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**Continuous Network Interdiction**

by

Alan Washburn

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# Continuous Network Interdiction

by

Alan Washburn

## ***Abstract***

This report deals with network interdiction campaigns that are expected to be carried out indefinitely in time. Two sides are always involved, one desiring the unimpeded movement of some commodity, while the other desires the opposite. We consider two distinct situations. The first is motivated by warfare involving Improvised Explosive Devices (IEDs) directed against the movement of materiel in convoys on roads. It will be assumed that this kind of warfare is low level in the sense of destroying only a negligible fraction of the shipped materiel, so the objective becomes the maximization or minimization of the rate at which the convoys take lethal hits. The second situation is an economic one where the interdictor attempts to capture a significant fraction of the shipped materiel, so much so that a profit cannot be made by shipping it. This leads to a Nash equilibrium where the shipper's quantity shipped is in equilibrium with the interdictor's budget for interdiction.

## **1. Introduction**

In competitive logistic situations, one side may desire to interrupt or delay traffic that moves over a network, while the other has the opposite motivation. A variety of abstractions of such situations have been explored, most of which take the form of an interdictor destroying or damaging parts of the network, after which a shipper uses the reduced network as best he can. The payoff may be the amount of materiel shipped over the network, the shortest path through the network, or some other scalar measure. Usually the interdictor moves first, after which the shipper assesses the damage and makes his move. One of the earliest papers in this vein is Wollmer [1964]. Wood [1993] provides a good review, or see the more recent Lim and Smith [2005]. Sometimes the two moves are simultaneous, with neither side knowing the action of the other. In this case, the optimal strategies are typically mixed (Washburn and Wood [1995]).

The interdictor's actions are generally constrained by the availability of some resource that prevents him from simply interdicting every arc. The amount of this resource is presumably the amount that the interdictor can assemble in the time period between shipments, but otherwise the notion of time is not involved. If there are multiple interdiction attempts, then there is implicitly assumed to be some pacing mechanism, perhaps diurnal, that makes them all independent of each other.

The work reported here is an abstraction of a different sort. There is still an interdictor and a shipper, but the shipments and interdiction attempts are multiple and all mixed up in time. There is no starting or ending time—we imagine an equilibrium situation where the network is used and interdicted indefinitely. Although competition

occurs on the arcs of the network, the network itself is not destroyed. The principle application areas we have in mind are zero-sum games like IED warfare (Figure 1 and Section 2), and non zero-sum economic situations where the interdicator attempts to make the shipment of some substance, such as drugs or illegally cut logs, unprofitable (Figure 2 and Section 3). Although the two situations are similar in some ways, distinct models are still called for.



**Figure 1: IED attack in Baghdad.**



**Figure 2: Illegally cut logs on a truck.**

### ***Mathematical Notation***

Boldface type is used to indicate vectors and matrices. Thus  $\mathbf{x}=(x_k)$  indicates a vector with components indexed by  $k$ . The sum of all the components of  $\mathbf{x}$  is  $\sum_k x_k$  — when the set of values for a summation index is not stated, “for all” should be understood. Small versions of several of our models are implemented in the Excel workbook *ied3.xls*, which the reader may find handy to have open while reading this.

## **2. IED Warfare**

The general idea in the models of this section is that IEDs are placed by Red on a road network with the objective of killing Blue traffic. Blue has some sweeping assets that are capable of safely removing or destroying IEDs. The IEDs, as well as Blue’s assets, must be allocated to the arcs of the network by the players of this two-person zero-sum game. Red tries to maximize the destruction of Blue traffic, while Blue simultaneously tries to minimize it.

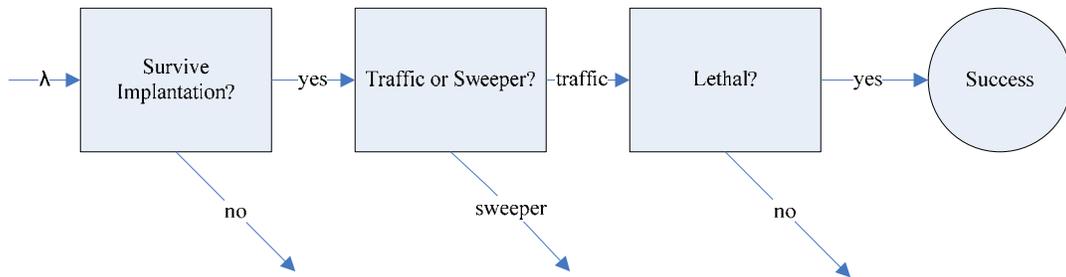
IEDs are essentially mines as far as this analysis is concerned. The essential features of both IEDs and mines are that:

- each type must be implanted, at some cost and risk to the interdicator;
- neither moves once implanted, relying instead on the motion of the transitor; and
- when either detonates, it destroys itself and possibly its target.

Mine warfare is usually imagined to proceed in discrete stages where the minefield is created, discovered, swept, and finally penetrated by traffic. Traffic is limited, so a mine can fail through lack of opportunity, as well as by being swept. As envisaged here, IED warfare differs in that the various activities occur continuously and indefinitely in time. New IEDs are being implanted even while old ones are being removed, and the question for each IED is whether it will be removed by sweeping (as Blue hopes) or by traffic (as Red hopes). The possibility of a mine’s failing for lack of opportunity is not recognized.

Although mines are also encompassed by our models, we will refer only to IEDs in the sequel.

The situation is as shown in Figure 3, where  $\lambda$  is the rate at which attempts are made to implant IEDs. In order to ultimately be successful, an IED must survive implantation, encounter traffic rather than a sweeper, and then prove to be lethal when the detonation actually occurs. “Up-arming” is an attempt to make IEDs less lethal, thus reducing the chances of success in the last (third) box. We will not deal with that tactic here, confining our interference with IEDs to the first and second boxes. The Basic Fixed Model deals only with the second box. The possibility of interfering in the first box through surveillance of the implantation activity is introduced in the second variant of that model.



**Figure 3: Success flowchart for IEDs.**

In modeling IED warfare, a basic question is whether routing of network traffic is to be part of Blue’s strategy. We will begin by assuming that traffic levels are given and known to both sides. This “fixed” assumption is appropriate if Blue’s losses to IEDs are small enough that other considerations determine traffic levels. At some level of loss, this assumption should be replaced by a “variable” assumption, where part of Blue’s strategy is the routing of traffic over the network. We begin by considering models based on the fixed assumption.

### ***Basic Fixed Model***

Blue traffic levels are assumed to be determined exogenously, and to not be significantly affected by losses to IEDs. Every unit of Blue traffic is assumed to complete its journey through the network, possibly accumulating multiple hits from IEDs as it does so. We do not recognize the possibility that a Blue unit, having taken one hit, will retreat rather than risk additional hits, nor do we distinguish between one unit taking two hits and two units taking one hit each. With these assumptions, the network becomes a simple set of arcs with no relevant structure. Each arc represents a separate opportunity

for interdiction, and conservation of flow constraints are not needed because they are presumably reflected in data about Blue traffic levels.

The required data include several quantities identified below as “rates”. These are to be interpreted in the sense of a continuous-time Markov process [Ross, 2000]. Neither the implantation of IEDs nor the timing of Blue sweeping is assumed to be affected by the specifics of any particular Blue convoy, but only by the general level of convoys on the road segment (arc). On any given arc, let

- $\lambda$  = rate at which functional IEDs are implanted (a decision variable for Red)
- $h$  = rate at which each IED is removed by Blue sweepers
- $t$  = rate at which each IED is removed by non-Blue traffic and other phenomena
- $b$  = rate at which Blue traffic enters the arc, counting both directions
- $a$  = actuation probability, the probability that any given IED is actuated by a Blue unit
- $d = ab$  = rate at which each IED is removed by Blue traffic
- $c$  = fraction of IED actuations caused by Blue traffic that are lethal to the traffic (other detonations are called “wasted fires”)

Neither  $b$  nor  $a$  is important except through their product, which is  $d$ . Blue traffic that does not actuate any IED is invisible in this analysis. It is possible that the value of  $a$  depends on exactly what a Blue traffic “unit” is—the actuation probability might be different for a large convoy of Blue trucks than it is for a small one, for example. Since the composition of Blue traffic is assumed to be determined by other considerations, we will not explore the implied tradeoff involving convoy size. All lethal IED actuations are taken to be equivalent in terms of Blue loss.

Defensive measures against IEDs will sometimes take the form of decreasing either  $a$  or  $c$ . Jamming decreases  $a$ , whereas up-armoring decreases  $c$ . If Blue traffic is itself capable of sometimes recognizing and destroying IEDs, then the effect is as if the IED is a wasted fire; that is, the effect is to decrease  $c$ . Losses are generally more sensitive to  $c$  than to  $a$ , since an IED that does not actuate can still damage future traffic, whereas a wasted IED will never damage anything. Nonetheless, reducing either  $a$  or  $c$  will reduce losses in a measurable way.

The clearance rate  $h$  might be determined by more fundamental quantities. Suppose that clearance consists of Blue sweeping units moving at fixed speed on an arc, removing each IED with a certain probability as the IED is passed. Let

- $L$  = arc length
- $w$  = speed of Blue sweeping units on the arc
- $y$  = number of Blue sweeping units assigned to the arc (a decision variable for Blue)
- $f$  = fraction of the time that a Blue sweeping unit is actually sweeping
- $p$  = probability that an IED will be removed by a Blue sweeping unit passing it

Then  $h = \beta y$ , where  $\beta \equiv wpf/L$ . Whether  $h$  is determined in this way or not, the crucial assumption is that  $h$  is proportional to  $y$ . We assume that the total number of sweeping units available to Blue on all arcs is some known quantity  $Y$ . While  $\beta$  is outside

of Blue's control, determination of  $y$  for each arc is part of Blue's strategy, subject to the total for all arcs not exceeding  $Y$ , the number of sweeping units available for the whole network.

Likewise, suppose that Red's efforts to implant IEDs on the network are constrained by the availability of some resource (possibly, but not necessarily, the IED supply) that becomes available at rate  $X$ . Let

$r$  = amount of resource required for one IED attempt, and  
 $q$  = the fraction of IED implantation attempts that succeed, and  
 $x$  = amount of the resource that Red devotes to the arc.

Then  $\lambda=xq/r$ , or, letting  $\alpha=cq/r$ ,  $c\lambda=\alpha x$ . While  $\alpha$  is outside of Red's control, determination of  $x$  is part of Red's strategy, subject to the total rate of IED implantation on all arcs not exceeding  $X$ .

The number of active IEDs present on the subject arc will fluctuate with time. Let  $N$  be the random number of IEDs present. Then  $N$  is a birth-death process where the rate of going up from state  $i$  is  $\lambda$ , while the rate of going down from state  $i$  is  $(d+h+t)i$ . Note that  $d$  is simply added to  $h$  as far as determining  $N$  is concerned, since Blue traffic and sweepers each have the effect of removing the IED, as do other phenomena represented by  $t$ . These rates are characteristic of an M/M/infinity queue, so  $E(N)=\lambda/(d+h+t)$ . The average rate of killing Blue traffic is then  $cdE(N)$ .

A simpler argument leading to the same formula is that the fraction of IEDs removed by Blue traffic is  $d/(d+h+t)$ . If this fraction is multiplied by the rate of IED implantation and the lethal fraction  $c$ , the same formula for the rate of killing Blue traffic results. Since  $t$  will in general be small, the crucial comparison is between  $d$  and  $h$ . If the two are equal, then about half of the IEDs will be removed by Blue traffic. If the fraction is to be substantially less than one-half, then  $h$  must be substantially greater than  $d$ . While it is true that some IEDs removed by Blue traffic are not lethal, the general principle nonetheless holds—if there is lots of traffic, then there needs to be lots of sweeping.

Now, subscript all variables by an arc index  $k$ , define the decision vectors  $\mathbf{x}=(x_k)$  and  $\mathbf{y}=(y_k)$ , and let  $L(\mathbf{x},\mathbf{y})$  be the total rate at which Blue traffic suffers lethal hits in the network. Then

$$L(\mathbf{x}, \mathbf{y}) = \sum_k \frac{\alpha_k x_k d_k}{d_k + \beta_k y_k + t_k}, \text{ to be maximized by Red subject to the constraint that}$$

$\sum_k x_k \leq X$ , while simultaneously being minimized by Blue subject to the constraint that

$\sum_k y_k \leq Y$ . This is a concave-convex game, so it has a saddle point that is easily

computed. It does not matter whether Blue first determines  $\mathbf{y}$ , after which Red (knowing  $\mathbf{y}$ ) determines  $\mathbf{x}$ , or vice versa. Washburn [2003] refers to such games as "logistics games", and gives a specialized solution technique. The value and optimal strategy for Blue can be obtained by solving minimization problem M1 in nonnegative variables  $\mathbf{y}$ . M1 is:

$$\begin{aligned}
& \text{Minimize } v \\
& \text{subject to } \sum_k y_k \leq Y, \text{ and} \\
& \frac{\alpha_k d_k}{d_k + \beta_k y_k + t_k} \leq v \text{ for all } k.
\end{aligned}$$

In words, Blue's problem is to minimize his vulnerability to IEDs on the arc for which his vulnerability is largest. The optimized objective function  $v$  is the minimal rate of Blue casualties per unit of Red resource  $X$ . Note that Blue's optimal strategy is independent of  $X$ ; that is, Blue's sweeping assets can be allocated optimally without knowing Red's overall level of effort.

Although M1 is nonlinear, there is a closely related linear problem where  $v$  is given and the object is to minimize the total Blue sweeping assets required to achieve it. If the object of analysis is to map out the relationship between  $v$  and  $Y$ , then the linear version is surely the right one to solve. In addition to being linear, the arcs all decouple when  $v$  is fixed (Washburn [2003]).

### ***Basic Fixed Model (Variant 1)***

We might also introduce "missions" for the Blue sweeping units, indexed by  $j$ , with  $f_{jk}$  being the fraction of the time that a Blue sweeping unit on mission  $j$  can spend actually sweeping on arc  $k$ . Other parameters (except for  $f$ ) remain the same as in the basic model. This variant might be natural for a situation where the geographic extent of the network is significant compared to its distance from the bases at which Blue sweeping units are stationed. The typical mission might be "move over segments 2, 5, and 6, doing only a cursory inspection for IEDs, then slow down and do a thorough inspection of arcs 4 and 9, and then return to base while making a cursory inspection of arcs 6 and 2". If this were mission  $j$ , then we would have  $f_{j1}=0$  and  $f_{j2}>0$ , since arc 1 is not covered at all, while arc 2 is covered twice.

Let  $z_j$  be the rate at which type  $j$  missions are undertaken, with  $Z$  being the total mission rate. Then  $y_k = \sum_j f_{jk} z_j$ , which can be substituted into the above expression for  $L(\mathbf{x}, \mathbf{y})$ . The essential concave-convex feature of the resulting game remains, and leads to minimization problem M2 in nonnegative variables  $z_j$ :

$$\begin{aligned}
& \text{Minimize } v \\
& \text{subject to } \sum_j z_j \leq Z, \text{ and} \\
& \frac{\alpha_k d_k}{d_k + \beta_k \sum_j f_{jk} z_j + t_k} \leq v \text{ for all } k.
\end{aligned}$$

In this case, the problem of minimizing  $\sum_j z_j$  when  $v$  is fixed is again a linear program, but the arcs do not decouple.

## Basic Fixed Model (Variant 2)

This is the same as the first variant, except that, in addition to  $Y$  sweepers, Blue has  $S$  surveillance units available. Surveillance units cannot detect IEDs once they are implanted, but may detect the operation of implanting them. Detected IEDs are assumed to be rendered ineffective, as in box 1 of Figure 3. We will assume that Blue surveillance is not detectable by Red, so that a Red implanter cannot abort an operation and thereby avoid detection. This is an important assumption. In World War Two, the success of antisubmarine operations in the Atlantic hinged on whether submarines could detect the radars of searching aircraft in time to submerge (abort). In practice it would also be important whether the implanter were eliminated along with the IED, but the fate of the implanter is not important here because the total implantation rate is assumed fixed.

Let

$u$  = speed of surveillance units (imagine UAVs)

$\delta$  = amount of time required to implant an IED

$\Delta$  = field of view of surveillance unit (linear measure)

$L$  = length of the arc under consideration

$g$  = fraction of the time that a Blue surveillance unit is actually working

$s$  = number of Blue surveillance units assigned to the arc (decision variable)

We will take  $\Delta$  to be 0. If it is not, simply replace  $\delta$  by  $\delta + \Delta/u$ , the effective IED vulnerability time per pass of a surveillance unit.

Blue surveillance units will cover arc length at an average rate of  $ugs$ , so the rate of making passes on the arc is  $ugs/L$ . The average number of detections in time  $\delta$  is  $ugs\delta/L$ . We assume that the actual number of detections is a Poisson random variable with that mean (the “random search” assumption), so the probability of 0 detections is  $\exp(-ugs\delta/L)$ . This exponential factor is the probability that any given implantation attempt is not detected by the surveillance units. This factor is multiplicative on  $\lambda$  and therefore on  $x$  in the basic model; since any surveillance detection is assumed to prevent a successful implantation, the rate of implantation is effectively reduced. Thus, letting  $\gamma$  stand for the quantity  $ugs\delta/L$ , and attaching an arc subscript as before, the revised payoff function is

$$L(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \sum_k \frac{\alpha_k x_k d_k \exp(-\gamma_k s_k)}{d_k + \beta_k y_k + t_k}. \text{ This function is linear in } \mathbf{x} \text{ and convex in } (\mathbf{y}, \mathbf{s}),$$

so another concave-convex game results. Because of the linearity in  $\mathbf{x}$ , the value of the game  $v$  and the optimal strategy  $(\mathbf{y}, \mathbf{s})$  for Blue can be found by solving M3:

Minimize  $v$ , subject to the constraints

$$\frac{\alpha_k d_k \exp(-\gamma_k s_k)}{d_k + \beta_k y_k + t_k} \leq v, \text{ for all } k, \text{ and}$$

$$\sum_k y_k \leq Y \text{ and } \sum_k s_k \leq S.$$

As usual, variables  $\mathbf{y}$  and  $\mathbf{s}$  are taken to be nonnegative. The value of the game is then  $vX$ ; that is,  $v$  is the probability that a unit of Red resource will ultimately result in a lethal hit on Blue traffic. M3 is an essentially nonlinear optimization, but not a difficult one. Surveillance units will be used on those arcs for which  $\gamma_k$  is relatively large compared to  $\beta_k$ , or otherwise sweeping units will be used.

We could also introduce a mission vector  $\mathbf{z}$  to reflect the idea that UAVs are more efficient at patrolling some arcs than others, with  $z_j$  being the number of UAVs assigned to mission  $j$ . Let  $g_{jk}$  be the fraction of the time that a UAV following mission  $j$  spends on segment  $k$ . Then the average number of UAVs on segment  $k$  is  $\sum_j z_j g_{jk}$ , and we have

minimization problem M4:

Minimize  $v$ , subject to the constraints

$$\frac{\alpha_k d_k \exp(-\gamma_k s_k)}{d_k + \beta_k y_k + t_k} \leq v, \text{ for all } k, \text{ and}$$

$$s_k = \sum_j z_j g_{jk}, \text{ for all } k, \text{ and}$$

$$\sum_k y_k \leq Y \text{ and } \sum_j z_j \leq Z.$$

When Blue's assets are used optimally, Red will find that the number of Blue casualties per unit of Red resource never exceeds  $v$ . See sheet "M4" of the spreadsheet *ied3.xls* for the solution of a small version of this problem with six arcs.

### ***A Variable Model Where Blue Optimizes Routing***

Routing of Blue traffic can be added to the collection of things that Blue optimizes, as long as Blue is constrained to deliver materiel at a required rate. Let  $B_i$  be the minimal amount of materiel to be delivered (positive) or supplied (negative) at node  $i$ , per unit time. Let  $\mathbf{B}=(B_i)$ , and let  $\mathbf{b}=(b_k)$  be Blue's flow vector over the arcs of the network. It is assumed that  $\sum_i B_i \leq 0$ , since otherwise supply is insufficient. The net inflow to node  $i$  is a linear expression in  $\mathbf{b}$ , with the coefficient of  $b_k$  being 1 if  $k$  points into  $i$ ,  $-1$  if  $k$  points out of  $i$ , or otherwise 0. Let  $\mathbf{F}$  be a node-by-arc dimensioned incidence matrix where  $(\mathbf{Fb})_i$  is the net inflow to node  $i$ . Then the flow vector  $\mathbf{b}$  is logistically feasible if  $\mathbf{Fb} \geq \mathbf{B}$ . Except for the necessity of being nonnegative and satisfying these inequalities, Blue's flow vector is at his discretion. We next describe a modification of the Basic Fixed Model that includes Blue's additional freedom.

Recall that  $d_k$  is the product of the actuation rate  $a_k$  and the traffic flow  $b_k$ . Substituting  $a_k b_k$  for  $d_k$  in the expression for  $L(\mathbf{x}, \mathbf{y})$ , the objective function becomes

$$L(\mathbf{x}, \mathbf{y}, \mathbf{b}) = \sum_k \frac{\alpha_k x_k a_k b_k}{a_k b_k + \beta_k y_k + t_k}. \text{ The minimizer controls } \mathbf{b} \text{ and } \mathbf{y}, \text{ while the maximizer}$$

controls  $\mathbf{x}$ . This game is concave in  $\mathbf{x}$  and convex in  $\mathbf{y}$ , but it is not convex in  $\mathbf{b}$ , and does not generally have a saddle point. The maxmin and minmax values can differ substantially, as the following example makes clear.

**Example:** Consider a network where two parallel arcs connect node  $s$  to node  $t$ .

For simplicity, set all parameters to 1, including the shipment rate from  $s$  to  $t$ . The payoff function is  $L(\mathbf{x}, \mathbf{y}, \mathbf{b}) = \frac{x_1 b_1}{b_1 + y_1 + 1} + \frac{x_2 b_2}{b_2 + y_2 + 1}$ . Red achieves the maxmin value of  $1/6$  by

making  $\mathbf{x}=(1/2, 1/2)$ , after which Blue should either make  $\mathbf{b}=\mathbf{y}=(1,0)$  or  $\mathbf{b}=\mathbf{y}=(0,1)$ . Note that Blue must not set each of the two vectors to  $(1/2, 1/2)$ , since this would result in the larger payoff of  $1/4$ , which turns out to be the minmax value. If he has to move first, Blue should set  $\mathbf{b}=\mathbf{y}=(1/2, 1/2)$ , since any imbalance would be exploited by Red to make the payoff even larger than  $1/4$ . The value of the game is the same as the minmax value, since the payoff is concave in  $\mathbf{x}$ . In playing the game where neither side moves first, Blue's optimal mixed strategy is to flip a coin to decide whether to make  $\mathbf{b}=\mathbf{y}=(1,0)$  or  $\mathbf{b}=\mathbf{y}=(0,1)$ . The maxmin value is 33% smaller than the minmax value.

As a practical matter, any attempt to apply mixed strategies by Blue in this example would be problematic. If we are to continue to interpret  $b_k$  as the constant rate at which arc  $k$  is utilized by Blue, then it stretches credibility to suppose that Red will not eventually notice that Blue is using only one of the arcs, adapting his IED distribution to concentrate on that arc. On the other hand, the derivation of the objective function relies on exactly that interpretation of  $b_k$ . The correct point of view here should be that of a multistage game, but the definition of "stage" would have to depend on the nature of the command-and-control loops employed by Red and Blue. We choose not to undertake that analysis here, and must therefore abandon any hope of making general statements about the value of secrecy and route randomization to Blue (however, the next subsection will consider a special case where those issues can be considered).

The computational goal in the rest of this subsection is merely to compute  $\min_{\mathbf{y}, \mathbf{b}} \max_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}, \mathbf{b})$ , rather than the value of the game. This is a conservative view of the problem from Blue's viewpoint. The interpretation of  $b_k$  is consistent with the derivation of  $L()$ , and Red is assumed to have observed the distribution of Blue traffic before committing himself to an IED distribution plan. The maximization is again trivial, so the minmax value is  $v$ , the solution of problem M5:

$$\begin{aligned} & \text{Minimize } v \\ & \text{subject to } \sum_k y_k \leq Y, \text{ and} \\ & \frac{\alpha_k a_k b_k}{a_k b_k + \beta_k y_k + t_k} \leq v \text{ for all } k, \text{ and} \\ & \mathbf{Fb} \geq \mathbf{B} \text{ and } \mathbf{b} \geq \mathbf{0}. \end{aligned}$$

Problem M5 is nonlinear. As in the Basic Fixed Model, there is a closely related linear program where  $v$  is fixed while  $Y$  is minimized. Note that Blue's motivation in M5 is to minimize  $\mathbf{b}$ , rather than to maximize it; if  $\mathbf{B}$  were 0, Blue would happily make  $\mathbf{b}=\mathbf{0}$  and  $\mathbf{y}=\mathbf{0}$ , thus achieving invulnerability at no cost. It is only the need to transport material as expressed in  $\mathbf{B}$  that makes Blue vulnerable in the first place.

Even when  $\mathbf{B}$  is not identically 0, Blue may choose to make  $b_k=y_k=0$  on some arcs. If in that case also  $t_k=0$ , how are we to interpret the ratio  $0/0$ ? Plainly Red is not motivated to attack such arcs, since there is no Blue traffic, so the proper interpretation is that the ratio is 0. Nonetheless, to preclude mathematical problems associated with dividing by 0, it is best to make  $t_k$  positive on every arc.

Variants of this model that include surveillance and missions can also be considered, as in the Fixed case. See sheet “M5” of *ied3.xls* for the solution of a variant of this model that includes the possibility of surveillance by a constrained number of UAVs, as in Variant 2 of the Fixed case.

### ***A Light Traffic Model Where Blue Optimizes Routing***

In this section, we consider a case where Blue traffic is so light that it has a negligible effect on clearance. Blue’s assignments of clearance assets are still assumed to be observable by Red, but likewise Blue can observe the resulting distribution of Red IEDs before routing convoys. Thus Red has no way of predicting how any given convoy will be routed. The number of convoys is not relevant except for the assumption that it is small, since convoys have a negligible effect on clearance. It therefore suffices to consider a single “test” convoy in this subsection. The criterion is assumed to be the average number of lethal hits that the test convoy takes in going from source node  $s$  to termination node  $t$ , to be maximized by Red and minimized by Blue.

With parameters as defined in the Basic Fixed Model, the average number of lethal hits to a convoy on arc  $k$ , given that it traverses that arc, is  $(\alpha_k x_k a_k) / (\beta_k y_k + y_k)$ . The average number of lethal hits on all of path  $p$  from  $s$  to  $t$  is therefore

$$L(\mathbf{x}, \mathbf{y}, p) = \sum_{k \in p} \frac{\alpha_k x_k a_k}{\beta_k y_k + t_k}. \text{ This is similar to the criterion in the Basic Fixed Model,}$$

except that there is a reference to the path  $p$ , rather than to traffic levels. Our object is to compute  $\min_{\mathbf{y}} \max_{\mathbf{x}} \min_p L(\mathbf{x}, \mathbf{y}, p)$ . For convenience, let  $\gamma_k = \alpha_k a_k / (\beta_k y_k + t_k)$ , and consider the problem with  $\mathbf{y}$  temporarily fixed. The inner maxmin problem is then one of maximizing the length of the shortest path from  $s$  to  $t$ , with the length of arc  $k$  being determined by the maximizer as  $\gamma_k x_k$ . Fulkerson and Harding [1977] show that its value is the value of the following maximization problem M6:

$$\begin{aligned} &\text{Maximize } w_t, \text{ subject to} \\ &w_i - w_j + \gamma_k x_k \geq 0 \text{ for all } k = (i, j) \\ &w_s = 0, \\ &\sum_k x_k \leq X, \text{ and } \mathbf{x} \geq 0. \end{aligned}$$

In M6, variable  $w_i$  can be interpreted as the shortest distance from  $s$  to  $i$ , with each constraint saying that the shortest distance to  $j$  cannot exceed the shortest distance to  $i$  plus the distance from  $i$  to  $j$ .

Since M6 is a linear program, it has the same value as its dual, which is a minimization problem. Let the dual variable for the constraint on arc  $k$  be  $b_k$ , and the dual variable for the constraint involving  $X$  be  $u$ . Also let  $\mathbf{F}$  be as in the previous section, and, since a single convoy must move from  $s$  to  $t$ , let  $\mathbf{B}$  be a vector that is 0 except that  $B_s = -1$  and  $B_t = 1$ . Then the dual of M6 is M7:

Minimize  $uX$ , subject to  
 $0 \leq \gamma_k b_k \leq u$  for all  $k$ ,  
 $\mathbf{Fb} \geq \mathbf{B}$  and  $\mathbf{b} \geq 0$ .

Variable  $u$  can be interpreted as average number of lethal hits per unit of Red resource. Variables  $\mathbf{b}$  and  $\mathbf{y}$  should be selected to minimize  $u$ . This can be done by writing out the formula for  $\gamma_k$  in M7 and adding a constraint on  $\mathbf{y}$ . The result is a nonlinear program with variables  $u$ ,  $\mathbf{y}$  and  $\mathbf{b}$ . Program M7 is nonlinear because of the dependence of  $\gamma_k$  on  $y_k$ . A similar problem where  $\sum_k y_k$  is minimized subject to  $u$  being fixed is a linear program, M8:

Minimize  $\sum_k y_k$ , subject to  
 $\alpha_k a_k b_k \leq u(\beta_k y_k + t_k)$  for all  $k$ ,  
 $\mathbf{Fb} \geq \mathbf{B}$  and  $\mathbf{b} \geq 0$  and  $\mathbf{y} \geq 0$ .

See sheet “MinMaxMin” of *ied3.xls* for an example.

Analytically speaking, variable  $b_k$  is the dual variable for the constraint on arc  $k$  in M6. Variable  $\mathbf{b}$  can also be interpreted as a mixed strategy for Blue where  $b_k$  is the probability that arc  $k$  is included in the path; use of this mixed strategy would guarantee the value  $uX$  without the necessity of Blue paying attention to Red’s IED distribution. Thus, although we have assumed above that Blue can observe the IED distribution before routing the test convoy, it turns out that this knowledge is not necessary. Blue can assure the same value by employing the mixed strategy represented by  $\mathbf{b}$ , without paying any attention to the IED distribution, as long as Red cannot discover the routing of the test convoy before deciding how to attack the network. As long as Blue’s routing choice remains secret, Blue can do just as well as if he could observe the IED distribution.

A bound on the value of secrecy to Blue can be obtained by assuming instead that Red moves last, after discovering the convoy routing as well as the clearance plan  $\mathbf{y}$ . In that case, Blue needs additional sweeping assets to assure the same value of  $u$ . The corresponding MinMinMax problem is not difficult to solve. If arc  $k$  is to be included in Blue’s path, then Blue must assure that  $\gamma_k$  does not exceed  $u$ , which implies a required amount of sweeping assets  $y_k$ . These required amounts are “distances” in a shortest path problem for Blue. See sheet “MinMinMax” of *ied3.xls* for an example. In artificial examples worked by the author, the value of secrecy in terms of sweeping assets required is positive, but not large.

### 3. **Economic Network Interdiction**

Interdiction is sometimes undertaken with the object of interrupting trade in some controlled or illegal commodity. The commodity might consist of cigarettes or alcohol for which the proper taxes have not been paid, illegal drugs such as cocaine, illegally cut lumber, or body parts of protected species. Since such trade is undertaken in the expectation of making a profit, it seems reasonable to keep economic considerations in

mind while constructing an interdiction plan. That is the general idea in this section.

The illegal commodity is to be shipped over a network that consists of a set of nodes interconnected by directed arcs, with  $(i,j)$  being the arc directed from node  $i$  to node  $j$ . Any two-way network segment can be represented by a twinned pair of one-way arcs  $(i,j)$  and  $(j,i)$ , but it is necessary to keep track of these twinned pairs for accounting purposes, since interdiction of one will normally imply interdiction of the other at the same level. Let  $T$  be the set of all such pairs of arcs; that is, if  $m$  and  $n$  are twins, then  $\{m,n\} \in T$ .

The shipper purchases the product for  $c$  per unit, ships it from node  $s$  to node  $t$  by choosing a path through the network, and sells any amount not confiscated for a larger price  $C$  at node  $t$ . We assume that there is at least one path from  $s$  to  $t$ , and, without loss of generality, that there are no arcs of the form  $(i,s)$  or  $(t,j)$ .

### ***The Minimum Required Interdiction Budget***

Let  $r=c/C$ , and let  $w_s$  be the fraction of the product that makes it from  $s$  to  $t$  without interdiction. Shipping will be profitable if and only if  $w_s > r$ , so the interdictor's object in this subsection is to make the reverse equality be true at minimum cost.

Let  $x_k$  be the amount of interdiction effort allocated to arc  $k$ , and let  $\mathbf{x}=(x_k)$ . We adopt the costing rule that, if  $m$  and  $n$  are twins, the total cost of interdicting the pair is  $\max(x_m, x_n)$ . It should be clear that it will always be in the interdictor's interest to equalize the efforts on such a pair, but, for later purposes, it is convenient to allow the two to be different. Let  $E(\mathbf{x}) = \sum_{\{m,n\} \in T} \min(x_m, x_n)$ , and let  $S(\mathbf{x}) = \sum_k x_k - E(\mathbf{x})$  be the total

cost of allocation  $\mathbf{x}$ . For twinned pairs, the first term in  $S(\mathbf{x})$  includes the allocations to both arcs, but the smaller of the two is then removed in the second term.

Let  $q_k(x_k)$  be the probability that a shipment over arc  $k$  is not confiscated. An alternative interpretation of  $q_k(x_k)$  is the fraction of each shipment that is not confiscated. Since we are interested only in averages, either interpretation leads to the same mathematical formulation. We assume, of course, that  $0 \leq q_k(x_k) \leq 1$ . The special "exponential" case is one where  $q_k(x_k) = \exp(-(\mu_k + \lambda_k x_k))$  for all  $k$ , for given nonnegative parameters  $\mu_k$  and  $\lambda_k$ . The exponential case might correspond to random surveillance of the arc by the interdictor.

The amount of product shipped is not important in this subsection, so we consider only a single unit shipment. The shipper must choose a path  $p$  from  $s$  to  $t$  through the network, and his payoff is the probability that the shipment arrives at  $t$  without being confiscated (alternatively, his payoff is the fraction of the shipment that is not confiscated). We assume that interdiction attempts on different arcs are independent, so the payoff probability is  $Q(p, \mathbf{x}) = \prod_{k \in p} q_k(x_k)$ . Anticipating that the shipper will use a

mixed strategy in this two-person zero-sum game, let  $\mathbf{z}=(z_p)$ , where  $z_p$  is the probability of using path  $p$ . Slightly overloading the payoff notation, we define the average payoff  $Q(\mathbf{z}, \mathbf{x}) = \sum_p z_p Q(p, \mathbf{x})$ . If the interdictor can choose  $\mathbf{x}$  in such a manner that  $Q(\mathbf{z}, \mathbf{x}) \leq r$  for

all  $\mathbf{z}$ , then the shipper cannot make a profit. This will be true if and only if  $Q(p, \mathbf{x}) \leq r$  for all  $p$ , so we are led to the following minimization problem M9 for the interdictor, which

is the problem of making the maximum value of  $Q(p, \mathbf{x})$  be as small as possible:

$$\begin{aligned} & \text{Minimize } v, \text{ subject to} \\ & Q(p, \mathbf{x}) \leq v \text{ for all } p, \\ & S(\mathbf{x}) \leq X, \text{ and } \mathbf{x} \geq 0. \end{aligned}$$

If the minimized  $v$  is  $r$  or smaller, then the budget  $X$  is sufficient to prevent profits.

The value of M9 is the solution of a minmax problem, but is it also the value of the game  $Q(\mathbf{z}, \mathbf{x})$ ? If  $Q(p, \mathbf{x})$  is a convex function of  $\mathbf{x}$  for all  $p$ , then  $Q(\mathbf{z}, \mathbf{x})$  is also a convex function of  $\mathbf{x}$  for all  $\mathbf{z}$ , therefore (being concave in  $\mathbf{z}$  in any case) a concave-convex payoff function, and therefore one with a saddle point [Washburn, 2003]. We therefore define the ‘‘convex’’ case to be one where  $Q(p, \mathbf{x})$  is a convex function of  $\mathbf{x}$  for all  $p$ . Convexity is sufficient for  $v$  to be the value of a game, as well as a minmax value. In the minmax computation, we are effectively assuming that the shipper knows  $\mathbf{x}$  before determining  $\mathbf{z}$ . In the convex case, this knowledge is really of no value to the shipper, since we are assured that there is a  $\mathbf{z}$  that will guarantee at least  $v$  regardless of  $\mathbf{x}$ . Otherwise there may not exist such a  $\mathbf{z}$ ; that is, it could be that smaller values than  $v$  are obtainable by the interdictor using a mixed strategy to determine  $\mathbf{x}$ . In other words,  $v$  is the value of the game in the convex case, but otherwise may be larger than the value of the game. Only the convex case will be pursued further here.

The convex case includes the previously identified exponential case, but there are also some important cases that are not convex. For example, any case that includes an arc  $k$  where the interdiction probability is 0 unless  $x_k$  exceeds some positive threshold is not convex.

Since the convex case is best approached by solving the minmax problem, we continue the minmax development. The difficulty with M9 is that it has one constraint for each path, and there may be a prohibitively large number of paths. Our object is to find a minimization problem with fewer constraints. Consider the problem of minimizing  $Q(\mathbf{z}, \mathbf{x})$  for fixed  $\mathbf{x}$ . Since  $\mathbf{x}$  is fixed, temporarily refer to  $q_k(x_k)$  as  $q_k$ . For each node  $i$  in the network, define the sets  $FS(i) = \{k | k = (i, j) \text{ for some } j\}$  and  $BS(i) = \{k | k = (j, i) \text{ for some } j\}$ . These sets are the sets of arcs pointing out of  $i$  and into  $i$ , respectively, either of which may be empty. For example,  $BS(s)$  and  $FS(t)$  are empty. For any mixed strategy  $\mathbf{z}$ ,

define  $b_k = \sum_{p \text{ such that } k \in p} z_p \prod_{m < k} q_m$ . Here the notation  $m < k$  means that only arcs before  $k$  in

the path  $p$  are to be included in the product. The quantity  $b_k$  can be interpreted as the fraction of a unit shipment that actually flows into arc  $k$ , taking account of what has been interdicted beforehand. The fraction flowing out of arc  $k$  is the same thing multiplied by  $q_k$ . We have the following conservation law at every node  $i$ , expressing the idea that, except at nodes  $s$  and  $t$ , the amount of product entering node  $i$  is the same as the amount leaving:

$$- \sum_{k \in BS(i)} q_k b_k + \sum_{k \in FS(i)} b_k = 1 \text{ if } i=s, \text{ } v \text{ if } i=t, \text{ else } 0.$$

The (average) amount delivered to node  $t$  is  $v$ , the quantity that the shipper wishes to maximize using  $\mathbf{z}$ . These conservation constraints are implied by the definition of  $\mathbf{b}$

and the assumption that every path connects  $s$  to  $t$ .

We next consider problem M10, which is to maximize  $v$ , subject to the conservation constraints and  $\mathbf{b} \geq 0$ . The equations that define  $b_k$  in terms of  $\mathbf{z}$  are ignored in M10, so the variable  $\mathbf{z}$  is not present. Since the conservation constraints are implied by the assumption that the shipper's options are limited to paths connecting  $s$  to  $t$ , M10 is a relaxation of the problem of maximizing  $Q(\mathbf{z}, \mathbf{x})$  with  $\mathbf{z}$ . The relaxation is comparatively easy computationally because  $\mathbf{b}$  is indexed by arcs, instead of paths. Fortunately, the two problems have exactly the same optimized objective function. In fact, there are methods for finding a mixed strategy  $\mathbf{z}$  for which a given optimizing flow  $\mathbf{b}$  can be achieved [Lawler, 1976].

Program M10 is a Generalized Flow problem for a lossy network [Ahuja, Magnanti, and Orin, 1993; Lawler, 1976]. Jewell [1962] and successors have introduced a variety of techniques that are especially efficient at solving this special case of a linear program. Our immediate goal here is to cast it as a minimization problem, rather than a maximization problem, on account of our overall goal of eventually solving a game.

Since M10 is a finite linear program, it has the same value as its dual. Letting  $w_i$  be the dual variable for the conservation constraint at node  $i$ , the dual of M10 is:

$$\begin{aligned} &\text{Minimize } w_s, \text{ subject to} \\ &w_i - q_k w_j \geq 0 \text{ for all } k = (i, j) \\ &w_t = 1. \end{aligned}$$

Variable  $w_i$  can be interpreted as the fraction of the product reaching node  $i$  that makes it to node  $t$ . No variable can be smaller than  $w_s$  or larger than  $w_t$  in an optimal solution, so, for all nodes  $i$ ,  $0 \leq w_s \leq w_i \leq 1$ .

Since this a minimization problem, and since the interdictor also wishes to minimize, the overall problem of finding the smallest budget for which the shipper cannot make a profit is M11, with variables  $\mathbf{w}$  and  $\mathbf{x}$ :

$$\begin{aligned} &\text{Minimize } S(\mathbf{x}), \text{ subject to} \\ &w_i - q_k(x_k)w_j \geq 0 \text{ for all } k = (i, j) \\ &w_t = 1, w_s \leq r, \text{ and } \mathbf{x} \geq 0. \end{aligned}$$

Program M11 is the desired alternative to M9. M11 can be further simplified by omitting all reference to twinned arcs. Let  $(\mathbf{w}, \mathbf{x})$  be an optimal solution of M11, and suppose that there is a pair of twinned arcs  $m$  and  $n$  such that both  $x_m > 0$  and  $x_n > 0$ . Let  $m$  connect node  $i$  to node  $j$ , suppose  $w_i \geq w_j$ , and let  $\mathbf{x}'$  be the same as  $\mathbf{x}$  except that  $x_m' = 0$ . The revised solution  $(\mathbf{w}, \mathbf{x}')$  is still feasible in M11, since  $q_m(0)$  cannot be larger than 1, and the revised solution has a smaller objective function. This contradicts the hypothesis of optimality, so there cannot be any optimal solution of M11 where  $x_m > 0$ ,  $x_n > 0$  and  $w_i \geq w_j$ . But if  $w_i \leq w_j$ , the same contradiction can be obtained by reducing  $x_n$  to 0. We conclude that there is no optimal solution of M11 where twinned arcs both have a positive allocation; that is,  $E(\mathbf{x})$  is 0 for any optimal solution of M11, so the  $E(\mathbf{x})$  term can be omitted in defining  $S(\mathbf{x})$  in the first place. This makes the objective function of M11

equal to  $\sum_k x_k$ , omitting all reference to twinned arcs.

In cases such as the exponential case, where  $q_k(x_k) > 0$  for all  $x_k$ , it may be convenient to replace variable  $w_i$  with variable  $u_i = -\ln(w_i)$ , simultaneously replacing the function  $q_k(x_k)$  with the function  $d_k(x_k) = -\ln(q_k(x_k))$  and the cost ratio  $r$  with the “distance”  $D = -\ln(r)$ . Taking logarithms and making the substitutions, we finally arrive at program M12 in variables  $\mathbf{u}$  and  $\mathbf{x}$ :

$$\begin{aligned} &\text{Minimize } \sum_k x_k, \text{ subject to} \\ &u_i \leq u_j + d_k(x_k) \text{ for all } k = (i, j) \\ &u_t = 0, u_s \geq D, \text{ and } \mathbf{x} \geq 0. \end{aligned}$$

This program has the interpretation that the interdictor controls the length of arc  $k$  by spending money on it, with the object of making the shortest route from  $t$  to  $s$  be longer than  $D$  as cheaply as possible. Program M12 is surely the right computational framework in the exponential case, since the fact that  $d_k(x_k) = \mu_k + \lambda_k x_k$  makes M12 a linear program. See sheet “M12” of *ied3.xls* for an example. Fulkerson and Harding [1977] consider a specialized algorithm for solving such linear programs efficiently.

In the special case where  $\mu_k = 0$  for all  $k$ , the minimized  $X$  will be proportional to  $D$ . Since  $r = \exp(-D)$ , we then have  $r = \exp(-\lambda X)$  for some parameter  $\lambda$  that can be determined by solving M12 once for any  $D$ . This is an especially simple relationship between the interdiction budget and the minimal transmission fraction.

The “mission trick”, where the interdictor’s budget is assigned to missions instead of arcs, can also be employed here. In the exponential case, the equivalent of M12 will still be a linear program.

## **Elastic Demand**

In this subsection, let the amount shipped be  $Q$ , not necessarily unity. The shipper is indifferent to  $Q$  when  $\mathbf{x}$  is determined as in the previous subsection, since the marginal profit is exactly 0 when the interdiction plan permits exactly the fraction  $r = c/C$  of the commodity to be shipped successfully. The trouble is that the selling price  $C$ , at least, is likely to be a function of  $Q$ . If the interdictor bases his strategy on an observed  $C$ , and if, as he hopes, the interdiction reduces the amount shipped over the network, then  $C$  is likely to increase, thus once again permitting profitable shipments. We assume here that the shipper will set  $Q$  to be whatever level maximizes his profits, recognizing that  $C$  is a function of  $Q$ . We investigate how the optimized  $Q$  depends on  $r$ , the transmission fraction that is under the interdictor’s control.

We take the relationship between  $C$  and  $Q$  to be  $C(Q) = C_0(Q/Q_0)^{-\beta}$ , where  $(Q_0, C_0)$  is some known realistic pair and  $\beta$  is the elasticity of price with respect to supply. For cocaine in the United States, for example, we might have  $Q_0 = 10^6$  kg/year,  $C_0 = \$25,000/\text{kg}$ , and  $\beta = 0.5$ . See Caulkins and Renter [1998] for a more detailed description of the cocaine market, and note that our  $\beta$  is the reciprocal of the usually estimated elasticity of demand with respect to price. It is a simple matter to calculate the

optimal  $Q$  when the transmission fraction is  $r$ . The shipper's profit is  $P=Q(rC(rQ)-c)$ . If  $\beta>1$ , infinite profits are in theory possible by letting  $Q$  approach 0. Otherwise, the best  $Q$  can be found by equating the derivative of  $P$  with respect to  $Q$  to 0. The solution is

$$Q^* = \frac{Q_0}{r} \left( \frac{r(1-\beta)C_0}{c} \right)^{1/\beta} \text{ and } P^* = \frac{\beta Q^* c}{1-\beta}.$$

As  $\beta$  approaches 1,  $Q^*$  approaches 0 and  $P^*$  approaches  $Q_0C_0$ . As long as  $\beta<1$ ,  $Q^*$  is an increasing function of  $r$ ; that is, interdiction has the anticipated effect of reducing the amount shipped, as well as the amount delivered and the resulting profit to the shipper.

Of course, these reductions may come at significant cost to the interdictor. The previous subsection essentially defines  $r(X)$ , the transmission fraction that is possible with budget  $X$ . It is only by making  $X$  approach infinity that  $r(X)$  can be made to approach 0, thereby asymptotically achieving the interdictor's goal of stopping shipments. Since infinite budgets are not possible, the interdictor's efforts will have to stop short of perfection. Is there any rational method for determining the "right" value for  $X$ ? One possible principle for answering this question is that  $X$  should bear some relationship resembling equality with the value of the shipments that are interdicted. The next section is a formalization of this idea that results in one particular "optimal" interdiction budget.

## ***An Equilibrium***

Much of tropical rainforest deforestation is due to illegal logging conducted for profit [Chatham House, 2006]. Logging is an activity that cannot be conducted without heavy equipment, including loaded logging trucks that are easy to identify (see Figure 2), and yet the antilogging laws are often not enforced. There are many reasons for this, one of which is that the law enforcement agency simply lacks the budget required for enforcement. In that circumstance, it is tempting to explore providing the interdiction funds, at least partially, by selling any captured logs. While objections can be raised to the idea, it does have the attraction that the enforcement agency's funding increases when it is most needed; that is, when illegal logging is common and therefore easy to interdict. The enforcement agency essentially functions as a legally sanctioned highwayman. This section develops a game-theory based model of the situation where the highwayman (interdictor) spends money on interdiction as long as doing so is profitable. The game is *not* zero-sum. It is also *not* cooperative. A cooperative version would have the loggers buy off the enforcement agency and sell the logs as usual.

There seems to be a continuing debate among national authorities about the proper fate of confiscated illegal shipments of otherwise legal products. The co-chairpersons of the UN International Conference on Illicit Tobacco Trade [UN, 2002] report that "... some delegates favored destroying all contraband while others felt that the product was otherwise a legal product but for the payment of taxes." While the best way of proceeding is unclear, it should at least be of interest to explore the consequences of viewing the problem from the highwayman's perspective.

Illegal logging is perhaps the most obvious application of the highwayman idea because no one objects to having a lumber market, but there are also other products such as tobacco and alcohol that have established legal markets. Even illegal shipments of things like ivory and cocaine might be dealt with in the same way. Of course, markets in

ivory and cocaine are not desired, at least not by most people, especially if the government acts as a supply source. There are clear possibilities for corruption, and the interdictor's actions in selling the captured product do not seem to correspond to his announced goal of reducing sales. Nonetheless, a careful analysis of the idea can still be instructive. If cocaine is to be interdicted, then what should be the budget for doing so, and how should it be spent? Both of those questions have definite answers from the highwayman's point of view, and the answers may be relevant even if one does not contemplate an actual highwayman. It does not make sense, does it, to spend \$100 on an operation that only captures \$1 in drugs? In searching for the "right" interdiction budget, one might suggest that the interdictor behave as if he were a highwayman, even though the product is actually destroyed when captured.

When captured product is sold, rather than destroyed, the shipper's profit becomes  $P=Q(rC(Q)-c)$ . This differs from the formula in the previous subsection in that the argument of  $C()$  is the total amount shipped  $Q$ , rather than the smaller amount  $rQ$  that is not interdicted. The effect is to depress the price and cause the shipper to ship less for any given  $r$ . The optimal quantity to ship and the resulting profit are now  $Q'$  and  $P'$ , where  $Q' = Q_0 \left( \frac{r(1-\beta)C_0}{c} \right)^{1/\beta}$  and  $P' = \frac{\beta Q' c}{1-\beta}$ .

The interdictor's profit is  $(1-r)Q'C(Q')-X$ . Since the interdictor can always make this 0 by choosing not to interdict ( $X=0, r=1$ ), the maximized value will always be nonnegative. A numerical exploration where  $r$  is varied will reveal the maximum profit. This value of  $r$ , together with the corresponding  $Q'$ , constitutes an equilibrium (Nash[1951]) in the sense that neither the interdictor nor the shipper is motivated to change  $r$  (the interdictor) or  $Q'$  (the shipper). Figure 4 shows the numerical exploration in a simple example where the network has only a single arc.

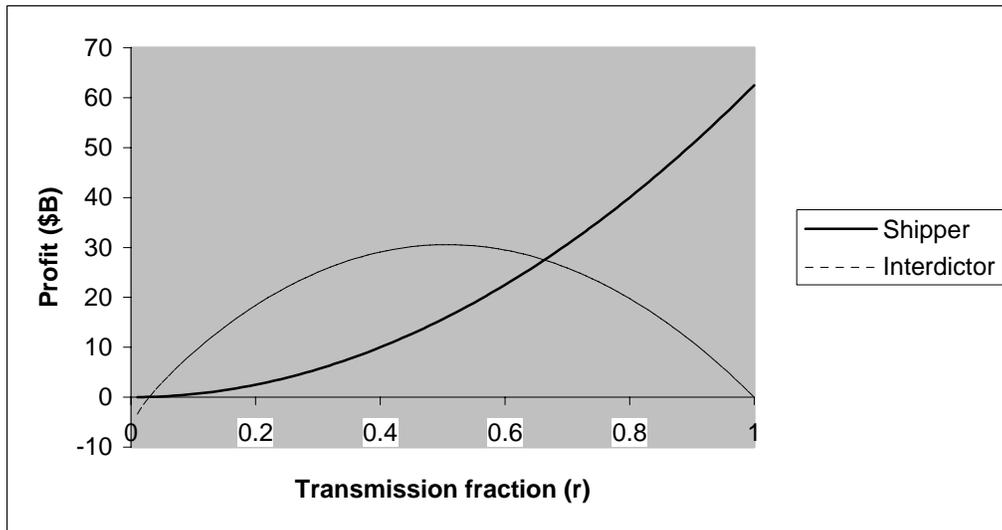


Figure 4: An example where the interdictor's optimal transmission fraction is 0.5. In equilibrium, the profits are approximately \$30B (interdictor) and \$15B (shipper). See sheet "Equilibrium" of *ied3.xls* for details.

## **4. Summary**

We have considered two distinct, continuous-time interdiction situations. The first is motivated by IED warfare. A variety of models are outlined, depending mainly on the type of assets available to combat interdiction. The second is motivated by interdiction of illegally shipped substances, a fundamentally economic activity for the shipper. The conventional point of view has an arbitrary budget for the interdictor. We consider an alternative point of view that results in a Nash equilibrium.

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