Relative Equilibria for Two Rigid Bodies

by L.S. Wang and P.S. Krishnaprasad
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RELATIVE EQUILIBRIA FOR TWO RIGID BODIES
CONNECTED BY A BALL-IN-SOCKET JOINT *

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ABSTRACT. For simple mechanical systems with symmetry, a variational principle on configuration space determines relative equilibria. Here, this principle of symmetric criticality is applied to a problem of coupled rigid bodies. Numerical optimization by CONSOLE (a package intended originally for optimization based control system design) is shown to be an effective technique to search for some of the relative equilibria.

1 Introduction

This paper is part of an ongoing program to understand the dynamics and control of multibody systems from a modern point-of-view. In recent years, engineering applications have brought into focus, questions concerning the dynamics of systems of kinematically

* This work was supported in part by the AFOSR University Research Initiative Program under grant AFOSR- 87-0073 and by the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012.
coupled rigid and flexible bodies. In [16], these are referred to as Eulerian many-body problems to emphasize the role of Euler forces (or frame forces) in determining the nature of inter-body interactions. Eulerian many-body problems arise as models of robotic manipulators, high speed mechanical machinery, complex spacecraft with articulated components, space-based sensors, etc. See Wittenburg[31] and [5],[7] for treatments of engineering questions and formulation of equations of motion.

In recent work [3][4][12][15–17][21][24–27][29][30], modern geometric techniques have been brought to bear on certain classes of Eulerian many-body problems. Included among the classes of problems investigated are rigid bodies carrying rotors, planar many-body systems, three dimensional rigid bodies coupled by ball-in-socket joints, and rigid bodies with flexible attachments.

In the present paper, we investigate the structure of relative equilibria in the dynamics of two rigid bodies connected by a ball-in-socket joint. We follow the framework of [12] and obtain a variational characterization of relative equilibria using a theorem of Smale. A key geometric condition is derived. Numerical search for extremal critical points is carried out using CONSOLE, a software package originally intended for optimization-based design.

The results of this paper can be considered as part of a program to determine the phase portrait of coupled rigid body systems. In his ongoing doctoral dissertation work at Berkeley (preliminary results presented at an AMS Summer Research Conference on Control Theory and Multibody System at Bowdoin, August 1988), George Patrick has computed relative equilibria for certain restricted classes of coupled rigid body problems. The restrictions involve material symmetry and the present paper does not require such symmetries. Patrick's methods involve direct symbolic calculations while here we use a variational principle.

The model problem treated in this paper should be taken as illustrative (partly inspired by early examples of multibody communication satellite design [11][23]), but the techniques apply to other configurations as well. Attitude control of a system such as the one in this paper is under investigation and we hope to report on this in a later paper.
2 Mechanical Setting

In this section we describe the kinematics of a mechanical system. Two bodies, with masses $m_1$, $m_2$, are free to move in three dimensional Euclidean space, subject to a (three degrees of freedom) ball and socket coupling (See Figure 1).

![Diagram of Rigid Bodies connected by the Ball-in-Socket Joint](image)

Figure 1 Rigid Bodies connected by the Ball-in-Socket Joint

We introduce the following notations.

$\Gamma_0$: inertial frame of reference in space.

$O$: origin of the inertial reference system.

$M_1$: center of mass of body 1.

$M_2$: center of mass of body 2.

$\Gamma_1$: orthonormal frame on body 1 with origin at $M_1$.

$A_1$: rotational coordinate transformation matrix from $\Gamma_1$ to $\Gamma_0$. 
$\Gamma_2$: orthonormal frame on body 2 with origin at $M_2$.

$A_2$: rotational coordinate transformation matrix from $\Gamma_2$ to $\Gamma_0$.

$d_1$: vector from the joint to $M_1$ in the frame $\Gamma_1$.

$d_2$: vector from the joint to $M_2$ in the frame $\Gamma_2$.

$r_1$: vector from $O$ to $M_1$ in frame $\Gamma_0$.

$r_2$: vector from $O$ to $M_2$ in frame $\Gamma_0$.

$r_0$: vector from $O$ to the system center of mass in frame $\Gamma_0$.

$m$: total mass ($= m_1 + m_2$).

$Q_1$: vector from $M_1$ to a point of body 1 in the frame $\Gamma_1$.

$q_1$: vector from $O$ to the same point of body 1 as $Q_1$ above in the frame $\Gamma_0$.

$Q_2$: vector from $M_2$ to a point of body 2 in the frame $\Gamma_2$.

$q_2$: vector from $O$ to the same point of body 2 as $Q_2$ above in the frame $\Gamma_0$.

$w$: vector from $O$ to the joint in the frame $\Gamma_0$.

We have the kinematic relations.

\begin{align}
q_1 &= r_1 + A_1 Q_1. \tag{1} \\
q_2 &= r_2 + A_2 Q_2. \tag{2} \\
m r_0 &= m_1 r_1 + m_2 r_2. \tag{3} \\
r_1 &= w + A_1 d_1. \tag{4} \\
r_2 &= w + A_2 d_2. \tag{5}
\end{align}

Also we know that $A_1$ and $A_2$ belong to the special orthogonal group $SO(3)$.

Let $\mu_1(\cdot)$ denote the mass measure of body 1 in the frame $\Gamma_1$ and $\mu_2(\cdot)$ denote the mass measure of body 2 in the frame $\Gamma_2$. The kinetic energy of body 1 can be thus written as

$$K_1 = \frac{1}{2} \int_{B_1} \|\dot{q}_1(Q_1)\|^2 d\mu_1(Q_1).$$

Expanding the above by using (1), (2) and the formula $\|x\|^2 = tr(xx^T)$, we have the form

$$K_1 = \frac{m_1}{2} \|\dot{r}_1\|^2 + \frac{1}{2} tr(\dot{A}_1 I_1 A_1^T).$$
where $I_1$ is the coefficient of inertia of body $1$, defined by

$$I_1 = \int_{B_1} Q_1 Q_1^T d\mu_1(Q_1),$$

and $tr(\cdot)$ denotes the trace of a matrix.

The kinetic energy of body $2$ has a similar form. We thus have the total kinetic energy expressed as

$$K = K_1 + K_2$$

$$= \frac{m_1}{2} \|\dot{r}_1\|^2 + \frac{1}{2} tr(\dot{A}_1 I_1 \dot{A}_1^T) + \frac{m_2}{2} \|\dot{r}_2\|^2 + \frac{1}{2} tr(\dot{A}_2 I_2 \dot{A}_2^T).$$

By (3)-(5), we may write the total kinetic energy in terms of the total linear momentum $p = m\dot{r}_0$ of the system.

$$K = \frac{1}{2} tr(\dot{A}_1 I_1 \dot{A}_1^T) + \frac{1}{2} tr(\dot{A}_2 I_2 \dot{A}_2^T) + \frac{\epsilon}{2} \|\dot{A}_1 d_1 - \dot{A}_2 d_2\|^2 + \frac{1}{2m} \|p\|^2.$$

Here $\epsilon = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. Since there is no potential assumed, this is also the Lagrangian of the system.

The configuration space is $SO(3) \times SO(3) \times R^3$. The system is invariant under translation of the inertial reference frame, i.e. we have a symmetry group action on the configuration space

$$\Phi: R^3 \times (SO(3) \times SO(3) \times R^3) \rightarrow SO(3) \times SO(3) \times R^3$$

$$(\lambda, (A_1, A_2, r)) \mapsto (A_1, A_2, \lambda + r).$$

We can symplectically reduce the system by $R^3$ (see Marsden and Weinstein[20], Abraham and Marsden[1]) which in turn corresponds to jumping to the center of mass frame. This is also done in [12] and for planar problem in [21][29][30]. After this reduction, the reduced Lagrangian is

$$L = \frac{1}{2} tr(\dot{A}_1 I_1 \dot{A}_1^T) + \frac{1}{2} tr(\dot{A}_2 I_2 \dot{A}_2^T) + \frac{\epsilon}{2} \|\dot{A}_1 d_1 - \dot{A}_2 d_2\|^2.$$

which is a function on $T(SO(3) \times SO(3)).$

Although the mechanical system considered here is exactly the same as in [12], the Lagrangian is expressed in terms of coefficients of inertia referred to different body
frames than the one they use. Ours is based on the body frames affixed to centers of mass. By applying the formula for change of coefficient of inertia by translation, one checks that the results are the same. In the next section, we outline the general theory of simple mechanical system with symmetry to the point of characterizing relative equilibria. Standard references for the next section are Smale[28] and Abraham and Marsden[1]. See also Libermann and Marle[18], Arnold[2], and Guillemin and Sternberg[13].

3 Mechanical Systems with Symmetry

Let \((Q, \ll \cdot, \cdot \gg)\) be a Riemannian manifold with the Riemannian metric \(\ll \cdot, \cdot \gg\). Let \(G\) be a Lie group. \(\Phi : G \times Q \to Q\) is a group action on the manifold \(Q\) that leaves the Riemannian metric invariant. Let \(V : Q \to R\) be a \(G\)-invariant function on the manifold, i.e.

\[ V(\Phi_g(x)) = V(x) \quad \forall g \in G, \]

where \(\Phi_g : M \to M\) is defined by

\[ \Phi_g(x) \equiv \Phi(g, x). \]

Let \(\tau : TQ \to Q\) be the canonical projection. We now define a Lagrangian \(L : TQ \to R\) to be, for \(v_x \in TQ\) with \(\tau(v_x) = x\),

\[ L(v_x) = \frac{1}{2} \ll v_x, v_x \gg - V(x). \]

It follows that the Lagrangian is \(G\)-invariant and hyperregular. We thus have the corresponding invertible Legendre transformation \(FL : TQ \to T^*Q\) given by

\[ FL(v_x)(w_x) = \ll v_x, w_x \gg. \]

Let \(\omega_0\) be the canonical symplectic two form on \(T^*Q\), and define

\[ \omega_L = (FL)^*\omega_0. \]

6
Then it is easy to see that $\omega_L$ is a symplectic form on $TQ$. The action $A: TQ \to R$ is now

$$A(v_x) = \left\langle v_x, v_x \right\rangle$$

and the energy $E = A - L$ can be found subsequently. Thus we have two equivalent hamiltonian systems $(TQ, \omega_L, E)$ and $(T^*Q, \omega_0, H = E \circ FL^{-1})$. These are simple mechanical systems with symmetry in the sense of Smale[28][1]. The hamiltonian function on $T^*Q$ can be written as (for $\alpha_x \in T^*Q$ with $\tau^*(\alpha_x) = x$ where $\tau^*$ is the canonical projection on $T^*Q$),

$$H(\alpha_x) = \frac{1}{2} \left\langle FL^{-1}(\alpha_x), FL^{-1}(\alpha_x) \right\rangle + V(x).$$

We define an inner product on $T^*_xQ$ by

$$\left\langle \alpha_x, \beta_x \right\rangle_{T^*Q} = \left\langle FL^{-1}(\alpha_x), FL^{-1}(\beta_x) \right\rangle.$$

Then we may write

$$H(\alpha_x) = \frac{1}{2} \left\langle \alpha_x, \alpha_x \right\rangle_{T^*Q} + V(x).$$

Let $\mathfrak{g}$ denote the Lie algebra of $G$ and $\mathfrak{g}^*$ is the dual of the Lie algebra. By the Corollary 4.2.11 in [1], we know that the lifted action

$$\Phi^*: G \times T^*Q \to T^*Q$$

is symplectic and has an $Ad^*$-equivariant momentum mapping given by $J: T^*Q \to \mathfrak{g}^*$,

$$J(\alpha_x)(\xi) = \alpha_x(\xi_Q(x)) \tag{6}$$

where $\xi \in \mathfrak{g}$ and $\xi_Q$ is the associated infinitesimal generator of $\Phi$ on $Q$.

Now we introduce the notion of relative equilibrium in a general setting. Assume $G$ acts on a symplectic manifold $(P, \omega)$ freely and properly. Then the quotient space $P/G$ is a smooth manifold with an induced Poisson structure. For any $G$-invariant hamiltonian function $H$ on $P$, we find the induced function $\tilde{H}: P/G \to R$ in the following way. Letting $\tilde{\tau}$ be the projection from $P$ to $P/G$, we have

$$\tilde{H} \circ \tilde{\tau}(x) = H(x).$$
Since $H$ is $G$-invariant, $\tilde{H}$ is well-defined. The quotient $P/G$ carries an induced Poisson structure[19]. Given $f, g \in C^\infty(P/G)$, the induced Poisson bracket of $f$ and $g$ is
\[
\{ f, g \} \circ \tilde{\tau} = \{ f \circ \tilde{\tau}, g \circ \tilde{\tau} \}_0.
\]
where $\{ , \}_0$ is the standard Poisson structure on the symplectic manifold $(P, \omega)$.

With this induced Poisson structure and the induced hamiltonian function, we define the projected hamiltonian vector field $X_{\tilde{H}}$ on $P/G$. For any $f \in C^\infty(P/G)$,
\[
X_{\tilde{H}}[f] = \{ f, \tilde{H} \}.
\]

**Definition (Relative Equilibrium)**

$z_e \in P$ is a relative equilibrium for $X_H$ if
\[
X_{\tilde{H}}(\tilde{\tau}(z_e)) = 0.
\]

Assume there is an $Ad^*$-equivariant momentum mapping $J$ on $P$. Then we have the following characterization of a relative equilibrium.

**Theorem (Relative Equilibrium)**

$z_e \in P$ is a relative equilibrium for $X_H$ iff there exists a $\xi \in \mathfrak{g}$ such that $z_e$ is a critical point of
\[
H_\xi \equiv H - \langle J, \xi \rangle,
\]
where $\langle J, \xi \rangle : P \rightarrow \mathbb{R}$ is given by $x \mapsto J(x)(\xi)$.

**Proof**

Let $F_t : P \rightarrow P$ be the flow of $H$ on $P$. We have the induced flow $\tilde{F}_t : P/G \rightarrow P/G$ satisfying
\[
\tilde{F}_t \circ \tilde{\tau} = \tilde{\tau} \circ F_t.
\]

Thus $z_e$ is a relative equilibrium iff $X_{\tilde{H}}(\tilde{\tau}(z_e)) = 0$ iff
\[
\frac{d}{dt} \tilde{F}_t(\tilde{\tau}(z_e)) = 0
\]
iff
\[
\frac{d}{dt} \tilde{\tau} \circ F_t(z_e) = 0.
\]
That is
\[ \tilde{\tau} \circ F_t(z_e) = \tilde{\tau}(z_e). \quad \forall t \]

Thus, for all \( t \), \( F_t(z_e) \) must belong to the same orbit. And there exists a one-parameter subgroup \( g(t) \in G \) such that
\[ F_t(z_e) = \Phi_{g(t)}(z_e). \]

By the one-to-one correspondence between one-parameter subgroups in \( G \) and its Lie algebra, we know that there exists a \( \xi \in \mathfrak{g} \) such that
\[ F_t(z_e) = \Phi_{\exp t\xi}(z_e). \]

Differentiate both sides with respect to \( t \) and set \( t = 0 \), we get
\[ X_H(z_e) = \xi_P(z_e). \]

On the other hand, by the definition of a momentum mapping, \( X_{<J,\xi>} = \xi_P \). Thus
\[ X_H(z_e) = X_{<J,\xi>}(z_e), \]

which implies
\[ X_{H-<J,\xi>}(z_e) = 0. \]

By the nondegeneracy of \( \omega \), we have
\[ d(H-<J,\xi>)(z_e) = 0, \]
i.e. \( z_e \) is a critical point of \( H_{\xi} = H-<J,\xi> \).

Conversely, if \( z_e \) is a critical point of \( H_{\xi} \), also from the nondegeneracy of \( \omega \),
\[ X_{H-<J,\xi>}(z_e) = 0. \]

which implies
\[ X_H(z_e) = X_{<J,\xi>}(z_e) = \xi_P(z_e). \]

Thus, by uniqueness of the integral curve,
\[ F_t(z_e) = \Phi_{\exp t\xi}(z_e). \]
It follows that \( z_e \) is a relative equilibrium for \( X_H \) from the previous arguments.

\[ \text{QED} \]

**Remark 1.**

This theorem can be considered a corollary to the Souriau-Smale-Robbin Theorem (see Abraham and Marsden[1]). It is also discussed in [27].

**Remark 2.**

We note that if a point \( z_e \) is a relative equilibrium point, the motion \( F^t_{X_H}(z_e) \) is a stationary motion, i.e. it corresponds to a group orbit. If for instance the group \( G = SO(3) \), this would imply that \( F^t_{X_H}(z_e) \) corresponds to a uniform rotation about a fixed axis \( \xi \) in space.

We now apply this theorem to the setting of simple mechanical systems with symmetry discussed before. By the hyperregularity of \( FL \), for any \( \alpha_x \in T^*Q \), there is an element \( v_x \in TQ \) such that \( \alpha_x = FL(v_x) \) and

\[ \alpha_x(w) = \langle v_x, w \rangle \quad \forall w \in T_zQ. \]

Thus the momentum mapping, which is given by (6), can be written as

\[ \alpha_x(\xi_Q(x)) = \langle FL^{-1}(\alpha_x), \xi_Q(x) \rangle. \]

Defining \( B_\xi(x) = FL(\xi_Q(x)) \), we have

\[ \langle J, \xi \rangle (\alpha_x) = \langle \alpha_x, B_\xi(x) \rangle_{T^*Q}. \]

We can now write

\[ H_\xi(\alpha_x) = \frac{1}{2} \langle \alpha_x, \alpha_x \rangle_{T^*Q} + V(x) - \langle \alpha_z, B_\xi(x) \rangle_{T^*Q}. \]

By determining the norm on \( T^*_zQ \) through the inner product and completion of squares, we may express \( H_\xi \) as

\[ H_\xi(\alpha_x) = \frac{1}{2} \| \alpha_x - B_\xi(x) \|^2 + V(x) - \frac{1}{2} < B_\xi(x), B_\xi(x) >_{T^*Q}. \]
By the above theorem, computing the relative equilibria is equivalent to finding critical points of \( H_\xi \). Letting \( \alpha_x = (x, p) \) we have

\[
H_\xi(x, p) = \frac{1}{2} \|p - B_\xi(x)\|^2 + V(x) - \frac{1}{2} \langle B_\xi(x), B_\xi(x) \rangle_{\mathcal{T}^* Q}.
\]

It is then easy to check that the necessary conditions for \( (x_\varepsilon, p_\varepsilon) \) to be a critical point of \( H_\xi \) are

\[
p_\varepsilon = B_\xi(x_\varepsilon) \tag{7}
\]

and

\[
d_{x_\varepsilon} [V(x) - \frac{1}{2} \langle B_\xi(x), B_\xi(x) \rangle_{\mathcal{T}^* Q}] = 0
\]

We summarize the algorithm (principle of symmetric criticality) to find relative equilibria.

**Algorithm**

0. Pick \( \xi \in \mathcal{X} \).

1. Search for the critical points \( x_\varepsilon \) of the function

\[
V_\xi : Q \to R
\]

\[
V_\xi(x) = V(x) - \frac{1}{2} \langle \xi Q(x), \xi Q(x) \rangle
\]

2. Put \( x_\varepsilon \) in (7) to find the corresponding \( p_\varepsilon = B_\xi(x_\varepsilon) \).

We note that the computation in step 1 is fully on the configuration space.

**Remark 3.** (Historical)

The principle of symmetric criticality as stated here appears as Theorem 1.1 in Part II of Smale[28]. Smale also notes that special versions have been known earlier, e.g. in the study of symmetric geodesics. See also pp. 355 of [1], Theorem 16.7 in Hermann[14], and Palais[22].

**Remark 4.** (Symmetry of \( V_\xi \))

It should be noted, for a given \( \xi \in \mathcal{X} \), \( V_\xi \) has the symmetry,

\[
V_\xi(\Phi_g(x)) = V_\xi(x),
\]

for all \( g \in G_\xi = \{ g \in G | Ad_g(\xi) = \xi \} \), the stabilizer of \( \xi \). Thus \( V_\xi \) induces a function \( \hat{V}_\xi \) such that the diagram in Fig. 2 commutes.
Typically, $\hat{V}_\xi$ is a Morse function on $Q/G_\xi$ and $\pi_\xi^{-1}(\hat{x}_\epsilon)$ is a nondegenerate critical manifold in the sense of Bott[6], if $\hat{x}_\epsilon$ is a critical point of $\hat{V}_\xi$.

Example

One application of the principle here is to find the relative equilibria of the planar three-body system discussed in [29][30]. If we plot the function $\hat{V}_\xi$ (for particular kinematic parameters) on the joint space, we get the picture in Fig. 3, from which the fundamental equilibria defined in [29][30] can be easily seen. These are the relative (joint) configurations $(0,0)$, $(0,\pi)$, $(\pi,0)$, $(\pi,\pi)$.

4 Relative Equilibria for Coupled Rigid Bodies

For the mechanical system described in section 2, the Riemannian metric on $T(SO(3) \times SO(3))$ is given by the (symplectically) reduced Lagrangian as

$$\langle (W_1,W_2), (W_1,W_2) \rangle = tr(W_1 J_1 W_1^T) + tr(W_2 J_2 W_2^T) + \epsilon \|W_1 d_1 - W_2 d_2\|^2,$$

where $(W_1,W_2)$ belongs to $T(SO(3) \times SO(3))$. We know that every element in $T(SO(3) \times SO(3))$ can be represented as

$$T(SO(3) \times SO(3)) = \{(A_1,A_2,\hat{\omega}_1 A_1,\hat{\omega}_2 A_2): A_1,A_2 \in SO(3), w_1,w_2 \in R^3\},$$

where $^{\wedge}: R^3 \rightarrow so(3)$ is the one-to-one map from $R^3$ to the skew-symmetric matrices,

$$\hat{\omega} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$
In terms of \( w_1, w_2 \), we have

\[
\langle (W_1, W_2), (W_1, W_2) \rangle = tr(\dot{w}_1 A_1 I_1 A_1^T \dot{w}_1^T) + tr(\dot{w}_2 A_2 I_2 A_2^T \dot{w}_2^T) + \epsilon \| \dot{w}_1 A_1 d_1 - \dot{w}_2 A_2 d_2 \|^2.
\]

It is a straightforward calculation to show that

\[
tr(\dot{w} I \dot{w}^T) = \langle \omega, \dot{I} \omega \rangle_E
\]

where \( I \) is a coefficient of inertia tensor and \( \dot{I} \) is the associated moment of inertia tensor and \( \langle, \rangle_E \) is the Euclidean inner product. Upon further simplifications and rearrangements, we get

\[
\langle (W_1, W_2), (W_1, W_2) \rangle = ((A_1^T w_1)^T (A_2^T w_2)^T) \begin{pmatrix} J_1 & J_{12} \\ J_{12}^T & J_2 \end{pmatrix} \begin{pmatrix} A_1^T w_1 \\ A_2^T w_2 \end{pmatrix},
\]

where

\[
J_1 = \dot{P}_1 + \epsilon \dot{d}_1 \dot{d}_1
\]

\[
J_2 = \dot{P}_2 + \epsilon \dot{d}_2 \dot{d}_2
\]

\[
J_{12} = \epsilon \dot{d}_1 A_1^T A_2 \dot{d}_2
\]
The group action to consider is defined on $SO(3) \times SO(3)$, the configuration space, relative to an observer at the system center of mass. The diagonal action of the group $G = SO(3)$ is given by

$$\Psi: G \times (SO(3) \times SO(3)) \to SO(3) \times SO(3)$$

$$(R, (A_1, A_2)) \mapsto (RA_1, RA_2).$$

Letting $\hat{\xi} \in \mathfrak{g}$, the corresponding infinitesimal generator can be found as

$$\xi_Q(A_1, A_2) = \frac{d}{dt} \bigg|_{t=0} (\exp t \hat{\xi})(A_1, A_2)$$

$$= \frac{d}{dt} \bigg|_{t=0} ((\exp t \hat{\xi})A_1, (\exp t \hat{\xi})A_2)$$

$$= (\hat{\xi}A_1, \hat{\xi}A_2).$$

Since here the potential energy $V$ is identically 0, the function $V_\xi$ is

$$V_\xi(A_1, A_2) = -\frac{1}{2} (A_1^T \xi)^T (A_2^T \xi) \begin{pmatrix} J_1 & J_{12} \\ J_{12} & J_2 \end{pmatrix} \begin{pmatrix} A_1^T \xi \\ A_2^T \xi \end{pmatrix}.$$

It is clear that $V_\xi$ is invariant under $G_\xi = \{ R \in G : R \xi = \xi \}$ which is isomorphic to $S^1$.

By the compactness of $SO(3) \times SO(3)$ we know that for each $\xi$, $V_\xi$ has critical points. We need to find the conditions on $A_1, A_2$ so that the gradient of $V_\xi$ with respect to $A_1, A_2$ is 0. Equivalently one can check the vanishing of the differential $dV_\xi$ on the space $T(SO(3) \times SO(3))$.

Given $f : M \to R$ a smooth function on a smooth manifold $M$, $v_x \in T_x M$ which generates a curve $\phi : R \to M$ with $\phi(0) = x$, the differential $df$ at $x$ is defined by

$$df(x)(v_x) \equiv \frac{d}{dt} \bigg|_{t=0} f(\phi(t)).$$

Let $W \in T(SO(3) \times SO(3))$,

$$W = (A_1, A_2, \omega_1 A_1, \omega_2 A_2).$$

The curve in $SO(3) \times SO(3)$ generated by $W$ is $(e^{t\omega_1}A_1, e^{t\omega_2}A_2)$. Thus we have the formula

$$dV_\xi(A_1, A_2)(W) = \frac{d}{dt} \bigg|_{t=0} V_\xi(e^{t\omega_1}A_1, e^{t\omega_2}A_2),$$
Explicitly, we get the following final form (here $A = A_1^T A_2$),

\[
dV_\xi(A_1, A_2)(W) = < w_1, \xi A_1 J_1 A_1^T A_2 \xi >_E + < w_2, \xi A_2 J_2 A_2^T A_1 \xi >_E \\
+ \epsilon < w_1, \xi A_1 \hat{d}_1 A \hat{d}_2 A_2^T \xi >_E + \epsilon < w_2, \xi A_2 \hat{d}_2 A^T \hat{d}_1 A_1^T \xi >_E \\
+ \epsilon < w_1, A_1 \hat{d}_1 \xi A_2 \hat{d}_2 A_2^T \xi >_E + \epsilon < w_2, A_2 \hat{d}_2 \xi A_1 \hat{d}_1 A_1^T \xi >_E
\]

Thus we know that the necessary conditions for a critical point of $V_\xi$ are

\[
\begin{align*}
\hat{\xi} A_1 J_1 A_1^T \xi + \epsilon \hat{\xi} A_1 \hat{d}_1 A \hat{d}_2 A_2^T \xi + \epsilon A_1 \hat{d}_1 \xi A_2 \hat{d}_2 A_2^T \xi &= 0, \\
\hat{\xi} A_2 J_2 A_2^T \xi + \epsilon \hat{\xi} A_2 \hat{d}_2 A^T \hat{d}_1 A_1^T \xi + \epsilon A_2 \hat{d}_2 \xi A_1 \hat{d}_1 A_1^T \xi &= 0.
\end{align*}
\]

Now if we define $\Omega_1 \equiv A_1^T \xi$, and $\Omega_2 \equiv A_2^T \xi$, we get the conditions (in terms of cross products in $R^3$)

\[
\begin{align*}
\Omega_1 \times J_1 \Omega_1 + \epsilon d_1 \times (\Omega_1 \times A(d_2 \times \Omega_2)) &= 0, \quad (8) \\
\Omega_2 \times J_2 \Omega_2 + \epsilon d_2 \times (\Omega_2 \times A^T(d_1 \times \Omega_1)) &= 0, \quad (9)
\end{align*}
\]

which are exactly the conditions found by Poisson reduction in [unpublished notes of P.S. Krishnaprasad].

In step 2 of the algorithm of section 3, we put in the $A_1$, $A_2$ found by solving the above conditions into

\[
p = B_\xi(A_1, A_2) \\
= FL(\xi_Q(A_1, A_2)).
\]

Let $p \in T^*(SO(3) \times SO(3))$ be represented as

\[
p = (\hat{a}_1 A_1, \hat{a}_2 A_2).
\]

We find that $\alpha_1$, $\alpha_2$ can be expressed as

\[
\begin{align*}
\alpha_1 &= A_1 J_1 \Omega_1 + \epsilon (A_1 d_1 \times A_2 (d_2 \times \Omega_2)), \\
\alpha_2 &= A_2 J_2 \Omega_2 + \epsilon (A_2 d_2 \times A_1 (d_1 \times \Omega_1)).
\end{align*}
\]
Also we know that the relation between $\Omega_1$ and $\Omega_2$ is

$$\Omega_1 = A\Omega_2.$$ 

If we now let $s_1 = A_1 d_1$, $s_2 = A_2 d_2$, from (8), we get

$$A_1^T \xi \times J_1 A_1^T \xi + \epsilon A_1^T s_1 \times (A_1^T \xi \times A_2^T s_2 \times A_2^T \xi)) = 0,$$

which implies

$$\xi \times A_1 J_1 A_1^T \xi + \epsilon s_1 \times (\xi \times (s_2 \times \xi)) = 0. \quad (10)$$

Taking the inner product of (10) with $\xi$, we obtain a key necessary condition for a relative equilibrium

$$\xi \cdot (s_1 \times s_2) = 0. \quad (11)$$

We note that $\xi$ is the axis of rotation of the whole body, $s_1$, $s_2$ are the spatial vectors from joint to body 1 and 2, respectively. From (11), we conclude that, at relative equilibria, $\xi$, $s_1$, $s_2$ must lie on the same plane, no matter what the inertias are.

5 Numerical Method

Although we can get the same critical conditions (8) (9) by other methods, the principle of symmetric criticality provides more information. Notice that in the first step of the algorithm in Section 2, we simply try to find the critical points of $V_\xi$ on $SO(3) \times SO(3)$ without any additional constraint. Thus one has an associated unconstrained optimization problem. Numerical optimization schemes can be used to find extremal relative equilibria. This issue is discussed in this section.

By the symmetry of the system, we know that the function $V_\xi$ is invariant in the direction tangent to the orbit of $G_\xi$. Thus in the search for critical points, we should avoid these directions. It turns out that the usual gradient-type method is a good choice. Here, we use an optimization package named CONSOLE which was developed at the University
of Maryland[8]. The current version of CONSOLE basically uses the steepest descent method and is thus applicable to our circumstances.

In formulating the optimization problem, in order to avoid other constraints arising from the restrictions on $SO(3)$, e.g. $A^TA = Identity$, we use Cayley’s parametrization. That is, any element $A \in SO(3)$ can be represented by

$$A = \frac{1}{1 + a_1^2 + a_2^2 + a_3^2} \begin{pmatrix}
1 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_3) & 2(a_1a_3 + a_2) \\
2(a_1a_2 + a_3) & 1 - a_1^2 + a_2^2 - a_3^2 & 2(a_2a_3 - a_1) \\
2(a_1a_3 - a_2) & 2(a_2a_3 + a_1) & 1 - a_1^2 - a_2^2 + a_3^2
\end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. The problem can now be written as

$$\text{extremize } V_\xi(A_1, A_2)$$

$$\{ a_1, a_2, a_3 \}$$

$$\{ b_1, b_2, b_3 \}$$

where $(a_1, a_2, a_3), (b_1, b_2, b_3)$ are the parameters for $A_1, A_2$, respectively.

The CAD package CONSOLE is composed of two main programs: CONVERT, SOLVE. CONVERT reads a problem description file which describes the optimization problem to be solved. SOLVE then performs the optimization process with the interaction of user and/or some simulator. For more details, see Fan et al.[9][10]. The problem description file for our problem is easily formulated as follows.

```plaintext
design_parameter a1 init=0
design_parameter a2 init=0
design_parameter a3 init=0
design_parameter b1 init=0
design_parameter b2 init=0
design_parameter b3 init=1

objective "V-\xi"
  minimize {
    import a1, a2, a3;
    import b1, b2, b3;
    
    double cost();
    
    return cost( a1, a2, a3, b1, b2, b3 );
  }
  good_value=0
  bad_value=100
```

where the subroutine `cost()` reads a system description file containing the information of $I_1, I_2, d_1, d_2, \xi, m_1, m_2$ and then returns the value of the function $V_\xi$. By choosing
different moments of inertia and initial structure, we can perform the optimization. In the process, one thing we learned is that if the augmented inertia is diagonal, the rate of convergence is faster. Thus preliminary diagonalizations should be performed to get speed up.

In the particular case that

\[ m_1 = 3.0 \]
\[ m_2 = 2.0 \]
\[ d_1 = (0 \ 0 \ 1) \]
\[ d_2 = (-1 \ 1 \ 1) \]
\[ \xi = (0 \ 0 \ 1) \]
\[ I_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \]
\[ I_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} , \]

the relative equilibrium we found by numerical methods was

\[ A_1 = \begin{pmatrix} 0.0 & -0.939 & 0.344 \\ 0.0 & -0.344 & -0.939 \\ 1.0 & 0.0 & 0.0 \end{pmatrix} \]
\[ A_2 = \begin{pmatrix} 0.007 & 0.350 & -0.937 \\ -0.528 & 0.796 & 0.294 \\ 0.849 & 0.493 & 0.191 \end{pmatrix} \]
\[ s_1 = (0.344 \ -0.939 \ 0.0) \]
\[ s_2 = (-0.593 \ 1.618 \ -0.165) \]
\[ \Omega_1 = (1.0 \ 0.0 \ 0.0) \]
\[ \Omega_2 = (0.849 \ 0.493 \ 0.191) \]
\[ \alpha_1 = (0.0 \ 0.0 \ 10.269) \]
\[ \alpha_2 = (-0.004 \ 0.007 \ 14.321) . \]

Several relative equilibria corresponding to different choices of parameters are shown in Fig. 4. Case 1 in that figure corresponds to the above numerical result.
Note that in all cases, either $s_1 = A_1d_1$, $s_2 = A_2d_2$ are on a straight line or they and $\xi$ are on one plane. It matches the conclusion we made at the end of section 4.

6 Conclusion

We have shown how to determine relative equilibria by numerical search. A key geometric condition (Equation 11) appears as a consequence of the variational formulation. Dynamic simulations are being carried out to determine further details of the phase-portrait.
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