TECHNICAL RESEARCH REPORT

SYSTEMS RESEARCH CENTER

Supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), Industry and the University

Range of the *k*-Dimensional Radon Transform in Real Hyperbolic Spaces

by C.A. Berenstein and E.C. Tarabusi

TR 91-101

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE **1991**	2. REPORT TYPE	3. DATES COVERED **00-00-1991 to 00-00-1991**
4. TITLE AND SUBTITLE **Range of the k-Dimensional Radon Transform in Real Hyperbolic Spaces**		5a. CONTRACT NUMBER
		5b. GRANT NUMBER
		5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER
		5e. TASK NUMBER
		5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **University of Maryland,Systems Research Center,College Park,MD,20742**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**		
13. SUPPLEMENTARY NOTES		
14. ABSTRACT **see report**		
15. SUBJECT TERMS		

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**		**19**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

Range of the k-Dimensional Radon Transform in Real Hyperbolic Spaces

CARLOS A. BERENSTEIN* AND ENRICO CASADIO TARABUSI†

Abstract. Characterizations of the range of the totally geodesic k-dimensional Radon transform on the n-dimensional hyperbolic space are given both in terms of moment conditions and as the kernel of a differential operator.

1. INTRODUCTION

The recent interest on the k-dimensional totally geodesic Radon transform R in the real n-dimensional hyperbolic space $\mathbf{H}^n$ has arisen in part due to its applications to Electrical Impedance Tomography (EIT). This transform was introduced by Helgason in [**H1**], who found two kinds of inversion formulas, the former valid for k even [**H1**], [**H2**], the latter for any k [**H3**]. Since the approximate inversion algorithm for EIT proposed by Barber and Brown [**BB1**], [**BB2**], [**SV**], amounts to using the backprojection R^* as an approximate inverse, the authors searched for a filtered backprojection inversion [**BC**]. The crucial step is Helgason's observation that R^*R acts on the space $\mathcal{S}(\mathbf{H}^n)$ as a convolution operator with a radial function, namely, up to a multiplicative constant, $\sinh^{k-n} r$. Using symbolic calculus one finds the algorithm proposed in [**BC**], which has the form

$$p_{n,k}(\Delta_H)\mathcal{S}_{n,k} * R^*R = I,$$

where $p_{n,k}$ is a polynomial in the Laplace-Beltrami operator Δ_H and $\mathcal{S}_{n,k}$ is a radial integrable function. By analogy with the Euclidean Radon transform, one expects that this algorithm will perform better than [**BB1**], [**BB2**], [**IC**], [**AS**], in solving the EIT defining equations.

Another question that arises frequently in applications is the characterization of the ranges of the Radon transforms. This question, whose answer was long known in the space $\mathcal{D}(\mathbf{R}^n)$ for the Euclidean Radon transform [**H2**, Corollary I.2.28] has only recently been completely settled in $\mathcal{S}(\mathbf{R}^n)$. We have found intertwining operators between Euclidean and hyperbolic Radon transforms, hence the characterization of the range of the Radon transform in $\mathcal{D}(\mathbf{H}^n)$ and $\mathcal{S}(\mathbf{H}^n)$ can be settled very easily. Another consequence of the existence of intertwining operators is that one finds other inversion formulas for the hyperbolic Radon transform. This was inspired by the works of Quinto [**Q**] and Kurusa [**Ku2**].

The authors would like to thank W. M. Goldman, E. L. Grinberg, E. T. Quinto, and R. S. Strichartz for informative conversations.

*Research partially supported by grants NSFDCDR-8803012, NSFDMS 90-00619, and AFOSR 90-0105.
†Research partially supported by the Ministero dell'Università e della Ricerca Scientifica e Tecnologica.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

2. Preliminaries

We recall here the definitions and notation for the Radon and Riesz transforms in $\mathbf{H}^n$ as given in [**H2**]. We shall use the 'conformal disk' model for $\mathbf{H}^n$, viz., the open unit ball $\mathbf{B}^n$ of $\mathbf{R}^n$ with the metric

$$ds^2 = \frac{4dx^2}{(1-\|x\|^2)^2} = \frac{4\sum_{j=1}^n dx_j^2}{(1-\sum_{j=1}^n x_j^2)^2},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^n$. Such metric is in fact conformal to the Euclidean one dx^2 and has constant curvature -1 (in some chapters of [**H2**] the curvature is -4). The induced distance between $x, y \in \mathbf{H}^n$ is

$$d(x,y) = 2\operatorname{arc\,sinh}\frac{\|x-y\|}{\sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}};$$

conversely, the Euclidean norm of x can be recovered by

$$\|x\| = \tanh\frac{d(x,o)}{2}.$$

The geodesics and the totally geodesic hypersurfaces of $\mathbf{H}^n$ are arcs of circle, respectively spherical caps, which intersect $\mathbf{S}^{n-1}$ perpendicularly. The spheres which are tangent to $\mathbf{S}^{n-1}$ and are contained in the unit ball are called *horocycles*.

In geodesic polar coordinates write $x \in \mathbf{H}^n$ as $x = (\omega, r)$, where $r = d(x,o)$ and $\omega \in \mathbf{S}^{n-1}$. The hyperbolic metric is then expressed by

$$ds^2 = dr^2 + \sinh^2 r\, d\omega^2,$$

where $d\omega^2$ is the usual metric in $\mathbf{S}^{n-1}$. Correspondingly, the $(n-1)$-dimensional area of a geodesic sphere of radius r is

$$A_n(r) = \Omega_n \sinh^{n-1} r, \qquad \text{where } \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \text{ is the Euclidean area of } \mathbf{S}^{n-1}. \tag{2.1}$$

The Laplace-Beltrami operator on $\mathbf{H}^n$ is

$$\Delta_H = \frac{(1-\|x\|^2)^n}{4}\sum_{j=1}^n \frac{\partial}{\partial x_j}\left[(1-\|x\|^2)^{2-n}\frac{\partial}{\partial x_j}\right],$$

which specializes to $\Delta_H = (1-|z|^2)^2\partial^2/\partial z\partial\bar{z}$ (with $z = x_1 + ix_2$) in the case of $n = 2$. In polar coordinates

$$\Delta_H = \frac{\partial^2}{\partial r^2} + (n-1)\coth r\frac{\partial}{\partial r} + \sinh^{-2} r\Delta_S,$$

where Δ_S is the Laplace-Beltrami operator on $\mathbf{S}^{n-1}$.

The space $\mathcal{D}(\mathbf{H}^n)$ denotes as usual the space of all $\mathcal{C}^\infty$ functions with compact support in $\mathbf{H}^n$, i.e., it coincides with the space $\mathcal{D}(\mathbf{B}^n)$ of $\mathcal{C}^\infty$ functions f on $\mathbf{R}^n$ whose support $\operatorname{supp} f$ is in $\mathbf{B}^n$. The Schwartz space $\mathcal{S}(\mathbf{H}^n)$ of fast decreasing functions in $\mathbf{H}^n$ is the space of $\mathcal{C}^\infty$ functions f on $\mathbf{H}^n$ such that for any positive integers m, k we have

$$\sup_{x\in\mathbf{H}^n} \left|\Delta_H^k f(x)\right| e^{md(x,o)} < \infty$$

(cf. [**H2**]). This is equivalent to the condition that for every multiindex $\alpha \in \mathbf{N}^n$ (where $\mathbf{N}$ is the set of nonnegative integers), the function $\mathbf{B}^n \ni x \mapsto \partial^{|\alpha|} f/\partial x^\alpha(x)$ has a continuous extension to the closed ball $\overline{\mathbf{B}}^n$ which vanishes of infinite order in $\partial\mathbf{B}^n$. In other words, the space $\mathcal{S}(\mathbf{H}^n)$ coincides with the space $\mathcal{D}(\overline{\mathbf{B}}^n)$ of $\mathcal{C}^\infty$ functions f in $\mathbf{R}^n$ such that $\operatorname{supp} f \subseteq \overline{\mathbf{B}}^n$. The two spaces coincide even topologically.

Fix k, with $1 \le k < n$: the space $\Gamma = \Gamma_{n,k}$ of totally geodesic k-dimensional submanifolds (*k-geodesics* for short) of $\mathbf{H}^n$ is a homogeneous space under an action of the group $SO(1,n)$ of isometries of $\mathbf{H}^n$. Each $\gamma \in \Gamma$ carries the k-dimensional area element $dm_{H,\gamma}$ induced by the volume element dm_H in $\mathbf{H}^n$. Hence the totally geodesic k-dimensional Radon transform $R_H = R_{H,n,k}$ is defined on the space $\mathcal{S}(\mathbf{H}^n)$ by

$$R_H f(\gamma) = \int_\gamma f(x)\, dm_{H,\gamma}(x) \quad \text{for all } \gamma \in \Gamma.$$

The family Γ_x of elements of Γ passing through a fixed point x is a homogeneous space for the isotropy group $SO(1,n)_x$ of x, which is isomorphic to $SO(n)$. Hence Γ_x carries a normalized measure $dm_{\Gamma,x}$ which is invariant under $SO(1,n)_x$, and is 'independent' of x in an obvious sense. For a continuous function ϕ on Γ we can define the backprojection operator R_H^* by

$$R_H^*\phi(x) = \int_{\Gamma_x} \phi(\gamma)\, dm_{\Gamma,x}(\gamma) = \int_{SO(n)} \phi(gh\cdot\gamma)\, dh \quad \text{for all } x \in \mathbf{H}^n,$$

where g is a fixed element of $SO(1,n)$ such that $g\cdot o = x$, while h runs in $SO(n)$ and dh is the normalized invariant measure in $SO(n)$. One of the uses of the backprojection operator is to find an inversion formula for the Radon transform. This is based on the fact that, denoting by $d\theta$ the area element on the geodesic sphere $S(x,r)$ of center x and radius r, we have

$$R_H^* R_H f(x) = \int_{\mathbf{H}^n} f(y)\mathcal{R}(d(x,y))\, dm_H(y) = \int_0^\infty \mathcal{R}(r)\left[\int_{S(x,r)} f(y)\, d\theta(y)\right] dr, \tag{2.2}$$

for a function $\mathcal{R} = \mathcal{R}_{n,k}$ on $[0,+\infty)$ (cf. [**H2**, Theorem I.4.5]): interpreting $\mathcal{R}$ as a radial function on $\mathbf{H}^n$ through $\mathcal{R}(x) = \mathcal{R}(d(x,o))$ (the same abuse of notation will be extended to all radial functions, with no possible confusion), we write this integral as $\mathcal{R} * f(x)$—note that the inner integral is not normalized—: in fact both $\mathcal{R}$ and f can be pulled back as functions on the group $SO(1,n)$, convolved there, and the result, pushed to $\mathbf{H}^n$ again, coincides with the middle term of (2.2).

The function $\mathcal{R}$ turns out to be [**H1**]

$$\mathcal{R}(r) = \pi^{(k-n)/2} \frac{\Gamma(n/2)}{\Gamma(k/2)} \sinh^{k-n} r.$$

In [**BC**] it is shown that if S is the operator of convolution associated with the radial function

$$\mathcal{S}(r) = \sinh^{k-n} r \cosh r,$$

then for an explicit polynomial p of degree k one has

$$p(\Delta) S R_H^* R_H = I$$

(where I is the identity operator), which is the filtered backprojection formula mentioned in the introduction. For instance if $(n,k) = (2,1)$ this formula reduces to $-(4\pi)^{-1} S R_H^* R_H = I$: the kernel $\mathcal{S}$ can be replaced by $\coth r - 1$ to obtain an integrable kernel. Other inversion formulas, which do not factor through $R_H^* R_H$, can be found in [**H3**].

The k-dimensional hyperbolic Radon transform was in fact modeled upon the Euclidean one, which is similarly defined in the space $\mathbf{R}^n$. Namely, if $f \in \mathcal{S}(\mathbf{R}^n)$ and $\phi \in \mathcal{S}(G)$, where $G = G_{n,k}$ is the Grassmannian space of all *k-planes*—viz., of k-dimensional affine subspaces of $\mathbf{R}^n$—, then

$$\begin{aligned} R_E f(\pi) &= \int_\pi f(y)\, dm_{E,\pi}(y) && \text{for all } \pi \in G, \\ R_E^* \phi(x) &= \int_{G_y} \phi(\pi)\, dm_{G,y}(\pi) = \int_{SO(n)} \phi(gh \cdot \pi)\, dh && \text{for all } y \in \mathbf{R}^n, \end{aligned}$$

where $dm_{E,\pi}$ is the k-dimensional Lebesgue measure on π, while $dm_{G,y}$ is the normalized $SO(n)$-invariant measure on $G_y = \{\pi \in G \colon y \in \pi\}$.

In order to characterize the range of the transform R_H in the hyperbolic space $\mathbf{H}^n$ we shall show that it can be intertwined with the Euclidean Radon transform. For that purpose we need to set up some additional notation. Let $\Xi = \Xi_{n,k}$ be the set of $(n-k-1)$-*circles*, intersections of $(n-k)$-dimensional subspaces of $\mathbf{R}^n$ with the unit sphere $\mathbf{S}^{n-1}$. Denote by G_B the space of k-planes that intersect $\mathbf{B}^n$. Naturally the closure $\overline{G}_B$ of G_B is the compact subset of G consisting of k-planes intersecting $\overline{\mathbf{B}}^n$.

One can parametrize G and Γ in a similar way. First take polar coordinates in $\mathbf{R}^n$, so that a point x is given by a pair $(\omega, s) \in \mathbf{S}^{n-1} \times [0,\infty)$. Then identify G with the set of triples $(\xi, \omega, s) \in \Xi \times \mathbf{S}^{n-1} \times [0,\infty)$ such that $\omega \in \xi$: such a triple represents the k-plane which is orthogonal at the point (ω, s) to the $(n-k)$-plane through o generated by ξ (observe that (ω, s) is the closest point of the k-plane to o). In particular $G_B = \{(\xi, \omega, s) \in G \colon s < 1\}$. In a completely analogous way Γ can be parametrized by the same set of triples, and the $(n-k)$-plane generated by ξ in $\mathbf{R}^n$ projects to an $(n-k)$-geodesic through o via the exponential map at o. Note that in both cases the ξ coordinate is redundant if $k = n-1$ (it must be $\xi = \{\pm\omega\}$).

In next lemma we find the explicit expressions in such coordinates of the measures on $\mathbf{R}^n$, $\mathbf{H}^n$, G, Γ. We use the following notation: if $\pi \in G$ let $d_\pi\omega$ be the k-dimensional

measure on $\{\omega \in \mathbf{S}^{n-1} : (\omega, s) \in \pi\}$; let $d(\xi, \omega')$ denote the normalized rotation-invariant measure on the manifold $P = \{(\xi, \omega') \in \Xi \times \mathbf{S}^{n-1} : \omega' \in \xi\}$ (notice that $\dim P = (n-k)(k+1) - 1$); finally, if $y \in \mathbf{R}^n$, let $d_y(\xi, \omega')$ be the $[(n-k)k]$-dimensional measure on $\{(\xi, \omega') \in P : (\xi, \omega', s') \in G_y\}$ (recall that $\dim G_y = \dim \Xi = (n-k)k$). Similarly define $d_\gamma \omega$ and $d_x(\xi, \omega')$ in the hyperbolic case.

LEMMA 2.1. *In the coordinates introduced above, the measures in the spaces in consideration are*

$$\begin{aligned}
dm_E(\omega, s) &= s^{n-1}\, ds\, d\omega && (\text{in } \mathbf{R}^n),\\
dm_H(\omega, r) &= \sinh^{n-1} r\, dr\, d\omega && (\text{in } \mathbf{H}^n),\\
dm_G(\xi, \omega', s') &= (s')^{n-k-1}\, ds'\, d(\xi, \omega') && (\text{in } G),\\
dm_\Gamma(\xi, \omega', r') &= \cosh^k r' \sinh^{n-k-1} r'\, dr'\, d(\xi, \omega') && (\text{in } \Gamma).
\end{aligned}$$

Moreover we have

$$\begin{aligned}
dm_{E,\pi}(\omega, s) &= (s^{k+1}/s')\, d_\pi \omega && (\text{on } \pi = (\xi, \omega', s') \in G),\\
dm_{H,\gamma}(\omega, r) &= \frac{\sinh^{k+1} r}{\sinh r'}\, d_\gamma \omega && (\text{on } \gamma = (\xi, \omega', r') \in \Gamma),\\
dm_{G,y}(\xi, \omega', s') &= d_y(\xi, \omega') && (\text{in } G_y \text{ for } y = (\omega, s) \in \mathbf{R}^n),\\
dm_{\Gamma,x}(\xi, \omega', r') &= \frac{\cosh^{k+1} r'}{\cosh r}\, d_x(\xi, \omega') && (\text{in } \Gamma_x \text{ for } x = (\omega, r) \in \mathbf{H}^n).
\end{aligned}$$

PROOF: The expression of dm_E is well-known, while that of dm_H follows from (2.1).

Let $y' = (\omega', s')$ and $x' = (\omega', r')$, the closest points of π, respectively γ, to o. If β is the angle $\widehat{y'yo}$ between straight line segments, then $\sin\beta = s'/s$; therefore $dm_{E,\pi}$ equals $s^k/\sin\beta = s^{k+1}/s'$. Analogously, if α is the angle $\widehat{x'xo}$ between geodesic line segments, then by hyperbolic trigonometry (cf., e.g., [C]) we have $\sin\alpha = \sinh r'/\sinh r$, and obtain $dm_{H,\gamma}$.

To proceed with the remaining measures in $\mathbf{H}^n$ we must make a preliminary computation. Let θ be the angle between the geodesic segment $\overline{x'o}$ and the normal to γ at x, and let $\Delta\omega = \widehat{\omega'\omega}$ be the angle between ω and ω'. Since θ is complementary of the angle α, by hyperbolic trigonometry we have $\sin\theta/\sin\Delta\omega = \cosh r'$ and $\cos\theta/\cos\Delta\omega = \cosh r'/\cosh r$, therefore

$$\frac{d\theta}{d\Delta\omega} = \frac{\cosh^2 r'}{\cosh r}; \tag{2.3}$$

this derivative relates the variation of the normal to γ at a fixed x—recall that the measure $dm_{\Gamma,x}$ is invariant under $SO(1,n)_x = SO(n)$—with that of ω'. (In the Euclidean case the angle θ simply equals $\Delta\omega$, so that the quotient in (2.3) just evaluates as 1.)

The factor $(s')^{n-k-1}$ in dm_G corresponds to the variations in the ω' coordinate of $\pi = (\xi, \omega', s')$: the exponent is the dimension of ξ. The same argument justifies the factor

$\sinh^{n-k-1} r'$ in dm_Γ. To account for the disparity between $\Delta\omega$ and θ we let $r' = r$ in (2.3) and obtain $\cosh r'$, which raised to the dimension of γ gives the other factor in dm_Γ: this corresponds to the variations of γ within the $(k+1)$-geodesic containing γ and o, keeping the point x fixed.

The measure $dm_{G,y}$ equals $d_y(\xi,\omega')$ by translation invariance. The factor in $dm_{\Gamma,x}$ is the product of $\cosh^2 r'/\cosh r$, given by (2.3), and $\dim\gamma - 1$ factors $\cosh r'$, which correspond to the variation of the normal to γ at x in the orthogonal space to the 2-geodesic containing o, x', x. ∎

With the above described measures the dual Radon transform R^* is (in both the Euclidean and the hyperbolic case) indeed the adjoint of the transform R: if $f \in \mathcal{D}(\mathbf{R}^n)$ and $\phi \in \mathcal{D}(G)$ we have

(2.4)
$$\begin{aligned}\langle R_E f, \phi\rangle &= \int_G \left[\int_\pi f(y)\, dm_{E,\pi}(y)\right] \phi(\pi)\, dm_G(\pi) = \int_V f(y)\,\phi(\pi)\, dm_V(y,\pi)\\ &= \int_{\mathbf{R}^n} f(y) \left[\int_{G_y} \phi(\pi)\, dm_{G,y}(\pi)\right] dm_R(y) = \langle f, R_E^*\phi\rangle,\end{aligned}$$

where $dm_V(y,\pi) = ds\,[s^{n-1}\,d\omega]\,d_y(\xi,\omega') = [(s')^{n-k-1}\,d(\xi,\omega')]\,[s^k\,d_\pi\omega]\,(s/s')\,ds'$ is a measure on the manifold $V = \{(y,\pi) \in \mathbf{R}^n \times G\colon y \in \pi\} = \{(\omega,s;\xi,\omega',s') \in \mathbf{S}^{n-1} \times [0,\infty) \times P \times [0,\infty)\colon s' = \langle\omega',\omega\rangle s,\ \mathrm{Span}(\omega',\omega) \perp \xi\}$ (here $\mathrm{Span}(\omega',\omega)$ denotes the linear span of ω', ω): by [**H2**, §I.3.2], the measures in Lemma 2.1 are characterized by (2.4). A similar computation can be carried out for $\mathbf{H}^n$.

3. Characterizations of the range

In the previous section we saw that the space $\mathcal{S}(\mathbf{H}^n)$ could be identified to $\mathcal{D}(\overline{\mathbf{B}}^n)$, and $\mathcal{D}(\mathbf{H}^n)$ to $\mathcal{D}(\mathbf{B}^n)$. In either case, a function f in $\mathcal{S}(\mathbf{H}^n)$ (respectively $\mathcal{D}(\mathbf{H}^n)$) can be identified to a $\mathcal{C}^\infty$ function in $\mathbf{R}^n$ with $\operatorname{supp} f \subseteq \overline{\mathbf{B}}^n$ (respectively $\operatorname{supp} f \subseteq \mathbf{B}^n$), and it makes sense to consider also the effect on f of the Euclidean Radon transform R_E, in which case only the integrals along elements of G_B can fail vanishing. We then have (see [**H2**]) the maps

$$R_H\colon \begin{cases} \mathcal{D}(\mathbf{H}^n) \to \mathcal{D}(\Gamma),\\ \mathcal{S}(\mathbf{H}^n) \to \mathcal{S}(\Gamma),\end{cases}$$
$$R_E\colon \begin{cases} \mathcal{D}(\mathbf{B}^n) \to \mathcal{D}(G_B),\\ \mathcal{D}(\overline{\mathbf{B}}^n) \to \mathcal{D}(\overline{G}_B) \subseteq \mathcal{D}(G).\end{cases}$$

Before we describe the intertwining operators between R_H and R_E we develop some auxiliary geometric results. Let $\eta\colon \mathbf{H}^n \to \mathbf{H}^n$ be the homothety of ratio 2 with respect to o, that is,

$$\eta(\omega,r) = (\omega,2r) \qquad \text{for all } (\omega,r) \in \mathbf{H}^n,$$

in particular

$$d(o,\eta(x)) = 2d(o,x);$$

the map η corresponds, via the exponential map, to multiplication by a factor 2 in the tangent space of $\mathbf{H}^n$ at o. If we let $\iota\colon \mathbf{H}^n \to \mathbf{B}^n$ be the identity map (which changes the metric from hyperbolic to Euclidean), so that

$$\iota(\omega, r) = (\omega, \tanh(r/2)) \qquad \text{for all } (\omega, r) \in \mathbf{H}^n,$$

the map η can be read in Euclidean polar coordinates as

$$\iota \circ \eta \circ \iota^{-1}(\omega, s) = (\omega, \tanh(2 \operatorname{arc\,tanh} s)) = (\omega, 2s/(1+s^2)) \qquad \text{for all } (\omega, s) \in \mathbf{B}^n.$$

The map

$$\tau = \iota \circ \eta\colon \mathbf{H}^n \to \mathbf{B}^n$$

is a diffeomorphism that satisfies

$$\tau(\omega, r) = (\omega, \tanh r) \qquad \text{for all } (\omega, r) \in \mathbf{H}^n;$$

we shall see that τ induces an isometry between the conformal (the one we are using) and Beltrami's model of hyperbolic space. This is shown by the following proposition:

PROPOSITION 3.1. *The image through τ of a k-geodesic of $\mathbf{H}^n$ is the intersection with $\mathbf{B}^n$ of a k-plane of $\mathbf{R}^n$.*

PROOF: Since τ preserves the tangential component $\omega \in \mathbf{S}^{n-1}$ we can assume $n = 2$, $k = 1$. Fixed a geodesic $\gamma = (\xi', \omega', r')$ (the ξ' component is useless here, as mentioned earlier), its closest point to o is $x' = (\omega', r')$; let $x = (\omega, r)$ be another point of γ. Let α be the angle $\widehat{x'ox} = \widehat{\omega'\omega}$, and observe that the angle $\widehat{xx'o}$ between geodesic segments is a right angle.

In a right hyperbolic triangle, the cosine of a non-right angle equals the ratio of hyperbolic tangents of the lengths of the adjacent sides (cf. [C, 12.99]). So

$$\tanh r' = \tanh r \, \cos \alpha.$$

Let $y' = \tau(x') = (\omega', \tanh r')$: then $\|y'\| = \tanh r'$. Similarly $y = \tau(x) = (\omega, \tanh r)$ and $\|y\| = \tanh r$. Moreover the angle $\widehat{y'oy}$ equals $\widehat{\omega'\omega} = \alpha$. Therefore

$$\|y'\| = \|y\| \cos \alpha$$

by the previous formula, and from Euclidean geometry one gathers that the angle $\widehat{yy'o}$, between straight line segments, must be a right angle. Hence y varies on the intersection with $\mathbf{B}^n$ of the straight line orthogonal at y' to the segment $\overline{y'o}$. ∎

REMARK 3.2. (The following description of models for $\mathbf{H}^n$ can be found in [Y]; see also [Kl].) Consider in $\mathbf{R}^{n+1}$ the signature $(1, n)$ Lorentz metric

$$g(x', y') = x_0 y_0 - \langle x, y \rangle, \qquad \text{for } x, y \in \mathbf{R}^n \text{ and } x' = (x_0, x),\ y' = (y_0, y),$$

where $\langle \cdot , \cdot \rangle$ is the standard scalar product in $\mathbf{R}^n$. The hyperboloid model for $\mathbf{H}^n$ is given by the 'upper' sheet

$$Q = \{g(x', x') = 1 \text{ and } x_0 > 0\}$$

of the two-sheeted hyperboloid $\{g(x', x') = 1\}$, endowed with the distance

$$d(x', y') = \operatorname{arc\,cosh} g(x', y'). \tag{3.1}$$

The group of automorphisms of $\mathbf{H}^n$ is thus $SO(1, n)$, which leaves Q invariant, and the k-geodesics are the intersections with Q of $(k+1)$-planes of $\mathbf{R}^{n+1}$.

Beltrami's model for $\mathbf{H}^n$ is obtained by identifying $\mathbf{B}^n$ with $D_1 = \{x_0 = 1 \text{ and } \|x\| \leq 1\}$ in $\mathbf{R}^{n+1}$, and projecting Q from the origin of $\mathbf{R}^{n+1}$: a point $x' \in Q$ gets mapped to $x'/x_0 \in D_1$, i.e., to $x/x_0 \in \mathbf{B}^n$. It is immediate that the k-geodesics are now intersections with $\mathbf{B}^n$ of k-planes of $\mathbf{R}^n$. The distance in D_1 is obtained by first making the argument of the right-hand side of (3.1) homogeneous, then restricting it to D_1: so

$$\begin{aligned} d(x, y) &= \operatorname{arc\,cosh} \frac{g(x', y')}{\sqrt{g(x', x')}\sqrt{g(y', y')}} \qquad (\text{where } x' = (1, x),\ y' = (1, y)) \\ &= \operatorname{arc\,cosh} \frac{1 - \langle x, y \rangle}{\sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}}. \end{aligned}$$

As explained, e.g., in [C], or [M], identifying $\mathbf{B}^n$ with $D_0 = \{x_0 = 0 \text{ and } \|x\| \leq 1\}$, the composition κ of the inverse stereographic projection of D_0 from one pole of the unit sphere $\{\|x'\| = 1\}$ in $\mathbf{R}^{n+1}$ onto the opposite hemisphere, with the orthogonal ('vertical') projection onto D_0, provides an isometry of the conformal model onto Beltrami's. The map κ is such that the image of each point lies on the same radius starting from o: therefore κ coincides with τ, that is, the composition of projections described above can also be obtained as a homothety in the conformal model (followed by a change of metric). ∎

As a consequence of Proposition 3.1, the map $\tau\colon \mathbf{H}^n \to \mathbf{B}^n$ naturally induces a diffeomorphism

$$\tau_k\colon \Gamma \to G_B.$$

Explicitly

$$\tau_k(\xi, \omega, r) = (\xi, \omega, \tanh r) \qquad \text{for all } (\xi, \omega, r) \in \Gamma.$$

LEMMA 3.3. *The jacobian of τ along the k-geodesic $\gamma = (\xi, \omega', r')$ at the point $x = (\omega, r) \in \gamma$ is $\cosh r' / \cosh^{k+1} r$.*

PROOF: From Lemma 2.1, setting $s = \tanh r$ and $\pi = \tau_k(\gamma)$, the desired jacobian is obtained as the ratio of $dm_{E,\pi}(\omega, \tanh r)$ and $dm_{H,\gamma}(\omega, r)$. ∎

We are now ready to define the intertwining operators between R_H and R_E.

PROPOSITION 3.4. *For $1 \leq k < n$ let $\rho\colon \mathbf{H}^n \to \mathbf{R}$ and $\sigma\colon \Gamma \to \mathbf{R}$ be defined by:*

$$\begin{aligned} \rho(\omega, r) &= \cosh^{k+1} r, \quad \textit{hence} \quad \rho \circ \tau^{-1}(\omega, s) = (1 - s^2)^{-(k+1)/2}; \\ \sigma(\xi, \omega, r) &= \cosh r, \quad \textit{hence} \quad \sigma \circ \tau_k^{-1}(\xi, \omega, s) = (1 - s^2)^{-1/2}. \end{aligned}$$

Then the transformations $\Phi\colon \mathcal{D}(\mathbf{H}^n) \to \mathcal{D}(\mathbf{B}^n)$ *and* $\Psi\colon \mathcal{D}(\Gamma) \to \mathcal{D}(G_B)$ *given by*

$$\Phi(f) = (\rho f) \circ \tau^{-1},$$
$$\Psi(\phi) = (\sigma\phi) \circ \tau_k^{-1}$$

are topological isomorphisms, and the diagram

$$\begin{array}{ccc} \mathcal{D}(\mathbf{H}^n) & \xrightarrow{R_H} & \mathcal{D}(\Gamma) \\ \Phi\big\downarrow & & \Psi\big\downarrow \\ \mathcal{D}(\mathbf{B}^n) & \xrightarrow{R_E} & \mathcal{D}(G_B) \end{array}$$

commutes, i.e., $\Psi R_H = R_E \Phi$.

(Note that ρ is independent of n, and σ is independent of n, k.)

PROOF: If $\gamma \in \Gamma$ and $\pi = \tau_k(\gamma)$, and if $f \in \mathcal{D}(\mathbf{H}^n)$, applying Lemma 3.3 we have

$$\Psi R_H f(\pi) = \sigma(\gamma) \int_\gamma f\, dm_{H,\gamma} = \int_\pi (\rho f) \circ \tau^{-1}\, dm_{E,\pi} = R_E \Phi f(\pi). \quad \blacksquare$$

Clearly the same proof furnishes the following result:

PROPOSITION 3.5. *The transformations* Φ *and* Ψ *defined in Proposition 3.4 also render commutative the following diagram:*

$$\begin{array}{ccc} \mathcal{S}(\mathbf{H}^n) & \xrightarrow{R_H} & \mathcal{S}(\Gamma) \\ \Phi\big\downarrow & & \Psi\big\downarrow \\ \mathcal{D}(\overline{\mathbf{B}}^n) & \xrightarrow{R_E} & \mathcal{D}(\overline{G}_B) \end{array} \tag{3.2}$$

(and are topological isomorphisms also between these spaces).

PROOF: The only observation that needs to be made is that Φ maps $\mathcal{S}(\mathbf{H}^n)$ into $\mathcal{D}(\overline{\mathbf{B}}^n)$. In fact it is clear that since

$$\mathcal{S}(\mathbf{H}^n) = \{f \in \mathcal{C}^\infty(\mathbf{H}^n)\colon \sup_{x \in \mathbf{H}^n} |D^\alpha f(x)| \cosh^m d(x,o) < \infty \text{ for every } \alpha \in \mathbf{N}^n,\ m \in \mathbf{N}\}$$

we have

$$\begin{aligned} \Phi(\mathcal{S}(\mathbf{H}^n)) &= \{g \in \mathcal{C}^\infty(\mathbf{B}^n)\colon \sup_{y \in \mathbf{B}^n} |D^\alpha g(y)|(1 - \|y\|)^{-m} < \infty \text{ for every } \alpha \in \mathbf{N}^n,\ m \in \mathbf{N}\} \\ &= \mathcal{D}(\overline{\mathbf{B}}^n). \quad \blacksquare \end{aligned}$$

(The term 'intertwining' used here is a slight abuse of language, since the operators Φ and Ψ do not preserve the group actions.)

REMARK 3.6. As an immediate consequence of Proposition 3.4 and Proposition 3.5 we obtain that any inversion of the Euclidean Radon transform R_E automatically yields a corresponding one for R_H, by the relation $R_H^{-1} = \Phi^{-1} R_E^{-1} \Psi$. ∎

In a very similar fashion one verifies that the map $\Phi_p \colon L^p(\mathbf{H}^n) \to L^p(\mathbf{B}^n)$ given by

$$\Phi_p(f) = (\rho_p f) \circ \tau^{-1},$$

where $\rho_p \colon \mathbf{H}^n \to \mathbf{R}$ is the function

$$\rho_p(\omega, r) = \cosh^{(n+1)/p} r,$$

is an isometry for every $p \geq 1$: in particular, the map Φ itself is an isometry of L^p-spaces for

$$p = \frac{n+1}{k+1}.$$

Dualizing the diagram (3.2) we obtain

$$\begin{array}{ccc} \mathcal{S}'(\Gamma) & \xrightarrow{{}^tR_H} & \mathcal{S}'(\mathbf{H}^n) \\ {}^t\Psi \uparrow & & {}^t\Phi \uparrow \\ \mathcal{E}'(\overline{G}_B) & \xrightarrow{{}^tR_E} & \mathcal{E}'(\overline{\mathbf{B}}^n) \end{array} \tag{3.3}$$

The space $\mathcal{S}(\Gamma)$ can be embedded into $\mathcal{S}'(\Gamma)$, and ${}^tR_H|_{\mathcal{S}(\Gamma)} = R_H^*$, whereas if restr_B is the restriction operator from $\mathbf{R}^n$ to $\overline{\mathbf{B}}^n$ then ${}^tR_E|_{\mathcal{D}(\overline{G}_B)} = \mathrm{restr}_B \circ R_E^*$: however, by abuse of notation the indication of the map restr_B will be henceforth suppressed. We thus obtain the diagram

$$\begin{array}{ccc} \mathcal{S}(\Gamma) & \xrightarrow{R_H^*} & \mathcal{S}' \cap \mathcal{E}(\mathbf{H}^n) \\ {}^t\Psi \uparrow & & {}^t\Phi \uparrow \\ \mathcal{D}(\overline{G}_B) & \xrightarrow{R_E^*} & \mathcal{E}(\overline{\mathbf{B}}^n) \end{array}$$

In (3.3) the maps ${}^t\Phi$, ${}^t\Psi$ are isomorphisms. We shall now proceed to make their inverses explicit when acting on smooth functions.

LEMMA 3.7. *The jacobian of τ at $x = (\omega, r) \in \mathbf{H}^n$ is $\cosh^{-n-1} r$, and that of jacobian of τ_k at $\gamma = (\xi, \omega', r') \in \Gamma$ is $\cosh^{-n-1} r'$.*

PROOF: Same as for Lemma 3.3, using Lemma 2.1. ∎

PROPOSITION 3.8. *For $1 \leq k < n$ let $\rho' \colon \mathbf{H}^n \to \mathbf{R}$ and $\sigma' \colon \Gamma \to \mathbf{R}$ be defined by:*

$$\begin{aligned} \rho'(\omega, r) &= \cosh r, && \textit{hence} \quad \rho' \circ \tau^{-1}(\omega, s) = (1 - s^2)^{-1/2}; \\ \sigma'(\xi, \omega, r) &= \cosh^{k+1} r, && \textit{hence} \quad \sigma' \circ \tau_k^{-1}(\xi, \omega, s) = (1 - s^2)^{-(k+1)/2}. \end{aligned}$$

Then the transformations $\Phi'\colon \mathcal{S}' \cap \mathcal{E}(\mathbf{H}^n) \to \mathcal{E}(\overline{\mathbf{B}}^n)$ *and* $\Psi'\colon \mathcal{S}(\Gamma) \to \mathcal{D}(\overline{G}_B)$ *given by*

$$\Phi'(f) = (\rho' f) \circ \tau^{-1},$$
$$\Psi'(\phi) = (\sigma' \phi) \circ \tau_k^{-1}$$

are topological isomorphisms, and the diagram

$$\begin{array}{ccc} \mathcal{S}(\Gamma) & \xrightarrow{R_H^*} & \mathcal{S}' \cap \mathcal{E}(\mathbf{H}^n) \\ \Psi' \big\downarrow & & \Phi' \big\downarrow \\ \mathcal{D}(\overline{G}_B) & \xrightarrow{R_E^*} & \mathcal{E}(\overline{\mathbf{B}}^n) \end{array}$$

commutes, i.e., $\Phi' R_H^* = R_E^* \Psi'$. *In other words,* $\Phi' = ({}^t\Phi)^{-1}$ *and* $\Psi' = ({}^t\Psi)^{-1}$.

PROOF: Recalling the value of the jacobian of τ from Lemma 3.7 one has

$$\langle \Phi f, \Phi' f' \rangle = \int_{\mathbf{B}^n} (\rho f \rho' f') \circ \tau^{-1} = \int_{\mathbf{H}^n} f f' = \langle f, f' \rangle,$$

and since Φ is an isomorphism we have $(\Phi^*)^{-1} = \Phi'$ in the dense subspace $\mathcal{S}' \cap \mathcal{E}(\mathbf{H}^n)$. With Ψ, Ψ' we proceed analogously, and conclude that $\Psi' = (\Psi^*)^{-1}$, which proves the desired commutativity. ∎

Observe that Φ, Ψ are never unitary: in fact they do not coincide with Φ', Ψ', respectively, for any n, k.

For functions $g \in \mathcal{D}(\mathbf{R}^n)$ Helgason proved a support theorem [**H2**, Corollary I.2.25], which characterizes the radius s_o of the smallest ball in $\mathbf{R}^n$ centered at o that contains the support of g: all is needed is that $R_E g$ vanish at every k-plane $\pi = (\xi, \omega, s)$ with distance s to the origin greater than s_o. Therefore one can see that among the functions $\psi \in \mathcal{S}(G)$ that are in the range of R_E one can easily distinguish those that are in $R_E(\mathcal{D}(\mathbf{B}^n))$ or in $R_E(\mathcal{D}(\overline{\mathbf{B}}^n))$, by the conditions (respectively)

$$\operatorname{supp} \psi \subseteq \{(\xi, \omega, s) \in G\colon s < 1\},$$
$$\operatorname{supp} \psi \subseteq \{(\xi, \omega, s) \in G\colon s \leq 1\}.$$

Simply notice that, under either of these support conditions, automatically $\psi \in \mathcal{D}(G)$. As to functions on $\mathbf{H}^n$, the support theorem is also valid for functions in $\mathcal{S}(\mathbf{H}^n)$ (cf. [**H2**, Theorem I.4.2]).

One way to characterize the range of the Euclidean Radon transform for $\mathcal{S}(\mathbf{R}^n)$ or $\mathcal{D}(\mathbf{R}^n)$, in case $1 \leq k < n-1$, has been to do it in terms of the solutions of a homogeneous system of linear differential equations. The idea of using differential equations originates with John [**J**], who found that the ultrahyperbolic operator in $\mathbf{R}^4$ could be used to characterize the range of the X-ray transform (i.e., $k = 1$) in $\mathbf{R}^3$. In general, Richter [**R**] exhibited a system of translation invariant, globally defined second-order differential operators which

characterize $R_H(\mathcal{S}(\mathbf{R}^n))$ for each $k < n-1$. Recently, other characterizations were provided by Kurusa [**Ku1**] with a system of ultrahyperbolic equations, and by Gonzalez [**Go1**] with a single fourth-order differential operator, invariant by all isometries of $\mathbf{R}^n$. We now derive the corresponding results in $\mathcal{S}(\mathbf{H}^n)$ (and $\mathcal{D}(\mathbf{H}^n)$). We will concentrate on Richter's and Gonzalez' operators and explain how they give rise to (second-, respectively) fourth-order equations characterizing the ranges of the hyperbolic Radon transforms, although the same procedure can be applied to other differential equations, such as Kurusa's.

Let $E = E_n$ be the Lie group of isometries of $\mathbf{R}^n$, let $\underline{e}$ be its Lie algebra, and let $\mathcal{A}$ be the universal enveloping algebra of $\underline{e}$. Denote by β the left action of E on G. We say that there is a *germ (at the identity) of left action* β of E on the open set G_B of G, by restriction: for every $\pi \in G_B$, only those $b \in E$ are allowed such that $\beta(b)\pi \in G_B$—these form an open set of E that depends on π and contains the identity. Denote by ν the left regular representation of E on $\mathcal{S}(G)$, so that $\nu(b)\psi = \psi \circ \beta(b)^{-1}$ for every $\psi \in \mathcal{S}(G)$, and extend to $\mathcal{A}$ the infinitesimal left regular representation $d\nu$ on $\mathcal{S}(G)$ in the usual way: thus

$$\begin{aligned} d\nu(V_1 \cdots V_m)\psi(\pi) &= \frac{\partial^m}{\partial t_1 \cdots \partial t_m}[\nu(\exp t_1 V_1 \cdots \exp t_m V_m)\psi(\pi)]_{t_1=\cdots=t_m=0} \\ &= \frac{\partial^m}{\partial t_1 \cdots \partial t_m}[\psi(\beta(\exp(-t_m V_m)\cdots\exp(-t_1 V_1))\pi)]_{t_1=\cdots=t_m=0} \\ &\qquad\qquad \text{for each } V_1,\ldots,V_m \in \underline{e},\ \psi \in \mathcal{S}(G),\ \pi \in G. \end{aligned}$$

The expression after the last equality sign can be used to define the 'infinitesimal left regular representation' $d\nu$ on $\mathcal{D}(G_B)$ (even though the 'finite' left regular representation ν on $\mathcal{D}(G_B)$ does not itself make sense). In fact, on each fixed $\pi \in G_B$ the element $\exp(-t_m V_m)\cdots\exp(-t_1 V_1)$ of E can act whenever $-a < t_1,\ldots,t_m < a$, for some positive a depending on π, $V_1, \ldots, V_m$.

Conjugating β by τ_k, we get a germ of left action α of E on Γ: that is, $\alpha(b) = \tau_k^{-1} \circ \beta(b) \circ \tau_k$. Notice that $\alpha(b)$ is an automorphism of $\mathbf{H}^n$ if and only if it keeps o fixed. (The other elements of E cannot act, through α, as hyperbolic isometries, since they are only defined on part of Γ, and push the remaining k-geodesics 'off' $\mathbf{H}^n$.) By analogy to the above situation, define the infinitesimal left regular representation $d\mu$ on $\mathcal{S}(\mathbf{H}^n)$ by

$$d\mu(V_1 \cdots V_m)\phi(\gamma) = \frac{\partial^m}{\partial t_1 \cdots \partial t_m}[\phi(\alpha(\exp(-t_m V_m)\cdots\exp(-t_1 V_1))\gamma)]_{t_1=\cdots=t_m=0}$$
$$\text{for each } V_1,\ldots,V_m \in \underline{e},\ \phi \in \mathcal{S}(\Gamma),\ \gamma \in \Gamma.$$

Of course $d\mu(V_1 \cdots V_m)\phi = [d\nu(V_1 \cdots V_m)(\phi \circ \tau_k^{-1})] \circ \tau_k$.

Let $T_j \in \underline{e}$ be the 'infinitesimal translation in the j-th coordinate' of $\mathbf{R}^n$, for $j = 1,\ldots,n$: identifying $\underline{e}$ with the Lie algebra of matrices $\begin{pmatrix} L & v \\ 0 & 0 \end{pmatrix}$, where $L \in so(n)$ and $v \in \mathbf{R}^n$, then T_j corresponds to the matrix all of whose entries all vanish except the $(j, n+1)$, which equals 1. Also, let X_{ij} be the 'infinitesimal rotation around the origin in the plane of the i-th and j-th coordinates', for $i,j = 1,\ldots,n$ with $i \neq j$: the only nonzero entries of its matrix are the (i,j), which equals 1, and the (j,i), which equals -1. Richter's 'pre-operators' $V_{ijl} \in \mathcal{A}$, for distinct $i,j,k = 1,\ldots,n$, are given by [**R**]

$$V_{ijl} = T_i X_{jl} + T_j X_{li} + T_l X_{ij},$$

whereas Gonzalez' is [**Go1**]

$$V = \sum_{1 \le i < j < l \le n} V_{ijl}^2.$$

Richter's differential operators $d\nu(V_{ijl})$ on $\mathcal{S}(G)$ are invariant by translations, but not by rotations, while Gonzalez' operator $d\nu(V)$ enjoys both invariances. Notice that V_{ijl} and V are independent of k, whereas the differential operators proper are not, as they act on functions defined on different Grassmannian manifolds. Stipulating that a differential operator D on $\mathcal{S}(\overline{G}_B)$ is said to be invariant by a motion $b \in E$, such that $\beta(b)G_B$ intersects G_B, if

$$D[\psi \circ \beta(b)] \equiv (D\psi) \circ \beta(b) \quad \text{on } \{\pi \in G_B \colon \beta(b)\pi \in G_B\} \text{ for every } \psi \in \mathcal{S}(\overline{G}_B)$$

(the equality would mean that D commutes with $\nu(b)$, if the latter made sense), the second-order operators $D_{ijl} = d\nu(V_{ijl})$ on $\mathcal{S}(\overline{G}_B)$ are, as well, invariant by translations but not by rotations, while the fourth-order $D = d\nu(V)$ is fully invariant under E.

Pulling back with Ψ, it is easy to see that the differential operators $C_{ijl} = \Psi^{-1} D_{ijl} \Psi$ on $\mathcal{S}(\Gamma)$ are given by

$$C_{ijl}\phi = \sigma^{-1} d\mu(V_{ijl})(\sigma\phi) \quad \text{for every } \phi \in \mathcal{S}(\Gamma). \tag{3.4}$$

It is easy to realize that C_{ijl} are neither invariant by rotations around o (because D_{ijl} are not), nor by other automorphisms of $\mathbf{H}^n$; yet, they are invariant under the germ of action α of the subgroup of E of Euclidean translations in $\mathbf{R}^n$.

Analogously Ψ pulls back D to the fourth-order operator $C = \Psi^{-1} D \Psi$ on $\mathcal{S}(\Gamma)$, given by an expression similar to (3.4). Now, C is fully invariant under the germ of action α of E, whereas the only hyperbolic isometries which leave it invariant are those obtainable as α of some element of E, i.e., the rotations around the origin.

The explicit computation of C_{ijl} and C is relegated to the appendix to this paper. We summarize the above remarks in the following statement.

PROPOSITION 3.9. *If $1 \le k < n-1$, a function ϕ in $\mathcal{S}(\Gamma)$ (or $\mathcal{D}(\Gamma)$) is in the range $R_H\mathcal{S}(\mathbf{H}^n)$ (respectively $R_H\mathcal{D}(\mathbf{H}^n)$) of the k-dimensional hyperbolic Radon transform if and only if it is annihilated by the second-order differential operator C_{ijl} for all $1 \le i < j < l \le n$; equivalently, if and only if it is annihilated by the fourth order differential operator C.* ∎

The remaining case, namely $k = n-1$, is covered by another kind of characterization. Namely, for all $1 \le k \le n-1$, the following *Euclidean moment conditions* characterize the functions $\psi \in \mathcal{D}(G)$ which are in the range of the Radon transform R_E acting on $\mathcal{D}(\mathbf{R}^n)$ [**H2**, Corollary I.2.28]:

For every $m \in \mathbf{N}$ there exists a homogeneous polynomial P_m on $\mathbf{R}^n$ of degree m such that

$$\int_{\omega \in \xi} \langle \omega, \omega' \rangle^m \left[\int_0^1 s^{m+n-k-1} \psi(\xi, \omega, s)\, ds \right] d\omega = P_m(\omega') \quad \text{for each } \xi \in \Xi \text{ and } \omega' \in \xi.$$

(Such conditions also characterize the range for functions of $\mathcal{S}(\mathbf{R}^n)$, but only in the case $k = n-1$.)

The corresponding result in the hyperbolic space is the following:

THEOREM 3.10. *Let $\phi \in \mathcal{D}(\Gamma)$, respectively $\mathcal{S}(\Gamma)$. Then necessary and sufficient condition for ϕ to belong to $R_H(\mathcal{D}(\mathbf{H}^n))$, respectively $R_H(\mathcal{S}(\mathbf{H}^n))$, is that for every $m \in \mathbf{N}$ there exists a homogeneous polynomial P_m on $\mathbf{R}^n$ of degree m such that*

$$\int_{\omega\in\xi} \langle\omega,\omega'\rangle^m \left[\int_0^\infty \frac{\tanh^{m+n-k-1} r}{\cosh r}\, \phi(\xi,\omega,r)\, dr\right] d\omega = P_m(\omega')$$

for each $\xi \in \Xi$ and $\omega' \in \xi$.

These are called the *hyperbolic moment conditions.*

PROOF: Recall that $\Psi(\phi)$ belongs to $\mathcal{D}(\mathbf{B}^n)$, respectively $\mathcal{D}(\overline{\mathbf{B}}^n)$. Hence a necessary and sufficient condition for $\Psi(\phi)$ to be in the range of the Euclidean Radon transform R_E is that $\Psi(\phi)$ satisfy the Euclidean moment conditions. Substituting (ξ,ω,s) by $\tau_k(\xi,\omega,r)$ in them one obtains a g in $\mathcal{D}(\mathbf{B}^n)$, respectively $\mathcal{D}(\overline{\mathbf{B}}^n)$, such that $\Psi(\phi) = R_E(g)$ if and only if ϕ satisfies the hyperbolic moment conditions. Let $f = \Phi^{-1}(g)$: then f is in $\mathcal{D}(\mathbf{H}^n)$, respectively $\mathcal{S}(\mathbf{H}^n)$, and

$$R_H(f) = \Psi^{-1} R_E \Phi(f) = \Psi^{-1} R_E(g) = \phi. \quad \blacksquare$$

APPENDIX: COMPUTATION OF THE RANGE-CHARACTERIZING DIFFERENTIAL OPERATORS

Euclidean case. For distinct indices a,b,c and distinct i,j,l let $\zeta = \zeta_{(a,b,c,i,j,l)}\colon \mathbf{R}^4 \to E$ be the map given by

$$\zeta(t_1,t_2,t_3,t_4) = \exp(-t_4 X_{ab})\exp(-t_3 T_c)\exp(-t_2 X_{ij})\exp(-t_1 T_l)$$

for $t_1,t_2,t_3,t_4 \in \mathbf{R}$. Let $\{e_1,\dots,e_n\}$ be the standard basis of $\mathbf{R}^n$. Let $t \mapsto r_{ij}(t)$ be the one-parameter subgroup in E generated by $-X_{ij}$: so $r_{ij}(t) = \exp(-tX_{ij}) = r_{ji}(t)^{-1}$ is the rotation of angle t in the (i,j) plane. Such subgroup has natural actions on Ξ and on $\mathbf{S}^{n-1}$, which will merely be denoted by juxtaposition. Let P_ξ be the orthogonal projection of $\mathbf{R}^n$ onto $\xi \in \Xi$. For every k-plane $\pi = (\xi,\omega,s) \in G$ and for t_1,t_2,t_3,t_4 small in absolute value we have

$$\begin{aligned}\zeta(t_1,t_2,t_3,t_4)(\pi) = (&r_{ab}(t_4)r_{ij}(t_2)\xi,\\ &r_{ab}(t_4)r_{ij}(t_2)\operatorname{dir}(s\omega - t_1P_\xi e_l - t_3P_\xi r_{ji}(t_2)e_c),\\ &\|s\omega - t_1P_\xi e_l - t_3P_\xi r_{ji}(t_2)e_c\|),\end{aligned}$$

where $\operatorname{dir}(v) = v/\|v\|$ for $0 \neq v \in \mathbf{R}^n$. Set

$$\zeta_1(\pi) = \frac{\partial}{\partial t_1}[\zeta(t_1,0,0,0)(\pi)]_{t_1=0},$$

$$\vdots$$

$$\zeta_{234}(\pi) = \frac{\partial^3}{\partial t_2\,\partial t_3\,\partial t_4}[\zeta(0,t_2,t_3,t_4)(\pi)]_{t_2=t_3=t_4=0},$$

$$\zeta_{1234}(\pi) = \frac{\partial^4}{\partial t_1\,\partial t_2\,\partial t_3\,\partial t_4}[\zeta(t_1,t_2,t_3,t_4)(\pi)]_{t_1=t_2=t_3=t_4=0},$$

for all possible combinations of order one through four.

Therefore

$$
\begin{aligned}
d\nu(T_l X_{ij} T_c X_{ab})\psi &= \frac{\partial^4}{\partial t_1\,\partial t_2\,\partial t_3\,\partial t_4}[\psi \circ \zeta(t_1,t_2,t_3,t_4)]_{t_1=t_2=t_3=t_4=0} \\
&= d^4\psi(\zeta_1;\zeta_2;\zeta_3;\zeta_4) \\
&\quad + d^3\psi[(\zeta_1;\zeta_2;\zeta_{34}) + (\zeta_1;\zeta_3;\zeta_{24}) + (\zeta_1;\zeta_4;\zeta_{23}) \\
&\qquad + (\zeta_2;\zeta_3;\zeta_{14}) + (\zeta_2;\zeta_4;\zeta_{13}) + (\zeta_3;\zeta_4;\zeta_{12})] \\
&\quad + d^2\psi[(\zeta_{12};\zeta_{34}) + (\zeta_{13};\zeta_{24}) + (\zeta_{14};\zeta_{23}) \\
&\qquad + (\zeta_1;\zeta_{234}) + (\zeta_2;\zeta_{134}) + (\zeta_3;\zeta_{124}) + (\zeta_4;\zeta_{123})] \\
&\quad + d\psi(\zeta_{1234}),
\end{aligned}
$$

where, denoting by $P = P_{\xi\cap\omega^\perp}$ the orthogonal projection of $\mathbf{R}^n$ onto $\xi \cap \omega^\perp$, and by $v \cdot w$ the ordinary scalar product of $v, w \in \mathbf{R}^n$,

$$
\begin{aligned}
\zeta_1(\pi) &= (0, -Pe_l/s, -\omega\cdot e_l), \\
\zeta_2(\pi) &= (-X_{ij}\xi, -X_{ij}\omega, 0), \\
\zeta_3(\pi) &= (0, -Pe_c/s, -\omega\cdot e_c), \\
\zeta_4(\pi) &= (-X_{ab}\xi, -X_{ab}\omega, 0), \\
\zeta_{12}(\pi) &= (0, X_{ij}Pe_l/s, 0), \\
\zeta_{13}(\pi) &= (0, -[(\omega\cdot e_l)Pe_c + (\omega\cdot e_c)Pe_l + (e_l\cdot Pe_c)\omega]/s^2, e_l\cdot Pe_c/s), \\
\zeta_{14}(\pi) &= (0, X_{ab}Pe_l/s, 0), \\
\zeta_{23}(\pi) &= (0, [X_{ij}P - PX_{ij}]e_c/s, -\omega\cdot X_{ij}e_c), \\
\zeta_{24}(\pi) &= (X_{ab}X_{ij}\xi, X_{ab}X_{ij}\omega, 0), \\
\zeta_{34}(\pi) &= (0, X_{ab}Pe_c/s, 0), \\
\zeta_{123}(\pi) &= (0, [(\omega\cdot e_c)X_{ij}Pe_l + (\omega\cdot e_l)[X_{ij}P - PX_{ij}]e_c - (\omega\cdot X_{ij}e_c)Pe_l \\
&\qquad - (e_l\cdot PX_{ij}e_c)\omega + (e_l\cdot Pe_c)X_{ij}\omega]/s^2, e_l\cdot PX_{ij}e_c/s), \\
\zeta_{124}(\pi) &= (0, -X_{ab}X_{ij}Pe_l/s, 0), \\
\zeta_{134}(\pi) &= (0, X_{ab}[(\omega\cdot e_l)Pe_c + (\omega\cdot e_c)Pe_l + (e_l\cdot Pe_c)\omega]/s^2, 0), \\
\zeta_{234}(\pi) &= (0, -X_{ab}[X_{ij}P - PX_{ij}]e_c/s, 0), \\
\zeta_{1234}(\pi) &= (0, -X_{ab}[(\omega\cdot e_c)X_{ij}Pe_l + (\omega\cdot e_l)[X_{ij}P - PX_{ij}]e_c - (\omega\cdot X_{ij}e_c)Pe_l \\
&\qquad - (e_l\cdot PX_{ij}e_c)\omega + (e_l\cdot Pe_c)X_{ij}\omega]/s^2, 0),
\end{aligned}
$$

where we stipulated that

$$
\begin{aligned}
-X_{ij}\xi &= \frac{\partial}{\partial t}[r_{ij}(t)\xi]_{t=0}, \\
X_{ab}X_{ij}\xi &= \frac{\partial^2}{\partial t_1\,\partial t_2}[r_{ab}(t_2)r_{ij}(t_1)\xi]_{t_1=t_2=0}.
\end{aligned}
$$

We obtain the operator D as

$$\begin{aligned} D = d\nu(V) = \sum_{1\le i<j<l\le n} & [d\nu(T_iX_{jl}T_iX_{jl}) + d\nu(T_iX_{jl}T_jX_{li}) + d\nu(T_iX_{jl}T_lX_{ij}) \\ & + d\nu(T_jX_{li}T_iX_{jl}) + d\nu(T_jX_{li}T_jX_{li}) + d\nu(T_jX_{li}T_lX_{ij}) \\ & + d\nu(T_lX_{ij}T_iX_{jl}) + d\nu(T_lX_{ij}T_jX_{li}) + d\nu(T_lX_{ij}T_lX_{ij})]. \end{aligned}$$

As to D_{ijl} we simply have

$$D_{ijl} = d\nu(V_{ijl}) = d\nu(T_iX_{jl}) + d\nu(T_jX_{li}) + d\nu(T_lX_{ij})$$

for $i, j, l = 1, \ldots, n$, where

$$d\nu(T_lX_{ij})\psi = \frac{\partial^2}{\partial t_1\,\partial t_2}[\psi \circ \zeta(t_1, t_2, 0, 0)]_{t_1=t_2=0} = d^2\psi(\zeta_1; \zeta_2) + d\psi(\zeta_{12}),$$

and ζ_1, ζ_2, ζ_{12} are given above.

Pullback to the hyperbolic case. As described in Section 3, the pullback of the operator D is $C = \Psi^{-1}D\Psi$, so its effect on $\phi \in \mathcal{S}(\Gamma)$ will be

$$C\phi = \sigma^{-1}D([\sigma\phi] \circ \tau_k^{-1}) \circ \tau_k.$$

The summands are

$$\begin{aligned} \Psi^{-1}d\nu(T_lX_{ij}T_cX_{ab})\Psi\phi = {} & d^4\phi(\theta_1; \theta_2; \theta_3; \theta_4) \\ & + d^3\phi[(\theta_1; \theta_2; \theta_{34}) + (\theta_1; \theta_3; \theta_{24}) + (\theta_1; \theta_4; \theta_{23}) \\ & \quad + (\theta_2; \theta_3; \theta_{14}) + (\theta_2; \theta_4; \theta_{13}) + (\theta_3; \theta_4; \theta_{12})] \\ & + d^2\phi[(\theta_{12}; \theta_{34}) + (\theta_{13}; \theta_{24}) + (\theta_{14}; \theta_{23}) \\ & \quad + (\theta_1; \theta_{234}) + (\theta_2; \theta_{134}) + (\theta_3; \theta_{124}) + (\theta_4; \theta_{123})] \\ & + d\phi(\theta_{1234}) \\ & + \tanh r\,[d^3\phi[(\theta_2; \theta_3'; \theta_4)\theta_1^r + (\theta_1'; \theta_2; \theta_4)\theta_3^r] \\ & \quad + d^2\phi[(\theta_2; \theta_{34})\theta_1^r + (\theta_3'; \theta_{24})\theta_1^r + (\theta_1'; \theta_{24})\theta_3^r \\ & \quad\quad + (\theta_4; \theta_{23}')\theta_1^r + (\theta_1'; \theta_4)\theta_{23}^r + (\theta_2; \theta_{14})\theta_3^r \\ & \quad\quad + (\theta_2; \theta_4)\theta_{13}^r + (\theta_4; \theta_{12})\theta_3^r] \\ & \quad + d\phi[(\theta_{24})\theta_{13}^r + (\theta_{14})\theta_{23}^r \\ & \quad\quad + (\theta_{234})\theta_1^r + (\theta_{124})\theta_3^r + (\theta_4)\theta_{123}^r]\,] \\ & + (1 + 2\tanh^2 r)\,[d^2\phi(\theta_2; \theta_4)\theta_1^r\theta_3^r \\ & \quad + d\phi[(\theta_{24})\theta_1^r\theta_3^r + (\theta_4)\theta_1^r\theta_{23}^r]\,], \end{aligned}$$

where θ_J, for each multiindex $J \subset \{1, 2, 3, 4\}$, is obtained from ζ_J by substituting in its expression the variable s with $\tanh r$ and multiplying by $\cosh^2 r$ its third component, which

is then denoted as θ^r_J; and θ'_J is obtained from θ_J by doubling its third component. Thus for instance

$$\begin{aligned}\theta_1 &= (0, -Pe_l/\tanh r, -\omega \cdot e_l \cosh^2 r),\\ \theta'_1 &= (0, -Pe_l/\tanh r, -2\omega \cdot e_l \cosh^2 r),\\ \theta^r_1 &= -\omega \cdot e_l \cosh^2 r.\end{aligned}$$

Observe that θ^r_J vanishes (and consequently $\theta'_J = \theta_J$) unless

$$J = \{1\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\};$$

the above sum does not appear symmetric in the indices $1,2,3,4$ because the terms that vanish were omitted, and because θ_J has been replaced by θ'_J only in case they were different. Furthermore the factor θ^r_J, when appearing in a summand, is meant to multiply the entire corresponding differential, not its argument: thus

$$d^3\phi[(\theta_2; \theta'_3; \theta_4)\theta^r_1 + (\theta'_1; \theta_2; \theta_4)\theta^r_3]$$

stands for

$$\theta^r_1 d^3\phi(\theta_2; \theta'_3; \theta_4) + \theta^r_3 d^3\phi(\theta'_1; \theta_2; \theta_4).$$

The whole operator C is now obtained in the same way as D in the preceding section. Analogously we obtain the pullback C_{ijl} of the operator D_{ijl} from

$$\Psi^{-1} d\nu(T_l X_{ij})\Psi\phi = d^2\psi(\theta_1; \theta_2) + d\psi(\theta_{12}) + \tanh r\, \theta^r_1 d\psi(\theta_2).$$

References

[**AS**] A. Allers, F. Santosa, *Stability and resolution analysis of a linearized problem in electrical impedance tomography*, 1990 preprint, Univ. of Delaware.

[**BB1**] D. C. Barber, B. H. Brown, *Applied potential tomography*, J. Phys. E.: Sci. Instrum. **17** (1984), 723–733.

[**BB2**] ———, *Recent developments in applied potential tomography—APT*, in "Information processing in medical imaging," S. Bacharach editor, Nijhoff, Dordrecht, 1986, pp. 106–121.

[**BC**] C. A. Berenstein, E. Casadio Tarabusi, *Inversion formulas for the k-dimensional Radon transform in real hyperbolic spaces*, Duke Math. J. **62** (1991), 613–631.

[**BZ**] C. A. Berenstein, L. Zalcman, *Pompeiu's problem on symmetric spaces*, Comment. Math. Helvetici **55** (1980), 593–621.

[**C**] H. S. M. Coxeter, "Non-Euclidean Geometry," Math. Expositions, vol. 2, Univ. of Toronto Press, Toronto, 1965.

[**Go1**] F. B. Gonzalez, *Invariant differential operators and the range of the Radon d-plane transform*, Math. Ann. **287** (1990), 627–635.

[**Go2**] ———, *On the range of the Radon d-plane transform and its dual*, Trans. Amer. Math. Soc. (to appear).

[**GR**] I. S. Gradshteyn, I. M. Ryzhik, "Table of Integrals, Series, and Products," corrected and enlarged edition, Academic Press, Orlando, 1980.

[**Gi**] E. L. Grinberg, *Spherical harmonics and integral geometry on projective spaces*, Trans. Amer. Math. Soc. **279** (1983), 187–203.

[**H1**] S. Helgason, *Differential operators on homogeneous spaces*, Acta Math. **102** (1959), 239–299.

[H2] _____, "Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions," Pure and Appl. Math., vol. 113, Academic Press, Orlando, 1984.

[H3] _____, *The totally-geodesic Radon transform on constant curvature spaces*, Contemp. Math. **113** (1990), 141–149.

[IC] D. Isaacson, M. Cheney, *Current problems in impedance imaging*, in "Inverse Problems in Partial Differential Equations," (D. Colton et al., eds.) 1989 Arcata Conference Proceedings, SIAM, Philadelphia, 1990, pp. 141–149.

[J] F. John, *The ultrahyperbolic differential equation with 4 independent variables*, Duke Math. J. **4** (1938), 300–322.

[Kl] F. Klein, "Vorlesungen über nicht-euklidische Geometrie," Grundlehren Math. Wiss., vol. 26, Springer, Berlin, 1928.

[Ku1] Á. Kurusa, *A characterization of the Radon transform's range by a system of PDEs*, J. Math. Anal. Appl. **161** (1991), 218–226.

[Ku2] _____, *The Radon transform on hyperbolic space*, preprint.

[M] J. Milnor, *Hyperbolic geometry: the first 150 years*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), 9–24. Reprinted in "The Mathematical Heritage of Henry Poincaré" (F. E. Browder, ed.) Proc. Sympos. Pure Math., vol. 39, A.M.S., Providence, 1983, part 1, pp. 25–40.

[Q] E. T. Quinto, *The invertibility of rotation invariant Radon transforms*, J. Math. Anal. Appl. **91** (1983), 510–522; *Erratum*, J. Math. Anal. Appl. **94** (1983), 602–603.

[R] F. Richter, "Differentialoperatoren auf Euklidischen k-Ebenraumen und Radon-transformationen," Dissertation, Humboldt Universität zu Berlin, 1986.

[SV] F. Santosa, M. Vogelius, *A backprojection algorithm for electrical impedance imaging*, SIAM J. Appl. Math. **50** (1990), 216–243.

[S] R. S. Strichartz, *Radon inversion—variations on a theme*, Amer. Math. Monthly **89** (1982), 377–384, 420–423.

[Y] I. M. Yaglom, "A Simple Non-Euclidean Geometry and Its Physical Basis," Heidelberg Science Library, Springer, Berlin, 1979.

Keywords. Radon transform, totally geodesic submanifolds, real hyperbolic space, invariant differential operators.

1980 *Mathematics subject classifications*: Primary 44A05; Secondary 51M10, 53C65.

C. A. Berenstein: Department of Mathematics and Systems Research Center, University of Maryland, College Park, Maryland 20742, U.S.A. E-mail: cab@math.umd.edu, carlos@src.umd.edu.

E. Casadio Tarabusi: Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento), Italy. E-mail: casadio@itnvax.cineca.it, casadio@itncisca.bitnet.