Diagonal Representation of Certain Matrices

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Abstract

An explicit expression is provided for the characteristic polynomial of a matrix $M$ of the form

$$M = D - \begin{pmatrix} 0 & ab^T \\ ba^T & 0 \end{pmatrix},$$

(1)

where $D$ is a diagonal matrix, and $a$ and $b$ are column vectors. Also, an explicit expression is provided for the matrix of normalized eigenvectors of $M$, in terms of the roots of the characteristic polynomial (i.e., in terms of the eigenvalues of $M$).

1 A Lemma, a Remark, and an Observation

The following lemma is verified by substituting into the left hand side of (7) the definitions of $P$ in (6) and $U$ in (9)–(16), and simplifying the result using (4). See [2] for similar results, and [3] and [1] for applications.

Lemma 1 Suppose that $m$ and $n$ are positive integers, $a = (a_0, a_1, \ldots, a_{m-2}, a_{m-1})^T$ and $b = (b_0, b_1, \ldots, b_{n-2}, b_{n-1})^T$ are real vectors, and $d_0, d_1, \ldots, d_{m+n-2}, d_{m+n-1}$ and $\lambda_0, \lambda_1, \ldots, \lambda_{m+n-2}, \lambda_{m+n-1}$ are real numbers such that

$$\lambda_j \neq d_k$$

(2)

for any $j, k$ ($j, k = 0, 1, \ldots, m + n - 2, m + n - 1$),

$$\lambda_j \neq \lambda_k$$

(3)

when $j \neq k$, and

$$\left(\sum_{k=0}^{m-1} \frac{(a_k)^2}{d_k - \lambda_j}\right) \left(\sum_{k=0}^{n-1} \frac{(b_k)^2}{d_{m+k} - \lambda_j}\right) = 1$$

(4)

(with $j = 0, 1, \ldots, m + n - 2, m + n - 1$).

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Standard Form 298 (Rev. 8-98) Prescribed by ANSI Std Z39-18
Suppose further that $D$ is the diagonal $(m + n) \times (m + n)$ matrix defined by the formula

$$D = \begin{pmatrix}
d_0 & 0 & \cdots & \cdots & 0 \\
0 & d_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & d_{m+n-2} & 0 \\
0 & \cdots & \cdots & 0 & d_{m+n-1}
\end{pmatrix}, \quad (5)$$

and $P$ is the $(m + n) \times (m + n)$ matrix defined by the formula

$$P = \begin{pmatrix} 0 & ab^T \\ ba^T & 0 \end{pmatrix}, \quad (6)$$

where $0$ denotes matrices consisting entirely of zeroes.

Then,

$$(D - P)U = U \Lambda, \quad (7)$$

where $\Lambda$ is the diagonal $(m + n) \times (m + n)$ matrix defined by the formula

$$\Lambda = \begin{pmatrix}
\lambda_0 & 0 & \cdots & \cdots & 0 \\
0 & \lambda_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \lambda_{m+n-2} & 0 \\
0 & \cdots & \cdots & 0 & \lambda_{m+n-1}
\end{pmatrix}, \quad (8)$$

and $U$ is the orthogonal $(m + n) \times (m + n)$ matrix defined by the formula

$$U = \begin{pmatrix} AVR \\ BWS \end{pmatrix}. \quad (9)$$

In (9), $A$ is the diagonal $m \times m$ matrix defined by the formula

$$A = \begin{pmatrix}
a_0 & 0 & \cdots & \cdots & 0 \\
0 & a_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & a_{m-2} & 0 \\
0 & \cdots & \cdots & 0 & a_{m-1}
\end{pmatrix}, \quad (10)$$

$B$ is the diagonal $n \times n$ matrix defined by the formula

$$B = \begin{pmatrix}
b_0 & 0 & \cdots & \cdots & 0 \\
0 & b_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & b_{n-2} & 0 \\
0 & \cdots & \cdots & 0 & b_{n-1}
\end{pmatrix}. \quad (11)$$
$V$ is the $m \times (m + n)$ matrix with entry $V_{j,k}$ defined by the formula

$$V_{j,k} = \frac{1}{d_j - \lambda_k}$$

(with $j = 0, 1, \ldots, m-2, m-1; k = 0, 1, \ldots, m+n-2, m+n-1$), $W$ is the $n \times (m + n)$ matrix with entry $W_{j,k}$ defined by the formula

$$W_{j,k} = \frac{1}{d_{m+j} - \lambda_k}$$

(with $j = 0, 1, \ldots, n-2, n-1; k = 0, 1, \ldots, m+n-2, m+n-1$), $S$ is the diagonal $(m+n) \times (m+n)$ matrix with the diagonal entries $S_{0,0}, S_{1,1}, \ldots, S_{m+n-2,m+n-2}, S_{m+n-1,m+n-1}$ defined by the formula

$$S_{j,j} = \frac{1}{\sqrt{\sum_{k=0}^{m-1} \left( \frac{a_k c_j}{d_k - \lambda_j} \right)^2 + \sum_{k=0}^{n-1} \left( \frac{b_k}{d_{m+k} - \lambda_j} \right)^2}},$$

and $R$ is the diagonal $(m+n) \times (m+n)$ matrix with the diagonal entries $R_{0,0}, R_{1,1}, \ldots, R_{m+n-2,m+n-2}, R_{m+n-1,m+n-1}$ defined by the formula

$$R_{j,j} = c_j S_{j,j}.$$  \hspace{1cm} (15)

In (14) and (15), $c_0, c_1, \ldots, c_{m+n-2}, c_{m+n-1}$ are the real numbers defined by the formula

$$c_j = \sum_{k=0}^{n-1} \frac{(b_k)^2}{d_{m+k} - \lambda_j}.$$ \hspace{1cm} (16)

**Remark 2** The equation (4) is equivalent to the characteristic (secular) equation

$$\det |\lambda_j I - (D - P)| = 0$$

for the eigenvalues $\lambda_j$ (with $j = 0, 1, \ldots, m+n-2, m+n-1$) of the matrix $D - P$.

**Observation 3** The upper block $AVR$ of the matrix $U$ defined in (9) has the form of a diagonal matrix ($A$) times a matrix of inverse differences ($V$) times another diagonal matrix ($R$). The lower block $BWS$ of the matrix $U$ defined in (9) also has the form of a diagonal matrix ($B$) times a matrix of inverse differences ($W$) times another diagonal matrix ($S$). Therefore, there exists an algorithm which applies such an $N \times N$ matrix $U$ (or its adjoint) to an arbitrary real vector of length $N$ in $O(N \log(1/\varepsilon))$ operations, where $\varepsilon$ is the precision of computations (see [3]).

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References

