. In particular, we assume that $\boldsymbol{J}$ is a surface current supported on a known smooth surface Σ enclosing a domain B located interior to Ω. Note that this assumption is often made in the MEG literature (see for example [22]) and Σ corresponds to the surface of the brain.

As we shall see we need to additionally assume that the surface current $\boldsymbol{J}$ is tangent to Σ and more precisely that $\boldsymbol{J} \in H^{-1/2}(\mathrm{Div};\Sigma)$. In this case it is no longer true that $\boldsymbol{H} \in H(\mathrm{curl};\Omega)$. Instead the analogue of equations (2.1)-(2.3) is to seek $\boldsymbol{E} \in H(\mathrm{curl};\Omega)$ and $\boldsymbol{H} \in L^2(\Omega)$ such that $\boldsymbol{H}|_B \in H(\mathrm{curl};B)$, $\boldsymbol{H}|_{\Omega\setminus\overline{B}} \in H(\mathrm{curl};\Omega \setminus \overline{B})$ and

$$-\mathrm{i}\omega\epsilon\boldsymbol{E} + \sigma\boldsymbol{E} - \nabla \times \boldsymbol{H} = 0 \text{ in } \Omega \setminus \overline{B} \text{ and } B, \tag{3.1}$$

$$-\mathrm{i}\omega\mu\boldsymbol{H} + \nabla \times \boldsymbol{E} = 0 \text{ in } \Omega, \tag{3.2}$$

$$\boldsymbol{H} \times \boldsymbol{n} = T(\boldsymbol{E} \times \boldsymbol{n}) \text{ on } \Gamma, \tag{3.3}$$

$$[\boldsymbol{H} \times \boldsymbol{n}] = -\boldsymbol{J} \text{ on } \Sigma, \tag{3.4}$$

where $\boldsymbol{n}$ is the outward normal to Σ and the jump $[\boldsymbol{H} \times \boldsymbol{n}]$ is defined by $[\boldsymbol{H} \times \boldsymbol{n}] = (\boldsymbol{H}|_{\Omega \setminus \overline{B}} - \boldsymbol{H}|_B)|_\Sigma \times \boldsymbol{n}$.

The above equations can be reduced to a variational problem on Ω using the Dirichlet-to-Neumann map T. First we can eliminate $\boldsymbol{H}$ on $\Omega \setminus \overline{B}$ and B respectively. Then multiplying the resulting electric field equations by a smooth test vector and integrating over the respective domains, we see that $\boldsymbol{E} \in H(\mathrm{curl}; \Omega)$ satisfies

$$\int_\Omega \mu^{-1} \nabla \times \boldsymbol{E} \cdot \nabla \times \overline{\boldsymbol{\xi}} - (\omega^2 \epsilon + \mathrm{i}\sigma\omega) \boldsymbol{E} \cdot \overline{\boldsymbol{\xi}}\, dV + \int_\Gamma \mathrm{i}\omega T(\boldsymbol{n} \times \boldsymbol{E}) \cdot \overline{\boldsymbol{\xi}}\, dA = \int_\Sigma \mathrm{i}\omega \boldsymbol{J} \cdot \overline{\boldsymbol{\xi}}\, dV \tag{3.5}$$

for all $\boldsymbol{\xi} \in H(\mathrm{curl}; \Omega)$. Existence of a unique solution to this problem follows in the standard way [19].

Now let us turn to the inverse problem. Let $\boldsymbol{\xi} \in H(\mathrm{curl}; \Omega)$ denote a solution of the adjoint Maxwell problem (i.e. satisfying (2.7)). Using (3.5) in place of (2.6) and the same arguments as in the previous section we see that the integral equation (2.8) characterizing $\boldsymbol{J}$ is modified to become

$$\mathrm{i}\omega \int_\Sigma \boldsymbol{J} \cdot \overline{\boldsymbol{\xi}}\, dA = \int_\Gamma \boldsymbol{n} \times \boldsymbol{E} \cdot \mu^{-1} \nabla \times \overline{\boldsymbol{\xi}} + \mathrm{i}\omega \boldsymbol{n} \times \boldsymbol{H} \cdot \overline{\boldsymbol{\xi}}\, dA. \tag{3.6}$$

where now $\boldsymbol{J} \in H^{-1/2}(\mathrm{Div}; \Sigma)$ and $\boldsymbol{\xi} \in H(\mathrm{curl}; \Omega)$ denotes any solution of the adjoint Maxwell problem (i.e. satisfying (2.7)). We are now in a position to state our uniqueness theorem in this case.

THEOREM 3.1. *Suppose $\boldsymbol{J} \in H^{-1/2}(\mathrm{Div}; \Sigma)$ and $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy (3.1)-(3.4) (i.e. $\boldsymbol{J}$ is a distribution supported on a smooth surface Σ in Ω). Assume also that Σ is a given surface with $\Sigma = \partial B$ and B is a domain in Ω with connected complement. Suppose in addition that the electromagnetic properties of the medium in B are constant. Let Σ_0 denote the support of $\boldsymbol{J}$ on Σ. Then, assuming ω is not a Maxwell eigenvalue for the adjoint Maxwell equation in B, Σ_0 and $\boldsymbol{J}|_{\Sigma_0}$ are uniquely determined by the surface data $\boldsymbol{E} \times \boldsymbol{n}$ and $\boldsymbol{H} \times \boldsymbol{n}$ on Γ.*

Remarks: 1) This theorem rules out non-radiating sources on smooth surfaces satisfying the conditions of the theorem. We note that for the Helmholtz equation Devaney [8] has also considered the problem of non-radiating surface sources. His examples show that for more general surfaces than considered here (in particular an infinite plane), uniqueness is lost since non-radiating sources can be constructed.

2) If $\sigma > 0$ in B the eigenvalue assumption may be dropped.

Proof. By virtue of the assumptions on ϵ and σ, the electric field can be uniquely continued from Γ to $\Omega \setminus B$. Thus, from the point of view of uniqueness, it suffices to apply (3.6) on ∂B. Now suppose that there are two currents $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ in $H^{-1/2}(\mathrm{Div}; \Sigma)$ giving rise to the same electromagnetic fields on ∂B. Then

$$\int_{\partial B} (\boldsymbol{J}_1 - \boldsymbol{J}_2) \cdot \boldsymbol{\xi}\, dA = 0 \tag{3.7}$$

for all $\boldsymbol{\xi}$ satisfying the adjoint equation (2.7) (which has constant coefficients in B).

Let $\boldsymbol{\xi}$ be a Herglotz wave function using plane waves that are solutions of the adjoint Maxwell system on B. It is shown in [5] that any weak solution of the constant coefficient homogeneous Maxwell system can be approximated in the $H(\mathrm{curl}; \Omega)$ norm by a Herglotz wave function for the adjoint system. Given a function in $H^{-1/2}(\mathrm{Curl}; \Sigma)$ we may use it as tangential boundary data for the adjoint Maxwell problem. Using the fact that ω is not a Maxwell eigenvalue for the adjoint Maxwell problem, a unique solution of this boundary value problem exists. Approximating the resulting solution of the adjoint problem by a Herglotz wave function in $H(\mathrm{curl}; \Omega)$, and taking the tangential trace of the Herglotz wave function shows that the tangential trace of the Herglotz wave functions are dense in $H^{-1/2}(\mathrm{Curl}; \Sigma)$ (see also [19]). Hence (3.7) shows that $\boldsymbol{J}_1 - \boldsymbol{J}_2 = 0$ and so $\boldsymbol{J}$ is uniquely determined by equation (3.6). We have thus proved the theorem. □

This theorem may give some support for the observation that rather good source reconstructions can be obtained from MEG data assuming the source is on the brain surface.

4. Uniqueness for dipole sources. The final case we shall consider is where the current source is known a priori to be the superposition of a finite number of dipole sources. For the brain these dipoles might represent active channels in the nerve membrane. Thus we have in mind that a finite number dipole sources with unknown position and polarization are to be detected in the interior of Ω (in particular the sources can not lie on Γ). This is not a restriction in practice since, when detecting sources in the brain, measurements

are taken on the skin surface, and sources are located in the brain within the skull. In this case the source takes the form

$$\boldsymbol{J}(\boldsymbol{x}) = \sum_{j=1}^{N} \boldsymbol{p}_j \delta_{z_j}(\boldsymbol{x})$$

for some $N < \infty$ where $|\boldsymbol{p}_j| \neq 0$ and $\boldsymbol{z}_j \in \Omega$ for all j (δ_z is the delta function located at $\boldsymbol{z}$). We assume also that the source points $\boldsymbol{z}_1, \cdots, \boldsymbol{z}_N$ are distinct.

In order to establish the unique determination of $\boldsymbol{J}$ in this case, we need an additional technical restriction on the medium near the source points. We assume that each source point is located in a region of constant electromagnetic properties. In particular, for each j, there is a ball centered at $\boldsymbol{z}_j$ such that ϵ and μ are constant in the ball. This assumption is used first to justify the variational formulation of the inverse problem (and later in our uniqueness result). Since a weak solution of the Maxwell system is a strong solution away from boundaries and discontinuities of the coefficients, we know that the solution $\boldsymbol{\xi} \in H(\text{curl}; \Omega)$ of (2.7) is an analytic function in the neighborhood of each source point and hence (2.8) still holds. In the case of dipole sources (2.8) gives

$$\sum_{j=1}^{N} \boldsymbol{p}_j \cdot \overline{\boldsymbol{\xi}(\boldsymbol{z}_j)} = \int_{\Gamma} \boldsymbol{n} \times \boldsymbol{E} \cdot \mu^{-1} \nabla \times \overline{\boldsymbol{\xi}} + \mathrm{i}\omega \boldsymbol{n} \times \boldsymbol{H} \cdot \overline{\boldsymbol{\xi}}\, dA. \tag{4.1}$$

This equation, which appears already in [1], will form the basis of our study of uniqueness results for dipole sources where we generalize the results of Ammari, Bao and Flemming [1] to more than one dipole. For one dipole in free space this theorem was also proved in [13] and [3].

We can now state our uniqueness result.

THEOREM 4.1. *Under the aforementioned restrictions on the electromagnetic parameters ϵ and σ and the source points, the source problem has a unique solution in that the number, position and non-zero polarization of the point sources are uniquely determined.*

Proof. The first part of the proof is an extension of the proof in [1] for a single dipole. We start by showing that the number and position of the dipoles is uniquely determined. Suppose that there are two sets of dipole positions $S_1 = \{\boldsymbol{z}_1, \cdots, \boldsymbol{z}_N\}$ and $S_2 = \{\tilde{\boldsymbol{z}}_1, \cdots, \tilde{\boldsymbol{z}}_M\}$ giving rise to the same measurements on Γ. By assumption the points in S_1 (respectively S_2) are mutually distinct. If $S_1 \neq S_2$ there must be a point in one set not in the other. We can assume $\boldsymbol{z}_1 \in S_1$ and $\boldsymbol{z}_1 \notin S_2$. By unique continuation, the electric field can be continued from the boundary Γ to a neighborhood of $\boldsymbol{z}_1$. Let $\boldsymbol{E}_1$ denote the electric field continued to the domain $\Omega \setminus S_1$ and $\boldsymbol{E}_2$ denote the field continued to the domain $\Omega \setminus S_2$. Then $\boldsymbol{E}_1 = \boldsymbol{E}_2$ in a punctured ball around $\boldsymbol{z}_1$.

Now suppose that $\boldsymbol{x}$ approaches $\boldsymbol{z}_1$. The field $\boldsymbol{E}_2$ is a continuous function of $\boldsymbol{x}$ since $\boldsymbol{z}_1$ is assumed to lie in a ball where ϵ and σ are constant, and $\boldsymbol{z}_1$ is not a source point for $\boldsymbol{E}_2$. Thus $|\boldsymbol{E}_2(\boldsymbol{x})| \to |\boldsymbol{E}_2(\boldsymbol{z}_1)|$ as $\boldsymbol{x} \to \boldsymbol{z}_1$. Considering now $\boldsymbol{E}_1$ there are two possibilities: either $\boldsymbol{p}_1 = 0$ and the source point should be dropped from S_1 since we have assume the the polarization at each source point is non-zero, or $\boldsymbol{p}_1 \neq 0$. In the latter case $|\boldsymbol{E}_1(\boldsymbol{x})|$ is unbounded as $\boldsymbol{x} \to \boldsymbol{z}_1$ since $\boldsymbol{z}_1$ is a source point for this field. This contradicts the previously noted equality of the fields and so $S_1 \subset S_2$. Reversing the roles of S_1 and S_2 shows that $S_1 = S_2$.

We now must show that the polarization at each source point is uniquely determined. We know already that the position of the non-trivial sources is uniquely determined, so we know that we can continue the electric field to a punctured ball about each source point. In particular we may continue the field to the surface of a ball centered at a source point containing media with constant electromagnetic parameters ϵ and σ. Thus we are faced with the problem of knowing $\boldsymbol{n} \times \boldsymbol{E}$ and $\boldsymbol{n} \times \boldsymbol{H}$ on the surface of a ball containing a single source point at it's center, and having constant electromagnetic properties inside. If B denotes the ball and ∂B denotes its boundary, the variational characterization of the source field (see (2.8) becomes, after moving the source point to the origin,

$$\boldsymbol{p} \cdot \overline{\boldsymbol{\xi}(0)} = \int_{\partial B} \left(\boldsymbol{n} \times \boldsymbol{E} \cdot \mu^{-1} \nabla \times \overline{\boldsymbol{\xi}} - \mathrm{i}\omega \boldsymbol{n} \times \boldsymbol{H} \cdot \overline{\boldsymbol{\xi}} \right) dV \tag{4.2}$$

where $\boldsymbol{\xi}$ is an $H(\text{curl}; B)$ solution of $\nabla \times \nabla \times \boldsymbol{\xi} - k^2 \boldsymbol{\xi} = 0$ in B and $k^2 = (\omega^2 \epsilon\mu - \mathrm{i}\sigma\omega\mu)$. We now know that $\boldsymbol{\xi} = \boldsymbol{\eta} \exp(\mathrm{i}k\boldsymbol{x} \cdot \boldsymbol{d})$ with $\boldsymbol{d} \in \mathbb{R}^3$ such that $|\boldsymbol{d}| = 1$ and $\boldsymbol{d} \cdot \boldsymbol{\eta} = 0$ is a solution of the above adjoint

Maxwell system. Using this solution in (4.2) shows that $\boldsymbol{p}\cdot\boldsymbol{\eta}$ is determined for any $\boldsymbol{\eta}$ and hence $\boldsymbol{p}$ is uniquely determined. This completes the proof. □

5. Measurements on a portion of Γ. In real applications the measurements of $\boldsymbol{n}\times\boldsymbol{E}$ and $\boldsymbol{n}\times\boldsymbol{H}$ are only made on a subset of the boundary. We now assume that $\boldsymbol{n}\times\boldsymbol{E}$ and $\boldsymbol{n}\times\boldsymbol{H}$ are known on $\Gamma_m\subset\Gamma$ where Γ_m is a "measurement" domain that may be multiply connected. In this case a variational characterization of $\boldsymbol{J}$ can still be derived. Let $\boldsymbol{\xi}\in H_{\text{loc}}(\text{curl};\mathbb{R}^3\setminus\Gamma_m)$ be any solution of the adjoint Maxwell system in $\mathbb{R}^3\setminus\Gamma_m$ together with the adjoint Silver-Müller radiation condition so that, in a weak sense,

$$\nabla\times\mu^{-1}\nabla\times\boldsymbol{E}-(\omega^2\epsilon-\mathrm{i}\omega\sigma)\boldsymbol{\xi}=0 \text{ in } \mathbb{R}^3\setminus\Gamma_m \tag{5.1}$$

$$\lim_{|\boldsymbol{x}|\to\infty}((\nabla\times\boldsymbol{\xi})\times\boldsymbol{x}+\mathrm{i}\omega\mu_0|\boldsymbol{x}|\boldsymbol{\xi})=0. \tag{5.2}$$

We then have the following lemma giving the extension of (2.8) to this case.

LEMMA 5.1. *Suppose $\boldsymbol{n}\times\boldsymbol{E}$ and $\boldsymbol{n}\times\boldsymbol{H}$ are known on a subdomain Γ_m of Γ. Then provided $\boldsymbol{\xi}$ satisfies the adjoint problem (5.1)-(5.2) on $\mathbb{R}^3\setminus\Gamma_m$ then*

$$\mathrm{i}\omega\int_\Omega \boldsymbol{J}\cdot\xi\,\mathbf{dV}=\int_{\Gamma_\mathbf{m}}\left(\boldsymbol{n}\times\boldsymbol{E}\cdot[\![\mu^{-1}\nabla\times\overline{\boldsymbol{\xi}}]\!]_\mathbf{T}+\mathrm{i}\omega\boldsymbol{n}\times\boldsymbol{H}\cdot[\![\overline{\boldsymbol{\xi}}]\!]_\mathbf{T}\right)\mathbf{dA} \tag{5.3}$$

where the previously introduced tangential jump $[\![\cdot]\!]_T$ is extended to a smooth function on Ω and $\mathbb{R}^3\setminus\overline{\Omega}$ in the obvious way so that for $\boldsymbol{x}\in\Gamma$

$$[\![\boldsymbol{u}]\!]_T(\boldsymbol{x})=\lim_{h\to 0}((\boldsymbol{n}(\boldsymbol{x})\times\boldsymbol{u}(\boldsymbol{x}-h\boldsymbol{n}(\boldsymbol{x})))\times\boldsymbol{n}(\boldsymbol{x})-(\boldsymbol{n}(\boldsymbol{x})\times\boldsymbol{u}(\boldsymbol{x}+h\boldsymbol{n}(\boldsymbol{x})))\times\boldsymbol{n}(\boldsymbol{x}))$$

where we recall that $\boldsymbol{n}$ is the unit outward normal to Ω. Analogues of equations (3.6) and (4.1) also hold replacing the right hand side of the respective equations by the above right hand side.

Proof. The proof proceeds similarly to the derivation of (2.8). Let B_R denote a ball of radius R with R chosen sufficiently large such that $\overline{\Omega}\subset B_R$. Let $\boldsymbol{\xi}$ denote a smooth test function on $B_R\setminus\Gamma_m$. Then multiplying (2.4) by $\overline{\boldsymbol{\xi}}$ and integrating by parts twice we arrive at

$$\begin{aligned}\mathrm{i}\omega\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{\xi}}\,dV=&\int_{B_R\setminus\Gamma_m}\boldsymbol{E}\cdot\left(\overline{\nabla\times\mu^{-1}\nabla\times\boldsymbol{\xi}-(\omega^2\epsilon-\mathrm{i}\omega\sigma)\boldsymbol{\xi}}\right)dV\\&+\int_{\Gamma_m}\left(\boldsymbol{n}\times\boldsymbol{E}\cdot[\![\mu^{-1}\nabla\times\overline{\boldsymbol{\xi}}]\!]_T+\mathrm{i}\omega\boldsymbol{n}\times\boldsymbol{H}\cdot[\![\overline{\boldsymbol{\xi}}]\!]_T\right)dA\\&+\int_{\partial B_R}\left(\boldsymbol{n}\times\boldsymbol{E}\cdot\mu^{-1}\nabla\times\overline{\boldsymbol{\xi}}+\mathrm{i}\omega\boldsymbol{n}\times\boldsymbol{H}\cdot\overline{\boldsymbol{\xi}}\right)dA\end{aligned}$$

where we have used the fact that $\boldsymbol{J}=0$ outside Ω. If we now choose $\boldsymbol{\xi}$ to satisfy (5.1) the volume integral term vanishes. The integral on ∂B_R may be rewritten as follows:

$$\begin{aligned}\int_{\partial B_R}\left(\boldsymbol{n}\times\boldsymbol{E}\cdot\mu^{-1}\nabla\times\overline{\boldsymbol{\xi}}+\mathrm{i}\omega\boldsymbol{n}\times\boldsymbol{H}\cdot\overline{\boldsymbol{\xi}}\right)dA=&\int_{\partial B_R}\left(\boldsymbol{E}\cdot\left(\overline{(\mu_0^{-1}(\nabla\times\boldsymbol{\xi})\times\boldsymbol{n}+\mathrm{i}\omega\mu_0\boldsymbol{\xi})}\right)\right.\\&\left.-\mathrm{i}\omega(\boldsymbol{H}\times\boldsymbol{n}-\boldsymbol{E})\cdot\overline{\boldsymbol{\xi}}\right)dA.\end{aligned}$$

Using the fact (see [4]) that

$$|\boldsymbol{E}|=O\left(\frac{1}{|\boldsymbol{x}|}\right)\text{ and }|(\nabla\times\boldsymbol{\xi})\times\boldsymbol{n}+\mathrm{i}\omega\mu_0\boldsymbol{\xi}|=O\left(\frac{1}{|\boldsymbol{x}|^2}\right)$$

for large $|\boldsymbol{x}|$ shows that as $R\to\infty$ the first term in the integral on the right hand side above vanishes. Similar estimates show that the second term vanishes. This proves the desired equality. □

It is now possible to use the characterization of $\boldsymbol{J}$ in this theorem to study the uniqueness of solutions of the inverse source problem with measurements not over the entire surface Γ. In addition the formula could be used as part of an inversion scheme.

Obviously volume currents cannot be reconstructed using (5.3). However the proofs of uniqueness in Sections 3 and 4 can be proved provided Γ_m is such that the unique continuation property holds.

6. Conclusion. We have examined the uniqueness problem for the inverse source problem in the case where the background medium is non-homogeneous. We have shown that, without additional a priori information, the data does not uniquely determine a volume source, nor does it even determine the support of a volume source (these results are well known for a constant background). However if the source is a priori known to have special features, in particular if it is a surface current on a known surface, or a a current due to a collection of dipoles, we show that the data does uniquely determine the current density. It would be useful to remove the many technical restrictions regarding the electromagnetic properties assumed in this work.

The variational characterization of the current distribution in each case can also be used as part of inversion scheme. We hope to test this approach in the future, in particular focusing on the case of measurements on a portion of the boundary.

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