

NPS-MA-99-001

# NAVAL POSTGRADUATE SCHOOL Monterey, California



## CONVOLUTION METHODS FOR MATHEMATICAL PROBLEMS IN BIOMETRICS

by

Christopher Frenzen

January 15, 1999

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Prepared for: Biometrics Identification  
College of Engineering  
San Jose State University  
One Washington Square  
San Jose, CA 95192-0205

1 9 9 9 0 2 1 8 0 8 4

**REPORT DOCUMENTATION PAGE**

Form approved

OMB No 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)

2. REPORT DATE

January 15, 1999

3. REPORT TYPE AND DATES COVERED

March 1997 – September 1997

4. TITLE AND SUBTITLE

Convolution Methods for Mathematical Problems in Biometrics

5. FUNDING

MIPR # NAFRLP62800493

6. AUTHOR(S)

Christopher L. Frenzen

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Postgraduate School  
Monterey, CA 93943

8. PERFORMING ORGANIZATION  
REPORT NUMBER

NPS-MA-99-001

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Biometrics Identification  
College of Engineering  
San Jose State University  
One Washington Square  
San Jose, CA 95192-0205

10. SPONSORING/MONITORING  
AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

Approved for public release; distribution is unlimited.

12a. DISTRIBUTION/AVAILABILITY STATEMENT

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words.)

Estimation of the impostor probability density function from the inter-template and template-sample probability density functions involves a multi-dimensional convolution of the latter density functions. We assume isotropy of the probability distribution functions and use the Fourier transform to express the convolution as a one-dimensional integral with a kernel given by a spherical Bessel function.

14. SUBJECT TERMS

Biometrics, Convolution

15. NUMBER OF  
PAGES

6

16. PRICE CODE

17. SECURITY CLASSIFICATION  
OF REPORT

UNCLASSIFIED

18. SECURITY CLASSIFICATION  
OF THIS PAGE

UNCLASSIFIED

19. SECURITY CLASSIFICATION  
OF ABSTRACT

UNCLASSIFIED

20. LIMITATION OF  
ABSTRACT

NAVAL POSTGRADUATE SCHOOL  
Monterey, California 93943-5000

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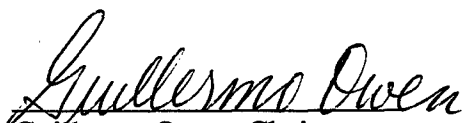
This report was prepared for Biometrics Identification, College of Engineering, San Jose State University.

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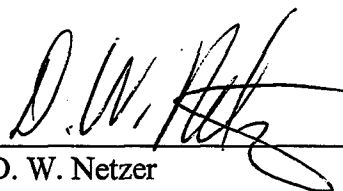
Christopher L. Frenzen  
Professor of Mathematics

Reviewed by:



Guillermo Owen, Chair  
Department of Mathematics

Released by:



D. W. Netzer  
Associate Provost and  
Dean of Research

## Convolution Methods for Mathematical Problems in Biometrics

C.L. Frenzen  
Department of Mathematics  
Naval Postgraduate School, Monterey, CA

**1. Introduction.** The problem we shall investigate can be formulated in the following way, due to Peter Bickel [1]: For  $M + 1$  individuals  $K$  measurements  $X_{ij}, i = 1, \dots, M + 1, j = 1, \dots, K$  are made. The  $X_{ij}$  are assumed to be vectors in  $\mathbf{R}^N$ .

We assume, following Bickel [1], that

$$X_{ij} = \mu_i + \varepsilon_{ij},$$

where  $\mu_1, \dots, \mu_{M+1}$  are individual effects vectors from  $\mathbf{R}^N$  which are i.i.d. (independently and identically distributed) with a distribution which is spherically symmetric about some  $\mu$  in  $\mathbf{R}^N$ . The  $\varepsilon_{ij}$  are i.i.d. independently of the  $\mu_i$  and their distribution is spherically symmetric about 0.

The  $K(M + 1)X_{ij}$ s represent  $K$  biometric measurements made on  $M + 1$  individuals; however the measuring device actually records only the data  $|\bar{X}_i - \bar{X}_{i'}|$  and  $|X_{M+1,j} - \bar{X}_{M+1}|$ ,  $1 \leq i < i' \leq M, 1 \leq j \leq k$ , where  $|x|$  is the length of  $x$  and

$$\bar{X}_i = \frac{1}{K} \sum_{j=1}^K X_{ij}$$

is the centroid of the  $K$  measurements taken on the the  $i$ th individual.

Let  $p$  be the density of  $|\bar{X}_1 - \bar{X}_2|$ ,  $q$  the density of  $|X_{M+1,1} - \bar{X}_{M+1}|$ , and  $r$  the density of  $|X_{M+1,1} - \bar{X}_1|$ . The basic problem is to estimate  $r$  given estimates  $\hat{p}, \hat{q}$  for  $p$  and  $q$ . As  $p$  and  $q$  are densities on  $[0, \infty)$ , the problem of estimating  $\hat{p}$  and  $\hat{q}$  is a standard problem in estimation theory which we do not consider further.

**2. Bickel's Approach.** What is the relationship between  $p$ ,  $q$ , and  $r$ ? Following Bickel [1], we write

$$X_{M+1,1} - \bar{X}_1 = \varepsilon_{M+1,1} + \mu_{M+1} - \mu_1 - \bar{\varepsilon}_1, \quad (1)$$

$$\bar{X}_{M+1} - \bar{X}_1 = \bar{\varepsilon}_{M+1} + \mu_{M+1} - \mu_1 - \bar{\varepsilon}_1, \quad (2)$$

$$X_{M+1,1} - \bar{X}_{M+1} = \varepsilon_{M+1,1} - \bar{\varepsilon}_{M+1}. \quad (3)$$

Note that addition of equations (2) and (3) yields equation (1). Further, the quantities on the left sides of (2) and (3) have spherically symmetric distributions. As the lengths of the left sides of (2) and (3) have densities  $p$  and  $q$  respectively, and the length of the left side of (1) is  $r$ , it seems that what is required is a formula for the density of the length of the convolution of two spherically symmetric (about 0) distributions given the densities of their lengths. However, as Bickel pointed out in [1], this is not quite correct since the use of the convolution assumes the independence of the densities  $p$  and  $q$ , and these two densities are not generally independent, since in the right sides of (2) and (3) the terms  $\varepsilon_{M+1,1} - \bar{\varepsilon}_{M+1}$  and  $\bar{\varepsilon}_{M+1}$  are only uncorrelated, and not in general independent unless  $\varepsilon_{11}$  has a Gaussian distribution.

However, for large  $K$ , we follow Bickel's argument in [1] to show that the terms  $\varepsilon_{M+1,1} - \bar{\varepsilon}_{M+1}$  and  $\bar{\varepsilon}_{M+1}$  are independent. Let

$$Z_K = (K-1)^{-\frac{1}{2}} \sum_{j=2}^K \varepsilon_{M+1,j}, \quad (4)$$

and

$$U_K = \varepsilon_{M+1,1} \left(1 - \frac{1}{K}\right). \quad (5)$$

Then,  $Z_K$  and  $U_K$  are independent and, if  $K$  is large,

$$\varepsilon_{M+1,1} - \bar{\varepsilon}_{M+1} = U_K - \frac{(K-1)^{\frac{1}{2}}}{K} Z_K \approx U_K \quad (6)$$

and

$$K^{\frac{1}{2}} \bar{\varepsilon}_{M+1} = \left(\frac{K-1}{K}\right)^{\frac{1}{2}} Z_K + U_K \frac{K^{\frac{1}{2}}}{(K-1)} \approx Z_K. \quad (7)$$

Thus, for large  $K$ , to a first approximation it is possible to ignore the dependence between (6) and (7). The terms  $U_K$  and  $Z_K$  on the right sides of (6) and (7) respectively are the first terms in an asymptotic expansion for large  $K$  of the left sides of those equations.

**3. Convolution and Fourier Transform.** Let  $U, V$  be random vectors in  $\mathbf{R}^N$  which are independent and have isotropic distributions with densities

$f_1, f_2$  respectively. We let  $g_1, g_2$  be the corresponding densities on  $\mathbf{R}^+$  of the lengths of  $U$  and  $V, |U|, |V|$ . Further, let  $f$  be the density of  $U + V$  and  $g$  be the density of  $|U + V|$ . Then by the independence of  $U, V,$

$$f(y) = \int_{\mathbf{R}^N} f_2(y - x) f_1(x) dx. \quad (8)$$

Our interest is to determine a formula for  $g$  in terms of  $g_1$  and  $g_2$ . To this end, we introduce the Fourier transform. If  $f$  is absolutely integrable on  $\mathbf{R}^N$ , then the Fourier transform of  $f$  is defined by

$$\hat{f}(t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} f(y) e^{it \cdot y} dy, \quad (9)$$

where the  $N$  vector  $t = (t_1, t_2, \dots, t_N)$  and  $t \cdot y = t_1 y_1 + \dots + t_N y_N$ . If  $f(y)$  is spherically symmetric, i.e.,  $f(y)$  is a function of  $r = |y|$  only, say  $f(y) = h(r)$ , then its Fourier transform  $\hat{f}(t)$  is also spherically symmetric; more specifically, we have

$$\hat{f}(t) = \rho^{(2-N)/2} \int_0^\infty r^{N/2} h(r) J_{(N-2)/2}(\rho r) dr, \quad (10)$$

where  $\rho = |t|$  and  $J_\nu(r)$  denotes the Bessel function of the first kind of order  $\nu$ . For a proof of this formula, we refer to Schwartz[2]. (Note that we have introduced the same letter  $r$  for  $|y|$  as we used for the density of the length of the left side of (1). From the context, there should be no confusion as to which meaning for  $r$  is intended.) The Fourier transform of the convolution in (8) yields

$$\hat{f}(t) = \hat{f}_2(t) \hat{f}_1(t), \quad (11)$$

and combining (10) and (11) gives

$$\hat{f}(t) = \rho^{2-N} \int_0^\infty r^{N/2} h_2(r) J_{(N-2)/2}(\rho r) dr \int_0^\infty r^{N/2} h_1(r) J_{(N-2)/2}(\rho r) dr, \quad (12)$$

where the functions  $h_1, h_2$  in (12) are defined by

$$h_1(r) = f_1(y), \quad h_2(r) = f_2(y), \quad (13)$$

with  $r = |y|$  since the distributions defined by  $f_1, f_2$  are isotropic.

Now if  $U$  is an isotropically distributed  $N$  vector with density  $f$  on  $\mathbf{R}^N$ , and  $g$  is the corresponding density on  $\mathbf{R}^+$  of the length  $|U|$ , where  $|x|^2 =$

$\sum_{i=1}^N x_i^2, x = (x_1, \dots, x_N)$ , then the relationship between the densities  $f$  and  $g$  is given by

$$f(y) = |y|^{-(N-1)} c_N^{-1} g(|y|), \quad (14)$$

where  $c_N$ , the surface 'area' of the unit sphere in  $\mathbf{R}^N$ , is

$$c_N = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})}. \quad (15)$$

Hence, upon substituting

$$h_1(r) = f_1(y) = r^{-(N-1)} c_N^{-1} g_1(r), \quad (16)$$

$$h_2(r) = f_2(y) = r^{-(N-1)} c_N^{-1} g_2(r),$$

into (12), we have

$$\hat{f}(t) = \rho^{2-N} c_N^{-2} \int_0^\infty r^{1-N/2} g_2(r) J_{(N-2)/2}(\rho r) dr \int_0^\infty r^{1-N/2} g_1(r) J_{(N-2)/2}(\rho r) dr. \quad (17)$$

If the dimension  $N$  of the space the measurement vectors  $X_{ij}$  belong to is even, say  $N = 2m$ , then  $J_{(N-2)/2}(\rho r) = J_{m-1}(\rho r)$  is a Bessel function of the first kind of integer order. If  $N$  is odd, say  $N = 2m + 1$ , then  $J_{(N-2)/2}(\rho r) = J_{m-1/2}(\rho r)$  is a Bessel function of the first kind of fractional order, and is closely related to the Spherical Bessel function of the first kind  $j_n(z)$ , defined by

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z). \quad (18)$$

More detailed information about Bessel functions may be in found in Abramowitz and Stegun [3].

**4. Convergence of the Integrals.** We now discuss convergence of the integrals in (17). Both integrals are functions of the variable  $\rho = |t|$ , hence  $\hat{f}(t)$  is a spherically symmetric function of the transform variable  $t$ . This means that the inverse Fourier transform of  $\hat{f}(t)$ , i.e.  $f(y)$ , can also be obtained as a one-dimensional integral with a Bessel function kernel.

For fixed  $\rho$  and small  $r$ ,  $J_{(N-2)/2}(\rho r) = O(r^{(N-2)/2})$ . Since we expect  $g_i^{(j)}(0) = 0$  for  $i = 1, 2$  for  $0 \leq j \leq N - 1$  (see Bickel [1]), it follows that both

integrals in (17) are convergent at the lower limit 0. At the upper limit  $\infty$ ,  $J_{(N-2)/2}(\rho r) = O(r^{-1/2})$ , and this by itself will not be enough to make the integral convergent. However the term  $r^{1-N/2}$  will also make the integrals converge at the upper limit if  $N$  is sufficiently large. In practice the densities  $g_1, g_2$  also tend to zero sufficiently rapidly to make the integrals converge at the upper limit. If these densities have compact support (i.e., are zero outside of a closed bounded subset of  $\mathbf{R}^+$ ), then the integrals in (17) no longer have infinite upper limits. The integrals in (17) can be evaluated accurately and efficiently by standard numerical quadrature methods.

**5. Inversion.** Note that the right side of (17) is a function of  $\rho = |t|$  only. Hence  $\hat{f}(t)$  is spherically symmetric. Let  $\hat{f}(t) = G(\rho)$ . The inverse Fourier transform of  $\hat{f}(t)$ , i.e.,  $f(y)$  from (9), is defined by

$$f(y) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} \hat{f}(t) e^{-it \cdot y} dt. \quad (19)$$

It follows by analogy with (10) that

$$f(y) = r^{(2-N)/2} \int_0^\infty \rho^{N/2} G(\rho) J_{(N-2)/2}(\rho r) d\rho, \quad (20)$$

where  $r = |y|$ . The relationship (14) between the density  $f$  of an isotropically distributed  $N$  vector and the corresponding density  $g$  of its length then implies

$$g(r) = r^{N/2} c_N \int_0^\infty \rho^{N/2} G(\rho) J_{(N-2)/2}(\rho r) d\rho \quad (21)$$

where  $G(\rho) = \hat{f}(t)$  is given by (17). Numerical evaluation of the integral in (21) proceeds similarly to the integrals in (17). With  $g_1, g_2$  taken as  $p, q$  introduced at the end of section 1, and  $g$  taken as  $r$  (the density function for  $|X_{M+1,1} - \bar{X}_1|$ , not  $|y|$ ), (17) and (21) together give the density  $r$  in terms of  $p$  and  $q$ .

**6. Conclusions.** We have shown that for large  $K$ , the left sides of (2) and (3) are approximately independent. Under this approximation, we



derived an expression for the density  $r$  of the length of the left side of (1) in terms of the densities  $p, q$  of the lengths of the left sides of (2) and (3). This result is contained in equations (17) and (21) of the previous section.

Future work could include determining corrections to the above result for a finite number of measurements  $K$ , and practical numerical implementation of the above result.

#### References

- 1.) NSA SAG Problem 97-2-1 Solution, Peter Bickel.
- 2.) *Mathematics for Physical Sciences*, L. Schwartz, Addison-Wesley, Reading, MA, 1966, pp. 201-203.
- 3.) *Handbook of Mathematical Functions*, A. Abramowitz and I. Stegun, National Bureau of Standards, 1964.

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