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13. ABSTRACT (*Maximum 200 words*)

This final technical report presents a summary of a three-year project in which the mathematical basis of failure criteria for metals and composite materials was investigated. The investigation covered criteria associated with crack formation and structural instability. The main results are: (a) General procedures for the determination of mathematical functions that model the conditions for crack formation at critical points were developed and implemented into a generally available computer code. (b) Mathematical models for buckling failure were formulated, implemented and their properties clarified.

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RESEARCH ON THE p- AND hp-VERSIONS OF THE FINITE ELEMENT METHOD

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1 Executive Summary

Multi-material interfaces are sites of failure initiation in composite materials. The development of reliable quantitative criteria for failure initiation in electronic components, adhesively bonded joints and laminated composites is obviously very important. At present there are no universally accepted procedures for the evaluation of fatigue and durability characteristics of structural, mechanical and electronic components made of composite materials.

This project was concerned with the development of mathematical methods and computational procedures for the determination of functionals which can be correlated with failure initiation events in composite materials subjected to thermal and mechanical loads. The approach is based on the assumption that failure initiation events are associated with the natural straining modes, analogously to the well established correlation between generalized stress intensity factors in linear elastic fracture mechanics and crack propagation events.

Failure criteria must be formulated in terms of functionals the exact values of which are finite. Stresses corresponding to the exact solution are usually infinity in singular points. The numerically computed stresses are, of course, finite but very sensitive to the discretization. Therefore stresses cannot be used for formulating failure criteria.

The specific objectives of the project were:

1. Develop procedures for the numerical determination of the eigenpairs λ_i and $\vec{\phi}_i$ that characterize the natural straining modes and natural flux states at singular points in heterogeneous bodies.

2. Develop a method for numerical determination of the generalized stress intensity and flux intensity factors in heterogeneous bodies subjected to thermal and mechanical loading.

In addition, development of methods for the estimation of limit loads for fiber-reinforced composites in compression was undertaken. The instability of fibers is an important consideration in the design of fiber-reinforced composites.

1.1 Summary of accomplishments

- A reliable numerical method for the determination of the flux and stress fields at multi-material interfaces in thermoelastic problems has been developed. The method involves numerical determination of the eigenpairs of the asymptotic expansion by a procedure called the modified Steklov method and determination of the coefficients by a procedure based on the complementary energy principle.

- A test implementation has been completed and the effectiveness of the method established through benchmark studies.

- Industrial application has been made possible by a Phase I STTR grant to Engineering Software Research and Development, Inc. (ESRD) and Washington University. The two-dimensional thermoelastic capabilities have been implemented in the commercial FEA code StressCheck. This is an unique capability which is now being used for investigation of the correlation of observed failure events in lap-shear test specimen with generalized stress intensity factors. This work, started on August 1, 1997 is being performed in collaboration with Raytheon TI Systems.

- The original scope of work was extended to include numerical simulation of failure of homogeneous and composite elastic materials through loss of stability. This work, performed in collaboration with Dr. Manil Suri of the University of Maryland and Dr. Ivo Babuška of The University of Texas, Austin, led to a clarification of some fundamental theorems related to the numerical simulation of problems of elastic stability. A doctoral dissertation has been completed [3].

1.2 Personnel supported

Faculty:
    Dr. Barna A. Szabo (PI)
Post-doctoral persons:
    Dr. Ricardo L. Actis (part time)
    Dr. Xian-Zhong Guo (part time)
    Dr. Gyorgy Kiralyfalvi
Graduate Students:
    Mr. Andre Tamagnini Noel (D.Sc. candidate, Graduated May, 1996)
    Mr. Gyorgy Kiralyfalvi (D.Sc. candidate, Graduated May, 1997)
    Ms. Li Zhang (D.Sc. candidate)

1.3 Consultative and advisory functions

The Principal Investigator presented briefings to eight Air Force contractors and one Navy laboratory:

1. McDonnell Douglas Aerospace St. Louis, MO (contact persons: Mr. Scott Fields, Mr. Daniel Dudley) April 25, 1996

2. Boeing Aerospace, Downey, CA. Dr. Saeed Paydarfar visited Washington University and was briefed on the scope and objectives of the project. May 19, 1997

3. Boeing Aircraft Co., Wichita, KS (contact person: Mr. Phillip Legate) June 4, 1997

4. Cessna Aircraft Co, Wichita, KS (contact person: Mr. Milan Radovanov) June 5, 1997

5. Raytheon TI Systems, Dallas, TX (contact person: Dr. Terry Baughn) July 29, 1997

6. Lockheed-Martin, Fort Worth, TX (contact person: Mr. Michael Barnhart) July 30, 1997

7. Allied Signal, Phoenix, AZ (contact person: Dr. Malak Malak 602-231-3701) September 5, 1997

8. Structures Division, NAWCAD, Patuxent River, MD (contact person: Dr. David Barrett 301-342-9360) September 8, 1997

9. Lockheed-Martin, Marietta, GA (contact person: Dr. Stephen P. Engelstad 770-494-9714) March 3, 1998. This presentation was made to members of a government/industry consortium known as the Composites Affordability Initiative. Members represent each of the major US aerospace companies and the Air Force and Navy. This particular meeting was hosted by Lockheed Martin.

1.4 Publications and presentations

[1] Noël, A. and B. Szabó. Formulation of geometrically non-linear problems in the spatial reference frame. *Int. J. Numer. Methods Eng.*, 40:1263–1280, 1997.

[2] Noël A. T. Spatial Formulation and Numerical Solution of Geometrically Nonlinear Problems in Finite Elasticity. D.Sc. Dissertation, Sever Institute of Technology, Washington University, St. Louis, Missouri, 1996.

[3] I. Babuška, and B. Szabó. New problems and trends in the finite element method. In J. R. Whiteman, editor, *The Mathematics of Finite Elements and Applications*, pages 1–33, Chichester, 1997. John Wiley and Sons.

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[5] I. Páczelt and T. Szabó, B. and Szabó. Solution of elastic contact problems by the p-version of the finite element method. 4th U.S. National Congress on Computational Mechanics, August, 1997.

[6] S. A. Prost-Domaski, B. A. Szabó, and G. I. Zahalak. Large-deformation analysis of nonlinear elastic fluids. *Computers and Structures*, 64:1281–1290, 1997.

[7] B. Szabó and Z. Yosibash. Numerical analysis of singularities in two dimensions. Part 2: Computation of generalized flux/stress intensity factors. *Int. J. Numer. Meth. Engng.*, 39:409–434, 1996.

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[9] B. A. Szabó. Hierarchic models and discretizations. Symposium on Advances in Computational Mechanics, The University of Texas at Austin, January 1997.

[10] B. A. Szabó and R. A. Actis. Failure analysis of composite materials. International Mechanical Engineering Congress and Exposition, November, 1996 Atlanta, GA.

[11] G. Királyfalvi. Linear Models of Elastic Stability. D.Sc. Dissertation, Sever Institute of Technology, Washington University, St. Louis, Missouri, 1997.

[12] B. A. Szabó and G. Királyfalvi. Mathematical models of buckling and stress stiffening. 4th U.S. National Congress on Computational Mechanics, August, 1997.

[13] R. Szabó, B. and Actis. The problem of model selection in numerical simulation. 4th U.S. National Congress on Computational Mechanics, August, 1997.

[14] Y. Volpert, T. Szabó, I. Páczelt, and Szabó B. Application of the space enrichment method to problems of mechanical contact. *Finite Elements in Analysis and Design*, 24:157–170, 1997.

[15] Y. Wang, P. Monk, and B. Szabó. Computing cavity modes using the p-version of the finite element method. *IEEE Transactions of Magnetics*, 32:1934–1940, 1996.

[16] Z. Yosibash and B. A. Szabó. A note on numerically computed eigenfunctions and generalized stress intensity factors associated with singular points. *Engineering Fracture Mechanics*, 54(4):593–595, 1996.

[17] Z. Yosibash and B. A. Szabó. Failure analysis of composite materials and multi-material interfaces *Proceedings, 1995 Design Engineering Technical Conferences, ASME DE-Vol. 83:133–139, 1995.*

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[19] Z. Yosibash Numerical thermo-elastic analysis of singularities in two-dimensions. *International Journal of Fracture*, 74:341–361, 1996.

[20] Z. Yosibash Computing edge singularities in elastic anisotropic three-dimensional domains *International Journal of Fracture*, 86:221–245, 1997.

[21] G. Kiralyfalvi and B. Szabó. Quasi-Regional Mapping for the p-Version of the Finite Element Method. *Finite Elements in Analysis and Design*, 27:85–97 1997.

1.5 Transitions

The new capabilities have been made available to Air Force laboratories and contractors through a professional quality software called StressCheck. StressCheck is being developed and marketed by Engineering Software Research and Development, Inc., located in St. Louis, MO. The current users of Stress Check include Boeing Aircraft Company (on several locations); Piper Aircraft Co., Northrop Grumman Corporation; Cessna Aircraft Co. NASA Johnson Space Center and others.

A collaborative effort was started with Raytheon TI System for an experimental investigation of realtionships between generalized stress intensity factors and failure initiation events at bonded interfaces.

McDonnell Douglas Aerospace St. Louis (now Boeing) funded a project with ESRD for a particular specialization of the material and geometric nonlinear analysis capabilites within the p-version of the finite element method for application to the analysis of cold-worked holes and attachment lugs. This technology was developed at Washington University under AFOSR sponsorship.

2 Technical description

Singular points are those points in a structural component where a reentrant corner occurs (like cracks and V-notches), material properties abruptly change along a free edge, interior points of three (or more) zones of different materials intersect, or an abrupt change in boundary conditions occurs, see Fig. 1.

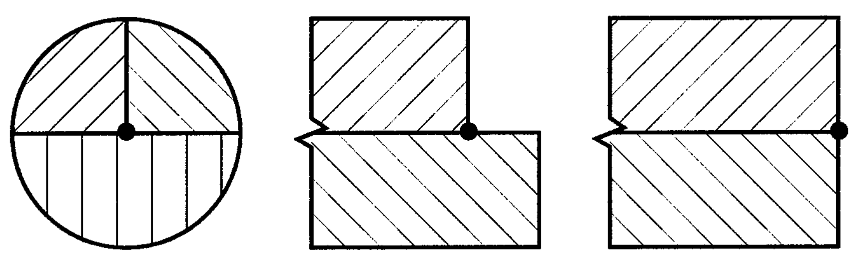

Figure 1: Typical singular points associated with multi-material interfaces.

In the vicinity of these singular points the exact solution of the problem of elasticity is of the form:

$$\vec{u}_{EX} = \sum_{i=1}^{\infty} A_i r^{\lambda_i} \vec{\phi}_i(\theta) \tag{1}$$

where $\vec{u}_{EX}$ is the exact displacement vector function, r and θ are polar coordinates centered on the singular point, $\vec{\phi}_i \stackrel{\text{def}}{=} \{\phi_{i|x}\ \phi_{i|y}\}$ is a piecewise smooth vector function, and A_i are coefficients. Eq. (1) is an *asymptotic expansion* of the exact solution at the singular point.

Within a radius of convergence the exact solution of the problem of elasticity can be written in this form. The exponents λ_i (numbered such that $\lambda_1 \le \lambda_2 \le \lambda_3 \ldots$) and the corresponding functions $\vec{\phi}_i(\theta)$ depend on the material properties and the geometric details at the singular point. These can be determined by solving an eigenvalue problem. Details are given in Section 2.3. A well known example is linear elastic fracture mechanics in two dimensions where $\lambda_1 = 1/2$ and

$$\phi_{1|x} = \frac{1}{2G}\left[\left(\kappa - \frac{1}{2}\right)\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right] \tag{2}$$

$$\phi_{1|y} = \frac{1}{2G}\left[\left(\kappa + \frac{1}{2}\right)\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{3\theta}{2}\right], \tag{3}$$

G is the shear modulus, $\kappa = 3 - 4\nu$ for plane strain, $\kappa = (3-\nu)/(1+\nu)$ for plane stress where ν is Poisson's ratio.

In linear elastic fracture mechanics λ_i and $\vec{\phi}_i$ have been determined by classical methods, and only the stress intensity factor, which is proportional to A_1, has to be determined by numerical means. For details see, for example, [6].

In the general case of multi-material singularities, such as those shown in Fig. 1, not only A_i but also λ_i and $\vec{\phi}_i(\theta)$ have to be determined by numerical means. The elastic stress is infinity in the singular point when $0 < \lambda_1 < 1$ and $A_1 \neq 0$ and/or $0 < \lambda_2 < 1$ and $A_2 \neq 0$, etc. A natural straining mode is the strain state associated with a particular term of the asymptotic expansion, eq. (1). As explained in Section 2.1, the natural straining modes provide a linkage between linear computations and observed failure initiation or failure propagation events.

In heat conduction the asymptotic expansion is analogous to eq. (1):

$$\mathcal{T}_{EX} = \sum_{i=1}^{\infty} A_i r^{\lambda_i} \phi_i(\theta) \tag{4}$$

where $\mathcal{T}_{EX}$ is the exact solution of the heat conduction problem and ϕ_i are piecewise smooth scalar functions. The coefficients A_i are called *flux intensity factors.*

Numerical accuracy is essential because unless the accuracy of the computed data is known it would not be possible to tell whether the working hypothesis is wrong or the numerical errors are too large, or both. In some cases a large error in the working hypothesis is nearly canceled by a similarly large numerical error, leading to false conclusions.

2.1 Basic principles and assumptions

Consider the neighborhood of a singular point. It is assumed in the following that the principles of continuum mechanics remain valid everywhere within the body up to the failure initiation event. The possibility of strongly nonlinear behavior in the neighborhood of singular points is not excluded, however.

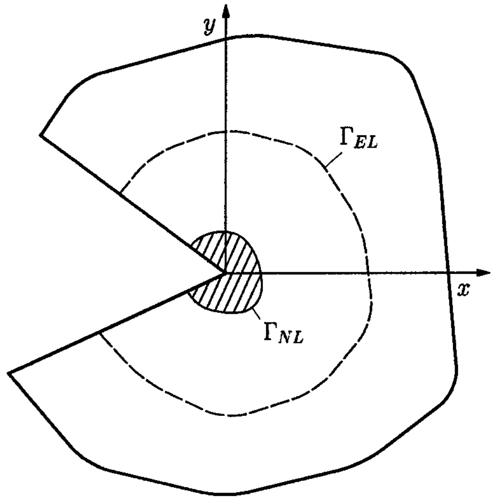

Figure 2: Definition of Γ_{NL} and Γ_{EL}.

Let $\vec{u}_{NL} = \{u_x \ u_y\}_{NL}$ be the solution of the general nonlinear continuum mechanics problem. It is expected that failure initiation will depend on $\vec{u}_{NL}$ (more precisely some functionals computable from $\vec{u}_{NL}$) in the strongly nonlinear region of the singular point bounded by a boundary Γ_{NL}, as shown in Fig. 2. This region is called the *process zone.* Let Γ_{EL} be a curve outside of Γ_{NL}, and let G be an operator which associates the solution $\vec{u}_{NL}$ of the nonlinear problem inside Γ_{NL} with the boundary condition $\vec{g}_{EL}$ specified on Γ_{EL}, that is:

$$G(\vec{g}_{EL}) = \vec{u}_{EX}, \qquad \vec{g}_{EL} = \vec{u}_{NL}|_{\Gamma_{EL}}$$

where $\vec{g}_{EL} = \vec{u}_{NL}|_{\Gamma_{EL}}$ denotes the trace of $\vec{u}_{EX}$ on Γ_{EL}. Denote the exact solution of the linear elastic problem by $\vec{u}_{EL}$. The basic assumptions (which are valid in linear elastic fracture mechanics) are stated in the following:

<u>Assumption A:</u>

Inside of Γ_{EL} the error $G(\vec{u}_{NL}|_{\Gamma_{EL}}) - G(\vec{u}_{EL}|_{\Gamma_{EL}})$ is so small that conclusions based on $G(\vec{u}_{EL}|_{\Gamma_{EL}})$ are sufficiently close to the conclusions based on $G(\vec{u}_{NL}|_{\Gamma_{EL}})$ for practical purposes. This assumption is expected to be valid whenever the nonlinear behavior is confined entirely to some small region inside Γ_{EL}.

Assumption A leads to the important conclusion that failure initiation, which depends on the solution of the nonlinear problem inside of Γ_{NL}, can be determined through the solution of the linear elastic problem, even though all basic assumptions of the linear theory may be violated inside of Γ_{NL}. Consequently failure initiation in the neighborhood of the singular point can be predicted on the basis of the linear theory of elasticity. (This is because $\vec{u}_{EL}$ defines $\vec{u}_{NL}|_{\Gamma_{EL}}$.)

Assumption B:

There exists a physical principle which establishes the relationship between crack initiation and the stress field on the basis of information obtained from the linear solution $\vec{u}_{EL}$ only. Linear elastic fracture mechanics (LEFM) is a typical application of Assumption B.

In general, the linear solution $\vec{u}_{EL}$ is not known, only an approximation to $\vec{u}_{EL}$, which will be denoted by $\vec{u}_{FE}$, is known. Therefore the following assumption is necessary:

Assumption C:

There exist a norm $\|\cdot\|$ such that when $\|\vec{u}_{EL}-\vec{u}_{FE}\|$ is sufficiently small then the physical principle of Assumption B is not sensitive to the replacement of $\vec{u}_{EL}$ with $\vec{u}_{FE}$. Of course, the norm $\|\cdot\|$ is expected to depend on the physical principle of Assumption B, which is material-dependent. If conclusions are to be based on $\vec{u}_{FE}$ then $\vec{u}_{FE}$ has to be close to $\vec{u}_{EL}$ in this particular norm.

Based on these assumptions linear computations can be used for the prediction of failure initiation and failure propagation even though failure processes are highly nonlinear in nature. There are two key elements of failure initiation analysis:

1. A hypothesis concerning the relationship between certain parameters of the stress or strain field and observed failure initiation or crack propagation events.
2. Convincing experimental confirmation that the hypothesis holds independently of variations in geometric attributes, loading and constraints.

It would not sensible to perform experiments without a hypothesis based on the functionals that characterize the stress or strain fields in the neighborhood of critical points and computations cannot provide useful information about the conditions under which failure occurs without experimental data.

For details on the algorithms developed for the computation of the natural straining modes at multi-material interfaces and the generalized stress intensity factors we refer to [12], [7], [8] [10], [11].

Remark 2.1 The assumption that the material is elastic on Γ_{EL} is not essential. Similar considerations apply to nonlinear elasticity and the deformation theory of plasticity. In fact, the methods of LEFM have been extended to the deformation theory of plasticity through the use of the J-integral [2], [5].

In the following the procedures developed for the computation of eigenpairs and their coefficients in heat conduction and elasticity are outlined and illustrated by examples. Additional information can be obtained from the references listed.

2.2 The problem of heat conduction

The index notation is used in the following. For two dimensional problems the range of the indices is 2, and for three dimensional problems the range is 3. The summation convention is used. The formulation of the mathematical is described for the problem of heat conduction.

The heat balance equation is analogous to the equilibrium equation in elasticity:

$$-q_{i,i} + Q = 0 \tag{5}$$

where: q_i is the flux vector (in W/m^2 units) and Q is the rate of heat generation per unit volume (in W/m^3 units). Multiplying eq. (5) by the scalar function v, integrating and applying Green's lemma, we have:

$$-\int_\Omega q_i v_{,i}\, dV = \int_\Omega Qv\, dV - \int_\Gamma q_i n_i v\, dS.$$

Using Fourier's law of heat conduction (which is analogous to Hooke's law):

$$q_i = -K_{ij}\mathcal{T}_{,j}$$

where K_{ij} is assumed to be independent of $\mathcal{T}$, we have for all $v \in H^1(\Omega)$:

$$\int_\Omega K_{ij} v_{,i} \mathcal{T}_{,j}\, dV = \int_\Omega Qv\, dV - \int_\Gamma q_i n_i v\, dS. \tag{6}$$

This is analogous to the principle of virtual work in elasticity. Alternatively, the exact solution of the heat conduction problem is the minimizer of the functional π:

$$\pi(\mathcal{T}) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega K_{ij}\mathcal{T}_{,i}\mathcal{T}_{,j}\, dV - \int_\Omega Q\mathcal{T}\, dV + \int_\Gamma q_i n_i \mathcal{T}\, dS \tag{7}$$

on the set of the admissible temperature fields. This is analogous to the principle of minimum potential energy in elasticity.

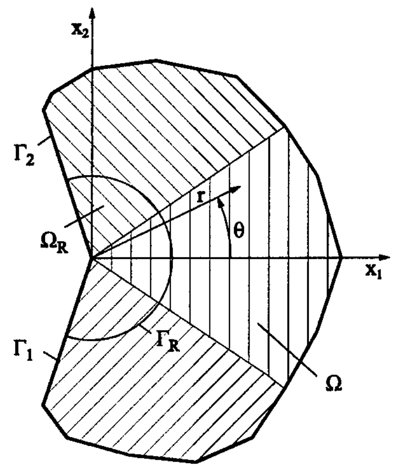

Figure 3: Typical singular point in 2D. Definition of Ω_R.

2.3 Computation of λ_i and $\phi_i(\theta)$

The procedure for the numerical approximation of the eigenpairs λ_i and $\phi_i(\theta)$ is briefly described for the problem of heat conduction. Additional details are available in [8].

Consider a small neighborhood of a singular point bounded by a circle of radius R and denoted by Ω_R, as shown in Fig. 3. Let $Q = 0$ on $\Omega_R \times t$ where t is the thickness, assumed to be constant, and $u = 0$ on Γ_1 and Γ_2 and seek solutions of the form:

$$\mathcal{T} = r^\lambda \phi(\theta).$$

Noting that

$$\begin{aligned} \mathcal{T}_{,1} &= \frac{\partial \mathcal{T}}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial \mathcal{T}}{\partial \theta}\frac{\partial \theta}{\partial x_1} = \frac{\partial \mathcal{T}}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \mathcal{T}}{\partial \theta}\sin\theta \\ \mathcal{T}_{,2} &= \frac{\partial \mathcal{T}}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial \mathcal{T}}{\partial \theta}\frac{\partial \theta}{\partial x_2} = \frac{\partial \mathcal{T}}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial \mathcal{T}}{\partial \theta}\cos\theta \end{aligned}$$

and $n_i = \{\cos\theta \;\; \sin\theta\}$, we write:

$$\begin{aligned} -\int_{\Gamma_R} q_i n_i v \, dS &= \lambda \int_{\Gamma_R} (K_{11}\cos^2\theta + K_{12}\sin 2\theta + K_{22}\sin^2\theta)\mathcal{T} v \, t d\theta \\ & \int_{\Gamma_R} ((K_{22} - K_{11})\sin\theta\cos\theta + K_{12}\cos 2\theta)\frac{\partial \mathcal{T}}{\partial \theta} v \, t d\theta. \end{aligned}$$

We are now in a position to apply eq. (6) which results in:

$$(\mathcal{B}(\mathcal{T}, v) - \mathcal{N}(\mathcal{T}, v)) - \lambda \mathcal{M}(\mathcal{T}, v) = 0 \tag{8}$$

where:

$$
\begin{aligned}
\mathcal{B}(\mathcal{T},v) &\stackrel{\text{def}}{=} \int_{\Omega_R} K_{ij} v_{,i} \mathcal{T}_{,j}\, dV \\
\mathcal{M}(\mathcal{T},v) &\stackrel{\text{def}}{=} \int_{\Gamma_R} (K_{11}\cos^2\theta + K_{12}\sin 2\theta + K_{22}\sin^2\theta)\mathcal{T} v\, t d\theta \\
\mathcal{N}(\mathcal{T},v) &\stackrel{\text{def}}{=} \int_{\Gamma_R} ((K_{22}-K_{11})\sin\theta\cos\theta + K_{12}\cos 2\theta)\,\frac{\partial\mathcal{T}}{\partial\theta} v\, t d\theta
\end{aligned}
$$

We seek $\lambda > 0$, $\mathcal{T} \in H^1(\Omega_R)$, $\partial\mathcal{T}/\partial\theta \in L_2(\Gamma_R)$ such that eq. (8) is satisfied for all $v \in H^1(\Omega_R)$. This non-symmetric eigenvalue problem can be solved numerically.

Remark 2.2 In the case of isotropic materials $\mathcal{N}(\mathcal{T},v) = 0$ hence the eigenvalue problem is symmetric. In the special case $K_{ij} = K\delta_{ij}$, where K is constant, we have:

$$\int_{\Omega_R} \mathcal{T}_{,i} v_{,i}\, dV - \lambda \int_{\Gamma_R} \mathcal{T} v t\, d\theta = 0.$$

The corresponding strong form is:

$$\Delta\mathcal{T} = 0$$

subject to the boundary conditions

$$u = 0 \text{ on } \Gamma_1,\, \Gamma_2;\;\; \frac{\partial\mathcal{T}}{\partial r} = \frac{\lambda}{R}\mathcal{T} \text{ on } \Gamma_R.$$

Remark 2.3 $\mathcal{N}(\mathcal{T},v)$ is non-symmetric, nevertheless all eigenvalues are real.

Remark 2.4 λ_i do not converge monotonically. No minimum principle is involved.

Remark 2.5 The formulation is analogous for any set of homogeneous boundary conditions on Γ_1, Γ_2.

The Steklov method on Ω_R requires hp-meshing. This is because the rate of convergence of p-extensions is low due to the presence of the singular point. Therefore it is better to use a modified domain $\Omega_R^\star$ shown in Fig. 4. Using $\Omega_R^\star$ is called the *modified Steklov method.* Detailed discussions on the procedures are available in references [8], [12], [7].

Remark 2.6 The modified Steklov method will find not only those eigenfunctions which lie in $H^1(\Omega_R)$ but also eigenfunctions corresponding to negative values of λ.

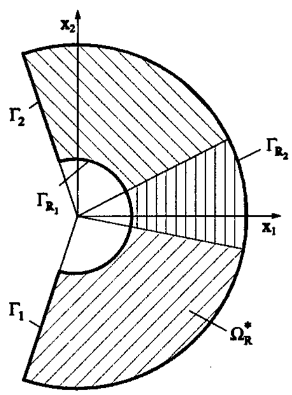

Figure 4: The domain $\Omega_R^\star$.

2.4 Extraction of the flux intensity factors

Analogously to eq. (1) the temperature field in the neighborhood of singular points is of the form:

$$\mathcal{T} = \sum_{i=1}^{\infty} A_i r^{\lambda_i} \phi_i(\theta) \tag{9}$$

where: A_i represents the *generalized flux intensity factors*, λ_i and ϕ_i are the eigenpairs characterized by the topological details at the singular point and the material properties.

Once the finite element solution is available, the flux intensity factors can be computed from the finite element solution by the contour integral method, the complementary energy method, and the L_2 projection method. These are briefly described in the following.

2.4.1 The contour integral method

The contour integral method is a procedure which utilizes the orthogonality of eigenfunctions and the fact that if λ_i is an eigenfunction then $-\lambda_i$ is also an eigenfunction. For details we refer to [1], [6].

2.4.2 The complementary energy method

Define:

$$\Pi_c(q_i) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\Omega_R} C_{ij} q_i q_j \, dV - \int_{\Gamma_R} q_i n_i \mathcal{T}_{FE} t \, d\theta$$

where C_{ij} is the inverse of K_{ij}; q_i satisfies the heat balance equation: $-q_{i,i} + Q = 0$ (and prescribed homogeneous flux boundary conditions); $\mathcal{T}_{FE}$ is the temperature computed by the finite element method.

Letting $q_i = -K_{ij}\mathcal{T}_{,j}$ and minimizing the complementary energy functional $\Pi_c(q_i)$ with respect to A_i, yields approximations to the generalized flux intensity factors.

2.4.3 The L_2 projection method

This method involves the projection of u_{FE} onto the space spanned by the eigenfunctions. Specifically, A_i are determined from the condition:

$$\int_{\Omega_R} \left(u_{FE} - \sum_{i=1}^{n} A_i r^{\lambda_i} \phi_i(\theta) \right) r^{\lambda_j} \phi_j(\theta)\, r dr d\theta = 0 \qquad j = 1, 2, \ldots, n$$

which yields n equations for A_i, $i = 1, 2, \ldots, n$.

2.5 Example: The slit domain problem

Consider the problem $\Delta\mathcal{T} = 0$ on a unit circle slit along the axis x_2 with the boundary conditions

$$\mathcal{T} = 0 \text{ on } \Gamma_1, \quad q_2 = 0 \text{ on } \Gamma_2, \quad q_n = q_i n_i = x_2 \text{ on } \Gamma_3.$$

In this case: $\lambda_1 = 0.25$, $\lambda_2 = 0.75$, $\lambda_3 = 1.25$. This is a very challenging problem from the point of view of numerical approximation by the finite element method, owing to the fact that the lowest eigenvalue is 1/4, hence the theoretical rate of convergence[1] of the p-version is 1/4 and the theoretical rate of convergence of the h-version is 1/8. The error is most effectively controlled by hp-extension, utilizing geometrically graded meshes [6].

The finite element mesh, consisting of 12 elements, and the temperature distribution at $p = 8$ (trunk space) is shown in Fig. 6. The p-convergence of the energy and A_1, A_2, A_3 are shown in Table 1. The extrapolated values are shown in the last row. It is seen that the rate of convergence is close to the theoretical rate of 0.25.

2.6 Example: Two-material internal interface problem

Consider the problem $\Delta\mathcal{T} = 0$ on a unit circle. On the first quadrant has the material properties are $K_{11} = K_{22} = 10.0$, $K_{12} = 0$ on the other three quadrants $K_{11} = K_{22} = 1.0$, $K_{12} = 0$. On the boundary $q_n = f(\theta)$ where $f(\theta)$ is a function given by Oh and Babuška in [4].

[1]The theoretical rate of convergence is given by β in the a priori estimate $\|u_{EX} - u_{FE}\|_{E(\Omega)} \leq kN^{-\beta}$ were k is a positive constant and N is the number of degrees of freedom. See [6].

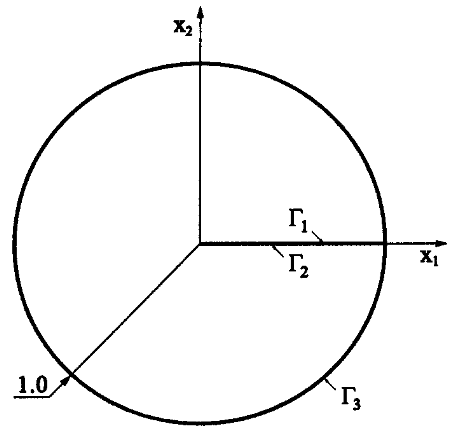

Figure 5: The slit domain problem.

2.7 The problem of thermo-elasticity

The equations of equilibrium are:

$$\sigma_{ij,j} + F_i = 0 \tag{10}$$

Multiplying (10) by a test function v_i and applying Green's lemma, the generic form of the principle of virtual work is obtained:

$$\int_\Omega \sigma_{ij}^{(u)} \epsilon_{ij}^{(v)}\, dV = \int_\Omega F_i v_i\, dV + \int_\Gamma T_i v_i\, dS$$

where

$$\epsilon_{ij}^{(v)} \stackrel{\text{def}}{=} \frac{1}{2}(v_{i,j} + v_{j,i})$$

is the small strain tensor corresponding to the virtual displacement v_i and

$$\sigma_{ij}^{(u)} = E_{ijkl}(\epsilon_{kl}^{(u)} - \alpha_{kl}\mathcal{T})$$

is the stress tensor corresponding to u_i; α_{kl} represents the coefficients of thermal expansion and $\mathcal{T}$ is the temperature.

Remark 2.7 The temperature field $\mathcal{T}$ is continuous but $\epsilon_{kl}^{(u)}$ does not have to be continuous.

Remark 2.8 E_{ijkl} may be a function of $\mathcal{T}$.

An alternative formulation is the principle of minimum potential energy:

$$\begin{aligned}\Pi(u) \ &\stackrel{\text{def}}{=}\ \frac{1}{2}\int_\Omega E_{ijkl}(\epsilon_{ij}^{(u)} - \alpha_{ij}\mathcal{T})(\epsilon_{kl}^{(u)} - \alpha_{kl}\mathcal{T})\, dV \\ &\quad - \int_\Omega F_i u_i\, dV - \int_\Gamma T_i u_i\, dS\end{aligned}$$

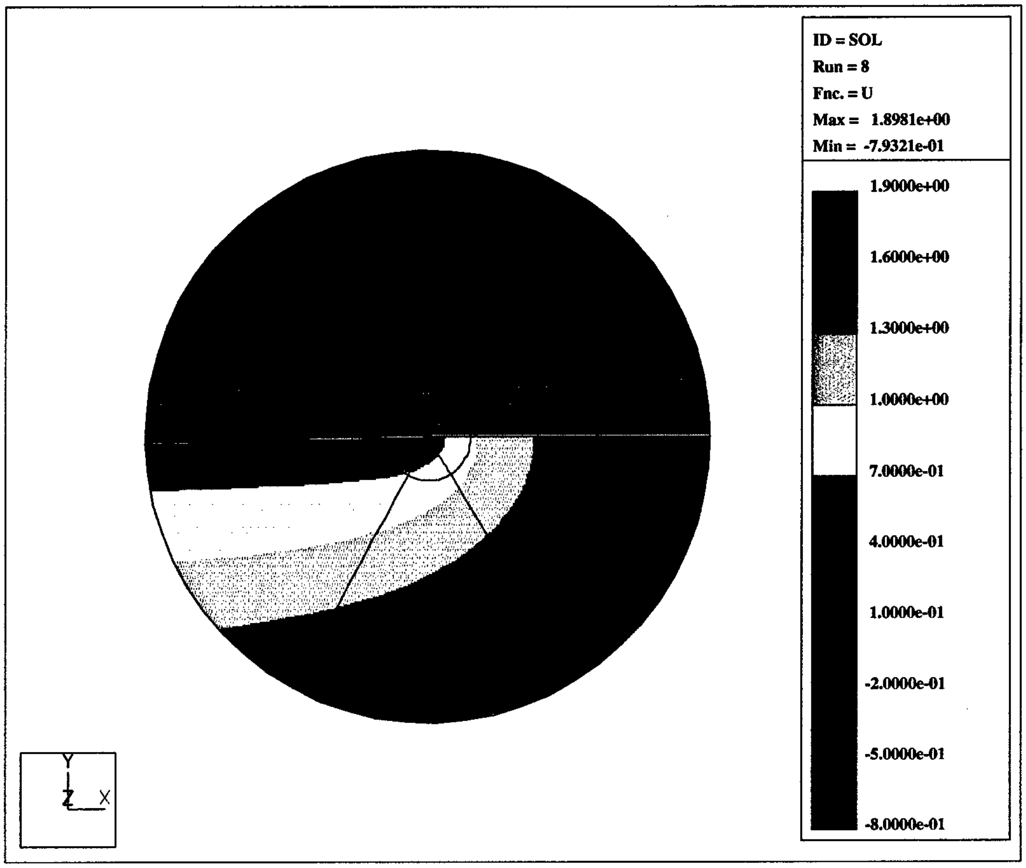

Figure 6: The slit domain problem: Mesh layout and contour plot of $\mathcal{T}$, p=8, trunk space.

The exact solution of the problem of elasticity minimizes the potential energy:

$$\Pi(u_{EX}) = \min_{u \in \tilde{E}(\Omega)} \Pi(u) \tag{11}$$

where Π is the potential energy and $\tilde{E}(\Omega)$ represents the space of admissible functions. In the case of isotropic materials the Euler equations are:

$$G\nabla^2 u_i + (\lambda + G)(u_{j,j})_{,i} = -F_i + \beta \mathcal{T}_{,i}$$

where

$$\beta \stackrel{\text{def}}{=} (3\lambda + 2G)\alpha.$$

where λ is the Lamé parameter, G is the shear modulus and α is the coefficient of thermal expansion.

Table 1: p-Convergence of the energy and A_1, A_2, A_3. Twelve-element mesh. The extrapolated values are shown in the last row.

p	N	Potential Energy	Rate of Conv.	Est.'d Rel. Err.	A_1	A_2	A_3
1	12	-1.977649	0.00	35.62	0.794380	-0.901532	0.373088
2	36	-2.141858	0.39	23.31	1.004779	-0.953011	0.457475
3	66	-2.178293	0.29	19.56	1.110997	-0.964365	0.455113
4	108	-2.196471	0.24	17.39	1.168927	-0.968574	0.453430
5	162	-2.208072	0.23	15.85	1.203440	-0.969836	0.452681
6	228	-2.216175	0.22	14.68	1.226336	-0.970035	0.452696
7	306	-2.222206	0.22	13.74	1.243025	-0.970053	0.452696
8	396	-2.226874	0.22	12.97	1.255925	-0.970047	0.452703
∞	∞	-2.264978	0.25	0	1.359910	-0.970047	0.452696

2.8 Computation of thermal stress intensity factors

Outline of the solution algorithm.

1. Assuming that the displacement field at the singular point is of the form

$$u_i = r^{\mu}\Phi_i(\theta)$$

compute the eigenpairs μ_j, $\Phi_{ij}(\theta)$ $j = 1, 2, \ldots$ using the modified Steklov method where the second index on Φ_{ij} represents the ordinal number of the eigenfunction. This involves the solution of a non-symmetric eigenvalue problem of the form:

$$(\mathcal{B}(u_i, v_i) - \mathcal{N}(u_i, v_i)) - \mu\mathcal{M}(u_i, v_i) = 0$$

The eigenvalues are usually complex. Both the real and imaginary parts must be considered.

2. Construct the homogeneous part of the statically admissible stress field $\sigma_{ij}^{(H)}$ from

$$u_i = \sum_{j=1}^{N} C_j r^{\mu_j}\Phi_{ij}(\theta)$$

where C_j represents the generalized stress intensity factors. Specifically,

$$\sigma_{ij}^{(H)} = \frac{1}{2}E_{ijkl}(u_{k,l} + u_{l,k}).$$

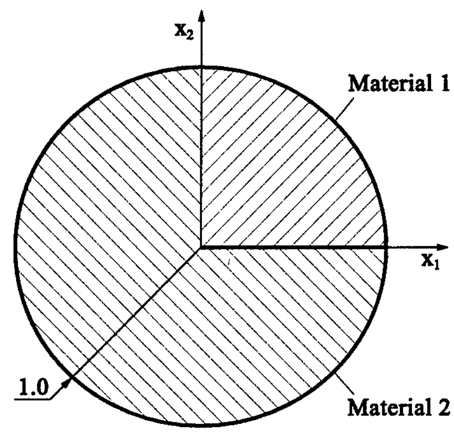

Figure 7: Problem definition: Two-material internal interface

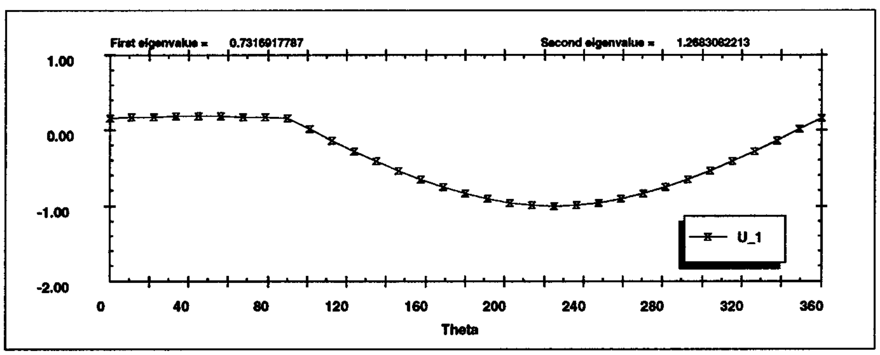

Figure 8: Two-material internal interface problem: The first four eigenfunction $\phi_1(\theta)$, corresponding to $\lambda_1 = 0.7317$.

3. The particular solution $\sigma_{ij}^{(P)}$ satisfies

$$\sigma_{ij,j}^{(P)} = E_{ijkl}\alpha_{kl}\mathcal{T}_{,j}$$

on Ω_R and the homogeneous traction boundary conditions on Γ_1, Γ_2. In general it is difficult to construct $\sigma_{ij}^{(P)}$. Note, however, that $\sigma_{ij}^{(P)}$ is of the order r^λ where $\lambda \stackrel{\text{def}}{=} \lambda_{\text{min}} > 0$ is the smallest eigenvalue of the thermal problem. On the other hand, $\sigma_{ij}^{(H)}$ is of order $r^{\mu-1}$ where $\mu \stackrel{\text{def}}{=} \mu_{\text{min}} > 0$ is the smallest eigenvalue of the elasticity problem.

4. Construct the complementary energy functional on Ω_R:

$$\Pi_c(\sigma_{ij}) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\Omega_R} C_{ijkl}\sigma_{ij}\sigma_{kl}\, dV - \int_{\Gamma_R} \sigma_{ij} n_j u_i^{(FE)} t\, ds$$

$$
\begin{aligned}
= & \underbrace{\frac{1}{2}\int_{\Omega_R} C_{ijkl}\sigma_{ij}^{(H)}\sigma_{kl}^{(H)}\,dV}_{O(R^{2\mu})} + \underbrace{\int_{\Omega_R} C_{ijkl}\sigma_{ij}^{(H)}\sigma_{kl}^{(P)}\,dV}_{O(R^{\lambda+\mu+1})} \\
+ & \underbrace{\frac{1}{2}\int_{\Omega_R} C_{ijkl}\sigma_{ij}^{(P)}\sigma_{kl}^{(P)}\,dV}_{O(R^{2\lambda+2})} - \int_{\Gamma_R} \sigma_{ij}n_j u_i^{(FE)} t\,ds
\end{aligned}
$$

This indicates that if R is sufficiently small then $\sigma_{ij,j}^{(P)}$ may be neglected.

5. Compute the thermal stress intensity factors $C_j^{(k)}$ by minimizing $\Pi_c(\sigma_{ij}^{(H)})$ on Γ_{R_k} for a sequence of decreasing radii R_k, $k = 1, 2, \ldots, n$.

6. Use Richardson extrapolation to find

$$C_j = \lim_{R_k \to 0} C_j^{(k)}.$$

2.9 Example: Cracked panel subject to thermal load

A centrally cracked panel is subjected to $\mathcal{T} = 100$ at the perimeter; $\mathcal{T} = 0$ on the crack faces. $K_{11} = K_{22} = 1.0$; $K_{12} = 0$; $E = 1.0$, $\nu = 0.3$, $\alpha = 0.01$, plane strain. $L = w = 50.0$, $a = 1$ is shown in Fig. 9. The p-convergence of the first thermal stress intensity factor A_1 computed with Richardson extrapolation. is shown in Fig. 11 and the results reported by Yosibash in [10] using direct computation are given in Table 2.

Table 2: Results reported by Yosibash using direct computation

R/a	0.5	0.1	0.01	0.001	0.0006
C_1	0.4908	0.3781	0.3528	0.3491	0.3481

2.10 Sources of errors

The methods used for computing the finite element solution, the eigenpairs and the generalized flux intensity factors, are approximate methods, hence certain errors are incurred:

1. In numerical work the asymptotic expansion is truncated to a few terms.

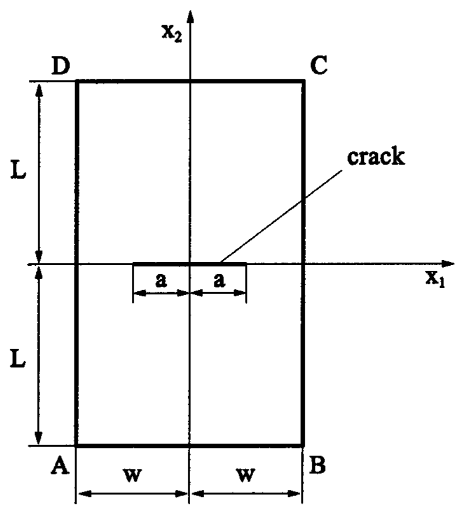

Figure 9: The cracked panel problem.

2. The eigenpairs are only approximations. Therefore q_i do not satisfy the heat balance equation exactly.

3. The prescribed temperature on Γ_R is approximate, computed from the finite element solution.

Nevertheless, numerical experience has indicated that this method of extraction is superconvergent well beyond the range of accuracy required in engineering work [8].

2.11 Stability problems

This topic is of substantial interest in aerospace engineering because the sizes of compression members are determined primarily by stability considerations. It is also of great importance in micromechanics where the strength of the composite materials is often determined by the buckling of fibers. The formulation and investigation of stability problems in the fully three-dimensional setting was undertaken. Details are available in a doctoral dissertation [3]. A brief outline is presented in the following. Define:

$$\sigma^0_{ij} \stackrel{\text{def}}{=} \lambda \sigma^\star_{ij}$$

where $\sigma^\star_{ij}$ is the pre-buckling stress state. We are interested in finding $\bar{u}_i \in \tilde{E}(\Omega)$ such that:

$$\int_\Omega C_{ijkl}\bar{u}_{i,j}v_{k,l}\,dV + \lambda \int_\Omega \sigma^\star_{ij}\bar{u}_{\alpha,i}v_{\alpha,j}\,dV =$$

$$\int_\Omega \bar{F}_i v_i\,dV + \int_{\Gamma_T} \bar{T}_i v_i\,dS + \int_\Omega \bar{\mathcal{T}} C_{ijkl}\alpha_{kl}\bar{v}_{i,j}\,dV$$

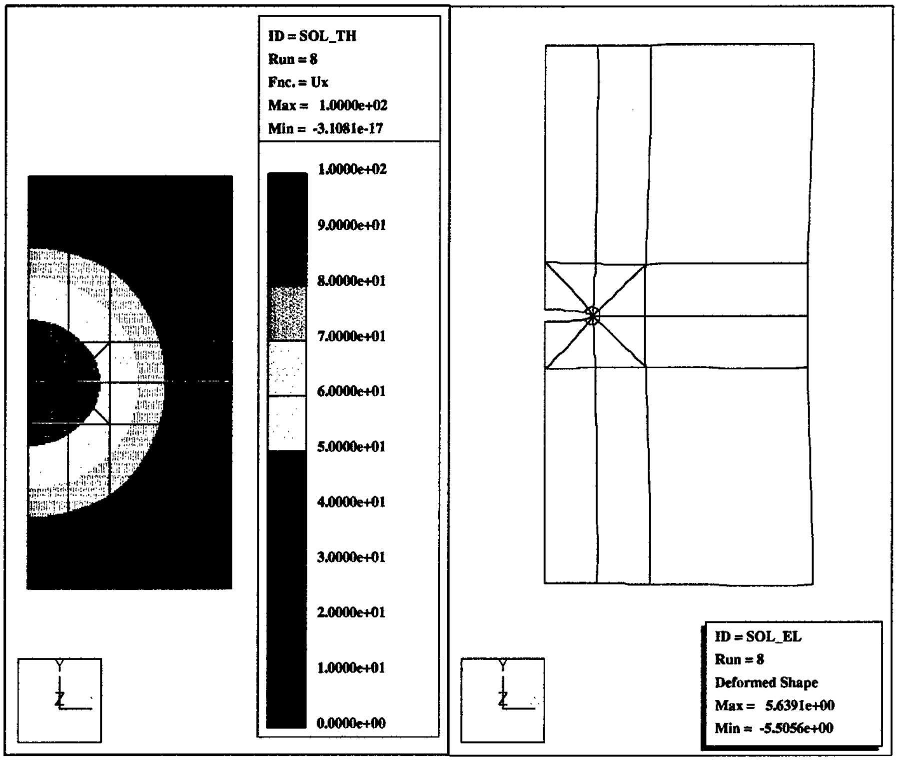

Figure 10: The mesh, temperature distribution and the resulting deformation.

for all $v_i \in \overset{\circ}{E}(\Omega)$. Note: σ^*_{ij} must be such that:

$$\left| \int_\Omega (C_{ijkl} + \lambda \sigma^*_{jl} \delta_{ik}) \bar{u}_{i,j} \bar{u}_{k,l} \, dV \right| < \infty$$

for all $\bar{u}_i \in \tilde{E}(\Omega)$. The set of λ for which a solution exists is the resolvent set. The complement is the spectrum. The spectrum may be point, continuous or residual.

The work done by the initial stress σ^0_{ij} due to the product terms of the Green-Lagrange strain tensor is incorporated in the strain energy:

$$U(\bar{u}_i) \stackrel{\text{def}}{=} \frac{1}{2} \int_\Omega C_{ijkl} \bar{\epsilon}_{ij} \bar{\epsilon}_{kl} \, dV + \underbrace{\frac{1}{2} \int_\Omega \sigma^0_{ij} \bar{u}_{\alpha,i} \bar{u}_{\alpha,j} \, dV.}_{\text{work of } \sigma^0_{ij}}$$

where $\bar{u}_i$ is a small increment of displacement and $\bar{\epsilon}_{ij}$ is the small strain.

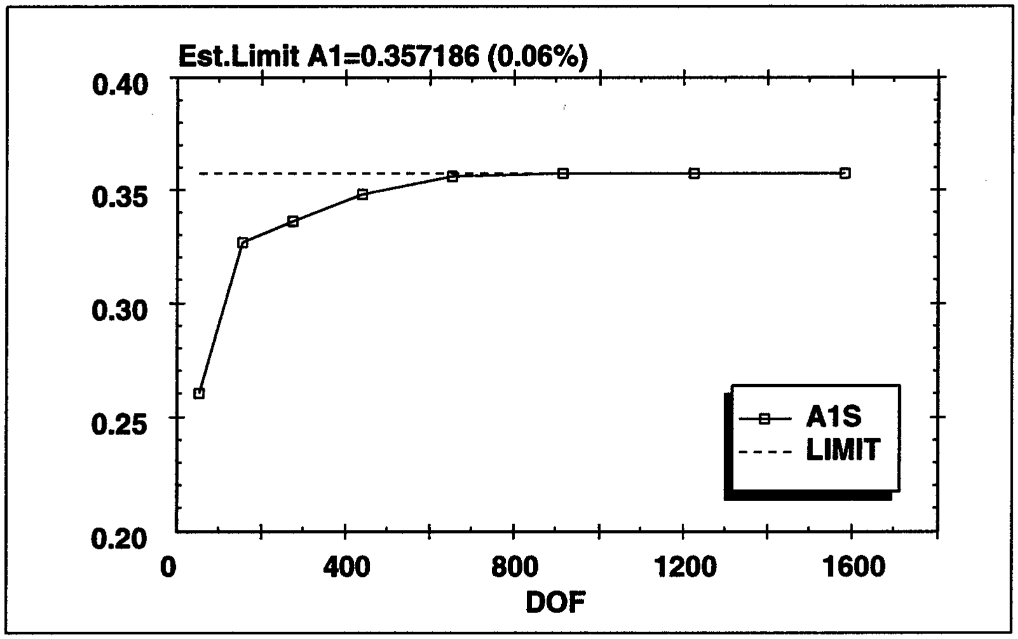

Figure 11: p-Convergence of the first thermal stress intensity factor C_1 computed with Richardson extrapolation.

Biot (1938) and Prager (1947) proposed a different formulation. Their definition is:

$$U(\bar{u}_i) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega C_{ijkl}\bar{\epsilon}_{ij}\bar{\epsilon}_{kl}\,dV + \frac{1}{2}\int_\Omega \sigma^0_{ij}(\bar{u}_{\alpha,i}\bar{u}_{\alpha,j} - \bar{\epsilon}_{\alpha i}\bar{\epsilon}_{\alpha j})\,dV$$

The potential energy is:

$$\Pi(\bar{u}_i) \stackrel{\text{def}}{=} U(\bar{u}_i) - \int_\Omega \bar{F}_i\bar{u}_i\,dV - \int_{\partial\Omega_T}\bar{T}_i\bar{u}_i\,dS + \int_\Omega \bar{\mathcal{T}}C_{ijkl}\alpha_{kl}\bar{u}_{i,j}\,dV$$

We seek $\bar{u}_i \in \tilde{E}(\Omega)$ such that the potential energy is stationary:

$$\delta\Pi(\bar{u}_i) \stackrel{\text{def}}{=} \left(\frac{\partial\Pi(\bar{u}_i + \varepsilon v_i)}{\partial\varepsilon}\right)_{\varepsilon=0} = 0 \qquad v \in \mathring{E}(\Omega).$$

The principle of virtual work in the case of initial stress:

$$\int_\Omega C_{ijkl}\bar{u}_{i,j}v_{k,l}\,dV + \int_\Omega \sigma^0_{ij}\bar{u}_{\alpha,i}v_{\alpha,j}\,dV =$$
$$\int_\Omega \bar{F}_i v_i\,dV + \int_{\partial\Omega_T}\bar{T}_i v_i\,dS + \int_\Omega \bar{\mathcal{T}}C_{ijkl}\alpha_{kl}\bar{v}_{i,j}\,dV$$

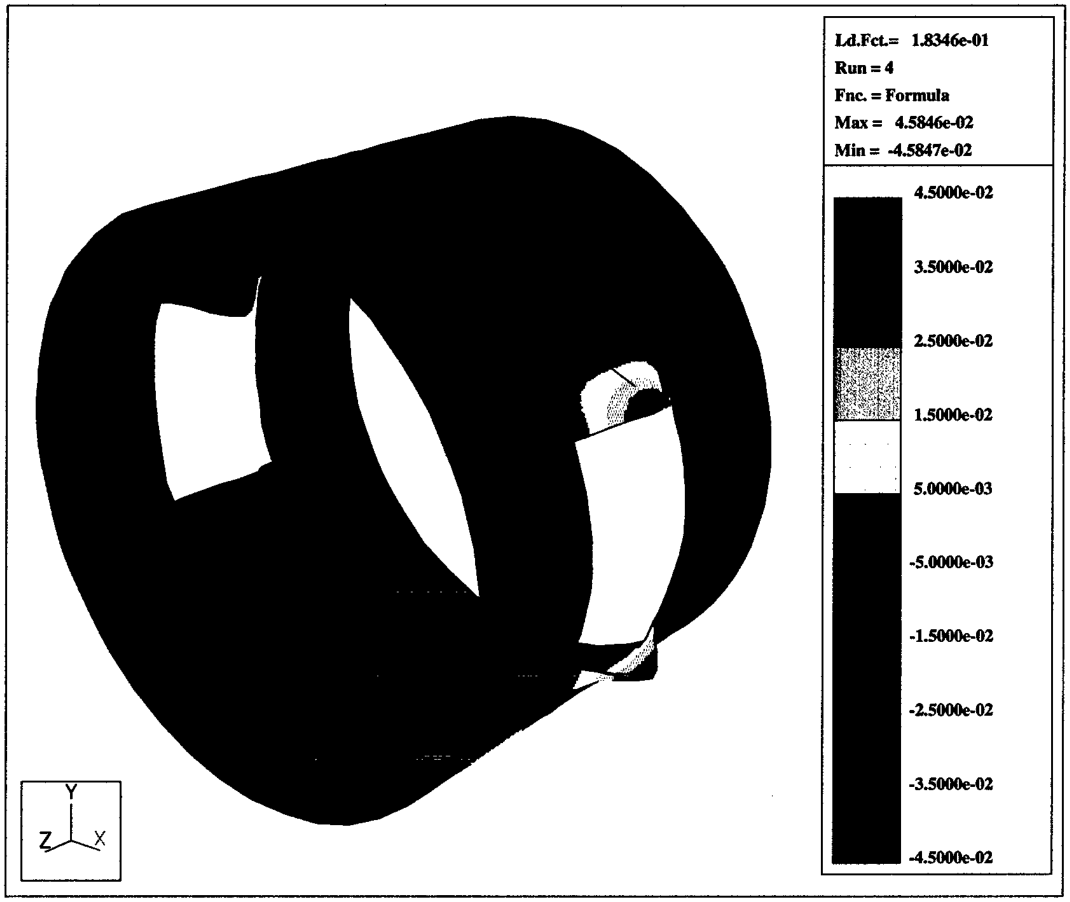

Figure 12: Lockheed test problem 2: The function $u_n \stackrel{\text{def}}{=} u_x n_x + u_y n_y$

for all $v_i \in \overset{\circ}{E}(\Omega)$.

Although two mathematical models of the general theory of elastic stability exist in the classical literature, some fundamental questions concerning the existence of a solution, the properties of the spectrum, and their relationship to loss of stability had not been investigated previously. Two working hypotheses were advanced:

1. The spectrum is a point spectrum, hence it is meaningful to consider the lowest nonzero eigenvalue as an indicator of the onset of instability;

2. The minimal eigenvalue of the finite dimensional problem converges to its infinite dimensional counterpart as the finite element space is enlarged (i.e., the degrees of freedom are increased).

Both classical formulations were implemented in fully three-dimensional setting so that numerical experiments could performed. For thin structures the results closely matched the classical results. It was found that the two classical models yield virtually identical results.

In a parallel investigation the first working hypothesis was proven by Professors Manil Suri and Ivo Babuska. At present it is not known whether the second hypothesis can be proven, but the available numerical results have not contradicted it.

The relationship between the limits of elastic stability estimated by the use of linear models and incremental models was investigated. It was found that for conservative loads a close relationship exists but the treatment of follower loads through the solution of linear eigenvalue problems does not appear possible, with the exception of very special cases, such as the buckling of circular rings. See, for example, [9].

The problem of modeling the elastic buckling of fibers in fiber-matrix composites was investigated. A model problem has been solved. For the investigation of the stability of a large number of fibers the use of periodic boundary conditions is necessary. Implementation of periodic boundary conditions and further investigation of the stability of fibers is being planned for 1998.

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