REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (*Leave blank*)

2. REPORT DATE
30 January 1995

3. REPORT TYPE AND DATES COVERED
Final Technical Report 1 Aug91-31 Dec94

4. TITLE AND SUBTITLE
Control of Nonlinear Systems

5. FUNDING NUMBERS
AFOSR-91-0346

6. AUTHOR(S)
Eduardo D. Sontag

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Rutgers The State University of New Jersey
New Brunswick, NJ 08903

8. PERFORMING ORGANIZATION REPORT NUMBER
AFOSR-TR- 95-0186

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AFOSR/NM, 110 Duncan Ave, Suite B115
Bolling AFB, DC 20332-0001

10. SPONSORING / MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

DTIC
ELECTE
MAR 30 1995
S B D

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release distribution unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

This report summarizes the accomplishments of the grant. Covered topics include stablization of nonlinear systems, control of saturated-actuator systems, discrete-time accessibility, input/output stability, path planning, and other related areas of research.

14. SUBJECT TERMS

15. NUMBER OF PAGES
22

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT
Unclassified

18. SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

19. SECURITY CLASSIFICATION OF ABSTRACT
Unclassified

20. LIMITATION OF ABSTRACT

CONTROL OF NONLINEAR SYSTEMS, AFOSR-91-0346

Final Technical Report, 1 Oct 1991 to 31 Dec 1994

Eduardo D. Sontag
SYCON – Rutgers Center for Systems and Control
Department of Mathematics, Rutgers University
New Brunswick, NJ 08903

ABSTRACT

This report summarizes accomplishments under the above grant, on several topics related to the control of nonlinear systems, including the following: State-Space Stabilization. Input/Output Stabilization. Partial-State and Set Stability. Classical Operator-Theoretic Stability Notions. Control-Lyapunov Functions: Constrained Control-Value Sets and Parametric Versions; Nonsmooth Case. Numerical Methods for Steering. Discrete-Time Control and Dynamical Systems. I/O Equations and Identification Questions.

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1 Summary

The PI feels that the work under the grant achieved great progress in settling important open questions (for instance, the complete characterization of the "input to state stability" property), as well as in making substantial advances towards the solution of many other problems. Besides the publication record, evidence of success is provided by the large number of researchers who are now pursuing applications (including in some cases numerical implementations of algorithms) as well as theoretical extensions of the results of work performed under this grant. Formal recognition of the work reported here includes several invitations to give plenary talks at major conferences, as well as an invited presentation at the 1994 International Congress of Mathematics, and the elevation of the PI to a Fellow of the Institute of Electronic and Electrical Engineers.

The topics to be discussed are all interrelated; the common denominator is the focus on nonlinear control. They can be organized roughly into these broad categories:

- Stabilization
  - State-Space Stabilization
  - Input/Output Stabilization; Feedback Equivalence
  - Partial-State and Set Stability
  - Classical Operator-Theoretic Stability Notions
- Control-Lyapunov Functions
  - Constrained Control-Value Sets and Parametric Versions
  - Nonsmooth Case
- Numerical Methods for Steering
- Other Areas
  - Discrete-Time Control and Dynamical Systems
  - I/O Equations and Identification Questions

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The rest of this Introduction will briefly outline the above areas. More details are provided in later sections. Most of the discussion in this report will be informal, with references to the PI's publications for detailed technical points.

Stabilization

The main objective of control is to modify the behavior of a dynamical system, typically with the purpose of regulating certain variables or tracking desired signals. Often, *stability* of the closed-loop system either is an explicit requirement or else the problem can be recast in a form that involves stabilization (e.g., of an error signal). For linear systems, the associated problems can now be treated fairly satisfactorily, but in the nonlinear case the area is still far from being settled. In fact, both of the late 1980s reports 'Challenges to control: A collective view" and "Future Directions in Control Theory: A Mathematical Perspective" identified the problem of

stabilization of finite-dimensional deterministic systems as basically the most important open problem in nonlinear control.

One of the main goals of this work was to deal with issues of stabilization in its two variants: *state-space* and *input/state (ISS)* (or more generally *input/output*) stability. The former term refers to the asymptotic stability of equilibria in the absence of (or subject to only *small*) external "disturbance" inputs. On the other hand, the latter term refers to various mathematical definitions all of which focus on bounds on state trajectories expressed in terms of bounds on forcing functions. Mathematically, state-space stabilization relates to classical dynamical systems ideas; in many practical situations, it can be achieved by means of linearization under feedback (Brockett, Hunt, Meyer, Su, Jakubczyk, Respondek), via control-Lyapunov ideas (Artstein, Jurdjevic-Quinn, Tsinias), via the study of zero-dynamics (Byrnes and Isidori), or, for more local studies, through Center Manifold Techniques (Abed, Aeyels, Bacciotti, Boothby, Marino, Crouch, and many others). Input/output stability, in contrast, has classically had a more operator-theoretic flavor and developed independently, often using techniques based on small-gain arguments (Desoer, Sandberg, Vidyasagar, Zames, and others).

Based on linear systems intuition, where all notions coincide, it is perhaps surprising that state-space and i/o stability are not automatically related. Even for feedback linearizable systems, this relation is more subtle than might appear: if one first linearizes a system and then stabilizes the equivalent linearization, in terms of the original system one does *not* in general obtain a closed-loop system that is input/output stable in any reasonable sense. However, and this has been the point of some of the PI's research starting with a paper in the 1989 Transactions in Automatic Control, it is always possible to make a choice of a –usually different– feedback law that achieves such stability, in the linearizable case as well as for all other smoothly stabilizable systems.

Recent work of the PI and his students have dealt with versions of these results for robust stability, as well as design techniques that apply to output stabilization problems and to the construction of observers. The unifying framework is that of stabilization (both in the state and i/o sense) with respect to arbitrary (typically noncompact) invariant sets. In this context, Lyapunov-like characterizations are very useful, and form the basis of applications of the PI's work in robust and adaptive control (e.g., in recent work by Praly and Teel, Kokotovic and Freeman, Tsinias, and others).

One of the main achievements of the project was in deriving a new converse Lyapunov Theorem in for robust stability ([**11**], with former students Lin and Wang) and in the application of this to obtaining complete characterizations of the ISS property. In the to-appear paper [**10**] with Wang, we were able to show that the ISS property is in fact equivalent to the existence of a nontrivial nonlinear stability margin, as well as to a natural characterization in terms of dissipation inequalities (connecting this work to so-called "nonlinear H^∞" work) and, also, solving an important and longstanding open problem, that ISS stability is equivalent to the existence of an ISS-Lyapunov function.

Control-Lyapunov Functions

A basic problem in feedback control theory –which also arises in many other areas; for instance, it is known as the "credit assignment problem" in artificial intelligence– is that of deciding on proper control actions at each instant in view of an overall, long-term, objective. Minimization of a cumulative (typically, integral) cost leads implicitly to such good choices, and this is the root of the Lagrangian or variational approach. Another possibility, which underlies the dynamic

programming paradigm, is to attempt to minimize at each instant a quantity that measures the overall "goodness" of a given state. In optimal control, this quantity is known as the *Bellman* (or the cost-to-go, or value, function); in stabilization, one talks about *Lyapunov* (or energy) functions; game-playing programs use *position evaluations* (and subgoals); in some work that falls under the rubric of learning control, one introduces a *critic*. Roughly speaking, all these are variants of the same basic principle of assigning a cost (or, dually, an expectation of success) to a given state –sometimes to a state and a proposed action– in such a way that this cost is a good predictor of eventual, long-term, outcome. Then, choosing the right action reduces to a simpler, nondynamic, pointwise-in-time, optimization problem: choose a control that leads to a next state with least cost.

Following ideas introduced by Artstein and himself in the early 1980s, the PI introduced in 1989 a systematic methodology for the use of such *control-Lyapunov functions*, which are "energy" functions that can be made to decrease pointwise by means of appropriate instantaneous controls. (This work is closely related to an approach standard in practice —but usually ah-hoc— to the control of nonlinear systems.) Since then, the "universal formulas" derived by the PI have been widely applied by many different authors, and are now part of both the textbook by Bacciotti on stabilizability and Isidori's (Third Edition) nonlinear control book. The research on this topic by the PI under this grant has dealt with (1) the generalization of the "universal formulas" to cases of bounded controls (work joint with the PI's graduate student Lin), and (2) the extension of these formulas to the nondifferentiable case (see [**25**]). This latter work makes contact with several less-explored areas, such as the use of viscocity solutions and nondifferentiable optimization in the context of control-Lyapunov functions.

Numerical Methods for Steering

The effective design of controls for the steering of nonlinear systems without drift, largely motivated by the study of nonholonomic mechanical systems, has attracted much attention during the past few years. Many sophisticated control strategies have been proposed, based on a nontrivial analysis of the structure of the Lie algebra of vector fields generated by the system, as in work by Brockett, Sastry and his students, Bloch and McClamroch, Sussmann and his students, and many others. In the recent work [**22**][**14**], the PI introduced an approach of an entirely different nature, the "generic nonsingular loop" method.

The PI's method is based on a simple-minded algorithm, in the style of classical numerical methods, requiring no symbolic computation to implement. Although fairly trivial to implement in principle, the *proof* of the general applicability of the method is based on not quite-so-trivial results asserting the genericity of nonsingular controls and establishing a duality with an observability problem. Obviously, as with any general procedure, it can be expected to be extremely inefficient, and to result in poor performance, when compared with techniques that use (when available) nontrivial information about the system being controlled. We expect our method to be useful mainly in conjunction with other techniques, to provide a first step of global control, to be followed by finer local control.

Discrete-Time and Dynamical Systems

Modern control techniques typically result in complex regulation mechanisms, which must be implemented using digital computers. Consequently, a fundamental area of research is that of studying the constraints imposed by computer control. The behavior of continuous-time plants

under digital control can be modeled by discrete-time systems. Starting in the mid-seventies in joint work with Rouchaleau, the PI had been interested in problems of control for nonlinear discrete time systems, and his Ph.D. thesis under Kalman was on this topic. During the early 1980s the focus of this work shifted largely to questions of accessibility and controllability; a summary is provided in the expository paper published in the January 1990 issue of SIAM J. Control by Jakubczyk and the PI.

During the last three years, in collaboration with graduate student Albertini, who completed her dissertation in mid 1993, the PI extended and greatly improved many of the basic results from the work with Jakubczyk regarding characterizations of nonlinear discrete-time controllability. Two fairly major journal papers resulted from these efforts, as well as a number of conference papers. Among other things, we were able to show that accessibility can be characterized in Lie algebraic terms, in much the same way as for continuous-time systems, under various dynamical-system assumptions on unforced dynamics, applying for instance to systems arising through the time-sampling of a Hamiltonian system. We also extended to discrete-time systems some of the basic results about "control sets" as introduced by Kliemann and Colonius for continuous-time (the extensions are nontrivial, as the basic enabling results were not available before our work), and in this way we managed to establish a correspondence, in perfect analogy to their work, between controllability of a given system and chaotic behavior of an "extended system" associated to it.

I/O Equations and Identification

We also continued the investigation of relations between state-space realizability and equations involving input/output data, as well as the associated structural questions about observation spaces. Much of this work in the past has been carried out jointly with the PI's former graduate student Wang, and this collaboration is ongoing (during the last two years, two major joint journal papers have appeared, and another one has been accepted for SIAM J. Control on the topic). It is motivated mostly by the interest in nonlinear identication algorithms (most such algorithms are based on establishing nonlinear dependences among i/o data, and hence one must ask when this implies that a state-space model can be built). Other motivations are given by the use of these results in understanding the connections between state-space theory and Willems' school's "behavioral" approach, as well as Fliess' differential-algebraic work (one recent joint conference paper with Wang provides various theorems in that regard). A particularly unexpected and interesting result was given in [**27**], which showed the existence of inputs that are sufficiently rich to permit the identification (in theory) of arbitrary nonlinear systems. This result constitutes probably the ultimate generalization of "universal input theorems" developed by many during the last 20 years (a journal paper is in preparation).

Next we provide some more details on the above topics.

2 State-Space and I/O Stabilization

A substantial part of this work dealt with the stabilization of nonlinear systems and the effect of input perturbations, and represents a long-standing direction of research pursued by the PI. As discussed in the introduction, problems of stabilization underlie most questions of control design, and many control objectives can be recast in those terms. Therefore a great deal of effort has been directed towards the understanding of the general problem of asymptotically stabilizing systems of the general form

$$\dot{x} = f(x, u) \tag{1}$$

with respect to an equilibrium point, or, more generally, with respect to a set of states of particular interest. (States $x(t)$ and inputs or controls $u(t)$ evolve in finite-dimensional spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively; as usual, we do not write the time-argument t in the equations. When global stabilization to equilibria is of interest, there is no point in studying more general systems in manifolds, since the existence of a globally asymptotically stable equilibrium forces the manifold to be diffeomorphic to Euclidean space. For questions of stabilization with respect to invariant sets, such as periodic orbits, systems on manifolds are of potential interest. To keep the exposition elementary, however, we restrict to states in $\mathbb{R}^n$. Furthermore, we omit from this informal discussion all technicalities regarding regularity properties of f; at least existence and uniqueness of trajectories for each bounded measurable control is assumed.)

In order to talk about stabilization, which deals with closed-loop behavior, it is first necessary to understand what is meant by "stability" of an open-loop system such as (1). Technicalities aside, there are two *conceptually* very different manners of defining this notion.

Notions of Stability

The most obvious approach to defining stability of control systems relies on the standard Lyapunov definition from dynamical systems theory, and is obtained when one disregards the control. Assume that 0 is an equilibrium state, that is, $f(0,0) = 0$. Then one simply requires the global asymptotic stability of the origin for the system which is obtained by setting $u \equiv 0$. (More generally, as discussed below, given a particular invariant set $\mathcal{A}$ of interest, not necessarily $\mathcal{A} = \{0\}$, one may want uniform asymptotic stability of this set, again in the standard sense of dynamical systems. Another variant is to consider the corresponding local definitions; for simplicity, the following discussion will be concerned only with the global case.) This notion of stability is often called *state-space* or *zero-input stability*.

In practice, control systems are affected by noise, expressed for instance as disturbances, actuator perturbations, and errors on observations. In this context, it is desirable for a system to display so-called "input/state" stability properties. This leads to the conceptual alternative of defining stability for (1) in an operator-theoretic sense. What is desired is not (only) that the system be zero-input stable as described above, but that, in addition, a bounded or decaying input $u(\cdot)$ should produce a similarly-behaved state trajectory $x(\cdot)$.

For suitably "small" inputs, this is often a consequence, and hence not different conceptually from, state-stability; much of the huge classical literature on total stability –Lefschetz called this "quasi-stability"– deals with such issues. But often one is interested in quantifying the effect of relatively "large" inputs. That is, one wishes to impose regularity (continuity, boundedness, etc) properties on the operator

$$\text{(initial state,control)} \quad \mapsto \quad \text{ensuing complete state trajectory.}$$

This *input/output* (if only partial observations are of interest) or *input/state stability* is central to the study of robustness with respect to actuator and sensor noise, as well as being the basis of approaches to controller parameterization.

Given the above intuitive description, there are many possible precise mathematical definitions of input/output stability. One possibility is to simply add the requirement *BIBS*: bounded inputs should produce bounded state trajectories (or more generally state trajectories that stay within a bounded distance of a desired invariant set $\mathcal{A}$). Even here, there are variants: The state bounds may be required to depend only on input and initial state bounds –but not on the actual values of these– and this dependence can expected to be linear; such a definition, amounting to boundedness of an operator, gives rise to "finite-gain" stability ("finite incremental gain" if the operator is to be also Lipschitz); see the discussion below on operator stability for saturated-control systems. Variants of the BIBS property are based on L_p norms instead of sup norms. Another possibility is to impose the condition *CICS*: convergent (to zero) inputs should produce convergent (to the desired equilibrium point or set $\mathcal{A}$) state trajectories.

The possibilities for i/o stability are many; the important issue is to provide a definition which (1) is potentially useful in practical applications and general enough, and (2) leads itself to nontrivial positive results. In the late 1980s, the PI introduced in a IEEE TAC paper a particular precise definition of *input/state stability*, or as it has since become known, "ISS" stability. The definition is reviewed in an appendix to this section, but it is not needed for the discussion to follow ; essentially it encompasses in one simple estimate both of BIBS and CICS stability as well as an ultimate boundedness property. Since its introduction, it and its obvious variants to deal with outputs (many of which were also described in the original reference and in Lin's Ph.D. thesis under the PI in 1992) have been the focus of several papers by many authors; just as examples one may cite the applications to observer design and new small-gain theorems in by Tsinias, Jiang, Praly, Teel, Freeman, Kokotovic, and others. An expository introduction to the topic can be found in [**16**].

Feedback Equivalence to ISS

We next illustrate explicitly one of many ways in which the ISS definition is useful and natural. Since much recent research in nonlinear control has been directed towards the development of techniques for state-space stabilization, the relation between state space and i/o stability is of great interest. However –contrary to the situation that holds for linear systems– state-space stability does not in general imply i/o stability. This is evidenced by the trivial example $\dot{x} = -x + ux^2$ with $m{=}n{=}1$; the system is zero-input stable but $u(t) \equiv 1$ makes every trajectory with $x(0) > 1$ escape to infinity in finite time. Given that the two broad notions of stability do not in general coincide, it makes sense to ask the *design* question: If one can make a system state-space stable by using feedback, can one also make it i/o stable? Mathematically, the stabilization problem is that of finding a feedback transformation of the type $u = k(x) + \beta(x)v$ so that the system with new control $v \in \mathbb{R}^m$

$$\dot{x} \;=\; F(x,v) \;=\; f(x, k(x) + \beta(x)v) \tag{2}$$

is stable in the desired sense. The new "closed-loop" system (2) is said to be *feedback equivalent* to the original one; the square matrix $\beta(x)$ is required to be invertible for all x so that no instantaneous controllability is lost (as usual in nonlinear feedback equivalence). Note that

the more restricted problem of state-space stabilization is that of finding just k (as β is now irrelevant) so that with respect to the origin (or a given invariant set of interest) the system with $v = 0$, $\dot{x} = F(x, 0) = f(x, k(x))$, be globally asymptotically stable; for i/o stability one wants that (2) be i/o stable.

It is perhaps surprising, given the gap between the two properties, that if a system is stabilizable in the state-space sense then it is also stabilizable in the i/o sense. In other words, from a differential equations viewpoint, the properties are very different, but from a *control* viewpoint they are equivalent modulo the standard group of feedback transformations. To be more precise, they become equivalent properties when one uses the PI's notion of ISS stability: *If there is a smooth k so that for $\dot{x} = f(x, k(x))$ the origin is globally asymptotically stable then there are also a (generally different) smooth k and an invertible β so that (2) is ISS.* Moreover, for systems *affine in controls*, that is, those of the special form

$$\dot{x} = f(x, u) = f_0(x) + G(x)\, u\,, \tag{3}$$

one may take β as the identity, resulting in a feedback law which is robust to input perturbations. (Systems affine in controls, possibly with control bounds, cover most interesting aerospace finite-dimensional control applications.) For stability with respect to sets, and for parametric versions with applications in robust control, this result has now been generalized, in Lin's thesis and further in [7], as mentioned below.

Perhaps more than the result itself, we view this equivalence as one more indication of the naturality and usefulness of the technical concept of ISS; other definitions are not conducive to similar results. Several very recent equivalent characterizations (see below) in terms of nonlinear stability margins and Lyapunov functions reinforce this view. Further, since the ISS property is well-behaved under cascade interconnections, it is especially suited as a systematic tool for the design of controllers.

Set Stability

Yuandan Lin's doctoral thesis, supported under this grant, and related papers, dealt with stabilization with respect to (not necessarily compact) sets. In this we were motivated by potential applications to a wide variety of areas.

As an illustration, consider problems of output feedback. One common definition of "detectability" (e.g. as done in the work of Vidyasagar) involves the existence of an observer for which the error satisfies Lyapunov estimates which depend only on the difference $\|x(t) - z(t)\|$, where $x(t)$ is the state of the plant and $z(t)$ is the state of the observer. For the joint plant/observer system, detectability becomes stabilization with respect to the set $\mathcal{A} := \{(x, z) | x = z\}$.

In other applications, one is interested in stabilization of an output variable as opposed to the complete state; this happens for instance in tracking or regulation problems, where the "output" is the error in a variable to be regulated, and the system model may have been enlarged to include an exosystem generating disturbances or reference trajectories. Qualitatively, things are very different from the full state case. Even local results, typically not an obstruction in the latter case, become false. For example, consider the following two-dimensional system:

$$\dot{x} = x,\ \dot{y} = -y + ux,$$

with the variable y taken as the output. Observe that when $u \equiv 0$ the y variable converges exponentially to zero, uniformly on the initial state $(x(0), y(0))$. However, for nonzero u, no

matter how small, the output diverges if $x(0) \neq 0$. This is in marked contrast to the case of state-space stability, where at least for small controls and small initial states, bounded states result, if the system was asymptotically stable for $u \equiv 0$, and shows the fundamental difference with the compact attractor case. (Similar examples are typically given to illustrate the nonrobustness of some adaptive control algorithms; mathematically, one is dealing with almost the same issue.) Thus it becomes of interest to try to make output variables ISS stable under perhaps additional feedback. This corresponds to ISS stability with respect to a set, in this example, the set where $y = 0$.

Another motivation arises in robust control, where one studies in effect stability of a control system with respect to the set $\mathcal{A} := \{(0, \lambda), \lambda \in \Lambda\}$, where λ is the vector of unknown parameters and "0" stands for the zero state. As yet another motivation, systems in which derivatives of controls appear can be reduced, adding integrators, to systems in which such derivatives do not appear, but at the cost of extra state variables which are not to be controlled.

The paper **[20]**, and related papers with Lin, concentrated on some basic questions related to set stability. We were able to extend the validity of many of the results previously known only for equilibria, such as the theorem on feedback equivalence to ISS systems. For instance, from this theory it is immediate how to modify by feedback the system $\dot{x} = x, \dot{y} = -y + ux$ mentioned above, so that the output satisfies an ISS-like property with respect to u (see **[20]**). (Of course, this can be solved directly as well, for this simple example, but the results are in full generality.) Several applications to robust control and to general problems of output regulation were given in the work with Lin.

At the core of the work with Lin was a converse Lyapunov theorem that does not assume compactness of the attracting set. Far more general is our (the PI, Lin, and Wang) paper **[11]**. There, motivated by problems of robust nonlinear stabilization, we gave a Converse Lyapunov Function Theorem which is in a form particularly useful for the study of such feedback control analysis and design problems. We provided a single (and natural) unified result that: (a) applies to stability with respect to not necessarily compact invariant sets; (2) deals with global asymptotic stability; (3) results in smooth (infinitely differentiable) Lyapunov functions; and most importantly, (4) applies to stability in the presence of bounded disturbances acting on the system.

Questions of stability for systems with disturbances are naturally modeled as problems of stability for differential inclusions. In that context, classical books such as Aubin and Cellina's deal with Lyapunov functions, but mostly in the role of providing sufficient conditions (one exception is the work in by Molchanov for the linear case). Our converse Lyapunov theorem can be restated in terms of differential inclusions, of course. But the conditions of applicability are different than the conditions usually found in that context, and in particular they may not be satisfied in many examples of interest, including many nonsmooth-control problems. It is quite likely that the results will indeed generalize, but mathematically the proofs are quite delicate and need considerable work.

Characterizations of ISS

One of the main applications of our converse Lyapunov theorem was in obtaining complete characterizations of the ISS property. In the paper **[10]**, we were able to show that the ISS property is in fact equivalent to the existence of a "nonlinear stability margin." By this we mean a $\mathcal{K}_\infty$ function ρ with the property that for each (possibly nonlinear and time varying) feedback law bounded by ρ ($|k(t,\xi)| \leq \rho(|\xi|)$), the closed-loop system $\dot{x} = f(x, k(t, x))$ is

globally asymptotically stable, with bounds uniform on k. Note that this is not a notion of local stability with respect to "small" perturbations. It is a global notion, and perturbations can be arbitrarily large (since the function ρ is in class $\mathcal{K}_\infty$). In some sense, this is analogous to exponential stability for linear systems, where a perturbation of the spectrum preserves global asymptotic stability. We also proved, solving an important and longstanding open problem, that ISS stability is equivalent to the existence of an ISS-Lyapunov function.

Classical Operator-Theoretic Notions

The notion of ISS was introduced in order to model i/o stability. Of course, ideally one would like to be able to obtain much stronger conditions. For instance, if the inputs are assumed to be integrable, one may ask that the input/state operator define a map into the space of integrable functions (of time) with values in $\mathbb{R}^n$. Moreover, one may want this map to be bounded or to satisfy other typical functional-analytic properties. If such stronger properties hold, one then can study stability problems by means of Banach space or (for instance for square integrable controls) Hilbert space techniques. Classical approaches (Willems, Desoer, Vidyasagar) indeed proceed in this fashion. More recent work such as that represented by Georgiou and Smith, then allows the application of powerful geometric notions (gap metric and so forth) to understand robustness questions. Related to this is also the so-called "nonlinear H^∞" work on minimizing L_2 norms of feedback interconnections.

In general, as mentioned earlier, it appears that it is not possible to strengthen the ISS conclusions of theorems such as the one on feedback equivalence to attain bounded operator norms. However, in one particular but still very important case, a special class of linear systems subject to actuator saturation, we were able to solve this problem, in recent work with Liu and Chitour; see **[23]** for preliminary results and **[12]** for the final version. We showed there how to obtain stability with respect to measurement and actuator noise, for the class of systems $(P)\,\dot{x} = Ax + Bs(u), y = Cx$ having the pair (A, B) controllable, A neutrally stable, and (A, C) observable, where s is a saturation type of nonlinearity. Specifically, we provided a linear controller C so that the operator $(u_1, u_2) \mapsto (y_1, y_2)$ defined by the standard systems interconnection (see Figure) is well-posed and finite-gain stable (with respect to any L_p norm, $1 \leq p \leq \infty$). Our work relates to, but is essentially independent of, other papers on computing

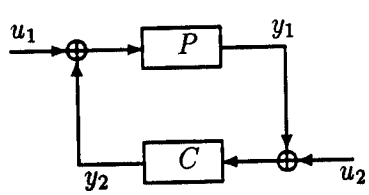

Figure 1: *Standard Closed-Loop*

norms for nonlinear systems in state space form, as done by van der Schaft, Isidori, Helton, and others. The proof relies upon techniques from dissipative systems theory, but the storage functions that we need to use are apparently completely new (in fact, they are not even smooth, contrary to usual energy functions). Motivated by our results, Saberi, Lin, and Teel have just obtained related ones for "semi-global" stability for the same class of systems. Among many problems suggested by this work, one of the most natural ones mathematically is the study of other properties (Lipschitz continuity, differentiability) for the operators in question. A fairly

complete understanding of these aspects was achieved towards the end of the grant period, in **[30]**-**[13]**.

Appendix: Definition of ISS

Recall that a function $\gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is: a *$\mathcal{K}$-function* if it is continuous, strictly increasing and $\gamma(0) = 0$; a *$\mathcal{K}_\infty$-function* if it is a $\mathcal{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$; a *positive definite* function if $\gamma(s) > 0$ for all $s > 0$, and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a *$\mathcal{KL}$-function* if: for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and for each fixed $s \geq 0$ it is decreasing to zero as $t \to \infty$.

We deal with nonlinear systems

$$\dot{x} = f(x, u) \tag{4}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is continuously differentiable and satisfies $f(0, 0) = 0$. *Controls* or *inputs* are measurable locally essentially bounded functions $u : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^m$. The set of all such functions, endowed with the (essential) supremum norm $\|u\| = \sup\{|u(t)|, t \geq 0\} \leq \infty$, is denoted by L^m_∞. (Everywhere, $|\cdot|$ denotes the usual Euclidian norm.) For each $\xi \in \mathbb{R}^n$ and each $u \in L^m_\infty$, we denote by $x(t, \xi, u)$ the trajectory of the system (4) with initial state $x(0) = \xi$ and the input u. This is defined on some maximal interval $[0, T_{\xi,u})$, with $T_{\xi,u} \leq +\infty$.

The following definition is intended as a nonlinear generalization of the bound $|x(t)| \leq |\xi|\, e^{-\alpha t} + c\|u\|$ which holds for linear systems $\dot{x} = Ax + Bu$ when the matrix A is asymptotically stable.

Definition 2.1 The system (4) is *(globally) input/state stable (ISS)* if there exist a $\mathcal{KL}$-function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, and a $\mathcal{K}$-function γ, such that, for each input $u \in L^m_\infty$ and each $\xi \in \mathbb{R}^n$, it holds that

$$|x(t, \xi, u)| \;\leq\; \beta(|\xi|, t) + \gamma(\|u\|) \tag{5}$$

for each $t \geq 0$. □

When there is no control, that is, we are considering an estimate $|x(t, \xi, u)| \leq \beta(|\xi|, t)$, this is *exactly* the same as the classical Lyapunov global asymptotic stability notion (it is not a difficult exercise to show the equivalence).

The next definition states that there is a proper (that is, what is sometimes called radially unbounded) positive definite function V on states with the property that its derivative $\dot{V}$ along trajectories is negative definite for large enough x, given any control magnitude.

Definition 2.2 A smooth function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is called an *ISS-Lyapunov function* for system (4) if there exist $\mathcal{K}_\infty$-functions α_1, α_2, and $\mathcal{K}$-functions α_3 and χ, such that

$$\alpha_1(|\xi|) \;\leq\; V(\xi) \;\leq\; \alpha_2(|\xi|) \tag{6}$$

for any $\xi \in \mathbb{R}^n$ and

$$\nabla V(\xi) \cdot f(\xi, \mu) \leq -\alpha_3(|\xi|) \tag{7}$$

for any $\xi \in \mathbb{R}^n$ and any $\mu \in \mathbb{R}^m$ so that $|\xi| \;\geq\; \chi(|\mu|)$. □

Observe that if V is an ISS-Lyapunov function for (4), then V is a Lyapunov function, in the usual sense, for the autonomous system $\dot{x} = f(x, 0)$ obtained when no controls are applied.

Given a closed invariant set $\mathcal{A}$, that is, a closed subset $\mathcal{A}$ of $\mathbb{R}^n$ with the property that trajefctories for $u \equiv 0$ cannot exit this set (for instance, $\{A\} = 0$ when $f(0,0) = 0$), one can define both of the above concepts, ISS and ISS-Lyapunov function, in an entirely analogous manner, simply using distance to the set instead of norms. This is done for instance in **[20]**, and we omit details here.

3 Control-Lyapunov Functions

As mentioned in the introductory section, a widespread, but typically ad-hoc, technique in nonlinear control (as well as in related areas) relies on the use of "energy" functions that can be made to decrease pointwise by means of instantaneous controls. Somewhat more technically, one assumes given a positive definite and proper (=radially unbounded) scalar continuous function V of states, with the property that V can be made to decrease infinitesimally by appropriate choice of controls. Then the control applied at each instant is one that forces this decrease. Of course, stated in this informal manner there are all kinds of technical complications –one needs a careful definition of "infinitesimal decrease" and it is not clear that a reasonably regular control selection should exist. But the paradigm is extremely powerful, and undelies not only feedback control design but also the optimal control approach of Bellman and even "artificial intelligence" techniques (position evaluations in games, "critics" in learning programs). Such functions V are generically called *control-Lyapunov functions*, in analogy to the Lyapunov functions classical in dynamical systems (when no control is present).

One way to make the above procedure effective computationally is if the function V is differentiable. In that case, the condition of infinitesimal decrease can be expressed simply as the requirement (the "clf property") that for each nonzero x there be some control value $u \in \mathbb{R}^m$ so that $\nabla V(x)\, f(x,u) < 0$. (This is if the objective is state-feedback stabilization to the origin; an alternative is to ask that this expression be uniformly negative away from a desired set $\mathcal{A}$, for stabilization to $\mathcal{A}$, or for instance that the control should depend on a fixed measurement function of x, for output feedback questions. To avoid complicating the exposition, we continue this discussion with stabilization to the origin.) In the early 1980s, Artstein and the PI produced independent and mathematically complementary theoretical justifications of this technique (the former work assumed more regularity; the latter required less smoothness but applied more generally). In 1989 the PI gave a systematic methodology for the use of smooth control-Lyapunov functions, resulting in "universal formulas" reviewed briefly below.

Universal Formulas

This work is especially easy to understand in the particular case of system affine in controls, those as in Equation (3). For simplicity, we assume that there is just a scalar control ($m = 1$), so $G(x) = g(x)$ is just a column vector. For such a system, the clf property is trivially shown to be equivalent to:

$$b(x) \neq 0 \;\Rightarrow\; a(x) < 0 \tag{8}$$

where we are denoting the Lie derivatives as $a(x) := \nabla V(x).f_0(x)$, $b(x) := \nabla V(x).g(x)$. On the other hand, giving a feedback law $u = k(x)$ for the original system, with the property that

the same V is a Lyapunov function for the obtained closed-loop system $\dot{x} = f_0(x) + k(x)g(x)$ is equivalent to asking that $\nabla V(x).(f_0(x) + k(x)g(x)) < 0$, that is,

$$a(x) + k(x)b(x) < 0$$

for all nonzero x. Thus finding a suitable feedback law reduces to a problem of finding continuous selections. The late-1980s work by the PI wasbased on the following observation: the condition (8) is equivalent to the statement that *the pair* $(a(x), b(x))$ *is stabilizable* seen as a one-dimensional parametric family of systems. And the condition on $k(x)$ is that, seen as a 1×1 matrix, it must be a constant linear feedback stabilizer for $(a(x), b(x))$, for each fixed x. Solving a Riccati equation for a simple linear-quadratic problem shows that the following feedback law: $k := -(a + \sqrt{a^2 + b^2})/b$ (with $k(x) := 0$ when $b(x) = 0$) works. This is in fact analytic, even algebraic, on a, b. (The apparent singularity due to division by b is removable.) Along trajectories of the corresponding closed-loop system, one has that $dV/dt = -\sqrt{a^2 + b^2} < 0$ as desired. This feedback law may fail to be continuous at zero, but under a natural "small control property" one can obtain continuity at the origin as well. Since the existence of a clf is also necessary whenever there is any continuous feedback stabilizer (this is an immediate consequence of the classical Lyapunov converse theorems), one has a necessary and sufficient condition for the existance of such feedback, together with an explicit fomula for feedback controls once that a clf is found. The formula shown above is an example of what we mean by a "univeral formula" in the context of clf's, meaning in general an analytic function of Lie derivatives which, when substituting a clf, provides a global stabilizer (there is also an extra technical requirement to guarantee continuity at the origin, which we omit in this overview; the references should be consulted for details).

Since clf's are as a general rule easier to obtain than the feedback laws themselves –after all, in order to prove that a given feedback law stabilizes, one typically has to exhibit a suitable Lyapunov function anyway– these techniques provide in principle a very powerful approach to nonlinear stabilization. In any case, the availability of universal formulas allows the search to be confined to just one scalar function, and could be the basis of numerical approaches. This had been observed before by the PI, but never implemented. A paper at the 1993 CDC by Long and Bayoumi provided one such experimental approach, using the formulas as a basis of a numerical technique.

Another important direction is in extending universal formulas to the case where the control value set is constrained. In work with his graduate Lin, the PI found formulas for bounded controls (see [**3**]) as well as nonnegative scalar controls ([**15**]).

Non-Smooth Cases

In an early (1983) paper, the PI had shown that asymptotic controllablility –which is the minimal possible condition for stabilizability– is sufficient to insure the existence of a control Lyapunov function V which is C^0 but not necessarily smooth; in the definition, derivatives of V are replaced by Dini derivatives. Though this is an extremely general result, Dini derivatives do not lend themselves to explicit checking of the clf property. In work recently began with Lafferriere (see the preliminary paper [**25**]), we revisited this general result from two different points of view. First of all, we observed that, by generalizing the clf condition "$(\forall x \neq 0)(\exists u)\,[\nabla V(x)\, f(x,u) < 0]$" to "$(\forall x \neq 0)(\exists u)(\forall p \in D^- V(x))\,[p\, f(x,u) < 0]$," where $D^- V(x)$ is the subdifferential of V at x, the same existence result holds. The proof is fairly straightforward, since the function V was obtained in the 1983 PI's paper from an optimal

control problem, and via a standard calculation it follows that then V must be a viscosity supersolution of the associated Hamilton-Jacobi-Bellman equation, from which the clf property follows. Unfortunately, this result is less interesting than it may appear at first sight, since there is no guarantee than V be Lipschitz, and in particular the subdifferential could be empty, and work in this area is still in progress.

A second, and from an immediate applied point of view more interesting, aspect aspect of our current work with Lafferriere is the study of clf's V which are *piecewise smooth*. This fits very nicely with work of Canudas and others in practical control problems (in particular in robotics) as well as work such as that recently reported by Helton on piecewise quadratic cost functions. For these V, obtained by "pasting together" smooth ones, the verification of the Lyapunov property can be carried out using gradients, just as in the smooth case. The main contribution of **[25]** is to give conditions under which the *same* universal formula given earlier produces for such a V a piecewise continuous globally stabilizing feedback law. (In our case, the feedback turns out to be continuous at the origin and smooth everywhere except on a hypersurface of codimension 1.) In particular, a classical example due to Artstein for which it is known that no differentiable clf (and no continuous feedback) exists is treated explicitly and a control law is derived for it automatically from the universal formula rather than in an ad-hoc manner.

4 Numerical Methods for Steering Nonlinear Systems

This part of the report deals with the problem of numerically finding controls that achieve (at least approximately) a desired state transfer. That is, for any given initial and target states ξ_0 and ξ_F in $\mathbb{R}^n$, one wishes to find a time $T > 0$ and a control u defined on the interval $[0, T]$, so that u steers ξ_0 sufficiently close to ξ_F, for the system (1).

In particular, it is of interest to consider the case of systems *without drift*, that is, systems for which the right-hand side is linear (not affine) in u, so the equations take the form $\dot{x} = G(x)u$. For such systems it is relatively straightforward to decide controllability, since it is equivalent to the standard Lie algebra rank condition, but the design of *explicit* control strategies has attracted considerable attention lately. Problems of steering systems without drift are in part motivated by the study of nonholonomic mechanical systems. Many sophisticated control strategies have been proposed, based on a nontrivial analysis of the structure of the Lie algebra of vector fields generated by the columns of G by many authors. The approach presented in the PI's paper **[22]** was of an entirely different nature; it is essentially a simple-minded algorithm, in the style of classical numerical methods as in Bryson and Ho's classical book and in its simplest form requires almost no effort to implement in a high-level language. Mathematically, the main contribution of the PI's work in this area was is in the formulation of the so-called "generic nonsingular loop" approach and the justification of the algorithm. The latter relies on a new result proving the existence (and genericity) of such loops with good controllability properties. This approach was motivated to a great extent by related work on time varying feedback laws by the French school during the past two years, especially Coron there and in **[22]**. The journal version of **[22]** has recently been accepted for the IEEE Transactions ([14]) and contains a worked-out numerical example.

The Basic Idea

Assume that f is continuously differentiable (later we will need to assume analyticity). Given a state $\xi_0 \in \mathbb{R}^n$ and a measurable and locally essentially bounded control $u : [0,T] \to \mathbb{R}^m$ so that the solution $x : [0,T] \to \mathbb{R}^n$ of the equation (3) with this control and the initial condition $x(0) = \xi_0$ is defined on the entire interval $[0,T]$, the state $x(t)$ at time $t \in [0,T]$ is denoted by $\phi(t, \xi_0, u)$. As discussed above, the objective, for any given initial and target states ξ_0 and ξ_F in $\mathbb{R}^n$, is to find a time $T > 0$ and a control u defined on the interval $[0,T]$, so that u steers ξ_0 to ξ_F, that is, so that $\phi(T, \xi_0, u) = \xi_F$, at least in an approximate sense. After a change of coordinates, one may assume without loss of generality that $\xi_F = 0$. Classical numerical techniques for this problem are based on variations of steepest descent. This is all very elementary, but we need to recall it for further reference. The basic idea is to start with a guess of a control, say $\overline{u} : [0,T] \to \mathbb{R}^m$, and to improve iteratively on this initial guess. More precisely, let $\overline{x} = \phi(\cdot, \xi_0, \overline{u})$. If the obtained final state $\overline{x}(T)$ is already zero, or is sufficiently near zero, the problem has been solved. Otherwise, we look for a perturbation $\Delta\overline{u}$ so that the new control $\overline{u} + \Delta\overline{u}$ brings us closer to our goal of steering ξ_0 to the origin. The various techniques differ on the choice of the perturbation; in particular, two possibilities are found most often.

The first one is basically Newton's method, and proceeds as follows. Denote, for any fixed initial state ξ_0, $\alpha(u) := \phi(T, \xi_0, u)$ thought of as a partially defined map from $\mathcal{L}^m_\infty(0,T)$ into $\mathbb{R}^n$. This is a continuously differentiable map, Theorem 1), so expanding to first order there results $\alpha(\overline{u} + v) = \alpha(\overline{u}) + \alpha_*[\overline{u}](v) + o(v)$ for any other control v uniformly near $\overline{u}$ ("$*$" as a subscript denotes differential). If we can now pick v so that

$$\alpha_*[\overline{u}](v) = -\alpha(\overline{u}) \tag{9}$$

then for small enough $h > 0$ real,

$$\alpha(\overline{u} + hv) = (1-h)\alpha(\overline{u}) + o(h) \tag{10}$$

will be smaller than the state $\alpha(\overline{u})$ reached with the initial guess control $\overline{u}$. In other words, the choice of perturbation is $\Delta\overline{u} := hv$, $0 < h \ll 1$. It remains to solve equation (9) for v. The operator $L : v \mapsto \alpha_*[\overline{u}](v)$ is the one corresponding to the solution of the variational equation (linearization along trajectory) $\dot{z} = A(t)z + B(t)v, z(0) = 0$, where $A(t) := \frac{\partial f}{\partial x}(\overline{x}(t), \overline{u}(t))$ and $B(t) := \frac{\partial f}{\partial u}(\overline{x}(t), \overline{u}(t))$ for each t, that is, $Lv = \int_0^T \Phi(T,s)B(s)v(s)\,ds$, where Φ denotes the fundamental solution associated to $\dot{X} = A(t)X$. The operator L maps $\mathcal{L}^m_\infty(0,T)$ into $\mathbb{R}^n$, and it is onto when the linearization is a controllable linear system on the interval $[0,T]$, that is, when $\overline{u}$ is a control *nonsingular for* ξ_0 relative to the system (3). In other words, ontoness of $L = \alpha_*[\overline{u}]$ is equivalent to first-order controllability of the original nonlinear system along the trajectory corresponding to the initial state ξ_0 and the control $\overline{u}$. The main point of our technique lies in showing that *it is not difficult to generate useful nonsingular controls*, at least for systems with no drift.

Assuming nonsingularity, there exist then many solutions to (9). Because of its use in (10) where a small v is desirable, and in any case because it is the most natural choice, it is reasonable to pick the least squares solution, that is, the unique solution of minimum norm, $v := -L^\# \alpha(\overline{u})$ where $L^\#$ denotes the pseudoinverse. The technique sketched above is well-known in numerical control. For instance, the derivation in pages 222-223 of Bryson and Ho's book, when applied to solving the optimal control problem having the trivial cost criterion $J(u) = 0$ and subject to the final state constraints $x = \psi(x) = 0$, results in this formula.

Alternatively, instead of solving (9) for v, one might use the steepest descent choice $v := -L^*\alpha(\overline{u})$ where L^* is the adjoint of L. This formula also results from the above-cited derivation, now when applied using the quadratic cost $J(u) = \|\alpha(u)\|^2$ but relaxing the terminal constraints ($\psi \equiv 0$). Now $\alpha(\overline{u}+hv) = (I - hLL^*)\alpha(\overline{u}) + o(h)$, where I is the identity operator. If again L is onto, that is, if the control $\overline{u}$ is nonsingular for ξ_0, then the symmetric operator LL^* is positive definite, so $0 < h \ll 1$ will give a contraction as earlier. (An advantage of using L^* instead of $L^\#$ is that no matrix inversion is required. On the other hand, one may expect Newton's method to behave better locally and steepest descent to be more effective globally.)

Of course, in general there are many reasons for which the above classical techniques may fail to be useful in a given application: the initial guess $\overline{u}$ may be singular for ξ_0, the iteration may fail to converge, and so forth. The main point of the PI's recent work was to show that, for a suitable class of systems, a procedure along the above lines can be guaranteed to work. More specifically, we showed that for analytic systems without drift there are always controls $\overline{u}$ with the property that they lead to "nonsingular loops" in the sense that: (1) $\overline{u}$ is nonsingular for every state x, and (2) $\phi(T, x, \overline{u}) = x$ for all x. The result is even more general: for all analytic systems that have the strong Lie accesibility property (no assumption that the system has no drift), there exist controls that are nonsingular for all states. Even more interestingly, generic controls (in the standard C^∞ topology sense) have this property. From here, one obtains the result for systems without drift: starting from such a control ω, defined on an interval $[0, T/2]$, one may now consider the control $\overline{u}$ on $[0, T]$ which equals ω on $[0, T/2]$ and is then followed by the antisymmetric extension: $\overline{u}(t) = -\omega(T - t), t \in (T/2, T]$. This $\overline{u}$ provides nonsingular loops. (The proof of the existence and genericity of nonsingular controls proceeds by first establishing the existence of nonsingular controls pointwise – a topological fixed point theorem is needed here– and then by dualizing Sussmann's universal input theorem for observability, which provides the required transversality result.) In **[22]** we also provide a convergence theorem that shows that, assuming $\overline{u}$ is picked as above, repeated application of the Newton or gradient algorithm starting from any fixed ball is guaranteed to get us within any desired epsilon-distance of the target state, for all small enough stepsizes h; a rate of convergence estiate is provided as well.

It is possible to combine the nonsingular loop technique with line searches over the scalar parameter h or, even more efficiently in practice, with conjugate gradient. Line search corresponds to leaving v fixed and optimizing on the step size h, only recomputing a variation v when no further improvement on h can be found. (The control applied at this stage is then the one for the "best" stepsize, not the intermediate ones calculated during the search.) The planning of motions with workspace obstacles is another interesting and important area. We proposed paper a method that allows including obstacles, essentially by multiplying the equations by a function which vanishes at the sets that are dissallowed for trajectories (this is somewhat related to the "potential" method in path planning).

5 Other Topics

In this section three other areas are discussed that represent recent work by the PI supported by this grant.

Discrete-Time Systems

Another thread of the PI's work focuses on controllability properties of discrete-time nonlinear systems. These are described by controlled difference equations of the following general form:

$$x(t+1) = f(x(t), u(t)), \quad t = 0, 1, 2, \ldots . \tag{11}$$

The sequences x and u take values in manifolds. Observe that both variables, the control u and the state x, take continuous values, which distinguishes this study from classical automata theory. In contrast to the linear case, where discrete and continuous-time systems are treated side by side, much of the literature regarding nonlinear systems has classically restricted itself to the analysis of the latter, namely, to the consideration of systems (1). With the exception of ad-hoc studies for special classes of systems, seldom are nontrivial results provided for discrete-time models, in spite of the fact that the latter arise very naturally in several contexts. This lack of study can be ascribed mostly to the lack of powerful geometric tools of analysis, when compared with the highly successful use of such techniques in the context of continuous-time problems.

One of the main sources of discrete-time questions in applications arises from time sampling in digital control. This is analogous to the consideration of "time 1 maps" in the study of uncontrolled differential equations. As far as a control algorithm is concerned, the physical system is a discrete-time system described by an equation of the type (11), where now $f(x, u)$ is the solution of the differential equation (1) at the end of the sampling interval assuming that the initial state was x and the control was held constantly equal to u. (There are of course variations of the basic sampling idea: A/D conversion involves also quantization, constant controls values may be smoothed out by a filter before being applied, multi-rate strategies are popular; but even without these complications, the study of discrete-time control systems appears naturally.) Another area in which results from discrete-time nonlinear control theory are of importance is in the study of Markovian systems (11) where the variables $u(t)$ are random, and together with the transitions f they characterize the probabilistic behavior of the process $x(\cdot)$. Reachability properties play a central role in establishing the existence and smoothness properties of equilibrium distributions, as in

In late-1980s work with Jakubczyk, and in other work together with various collaborators, most especially with his student Albertini, the PI followed an approach, based on the introduction of certain Lie algebras of vector fields, which permits obtaining results for discrete time which closely parallel some of those results known in the differential equation case. Roughly, the necessary vector fields are obtained by considering the infinitesimal effect of iterations of the mappings $f(\cdot, u)$. Formally, if instead of these iterates based on control sequences one would instead be considering actions of a Lie group G on a manifold, our vector fields would be the standard infinitesimal generators associated to the elements in the Lie algebra of G. Armed with these tools, the PI and coworkers obtained new results on controllability properties.

For a system (1) or (11), the *reachable set* or *forward-accessible set* from a state $x^0 \in X$ consists of those states to which one may steer x^0 using arbitrary –measurable essentially bounded in the continuous-time case– controls. The *orbit* or *forward-backward accessible set* from x^0 is defined as the set consisting of all states to which x^0 can be steered using both motions of the system as well as negative time motions. Negative time motions are in general not physically realizable, but the orbit is an extremely useful object to study, as it is always a submanifold of X whose dimension can be obtained from Lie-algebraic computations, and group-theoretic techniques are relevant. In contrast, the reachable set is not in general a

manifold, though one may define a "dimension" for it in various natural and equivalent ways, for instance meaning the largest dimension of a submanifold that it includes.

Controllability questions for continuous-time systems have been the subject of much research. In continuous-time, a fundamental fact in controllability studies is what may be called the *Chow-Krener (*) property: The dimension of the orbit from each* $x^0 \in X$ *equals the dimension of the reachable set from* x^0. (The property holds as stated for analytic systems; in the more general smooth case one would need an addditional Lie-algebraic assumption.) That is, the reachable set contains a submanifold of the state space and it is in turn contained in a submanifold of the same dimension, and this dimension can be computed from the rank of certain matrices formed by taking iterated Jacobians of the vector fields defining the system, evaluated at the state x^0. These Lie-theoretic characterizations are "direct" in that they do not involve integration of the differential equation, and they are closely related to more classical geometric material related to Frobenious' theorem. Perhaps the most important difference between (11) and (1) is that the (*) property fails for the former, even for systems obtained through the sampling of one-dimensional analytic systems.

The PI's work with Jakubczyk dealt with systems (11) that are time-reversible or *invertible*, that is, the maps $f(\cdot, u)$ are diffeomorphisms. (This is satisfied for systems that arise from the sampling of complete continuous-time systems.) For such systems, the paper derived several characterizations of accessibility and studied the geometric structure of accessible sets, and as a consequence a theorem was proved showing that the (*) property does hold provided that the state x^0 be an equilibrium state ($f(x^0, 0) = x^0$).

Much further progress has now been achieved in work with Albertini; see **[5]**,**[6]** and several conference papers published recently. Among results in this very recent work are the extension of the (*) property to many cases. For example, (*) holds for each state x^0 that satisfies a positive Poisson stability condition. (An equilibrium point satisfies this requirement, so the previously-known result is generalized.) It was also possible to prove that (*) does hold if the orbit from x^0 is compact. Moreover, it was shown that transitivity implies forward accessibility when there exists a transitive state $\bar{x}$ which is also a global attractor for the given system (i.e. $\bar{x} \in \mathrm{clos}\,(R(x))$ for each state x). Another interesting result was the proof that the set of points for which property (*) does *not* hold is very thin. In fact, it was established that if there is only one orbit (the system is transitive), then forward accessibility holds from an *open dense* set of states, assuming the state space to have at most finitely many connected components.

To prove these new results, one must associate to (11) various new Lie algebras of vector fields whose orbits are correlated to the geometry of the sets $R(x)$ and $O(x)$, providing new Lie-theoretic characterizations of the transitivity and the forward accessibility properties. This work also generalized several results to a larger class of systems, where the invertibility assumption was relaxed in various ways. More precisely, it dealt with partially defined dynamics, systems for which the functions $\{f(\cdot, u)\}$ are diffeomorphisms but possibly defined only on some open subsets of the state space, and also covered was the "submersive" case, systems for which the Jacobian of the function $f(\cdot, u)$ has full rank at each point. Unfortunately, for both classes many of the interesting results that hold in the invertible case do not longer hold; however, one can still give some theoretic characterizations of transitivity and forward accessibility. Also given were several examples showing the pathological behavior that can arise in these cases, and the main differences with the invertible case were described.

Another aspect of the PI's and Albertini's work dealt with connections between controllability of a given system and properties of an associated classical (uncontrolled) dynamical system. Recently, Colonius and Kliemann introduced the notion of ***controllability subsets*** of the state

space of continuous-time systems. These are essentially sets where "almost reachability" holds. Controllability sets have proved to be an extremely useful concept; in particular, Colonius and Kliemann have established an interesting relationship between such sets and chaotic behavior in subsets of an associated dynamical system. This work obtained analogs of these results in the discrete-time setting. This extension is not trivial, as it depends critically on the deep understanding of the forward accessibility properties of controllability sets. in fact the basic idea which underlies most of the results is the validity of property (*) for continuous-time models.

Identification Questions

The question of dynamical systems identification from input/output data is at the core of control theory, mostly in 'black box' approaches, where only qualitative a priori information is available about the process being controlled. (In other areas, such as in the study of dimensions of attractors for unforced dynamical systems, analogous issues appear, but mathematically the availability of inputs makes this a different topic; in particular, problems of choice of controls –persistent excitation and related concepts– take center stage in control theory.) Since his thesis work in the mid 1970s, the PI has had a continuing interest in a problem which is basic in this context, namely, the generalizations to nonlinear systems of the correspondence that exists in the linear case between "autoregressive moving average" representations

$$y^{(k)}(t) = a_1 y(t) + \ldots + a_k y^{(k-1)}(t) + b_1 u(t) + \ldots + b_k u^{(k-1)}(t) \tag{12}$$

(or in frequency-domain terms, rationality of transfer functions) and state-space representations. In the linear case, such representations form the basis of much of adaptive control and identification theory. The obvious generalization is to look at "input output equations" of the type

$$E\left(w(t), w'(t), w''(t), \ldots, w^{(r)}(t)\right) \;=\; 0 \tag{13}$$

where $w(\cdot) = (u(\cdot), y(\cdot))$ represent the i/o pairs of the system, connected by an operator F that maps input functions u to output functions $y = F[u]$. (If an initial state is not fixed, one has a family of such operators, one for each initial state.) In experiments, the functional relation E is usually estimated, for instance through least squares techniques, if a parametric general form is chosen, such as using polynomials of a fixed degree. A similar formalism can be developed for discrete-time systems, using difference equations instead; actually, the discrete-time case had been already developed by the PI as part of his thesis work, and it has been cited as a justification of identification algorithms by several other authors.

In work with his former graduate student Wang, the PI has extended to continuous-time the results obtained earlier by the PI for the discrete case; see especially [1]. This work, closely related and complementing studies by Crouch, Glad, Van der Schaft, and others, succeeded in showing that, *if one knows that the data is generated by a "well-posed" operator F*, then there is an algebraic i/o equation (i.e., all pairs $(u, F[u])$ satisfy (13)) *if and only if* it F realizable by a (singular) state space polynomial system (Precise definitions are given in the above-given references.) By "well-posedness" it is meant that F is a *Fliess operator*, i.e. one described by a convergent generating series. The i/o operators induced by convergent generating series form a very general class of causal operators, capable of representing a variety of nonlinear systems. In [2], the algebraic results are supplemented by a result linking analytic i/o equations to local internal realizability. To do this, one first constructs a "meromorphic" realization by studying the properties of meromorphically finitely generated field extensions, and then imposes

similar properties on the observation fields already introduced in the former paper. Finally, by a perturbation approach, together with the Fliess Lie rank condition for realizability, one concludes that around each point there is local analytic realization. That paper combines the techniques from differential algebra used in the polynomial case with techniques from differential geometry (Lie rank finiteness).

Central to this work is the analysis of *observation spaces*. One of the critical technical results used relates two different definitions of this space, one in terms of smooth controls and another in terms of piecewise constant ones. The first definition immediately relates to i/o equations, while the other is related to realizability; these two definitions were shown to coincide in a 1989 paper with Wang. The papers [17] and [21] started the study of *integral* as opposed to differential algebraic equations. For linear systems, it is standard to prefilter data, which amounts precisely to using with less-noise-sensitive integral equations. For nonlinear systems, similar motivations were proposed by A.E. Pearson. In the linear case, differential equations are always equivalent to integral ones, but it was shown in the above papers that this is not necessarily the case in general; several relationships between the two types of representations are provided. Another area deals with non-polynomial (and non-rational) realizability. In the textbook by Nijmeijer and van der Schaft it was shown that, under strict constant rank conditions, the outputs of a smooth state space system can be described by an equation of type (13) for which E is a smooth function. In the polynomial case, these conditions are not needed, and the converse holds too. In the analytic case, if there is an equation then there is a realization, but in principle the converse does not hold (unless there are nonsingularity conditions). That is, there are analytic systems and initial states for which the corresponding i/o operators satisfy no possible analytic equation. It is possible that there are partial converses to the above results, however. One possibility is to consider the mapping sending vectors $(x, u(0), u'(0), \ldots, u^{(n-1)}(0))$, consisting of states and $(n-1)$-jets of inputs, into n-jets of outputs and $(n-1)$-jets of inputs $(y(0), y'(0), \ldots, y^{(n)}(0), u(0), u'(0), \ldots, u^{(n-1)}(0))$. This map is analytic, and hence the image of each compact set is a finite union of embedded submanifolds (from results in subanalytic set theory). This implies that, for bounded states and bounded controls (in an appropriate Whitney topology), a finite number of local analytic equations (as opposed to a single global equation) are satisfied by i/o pairs. Preliminary work along these lines was reported in work with Wang (again see [17], [21]).

Our latest, and most unexpected result ([27]) in this general area showed the existence of inputs that are sufficiently rich to permit the identification (in a purely theoretical sense) of arbitrary nonlinear systems. This result constitutes a great generalization of the "universal input theorems" developed by Isidori, Sussmann, the author, and others during the last 20 years; a more detailed journal paper is in preparation.

Information-based complexity ideas

Finally on the topic of identification, we mention the work carried out in the context of information-based complexity In the joint paper [18]-[19], the PI and coauthors were able to extend to a nonlinear class of "fading-memory systems" the previous results known regarding the number of inputs needed for deterministic noisy identification of classes of stable linear systems.

6 List of PI's Publications Supported by this Grant

1. (With Y. Wang) "Algebraic differential equations and rational control systems," *SIAM J. Control and Opt.* **30**(1992): 1126-1149.

2. (With Y. Wang) "Generating series and nonlinear systems: analytic aspects, local realizability, and i/o representations," *Forum Mathematicum* 4(1992): 299-322.

3. (With Y. Lin) "A universal formula for stabilization with bounded controls," *Systems and Control Letters* **16**(1991): 393-397.

4. "Universal nonsingular controls," *Systems and Control Letters* **19**(1992): 221-224. Errata in **20**(1993): 77.

5. (with F. Albertini) "Discrete-time transitivity and accessibility: Analytic systems," *SIAM J. Control and Optimization* **31**(1993): 1599-1622.

6. (with F. Albertini) "Further results on controllability properties of discrete-time nonlinear systems," *Dynamics and Control*, 4 (1994): 235-253.

7. (with Y. Lin and Y. Wang) "Input to state stabilizability for parameterized families of systems," *Intern. J. Robust and Nonlinear Control*, to appear.

8. (with H.J. Sussmann and Y. Yang) "A general result on the stabilization of linear systems using bounded controls," IEEE Trans. Autom. Control, 39 (1994), pp.2411-2425.

9. (With Y. Wang) "Orders of input/output differential equations and state space dimensions," *SIAM J. Control and Optimization*, to appear.

10. (With Y. Wang) "On characterizations of the input-to-state stability property," *Systems and Control Letters*, to appear.

11. (with Y. Lin and Y. Wang) "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Control and Optimization*, to appear.

12. (with W. Liu and Y. Chitour) "On finite gain stabilizability of linear systems subject to input saturation," *SIAM J. Control and Optimization*, to appear.

13. (with W. Liu and Y. Chitour) "On the continuity and incremental-gain properties of certain saturated linear feedback loops," *Intern. J. Robust Nonl. Control*, to appear.

14. "Control of systems without drift via generic loops," *IEEE Trans. Autom. Control*, 1995, to appear.

15. (with Y. Lin) "Control-Lyapunov universal formulas for restricted inputs," submitted.

16. "State-space and i/o stability for nonlinear systems," *Feedback Control, Nonlinear Systems, and Complexity* (B.A. Francis and A.R. Tannenbaum, eds.), Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1995, to appear.

17. (with Y. Wang) "I/O equations for nonlinear systems and observation spaces," *Proc. IEEE Conf. Dec. & Control, Brighton, UK, Dec. 1991*, IEEE Publications, 1991, pp. 720-725.

18. (with M.A. Dahleh, D.N.C. Tse, and J.N. Tsitsiklis) "Worst-case identification of nonlinear fading memory systems," (summarized version) *Proc. Amer. Automatic Control Conference*, Chicago, June 1992, pp. 241-245.

19. (with M.A. Dahleh, D.N.C. Tse, and J.N. Tsitsiklis) "Worst-case identification of nonlinear fading memory systems," *Automatica*, **31**(1995), no. 3, March.

20. (with Y. Lin) "Stabilization with respect to noncompact sets: Lyapunov characterizations and effect of bounded inputs," in *Proc. Nonlinear Control Systems Design Symp., Bordeaux, June 1992* (M. Fliess, Ed.), IFAC Publications, pp. 9-14.

21. (With Y. Wang) "I/O equations in discrete and continuous time," *Proc. IEEE Conf. Dec. & Control, Tucson, Dec. 1992*, IEEE Publications, 1992, pp. 3661-3662.

22. "Numerical path planning based on nonsingular controls," *Proc. IEEE Conf. Dec. & Control, San Antonio, Dec. 1993*, IEEE Publications, 1993, pp. 2706-2711.

23. (with W. Liu and Y. Chitour) "Remarks on finite gain stabilizability of linear systems subject to input saturation," *Proc. IEEE Conf. Dec. & Control, San Antonio, Dec. 1993*, IEEE Publications, 1993, pp. 1808-1813.

24. (with Y. Lin and Y. Wang) "Lyapunov-function characterizations of stability and stabilization for parameterized families of systems," *Proc. IEEE Conf. Dec. & Control, San Antonio, Dec. 1993*, IEEE Publications, 1993, pp. 1978-1983.

25. (with G.A. Lafferriere) "Remarks on control Lyapunov functions for discontinuous stabilizing feedback," *Proc. IEEE Conf. Dec. & Control, San Antonio, Dec. 1993*, IEEE Publications, 1993, pp. 306-308.

26. (with Y. Lin and Y. Wang) "Recent results on Lyapunov-theoretic techniques for nonlinear stability," in *Proc. Amer. Automatic Control Conference, Baltimore, June 1994*, pp. 1771–1775.

27. (With Y. Wang) "Orders of I/O equations and uniformly universal inputs," in *Proc. IEEE Conf. Dec. & Control, Orlando, Dec. 1994*, IEEE Publications, 1994, pp. 1270-1275.

28. (With Y. Wang) "Notions equivalent to input-to-state stability," in *Proc. IEEE Conf. Dec. & Control, Orlando, Dec. 1994*, IEEE Publications, 1994, pp. 3438-3447.

29. (with Y. Lin) Y. Lin and E. Sontag "On control-Lyapunov functions under input constraints," in *Proc. IEEE Conf. Dec. & Control, Orlando, Dec. 1994*, IEEE Publications, 1994, pp. 640-645.

30. (with W. Liu and Y. Chitour) "On the continuity and incremental gain properties of certain saturated linear feedback loops," in *Proc. IEEE Conf. Dec. & Control, Orlando, Dec. 1994*, IEEE Publications, 1994, pp. 127-132.

31. Yang, Y.*, "Global stabilization of nonlinear affine systems with continuously differentiable feedback," *Control–Theo. Adv. Technology* **9**(1993): 1049-1070.

*Graduate student supported by grant but paper not coauthored by PI