Efficient Compilation of High-Level Data Parallel Algorithms

MS-CIS-94-17
LOGIC & COMPUTATION 78

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April 1994
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We present a high-level parallel calculus for nested sequences, NSC, offered as a possible theoretical "core" of an entire class of collection-oriented parallel languages. NSC is based on while-loops as opposed to general recursion. A formal, machine independent definition of the parallel time complexity and the work complexity of programs in NSC is given. Our main results are: (1) We give a translation method for a particular form of recursion, called map-recursion, into NSC, that preserves the time complexity and adds an arbitrarily small overhead to the work complexity, and (2) We give a compilation method for NSC into a very simple vector parallel machine, which preserves the time complexity and again adds an arbitrarily small overhead to the work complexity.
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Abstract

We present a high-level parallel calculus for nested sequences, \( \mathcal{NSC} \), offered as a possible theoretical "core" of an entire class of collection-oriented parallel languages. \( \mathcal{NSC} \) is based on while-loops as opposed to general recursion. A formal, machine independent definition of the parallel time complexity and the work complexity of programs in \( \mathcal{NSC} \) is given. Our main results are: (1) We give a translation method for a particular form of recursion, called map-recursion, into \( \mathcal{NSC} \) that preserves the time complexity and adds an arbitrarily small overhead to the work complexity, and (2) We give a compilation method for \( \mathcal{NSC} \) into a very simple vector parallel machine, which preserves the time complexity and again adds an arbitrarily small overhead to the work complexity.

1 Introduction

There are many advantages to programming in a high-level language. However, while sequential algorithms are most of the time designed and evaluated in reasonably high-level terms, the situation with parallel algorithms is - by necessity, so far - more complicated. The issue is intimately connected with the existing efforts to bridge the gap between the theoretical design of parallel algorithms and practical programming on massively parallel computers.

In the case of data parallelism, the work of Blelloch [Ble90, Ble93] and Blelloch and Sabot [BS90] has made substantial progress on this issue. For example, if we manage to represent an algorithm in a high-level language such as NESL with a certain work and time (a.k.a. element or step) complexity and if the representation satisfies certain restrictions then we are guaranteed an implementation of the same algorithm with the same asymptotic time and work complexity in terms of a low-level parallel vector model, which in turn admits efficient implementations on various architectures, for example the CM2. The present paper is proposing a different treatment of similar goals.

* The authors were partially supported by NSF Grant CCR-9117727.

We start with a somewhat abstract high-level language which represents and manipulates mostly nested sequences (lists) and so we called it \( \mathcal{NSC} \), for nested sequence calculus (section 3). We regard \( \mathcal{NSC} \) as a possible theoretical "core" of an entire class of collection-oriented parallel languages. In keeping with the tenets of data parallelism [HS86], \( \mathcal{NSC} \)'s only parallel operation is map (apply-to-all). We give a precise high-level definition of parallel complexity (in the work and time framework [Jaj92]) for \( \mathcal{NSC} \) programs.

Blelloch [Ble90, Ble93] gives convincing evidence that nested map's on nested sequences (what he calls nested parallelism) can enhance the expressiveness of a data parallel language. But these high-level features are quite removed from concrete parallel architectures or even the parallel vector model and need to be compiled away. Unnesting the nested parallelism is at the center of the compilation technique of [Ble90, BS90, Ble93]. However, in a language with general recursion, this technique is guaranteed to preserve the asymptotic parallel complexity only for programs that satisfy a certain semantic condition called containment.

\( \mathcal{NSC} \) is based on while-loops rather than general recursion. This will surely impose some limitations, although not that many: our first main result consists of showing that a large and practically relevant class of programs, called map-recursive, can be translated into \( \mathcal{NSC} \) while asymptotically preserving the time complexity and adding an arbitrarily small overhead to the work complexity (theorem 4.2). It even turns out that some recursive programs which are not contained in the sense of [Ble90] are in fact map-recursive. The major benefit however is that we can compile \( \mathcal{NSC} \) without the need for an unbounded stack of vectors, as general recursion would require. Avoiding the stack is a good idea because SIMD architectures associate a relatively small memory with each processor. A program that generates many entries in its vector stack will run out of memory even if the vectors are very short and hence much of the total amount of memory of the machine remains unused. We believe that our compilation technique can lead to better memory management. Of course, this needs to be tested in practice.

Following Blelloch, we define a simple parallel vector model in order to describe abstractly the class of target architectures for our compilation method (section 2). Our BVRAM (Bounded Vector Random Access Machine) differs from the VRAM [Ble90] primarily in that it has a finite number of vector registers. This emphasizes the absence of a runtime vector stack. Of course the number of registers needed
depends on the source program being compiled. Another important difference is that we need less powerful communication primitives. The BVRAM has no general permutation instruction, and its communication primitives can be implemented on a butterfly network with \( n \log n \) nodes in \( O(\log n) \) steps. The BVRAM can be efficiently implemented on SIMD architectures such as CM2 and MasPar MP-1, and it has the potential of efficient implementation on MIMD machines as well, such as CM5, Paragon XP/S, KSR1 etc.

Our second main result is a technique that compiles any \( \mathcal{NSC} \) program into a BVRAM program again while asymptotically preserving the time complexity and adding an arbitrarily small overhead to the work-complexity (theorem 7.1). Along the way we also give a simulation that allows us to understand \( \mathcal{NSC} \) complexity in terms of the complexity of computations on a certain flavor of PRAM (proposition 3.2), we show how to implement the BVRAM instructions on a butterfly network (proposition 2.1), we connect \( \mathcal{NSC} \) with some standard parallel complexity classes (proposition 6.2), we show how to represent in \( \mathcal{NSC} \) Valiant's \( O(\log n \log \log n) \) time sorting algorithm [Val75, Ja92] (section 5), and, as part of the compilation process, we define an intermediate abstract language - the sequence algebra - which has the same power as BVRAM's but may prove more flexible in connecting to the designs of the future (section 7).

\( \mathcal{NSC} \) borrows heavily from our experience with languages for collection types [BTS91, BBW92] and it is worthwhile mentioning that many of its operations make as much sense for sets and bags (multisets) as for lists (sequences). It matters to us, though it may not be so relevant to the goals of this paper, that \( \mathcal{NSC} \) is based on a clear, statically checkable type system, that we understand the meaning of this paper, that is based on a clear, statically checkable type system, that we understand the meaning of this paper, that is based on a clear, statically checkable type system, that we understand the meaning of

\section{The Target: Bounded Vector Random Access Machines}

To compile the higher level programming language described in section 3 only a very simple vector parallel model is needed. The Bounded Vector Random Access Machine, BVRAM, is a restriction of the VRAM introduced in [Ble90], in that it only admits a fixed number of registers, and has only particular communication primitives, not a general permutation. The BVRAM can be efficiently implemented on a wide range of parallel architectures, because: (1) only a simple, rather particular form of communication is needed to implement every instruction of the BVRAM, and (2) memory management at each processor is simplified by having only a bounded number of vector registers, as opposed to an unbounded number in the VRAM model.

A BVRAM, \( M \), consists of a fixed number of vector registers \( V_1, \ldots, V_r \). Each \( V_i \) can hold a sequence (a vector) of natural numbers of arbitrary, but finite length. To keep the model simple, we don't include scalar registers: a number is represented by a sequence of length 1. A program for \( M \) is a sequence of labeled instructions, from the following instruction set. For some of the instructions below, it is convenient to view a pair of registers \( V_i, V_j \) in which the length of the first equals the sum of the numbers in the second as a \textit{nested sequence}. E.g., intuitively we view \([x_0, z_1, z_0, z_1, z_2], [2, 0, 3] \) as standing for the nested sequence \([x_0, z_1], [z_0, z_1, z_2] \).

- Move instruction: \( V_i \rightarrow V_j \).
- Arithmetic operations, of the form \( V_i \rightarrow V_j \) op \( V_k \). Here op is an arithmetic operation from a set \( \Sigma \). \( V_j \) and \( V_k \) must be arrays of the same length, and the operation \( op \) is applied simultaneously on all all elements of \( V_j \) and \( V_k \) from the same positions, and the result is stored in \( V_i \). In general we leave \( \Sigma \) unspecified, but mention here that for theorems 4.2 and 7.1 \( \Sigma \) has to contain \( +, -, \ast, /, \text{right-shift}, \log_2 \), while for proposition 6.2 we require that all operations in \( \Sigma \) be in NC. \textit{Monus}, written \( m - n \), is defined as \( m - n \) when \( m \geq n \) and 0 otherwise.
- Sequence oriented operations: \( V_i \rightarrow \{ \} \) loads the empty sequence in \( V_i \). \( V_i \rightarrow \{ n \} \), where \( n \in \mathbb{N} \) loads the singleton sequence \([n]\) into \( V_i \). \( V_i \rightarrow V_j @V_k \) appends \( V_j \) and \( V_k \) and stores the result in \( V_i \). \( V_i \rightarrow \text{length}(V_j) \) computes the length of \( V_j \). \( V_i \rightarrow \text{enumerate}(V_j) \) loads the sequence \([0, 1, \ldots, n - 1]\) into \( V_i \), where \( n \) is the length of \( V_j \).
- Bounded monotone routing \( V_i \leftarrow \text{bm.route}(V_j, V_k, V_l) \); here \( V_j \) and \( V_k \) must have the same length. The effect is that each element in \( V_i \) is replicated a number of times equal to the corresponding number in \( V_j \). In addition, it is required that the result matches in length the sequence \( V_j \) (i.e. initially \( V_j, V_k \) represent a nested sequence). E.g. if \( V_j = [x_0, z_1, z_0, z_1, z_2], V_k = [2, 0, 3] \) and \( V_l = [a, b, c] \), then the instruction \( V_i \leftarrow \text{bm.route}(V_j, V_k, V_l) \) stores \([a, a, c, c, c]\) into \( V_i \).
- Segmented bounded monotone routing \( V_i \leftarrow \text{sbm.route}(V_j, V_k, V_l, V_m) \). Here, \( V_j, V_k \) and \( V_l, V_m \) must be nested sequences, and \( \text{length}(V_j) = \text{length}(V_k) \). Then, the subsequences of \( V_j \) are replicated according to the numbers in \( V_k \) and the result is stored in \( V_i \). E.g., suppose \( V_j = [x_0, z_1, z_0, z_1, z_2], V_k = [2, 0, 3], V_l = [a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2] \) and \( V_m = [2, 3, 3] \). Then, after \( V_i \leftarrow \text{sbm.route}(V_j, V_k, V_l, V_m) \), \( V_i \) will hold the value \([a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3, c_4, c_5]\). In the particular case when \( V_k, V_m \) have length one, this computes the cartesian product of \( V_j \) and \( V_l \). Note that the length of the output is \( \leq \text{length}(V_j) \times \text{length}(V_k) \) and that \( \text{bm.route} \) can be expressed with two \( \text{sbm.route} \) instructions.
- Selection \( V_i \rightarrow \sigma(V_j) \). The effect is that the nonzero values of \( V_j \) are packed and moved into \( V_i \). E.g. if \( V_j = [3, 0, 1, 0, 0, 4] \), then \([3, 1, 4]\) is stored in \( V_i \).
- The unconditional jump goto \( l \) and the conditional jump if empty?\( (V_i) \) then goto \( l \), where \( l \) is a label of some instruction. The conditional jump is taken iff \( V_i \) currently holds the empty sequence.
- halt, stops the program.

We associate with each BVRAM program \( P \) two numbers: \( r_i, r_o \), the number of input and output registers. \( P \) expects \( r_i \) inputs in the registers \( V_1, \ldots, V_{r_i} \), and returns \( r_o \) outputs, in \( V_1, \ldots, V_{r_o} \). For some input, the result of \( P \)
might be undefined, if \( P \) enters an infinite loop, or if an error occurs. For a terminating execution of \( P \), we define the parallel time complexity \( T \) to be the total number of instructions executed by \( P \), i.e. each instruction is considered to have parallel time complexity 1. Similarly, we define the work complexity \( W \) as the sum of the work complexities of all instructions executed by \( P \), where the work complexity of some instruction is defined to be the sum of the lengths of its input and output registers.

As opposed to VRAMs [Ble90] there is no general permutation instruction on a BVRA (but one can be computed with an increase in the time or work complexity). This may lead to more efficient implementations on fixed-connection networks, as exemplified by the following proposition.

**Proposition 2.1** Any BVRA instruction of work complexity \( W \) can be implemented in time \( O(\log n) \) on a butterfly network with \( n \log n \) nodes, where \( n = O(W) \), using only obvious routing algorithms.

**Proof.** (Sketch) The arithmetic operations involve no communication at all, thus can be implemented in \( O(1) \) steps. The append operation \( V_i \rightarrow V_{i+1} \) requires a monotone routing of the values in \( V_i \). This can be done in \( O(\log n) \) steps, using the greedy routing algorithm, see [Lei92], pp. 534. \( bm \cdot \text{route} \) is implemented by a monotone routing, and takes \( O(\log n) \) steps with the greedy algorithm. For \( s \cdot \text{route} \), suppose first that \( |s| = \log(n) \), i.e. \( s \cdot \text{route} \) computes the cartesian product of \( V_i \) and \( V_j \). Also, suppose that the length of \( V_i \) and \( V_j \) are powers of 2, namely \( 2^p \) and \( 2^q \) respectively. Take \( n = 2^{p+q} \); then we have \( 2^p \) packets residing in the first \( 2^q \) rows of a butterfly with \( 2^{p+q} \) rows, and we have to route the packet with address \( v_0 \ldots v_{p-1} \ldots v_{q-1} \ldots v_0 \) to all addresses of the form \( v_1 \ldots v_0 v_{p-1} \ldots v_{q-1} v_0 \). This is done in \( q \) stages, starting with the higher dimension, using the greedy algorithm. In the general case of \( s \cdot \text{route} \), we have to replicate a number of smaller sequences. First, round upwards to the closest power of 2 the length of each such subsequence, and spread the sequences such that each sequence of length \( m \) starts at an address divisible by \( m \). Next, perform in parallel all replications, as described above.

When the number \( n \) of available processors is less than the number \( W \) of elements in an array, then we group \( W \) adjacent elements of the array in the same processor. The above proposition can be extended to this case: some instruction of complexity \( W \) can be implemented on a butterfly network in \( O(\frac{W}{n} \log n) \) steps.

### 3 The Source: The Nested Sequence Calculus (N\( \Sigma \)C)

We use types to explain the structure of \( \Sigma \)C and classify its features. The types are given by the grammar \( t ::= unit | \mathbf{N} | t \times t | t + t | [t] \). unit has exactly one value: the empty tuple \( () \). \( \mathbf{N} \) is the type of nonnegative integers. The values of the product type \( s \times t \) are pairs \((x, y)\), with \( x \in s, y \in t \). \([t]\) is the finite sequences type over \( t \): it contains all sequences \( [x_0, \ldots, x_{n-1}] \), with \( n \geq 0 \) and \( x_0, \ldots, x_{n-1} \in t \). \( s + t \) is the disjoint union type of \( s \) and \( t \); its values are of the form \( \text{in}_1(x) \) with \( x \in s \) and \( \text{in}_2(y) \) with \( y \in t \). We define the boolean type \( \mathbf{B} \) to be \( \text{unit} + \text{unit} \), and identify its values \( \text{true} \) and \( \text{false} \) respectively. Extending the list of built-in types with reals, strings, etc., can be done while preserving all results.

The primitives of \( \Sigma \)C are chosen to be operations naturally associated to its types. Its expressions belong to one of two distinct syntactic categories: terms, denoted by \( M, N, P, U, V, \) etc., which have some type \( t \), and functions, denoted by \( F, G \) etc., have associated two types, the domain \( s \) and codomain \( t \). By abuse of the language we say in this case that the "type" of some function \( F \) is \( s \rightarrow t \). However \( s \rightarrow t \) is not a type per se, which makes constructs like \( s \rightarrow (t_1 \rightarrow t_2) \) or \( (s_1 \rightarrow s_2) \rightarrow t \) impossible. Both terms and functions may contain free variables. See appendix A for a full and formal description of the language:

- Variables \( x \), error \( \Omega \), constants \( n \) (where \( n \in \mathbb{N} \)), arithmetic operations \( M \odot N \), where \( \odot \in \Sigma \) (recall from section 2 that \( \Sigma = \{+, -, \cdot, /, \ldots \} \), and equality \( M = N \).
- Constructs associated with the product type: \( \langle \rangle, \pi_1, \pi_2, (M, N) \). Here \( \langle \rangle \) denotes the empty tuple, \((M, N)\) is a pair, while \( \pi_1, \pi_2 \) are the projections, with the meaning \( \pi_1(x, y) = x, \pi_2(x, y) = y \).
- Constructs associated with the sum type: \( \text{in}_1(M), \text{in}_2(N) \), and case \( M \) of \( \text{in}_1(x) \Rightarrow N \text{in}_2(x) \Rightarrow P \). The latter is defined to be equal to \( N[U/x] \) when \( M = \text{in}_1(U) \), and respectively to \( P[V/y] \) when \( M = \text{in}_2(V) \).
- Constructs associated with functions: \( \lambda x : s.M \) and \( F(M) \). The former is called a lambda abstraction, and is a function (as opposed to a term), of type \( s \rightarrow t \), provided that \( M \) is a term of type \( t \). The second construct, \( F(M) \), is a term called function application having type \( t \), provided that \( F \) is some function of type \( s \rightarrow t \), and \( M \) is a term of type \( s \). Although the type \( s \) is part of the syntax of \( \lambda x : s.M \), we shall drop it when it is clear from the context. Note that \( \lambda x : s.F \), where \( F \) is a function, is not a legal construct in \( \Sigma \)C, nor is \( \lambda x : s \rightarrow t.M \), i.e. no higher order functions are allowed.
- Iteration: \( \text{while}(P, F) \) is some function of type \( t \rightarrow t \), provided that \( P \) and \( F \) are functions of type \( t \rightarrow \mathbf{B} \) and \( t \rightarrow t \) respectively.
- Constructs associated with collections (these constructs work on sequences but also make sense for other kinds of collections, like sets and bags [BBW92]): \( \lbrace \rbrace, [M], M \odot N, \text{flatten}(M), \text{length}(M), \text{get}(M), \text{map}(F) \). Here \( \lbrace \rbrace \) denotes the empty sequence, \([M]\) is the singleton sequence, and \( \odot \) is the append operator. Next, \( \text{flatten}([x_0, \ldots, x_{n-1}]) = x_0 @ x_1 @ \ldots @ x_{n-1} \), and \( \text{length}(M) \) returns the length of some sequence. \( \text{get} \) is defined by \( \text{get}[[x_0, x_1, \ldots]] = x_0 \). Finally \( \text{map}(F) \) is a function of type \( [s] \rightarrow [t] \), provided that \( F \) is a function of type \( s \rightarrow t \). Its meaning is: \( \text{map}(F)([x_0, \ldots, x_{n-1}]) = [F(x_0), \ldots, F(x_{n-1})] \).
- Constructs associated only to sequences, and not to other kinds of collections: \( \text{zip}(M, N), \text{enumerate}(M), \) and \( \text{split}(M, N) \). The meanings are: \( \text{zip}(x_0, \ldots, x_{n-1}), \) 
\( y_0, \ldots, y_{n-1} \) \( = (\{x_0, y_0\}, \ldots, \{x_{n-1}, y_{n-1}\}) \) (zip is undefined if its two arguments have different lengths), \( \text{enumerate}(x_0, \ldots, x_{n-1}) = [0, \ldots, n - 1] \). Finally,
split(M, N) splits M according to the numbers contained in N; e.g. split([a, b, c, d, e, f], [3, 0, 1, 0, 2]) = [[a, b, c], [[d], [e, f]]]. It is defined only if the sum of elements in N equals the length of M.

Note that any function in NSC can, in a fixed amount of time, only increase the size of its input by some polynomial. Had we introduced as a primitive in the language something like [n] = [0, ..., n-1], which generates an arbitrarily long list out of a number, this property would fail. From this small set of primitives, we can derive a rich set of functions. Some examples:

**Database projections.** \( \Pi_i : [t_1 \times t_2] \rightarrow [t_i], \Pi_i \triangleq \text{map}(\pi_i) \)

**Conditions.** if \( x = y \) then M else N is expressed as \( \pi(x, y) \triangleq M \pi_2(x, y) \Rightarrow N \), where \( u, v \) are variables of type unit, not occurring in M, N.

**Broadcasting.** \( \rho_2(x, y, z, \ldots, y_n-1) \), which is defined to be \( [[x, y_0], \ldots, [x, y_{n-1}]] \), can be expressed as \( \rho_2(x, y) \triangleq \text{map}(\lambda(v)(x, v))(y) \) and has the type \( \rho_2 : s \times [t] \rightarrow s \times [t] \). When \( x \) itself is a sequence, \( \rho_2(x, y) \) essentially computes the cartesian product of \( x \) and \( y \). (The name \( \rho_2 \) is motivated by other considerations [BBW92].)

**Bounded monotone routing.** \( \text{bm.route}((u, d), x) \), expressed as \( \Pi_i(\text{flatten}(\text{map}(\rho_2)(\text{zip}(x, \text{split}(u, d)))) \), has type \( \text{bm.route} : (s \times [N]) \times [t] \rightarrow [t] \). \( \text{bm.route} \) is essentially the same operation as defined in section 2. E.g. \( \text{bm.route}([[\text{a}, \text{b}, \text{c}], \text{[a], \text{b}, \text{c}]] \rightarrow [\text{a, a, a, c}] \). The bound \( u \) prohibits us from constructing a very long sequence in constant parallel time. An unbounded monotone routing \( \text{m.route} : [N] \times [t] \rightarrow [t] \) can be defined in NSC (with while), but requires more than a constant number of parallel steps. This is indeed necessary, since \( \text{m.route}(n, [a]) \), produces the sequence \( [a, a, \ldots, a] \) of length \( n \), whose size is not polynomially bounded by the input. Finally, note that in the context of nested sequences, our bounded monotone routing is not truly “monotone”. Indeed, \( \text{bm.route}(([\text{a}, \text{b}, \text{c}], [\text{a}, \text{b}, \text{c}]), [\text{a}, \text{a}, \ldots, 1]) \), and the relative order of \( a, b \) and \( c \) has not been preserved. This forces us to introduce \( \text{sbm.route} \) in the BVRA model.

**Selections.** \( \sigma_i : [s+i] \rightarrow [t], \sigma_j : [s+j] \rightarrow [t], \sigma_k(x) \) selects from some sequence \( x \) only those elements which have the form \( \text{map}_2(x) \), while \( \sigma_k(x) \) selects only the elements of the form \( \text{map}_2(x) \). E.g. if \( x = [\text{map}_1(x), \text{in}_2(b), \text{in}_2(c), \text{in}_2(d), \text{in}_2(e), \text{in}_2(f)] \), then \( \sigma_1(x) = [\text{a}, \text{e}], \sigma_2(x) = [\text{b}, \text{c}, \text{d}, \text{f}] \). \( \sigma_k(x) \) is defined by \( \sigma_k(x) \triangleq \text{flatten}(\lambda(x). \text{case } u \text{ of } \text{in}_1(u') \Rightarrow [u'] \text{ in}_2(u'') \Rightarrow [])(x) \), and \( \sigma_k(x) \) is defined similarly.

**Operations on lists.** first and tail can be defined by:

\[
\text{first}(x) \triangleq \text{get(get(bm.route([[[]]], [1, 0]), split(x, [1, \text{length}(x) - 1])))}
\]

\[
\text{tail}(x) \triangleq \text{get(bm.route([[[[]]], [0, 1]]), split(x, [1, \text{length}(x) - 1])))}
\]

If \( x \) is empty, \( \text{split} \) will produce an error. Similarly we can define \( \text{last} \) and \( \text{remove.last} \), which return the last element, and delete the last element from a sequence, respectively. In general, we can access any element of some sequence of length \( n \) in \( O(1) \) parallel time, and with \( O(n) \) work complexity (we formally define below the time and work complexity). Using \( \text{map} \), we can produce an arbitrary permutation in \( O(1) \) parallel time, but with an increase of the work complexity to \( O(n^2) \). Using radix sort in base \( n^e \), for some arbitrary \( e > 0 \), we can even compute an arbitrary permutation in \( O(1) \) parallel time with \( O(n^{1+e}) \) work complexity. Alternatively, we can use an optimal sorting algorithm (see e.g. [Jaf92]), which reduces the work complexity to \( O(n) \) by increasing the time complexity (e.g. the sorting algorithm described in section 5 has \( T = O(n \log n \log \log n) \)). Thus, the cost of performing an arbitrary permutation is visible in the higher level language.

The compilation theorem 7.1 is robust enough to hold if NSC is extended with additional primitives, like a general permutation \( \text{permute} \) or scan operations, provided that corresponding instructions are added to the BVRA model. E.g. theorem 7.1 can be extended to prove that \( \text{NSC + permute} \) can be efficiently compiled into BVRA+permute. But in its present form theorem 7.1 is stronger, because it proves a general permutation is not necessary in a BVRA in order to compile efficiently a high-level language like NSC. This is of importance in view of the high cost of implementing a general permutation on existing massively parallel architectures [KGL90].

As promised, we will give a high-level definition of parallel time complexity \( T \) and work complexity \( W \) for NSC programs, in an machine independent way. The idea is for the parallel complexity of a program to be inferred from its structure in the same way in which the sequential complexity is inferred from the structure of a program in a sequential language. In our case, all primitive operations (including \( @ \) and \( \text{flatten} \) take one parallel step, while in a \( \text{map}(F)([c_0, \ldots, c_{n-1}]) \), the \( n \) executions of \( F \) are done in parallel. The iteration construct however may count for several steps hence our definition cannot be done solely by induction on programs. This is handled by providing a formal operational semantics and then counting the depth of derivations in it. The work complexity is tied to the size of the data that is being manipulated.

Formally, we start by defining \( \text{S-objects} \) by the grammar:\n
\[
C ::= () | n | (C, C) | \text{in}_1(C) | \text{in}_2(C) | (C, \ldots, C)
\]

where \( n \in \mathbb{N} \). We only consider typed \( \text{S-objects} \) objects. We adopt a unit size complexity measure, and define the size of some \( \text{S-object} \) by \( \text{size}() = \text{size}(n) = 1, \text{size}(C, D) = 1 + \text{size}(C) + \text{size}(D), \text{size}(\text{in}_1(C)) = \text{size}(\text{in}_2(C)) = 1 + \text{size}(C), \text{size}([C_0, \ldots, C_{m-1}]) = 1 + \sum_{i=0}^{m-1} \text{size}(C_i) \). We use \( \text{true} \) and \( \text{false} \) as abbreviations for \( \text{in}_1() \) and \( \text{in}_2() \).

Next, we define the \( \text{evaluation} \) of some term (also called the operational semantics) in a natural semantics style, as in [Kah87]. This consists of rules which simultaneously define a binary relation \( M \Downarrow C \) meaning that the term \( M \) evaluates to the \( \text{S-object} C \) and a ternary relation \( F(C) \Downarrow C' \) meaning that the function \( F \) applied to the \( \text{S-object} C \) evaluates to \( C' \). E.g. if \( F = \lambda x. \text{flatten}(x)@100 \) and \( C = ([3, 5], [2]) \), then \( F(C) \Downarrow [3, 5, 2, 100] \).

Some representative rules are:

\[
\begin{align*}
M \Downarrow m & \quad N \Downarrow n \\
M \Downarrow (C, D) & \quad M \Downarrow (C, D) \\
\pi_1(M) \Downarrow C & \quad \pi_2(M) \Downarrow D \\
(M, N) \Downarrow (C, D) & \quad (M, N) \Downarrow (C, D) \\
M \Downarrow C & \quad M \Downarrow C \\
M \Downarrow C & \quad M \Downarrow C \\
M \Downarrow C & \quad M \Downarrow C \\
M \Downarrow C & \quad M \Downarrow C \\
F(C) \Downarrow D & \quad F(C) \Downarrow D \\
\text{map}(F)([C_0, \ldots, C_{m-1}]) \Downarrow [D_0, \ldots, D_{m-1}] & \quad \text{map}(F)([C_0, \ldots, C_{m-1}]) \Downarrow [D_0, \ldots, D_{m-1}] \\
\text{permute}(C) \Downarrow \text{false} & \quad \text{permute}(C) \Downarrow \text{false} \\
\text{while}(P, F)(C) \Downarrow C & \quad \text{while}(P, F)(C) \Downarrow C
\end{align*}
\]
\[
P(C) \uparrow \text{true} \quad F(C) \uparrow C' \quad \text{while}(P, F)(C') \uparrow D \\
\text{while}(P, F)(C) \uparrow D
\]

The complete set of rules is given in appendix B where we explain a technical complication caused by the presence of bound variables (lambda abstraction) in the language, namely the need to use environments as in [Cur88].

Thus, to evaluate some closed term \(M\), one has to construct a proof tree, whose nodes are labeled with rules of the operational semantics, such that its root is labeled with some rule with conclusion \(M \downarrow C\). Based on this operational semantics, we now define the time and work complexity of \(\text{NSC}\) in a machine independent way.

**Definition 3.1** Consider some \(\text{NSC}\) term \(M\). The time and work complexity \(T(M), W(M)\) of \(M \downarrow C\) are defined by induction on the proof of \(M \downarrow C\). The induction is done simultaneously with the definitions of the time and work complexity \(T(F, C)\) and \(W(F, C)\) of some evaluation \(F(C) \downarrow D\), where \(F\) is a \(\text{NSC}\) function, and \(C, D\) are \(S\)-objects. Except for the rules for map and while, for every rule of the form:

\[
M_1 \downarrow C_1, \ldots, M_n \downarrow C_n, \quad M \downarrow C
\]

we define:

\[
T(M) \equiv 1 + \sum_{i=1}^{n} T(M_i) \quad W(M) \equiv \text{SIZE} + \sum_{i=1}^{n} W(M_i)
\]

where \(\text{SIZE}\) is the total size of all \(S\)-objects mentioned in the rule (in the premises and the conclusion, including the environments). For the map-rule, the definition of \(W\) remains the same, while the definition of \(T\) becomes:

\[
T(M) \equiv 1 + \max_{i=1}^{n} (T(M_i))
\]

(this corresponds to the fact that the function is applied in parallel on all objects in the sequence). For the while rule we do not include in \(\text{SIZE}\) the size of the output \(D\) (otherwise, the final output \(D\) of \(\text{while}\) would be counted as many times as many iterations are performed by \(\text{while}\)). More precisely, if the last rule of while of \((P, F)(C) \downarrow D\) was:

\[
P(C) \uparrow \text{true} \quad F(C) \uparrow C' \quad \text{while}(P, F)(C') \downarrow D \\
\text{while}(P, F)(C) \downarrow D
\]

then:

\[
T(\text{while}(P, F), C) \equiv 1 + T(P, C) + T(F, C) + T(\text{while}(P, F), C') \\
W(\text{while}(P, F), C) \equiv \text{size}(C) + \text{size}(C') + W(P, C) + W(F, C) + W(\text{while}(P, F), C')
\]

(i.e. size \((D)\) is not included explicitly in \(W(\text{while}(P, F), C))\)

The language \(\text{NSC}\) together with its notions of time and work complexity is a model of parallel computation in its own right but parallel algorithms are most commonly given in terms of one of the several known flavors of PRAM. To facilitate comparisons, we offer the following efficient simulation (\(\text{NSC}\)’s version of Brent’s scheduling principle, as it were):

**Proposition 3.2** Any \(\text{NSC}\) function of time complexity \(T\) and work complexity \(W\) can be simulated on a \(\text{CREW PRAM}\) with scan primitives using \(p\) processors with asymptotic complexity \(O(T + W/p)\).

**Proof.** (Sketch) Given some function \(f\) in \(\text{NSC}\), first flatten \(f\) for an extended version of a BVPRAM, with unbounded many vector registers and indirect addressing (essentially the VRAM of [Ble90], but with the communication primitives described in section 2). The resulting extended-BVPRAM program has the same time and work complexity as \(f\): see remark 7.3. Next use the simulation of an extended BVPRAM on a CREW with scan primitives, in the spirit of [Ble90]. We need a CREW instead of a EREW in order to simulate \(\text{bsm}._\text{route}\) and \(\text{sbm}._\text{route}\). \(\square\)

4 Expressing map-recursive functions in \(\text{NSC}\)

Although it is described in a concise, mathematical style (notice that we called it a "calculus" rather than a "language") \(\text{NSC}\) can be easily extended to a more user-friendly language, by allowing a certain amount of block structure: definitions of global/local variables and of nonrecursive functions. There is a straightforward translation of such an extension back into \(\text{NSC}\), which we omit from this extended abstract. Accommodating recursive functions though, is a more delicate problem, which we address here.

Consider the following limited form of recursion:

**Definition 4.1** A function definition is map-recursive if it has the form

\[
\text{fun } f(x) = c(x, \text{map}(f)(d(x)))
\]

First, it is easy for a compiler to check whether a recursive definition is of this form (in contrast, containment [Ble90] is an undecidable property). Second, this form is general enough to express many existing parallel algorithms: tail recursive definitions, and what is usually meant by divide-and-conquer recursion (for instance the worked example in section 2) are map-recursive. Here are some recursion schemata and a sketch of how to convert them into map-recursive form (and in the process "parallelize" them):

\[
\text{fun } g(x) = \text{if } p(x) \text{ then } s(x) \text{ else } c(g(d_1(x)), g(d_2(x)))
\]

\[
\text{fun } h(x) = \text{if } p(x) \text{ then } s(x) \text{ else } c(h(d(x)))
\]

\[
\text{fun } k(x) = \text{if } p'(x) \text{ then } c(k(d_1(x)), k(d_2(x))) \text{ else } c'(k(d_1(x)), k(d_2(x)))
\]

For \(g\), we construct a list of length 2, and recursively map \(g\) on it (QuickSort has this form). For \(h\), the list will have length 1 (tail recursion is a particularization of this form). \(k\) is more interesting, since it divides its input into either two or three subproblems. Note that it is not contained [Ble90], so the compilation techniques described here work on some cases on which those of [Ble90] don’t. In converting \(k\), the list will have length 1, 3 or 4, where the first element is a tag, and \(k\) is slightly modified to return the identity on the tag (a sum of types is used here).

The first of our two main results states that map-recursive can be translated (in a source-to-source manner) into a \(\text{NSC}\) expression, while preserving its time complexity and "almost" preserving its work complexity.
Theorem 4.2 Consider some function \( f \) defined in \( \text{NSC} \) extended with map-recursion, with time and step complexity \( T, W \). Then, for any \( \varepsilon > 0 \), one can construct a function \( f' \) in \( \text{NSC} \) which is equivalent to \( f \) and which has time and work complexity \( T' = O(T) \) and \( W' = O(W^{1+\varepsilon}) \) respectively. Moreover, if the divide and conquer tree of \( f \) is balanced, then \( W' = O(W) \).

Proof. For illustration, we consider only the function \( g \) from above. Suppose the types are: \( s : s \to t, d_1, d_2 : s \to s \), and \( c : t \times t \to t \). Not surprisingly, \( g \) can be expressed in \( \text{NRA} \), without recursion, in two steps, called divide phase and combine phase in [MH88]:

Divide Phase Start with the singleton sequence \( y = [x] \) of type \( s \), and apply repeatedly the function \( \text{flatten} \circ \text{map}(\lambda x, \text{if } p(x) \text{ then } [x] \text{ else } [d_1(x), d_2(x)]) \) having the type \( s \to [s] \), until all its elements satisfy the predicate \( p \). (We need to tag the elements resulting from \([x]\), to avoid applying \( p \) repeatedly on them; we omit the details.) Call \( y \) the resulting sequence.

Combine Phase Start by \( \text{map-ing} \) the function \( s \) on \( y \), and then apply repeatedly \( c \) to adjacent elements of \( y \). Some additional bookkeeping is necessary to make sure \( c \) is applied to the correct pairs (e.g., it suffices to store the depth in the divide and conquer tree for each element in \( y \), and only combine adjacent elements if they have the same depth). Stop when there is only one element in the resulting list.

Obviously, the translated \( g \) will have time complexity \( O(T) \). The work complexity is also preserved, in the case in which the divide an conquer tree for the computation of \( g(x) \) is perfectly balanced. When the is unbalanced, the leaves which are reached sooner have to coexists in the same sequence with those nodes which need more divide steps, thus adding to the total work complexity. Let \( \nu \) be the number of different levels in the divide and conquer tree which contain leaves. E.g., in an almost perfectly balanced tree, \( \nu = 1 \) or \( \nu = 2 \), while in a total "unbalanced" tree, \( \nu \) can be equal to the total number of leaves, but still \( \nu \leq W(g, z) \). We can compute \( \nu \) in time and work complexity \( O(T), O(W) \), by simulating only the divide phase, without retaining the results. Let \( \varepsilon > 0 \). We improve the divide phase, such that the time and work complexities of the translation of \( g \) into \( \text{NSC} \) become \( O(T) \) and \( O(\nu W) \) respectively. Namely, we start with \( \frac{1}{2} + 1 \) variables \( z_i, i = 0, \ldots, \frac{1}{2} \), initialized to \([\cdot]\), and with \( y \) initialized to the singleton \([x]\). We apply repeatedly the divide phase on \( y \); whenever some leaves are reached, we move them into \( z_0 \). We only allow \( z_0 \) to be touched \( \nu' \) times, after which we move its entire content into \( z_1 \), and empty \( z_0 \). We repeat this process, but only allow \( z_1 \) to be touched \( \nu'' \) times, at which point, we empty \( z_1 \), by moving everything into \( z_2 \). In general, we allow \( z_i \) to accumulate only \( \nu'' \) times, after which we empty it, by moving everything into \( z_{i+1} \). Obviously, a number of \( \nu'' \) levels of leaves must be discovered, before making one move into \( z_1 \); thus, \( z_2 \) will be filled exactly once, with the leaves from all \( \nu' \) levels. To compute the total additional work complexity, observe that each leaf travels exactly once through \( z_0, z_1, \ldots, z_2 \), and in each \( z_i \) is "touched" exactly \( \nu'' \) times. Thus, the total work complexity is bounded by \((\frac{1}{2} + 1)\nu W = O(\nu W)\). Of course, rather complicated bookkeeping is necessary to keep all elements in \( z_i \) sorted. The combine phase is done similarly, but in reverse.

The technique of theorem 4.2 seems to extend to more general recursion schema than the limited recursion. The main kind of recursion to which this technique does not apply is one in which some recursive call to \( f \) uses an argument which is computed with a recursive call itself, in the style of the Ackerman function: \( A(x, y) = A(x - 1, A(x, y - 1)) \). We argue that very few practical algorithms make indeed use of such recursion schemas.

5 An \( O(\log n \log \log n) \) Mergesort Algorithm Expressed in \( \text{NSC} \)

As evidence for the practical expressiveness of \( \text{NSC} \) we describe in it Valiant's fast mergesort algorithm [Val75, Jaj92], see the program in figures 1, 2, 3. As we have explained at the beginning of section 4 we are free to use block structure (we choose a syntax close to ML [MTH90]). More importantly, in view of theorem 4.2 we are free to use map-recursion definitions, or other recursive schemas which are convertible to map-recursion. The main function mergesort in figure 1 has the same recursion schema as the function \( g \) of section 4 and hence can be converted to a map-recursion form. Its parallel time complexity is \( O(\log n \log \log n) \).

The fast, \( O(\log \log m) \) time merge function exhibits a more complicated kind of map-recursion. To merge two sequences \( A = [a_0, \ldots, a_{m-1}] \), \( B = [b_0, \ldots, b_{m-1}] \), we divide \( A \) into \( \sqrt{m} \) subsequences of length \( \leq \sqrt{m} \); let \( AA = [A_0, \ldots, A_{\sqrt{m}-1}] \) be the resulting nested sequence. Next, we find for each subsequence \( A \); the corresponding subsequence \( B \) in \( B \), with which \( A \) has to be merged, and apply recursively merge on all pairs \((A_i, B_i)\); let \( BB = [B_0, \ldots, B_{\sqrt{m}-1}] \). Thus, the general structure of merge is:

\[
\text{fun merge}(A, B) = \\
\quad \text{if } \text{length}(A) \leq 2 \text{ then direct.merge}(A, B) \\
\quad \text{else } \ldots \text{ compute } AA, BB \text{ as explained in flatten(map(merge)(zip(AA, BB)))) end}
\]

which can be obviously translated into a map-recursion.

Figures 2 and 3 contain some auxiliary functions used in merge. The function \( \text{index.split}(C, I) \) expects a sorted sequence of indexes \( I = [i_0, \ldots, i_{k-1}] \) and, for \( C = [C_0, \ldots, C_{i_k-1}] \), returns the sequence \([C_{i_0}, \ldots, C_{i_k-1}]\); it has constant time complexity and work complexity = \( O(n + k) \). The function \( \text{index.split}(C, I) \) splits \( C \) according to the indexes in \( I \), again provided that \( I \) is sorted, with similar time and work complexity. We use the construct \( \text{filter}(P) : [i] \to [i] \), which for some predicate \( P : t \to \text{B} \) returns the sequence of all elements satisfying \( P \). It is expressible in \( \text{NSC} \) by:

\[
\text{filter}(P)(x) = \text{flatten(map}(\lambda u, \text{if } P(u) \text{ then}[u] \text{ else } [])(x))
\]

The functions \( \text{first}, \text{tail}, \text{last}, \text{remove.last} \) and \( \text{bm.route} \) are defined in section 3.

Using the techniques described in [Jaj92], the merge function can be transformed to become optimal, i.e. to reduce its work complexity from \( O(m + n) \log \log m) \) to \( O(m + n) \). This also gives us an optimal (i.e. \( O(n \log n) \) work complexity), \( O(\log n \log \log n) \)-time sorting function. The divide-and-conquer trees for both the sorting and the merging function are balanced, hence the translation of theorem 4.2 gives us an optimal \( O(\log n \log \log n) \)-time sorting function in \( \text{NSC} \).
fun mergesort(A) =
  if length(A) ≤ 1 then A
  else let val n = length(A)
      val AA = split(A, [n - n/2, n/2])
      in merge(mergesort(first(AA)),
               mergesort(last(AA)))
  end

fun merge(A, B) =
  if length(A) ≤ 2 then
dlirect_merge(A, B)
  else
      let val m = length(A)
          val ra = length(B)
          val R' = sqrt-positions(A)
          val S' = sqrt-positions(B)
          val R = direct-rank(A, B)
          val BB = sqrt-split(B)
          val a = map(rank-one)(zip(A', index(BB, R')))
          val RR' = map(X(x, y).(x - 1) * y + y)(zip(R', RR'))
          val AA = sqrt-split(A)
          val BB = index-split(B, R)
      in flatten(map(merge)(zip(AA, BB)))
  end

Figure 1: Valiant's O(log n log log n) sorting algorithm.

fun rank-one(a, B) =
  let val n = length(B)
      val A' = sqrt-positions(a)
      val B' = sqrt-positions(B)
      val R' = direct-rank(A, B')
      val BB = sqrt-split(B) (* split B into sqrt n blocks *)
      val a = zip(A', index(BB, R')) (* group each a' with its block *)
      val RR' = map(rank-one)(a) (* rank each a' in its block *)
  in map(X(x, y).(x - 1) * y + y)(zip(R', RR'))
end

fun sqrt-positions(C) =
  let val n = length(C)
      val I = filter(X x. i mod sqrt n = 0)(enumerate(C))
      in index(C, I)
  end

fun index(C, I) =
  let val n = length(C)
      val fc = length(I)
      val delta I = map(X x. (x - 1) * sqrt n + y)(zip(R', RR'))
      val P = sqrt-route((C, delta I), zero_to I)
      val R = map(X(x, y).(x - 1) * y + y)(zip(P, remove_last([0] @ P)))
      val AA = sqrt-split(A)
      val BB = index-split(B, R)
  in flatten(map(merge)(zip(AA, BB)))
end

fun sqrt-split(C) =
  index-split(C, enumerate(C))

fun direct-split(A, B) =
  let val m = length(A)
      val R = direct-rank(A, B)
      val BB = index-split(B, R)
  in first(BB) @ flatten(map(merge)(zip(A', index(BB, R))))
end

fun direct-rank(A, B) =
  map{X a. rank-one(a, B)}(A)

fun sqrt-positions(enumerate(C)) =
  index-split(C, sqrt-positions(enumerate(C)))

fun direct-merge(A, B) =
  let val m = length(A)
      val R = direct-rank(A, B)
      val BB = index-split(B, R)
  in first(BB) @ flatten(map(merge)(zip(A', index(BB, R))))
end

fun index_split(C, I) =
  let val n = length(C)
  in split(C, map(X x. (x - 1) * sqrt n + y)(zip([0] @ I)))
end

Figure 2: Auxiliary functions used in merge.

Figure 3: The functions index and index_split.

6 Theoretical Expressive Power

In this section we give evidence that NSc is not too restrictive, as a tool for designing parallel algorithms. Namely,
let CRCW-TIME-PROC(T(n)), P(n)) be the set of functions computable on a CRCW PRAM in time T(n) using P(n)
processors, and NSc-TIME-WORK(T(n), W(n)) the set of functions expressible in NSc with time and work complexity T(n), W(n).

Proposition 6.1 For T(n), W(n), that are suitable (in the sense of [SV84]), we have:

\[ \text{CRCW-TIME-PROC}(O(T(n)), O(W(n))) \subseteq \text{MSC-TIME-WORK}(O(T(n)), O(W(n))) \]

More, we get equality, if in the definition of NSc we restrict
the arithmetic operations to the set \( \Sigma = \{+, -\} \), and if we replace the unit size complexity (size(n) \( \equiv \) 1 - see section 2)
with the logarithmic size complexity (size(n) \( \equiv \) log n), in
the definition of the work complexity of NSc.

The proof uses a theorem in [SV84], credited to Ruzzo and Tompa, relating CRCW PRAM's to Alternating Turing
Machines, and is omitted from this extended abstract. Using
the above proposition and proposition 3.2 we can establish
that NC coincides with the functions in NSc with polylogarithmic time and polynomial work complexity. Recall that
NSc is parameterized by a set \( \Sigma \) of arithmetic operations.

Proposition 6.2 Suppose all arithmetic operations in \( \Sigma \) are
in NC. Then:

\[ \text{NC} = \text{NSc-TIME-WORK}(O(1) n, n O(1)) \]

7 Efficient Compilation of NSc to BVRAM

Theorem 7.1 (Compilation Theorem) For every function f in NSc with time and work complexity T, W, there is
a BVRAM, M, such that: \( \forall e > 0, \) there is some program P
for M, equivalent to f, having time complexity \( T' = O(T) \)
and \( W' = O(W^{1+e}) \).
Note that, in contrast to theorem 4.2, the number of registers only depends on \( f \) and not on \( \epsilon \). A \texttt{while}-construct can be rewritten as a tail recursive function, hence is contained, according to the definition in [Ble90], and therefore the compilation technique described there (for a VRAM, with unbounded many vector registers) preserves its step and work complexity. However, we cannot apply that compilation technique here. Indeed, when viewed as tail recursive function, the work complexity of \texttt{while} may increases significantly, because the final result after iterating \( n \) steps is touched additional times, as the tail recursive function returns from its calls. In the definition of the work complexity for \texttt{while}, these \( n \) additional touches are not counted (see definition 3.1). So the tail recursive translation has a higher work complexity than the original \texttt{while} construct.

We need a stronger compilation technique in order to stay within the lower work complexity. Moreover, we also only have a bounded number of vector registers. The proof goes through the following steps:

- **Variable Elimination.** We translate \( \mathcal{NSC} \) into a rather similar, but variable free language called Nested Relational Algebra, \( \mathcal{NSA} \). The new language only contains functions \( f : s \to t \), i.e. no terms. Some term \( M \in \mathcal{NSC} \), of type \( s \) and with free variables \( x_1 : s_1, \ldots, x_n : s_n \), will be translated into a function \( f_M : s_1 \times \ldots \times s_n \to t \) in \( \mathcal{NSA} \). The primitive functions and the constructs in \( \mathcal{NSC} \) correspond roughly to those in \( \mathcal{NSA} \), with only additional primitives: the function \( \rho_s : s \times [t] \to [s \times t] \) (see section 3 for its definition). The step and work complexity of functions expressed in \( \mathcal{NSC} \) and \( \mathcal{NSA} \) are the same. We omit the description of \( \mathcal{NSA} \) from this extended abstract; it can be found in appendix C.

- **Flattening.** We define a language for flat sequences, called Sequence Algebra \( \mathcal{SA} \), and translate \( \mathcal{NSA} \) into \( \mathcal{SA} \). Namely, for any \( \epsilon > 0 \), we show how to translate a function \( f \) of \( \mathcal{NSA} \) with time and work complexity \( T, W \) into an equivalent function in \( \mathcal{SA} \) (thus using only flat types), with time and work complexity \( O(T) \) and \( O(W^{1+\epsilon}) \). Of course, any function in \( \mathcal{SA} \) can be expressed in \( \mathcal{NSA} \) with the same time and work complexity.

- We show that \( \mathcal{SA} \) and BVRAM are equivalent, in the sense that any function in \( \mathcal{SA} \) can be simulated by a BVRAM with the same time and work complexity, and conversely. One direction of this equivalence helps us completing the compilation, while the other direction allows us to perform optimizations at the level of the language \( \mathcal{SA} \), instead of BVRAM.

### 7.1 The Sequence Algebra, \( \mathcal{SA} \)

The Sequence Algebra, \( \mathcal{SA} \), only has flat types. More precisely, we define first scalar types by the grammar: 

\[
\begin{align*}
s &::= \text{unit} \mid [N] \mid s \times s \mid s + s, \\
t &::= \text{unit} \mid [s] \mid [t] \mid [t] + [t].
\end{align*}
\]

\( \mathcal{SA} \) was designed by choosing some set of functions expressible in \( \mathcal{NSA} \) (or, equivalent, \( \mathcal{NSC} \)) over flat types, which seemed to be enough to allow the language \( \mathcal{NSA} \) to be translated (flattened) into \( \mathcal{SA} \). In addition, \( \mathcal{SA} \) is defined in an inductive way, which enables us to prove, by induction, properties about the functions expressible in \( \mathcal{SA} \), e.g. lemma 7.2. \( \mathcal{SA} \) stands in the same relationship to \( \mathcal{NSA} \) as the relational algebra stands to the nested relational algebra [AB88].

Similar to \( \mathcal{NSA} \), \( \mathcal{SA} \) is a variable-free language, containing some primitive functions, and a set of rules for combining them in order to get more complex functions. We briefly describe \( \mathcal{SA} \) below. A complete description of the language can be found in appendix D.

- **Error, viewed as a function \( \Omega : \text{unit} \to t \).**
- **map**'s of scalar functions, \( \text{map}(\varphi) : [s] \to [s] \), where \( \varphi : s \to s' \) is a scalar function, i.e., informally, a function defined in \( \mathcal{NSA} \) (or, equivalently \( \mathcal{NSC} \)) having only scalar types as input, output, and intermediate types, and without \texttt{while}.
- **Operations on sequences:** the empty sequence \([\ ]\), append \( @ \), length of a sequence, defined as \( \text{length}(z) = n \), where \( n \) is the length of \( x, \text{zip}, \text{bm.route}, \text{sm.route}, \) selections \( s_1, s_2 \) (see section 3), and the emptyness test \( \text{empty?} \), of type \([s] \to \text{B}\).
- **Functions over flat types:** the identity \( \text{id} : t \to t \), composition of functions \( g \circ f \), projections \( \pi_i : t_i \times t_2 \to t_i \), pairing of functions \( f, g \), injections \( \text{inj}_i : t_i \to t_i + t_2 \), and sum of functions \( f_1 + f_2 : t_1 + t_2 \to t \), where \( f_i : t_i \to t \) (an if construct can be derived from this).
  
  The latter is defined by: 
  \[
  (f_1 + f_2)(\text{inj}_1(x)) \equiv f_1(x) \\
  (f_1 + f_2)(\text{inj}_2(x)) \equiv f_2(x).
  \]
- **Iteration:** \texttt{while}(p, f) is a function of type \( t \to t \), whenever \( f : t \to t \) and \( p : t \to \text{B} \) (recall that \( \text{B} = \text{unit+unit} \) and, thus, is a type of \( \mathcal{SA} \)).

As for \( \mathcal{NSC} \) we define the the time and work complexity for some evaluation \( f(C) \), where \( f \) is a function in \( \mathcal{SA} \) and \( C \) is its input (a flat S-object). Note that in the absence of a general map there is no nested parallelism in \( \mathcal{SA} \).

Although \( \mathcal{SA} \) does not contain nested types, like \([N \times N] \), it is strong enough to allow such types to be encoded into flat types. The key technical tool for that is to encode some nonflat type \([t] \), where \( t \) is a flat type, by some flat type \( \text{SEQ}(t) \). For this we use segment descriptors, as in [Ble90]. Formally, we transform some flat type \( t \) into another flat type \( \text{SEQ}(t) \), defined by induction on \( t \):

\[
\begin{align*}
(1) \text{SEQ}([\text{unit}]) & \equiv [\text{N}] (2) \text{SEQ}([s]) \equiv [\text{N}] \times [s], (3) \text{SEQ}(t \times t') & \equiv \text{SEQ}(t) \times \text{SEQ}(t'), (4) \text{SEQ}(t + t') & \equiv [\text{B}] \times \text{SEQ}(t) \times \text{SEQ}(t').
\end{align*}
\]

The idea is that \( \text{SEQ}(t) \), although a flat type, can encode sequences of elements from \( t \), i.e. values of type \([t]\). The main technical fact enabling us to prove efficient compilation is the following map lemma.

**Lemma 7.2 (The Map Lemma).** Let \( f : t \to t' \) be some function in \( \mathcal{SA} \), and let \( T, W \) be the time and work complexity of \( \text{map}(f) \) (recall that map\( f \) is in \( \mathcal{NSC} \), but not in \( \mathcal{NSA} \)). Then, for every \( \epsilon > 0 \), there exists some function \( \text{SEQ}(f) : \text{SEQ}(t) \to \text{SEQ}(t') \) in \( \mathcal{SA} \), of time complexity \( O(T) \) and work complexity \( O(W^{1+\epsilon}) \), which simulates \( \text{map}(f) : [t] \to [t'] \). More, the structure of \( \text{SEQ}(f) \) is independent of \( \epsilon \), which implies that "number of vector registers" used by \( \text{SEQ}(f) \) is independent of \( \epsilon \).

**Proof.** (Sketch) This is done by induction on the structure of \( f \). When \( f \) is map of a scalar function, \( \text{SEQ}(f) \) is essentially the same map. When \( f \) is some operation on
is another

We describe

We will use

Suppose

also

which implies

Since

One

The simulation

Finally, we can prove:

Proposition 7.4 Let \( f : s \rightarrow s' \) be some function in \( NSA \) with time and work complexity \( T, W \). Then, for every \( \varepsilon > 0 \), there is some function \( COMPILE(f) : COMPILE(s) \rightarrow COMPILE(s') \) in \( SA \) which "simulates \( f \)". i.e. for every \( s \), \( COMPILE(f)(encode(x)) = \text{encode}(f(x)) \), with time and work complexity \( O(T), O(W^{1+\varepsilon}) \). Moreover, \( f \) requires "the same number of BVRAM registers" for every \( \varepsilon \).

Proof. (Sketch) By induction on the structure of \( f \).

All cases are straightforward, except for the case when \( f = \text{map}(g) \), where we use the Map lemma.

7.2 Equivalence of \( SA \) and BVRAM

The types in \( SA \) are slightly richer than those of the BVRAM. \( SA \) allows for types like \( \text{unit} + N \times N + N \times N \times [N] + \text{unit} \), while the types on the BVRAM are only of the form \( [N] \times \ldots \times [N] \). However, encoding of \( SA \) types into BVRAM types is straightforward.

Proposition 7.5 \( SA \) and BVRAM are equivalent, i.e. any function \( f \) in \( SA \) with time and work complexity \( T, W \) can be simulated on a BVRAM with the same time and work complexity, and conversely.

Proof. Simulating some function of \( SA \) by a BVRAM program is easily done by induction on the structure of that function. The converse is slightly more involved. Indeed, let \( r \) be the number of registers of a BVRAM program \( M \), and \( h \) some function in \( SA \) of type \([N] \times ([N])^r \rightarrow [N] \times ([N])^r \) performing one step of the program of \( M \) (where the program counter is encoded by a singleton sequence, on the first position). By iterating \( h \) we indeed achieve the desired time complexity, but not the work complexity, since at each step, the function \( h \) touches all \( r \) registers. To avoid this, we define a sequence of \( r \) functions \( f_i, i = 1, r \). The inputs and outputs for \( f_i \) are: the values of the \( i \) "smallest" registers, at some particular moment, the indexes of these \( i \) registers, the size \( S \) of the
next largest register, and the program counter. \( f_i \) iterates the one-step function as long as it only affects the \( i \) registers it sees, and as long as all the \( i \) sizes stay less than \( S \). If any of these conditions is violated, \( f_i \) stops. To do its job, \( f_i \) calls \( f_{i-1} \), which iterates steps on \( M \) by only looking at the smallest \( i - 1 \) registers: when \( f_{i-1} \) finishes, \( f_i \) tries to do one more step by taking into account the \( i \)'s smallest register as well, which \( f_{i-1} \) ignores. If it cannot, then it returns (to \( f_{i+1} \)). Else, it performs the operation, and calls \( f_{i-1} \) again, possibly with a different set of \( i - 1 \) registers, from the set of \( i \) registers it sees.

Although only one direction of proposition is actually needed for the compilation theorem 7.1, the converse is significant from the point of view of optimizations: it implies that any optimizations done for the BVRAM can also be performed at the level of the \( SA \) language.

8 Conclusions

We intend to use \( \mathcal{NSC} \) as a core for a "real" parallel language for querying nested collections, by adding proven features such as those encountered in functional languages like ML. Guaranteed complexity bounds such as those emerging from this paper can serve as useful guidelines for language design, especially in the database area. Of course, the techniques we have used in the translation of map-recursion and in the unnesting of nested parallelism need to be validated by practical implementations. Equally important is to continue to investigate the practical expressiveness of \( \mathcal{NSC} \) by attempting to represent various known efficient parallel algorithms. Another direction of investigation is to develop optimization techniques for this language by using ideas that have been proved useful in databases.

References


A The Nested Sequence Calculus \( \text{NSC} \)

We define a type context \( \Gamma \) to be a set of the form \( \Gamma = \{ x_1 : s_1, \ldots, x_n : s_n \} \), where \( x_i \) are variables and \( s_i \) are types. We write \( \Gamma \vdash M : t \), or \( \Gamma \vdash F : s \to t \), when we want to say that, under the type assumptions of \( \Gamma \), the term \( M \) has type \( t \), or the function \( F \) has type \( s \to t \). Below are the rules defining the language. Recall that \( \mathbb{B} \overset{\text{def}}{=} \text{unit + unit} \).

### Variables, Errors, Constants, Arithmetic

\[
\begin{align*}
\frac{x : t, \Gamma \vdash x : t}{\Gamma \vdash \Delta x : t} & \quad \frac{\Gamma \vdash M : \mathbb{N}}{\Gamma \vdash \Omega^i : t} & \quad \frac{\Gamma \vdash n : \mathbb{N}^- (n \in \mathbb{N})}{\Gamma \vdash M : \mathbb{N}} \\
\frac{\Gamma \vdash M : \mathbb{N}}{\Gamma \vdash \text{op} \, N : \mathbb{N}} & \quad \frac{(\text{op} \in \Sigma)}{\Gamma \vdash M : \mathbb{N}} & \quad \frac{\Gamma \vdash \text{M} : \mathbb{N}}{\Gamma \vdash \text{N} : \mathbb{N}}
\end{align*}
\]

### Type products

\[
\begin{align*}
\frac{}{\Gamma \vdash () : \text{unit}} & \quad \frac{\Gamma \vdash (M, N) : s \times t}{\Gamma \vdash \pi_1(M) : s} & \quad \frac{\Gamma \vdash \pi_2(M) : t}{\Gamma \vdash \pi_2(M) : t}
\end{align*}
\]

### Type sums

\[
\begin{align*}
\frac{}{\Gamma \vdash \text{in}_1(M) : s + t} & \quad \frac{\Gamma \vdash \text{in}_2(M) : s + t}{\Gamma \vdash \text{case} \ M \ \text{of} \ \text{in}_1(x) \Rightarrow N \ | \ \text{in}_2(y) \Rightarrow P : u}
\end{align*}
\]

### Functions

\[
\begin{align*}
\frac{x : s, \Gamma \vdash M : t}{\Gamma \vdash \lambda x : s. M : s \to t} & \quad \frac{\Gamma \vdash F : s \to t, M : s}{\Gamma \vdash F(M) : t}
\end{align*}
\]

### Iteration

\[
\frac{}{\Gamma \vdash P : t \to \text{bool}} & \quad \frac{}{\Gamma \vdash F : t \to t}
\]

### Collections

\[
\begin{align*}
\frac{}{\Gamma \vdash [ ] : [t]} & \quad \frac{\Gamma \vdash [M] : [t]}{\Gamma \vdash [M] : [t]} & \quad \frac{\Gamma \vdash M \oplus N : [t]}{\Gamma \vdash \text{flattened}(M) : [t]}
\end{align*}
\]

### Sequences

\[
\begin{align*}
\frac{}{\Gamma \vdash M : [s]} & \quad \frac{}{\Gamma \vdash N : [t]} & \quad \frac{}{\Gamma \vdash M : [t]}
\end{align*}
\]

### Weakening

\[
\frac{}{\Gamma \vdash M : t}
\]

\[
\frac{x : s, \Gamma \vdash M : t}{x : s, \Gamma \vdash M : t}
\]

11
B Operational Semantics

We define an environment to be a finite set of the form \( \rho = \{ x_1 = C_1, \ldots, x_n = C_n \} \), where \( x_1, \ldots, x_n \) are variables, and \( C_1, \ldots, C_n \) are S-objects. We say that \( \rho \) is associated to some type context \( \Gamma \) iff \( \rho \) and \( \Gamma \) mention exactly the same variables and if the type of \( C_i \) is the type of the variable \( x_i \) in \( \Gamma \).

The following rules define the ternary relation \( \rho \cdot M \Downarrow C \) and the 4-ary relation \( \rho \cdot F(C) \Downarrow C' \), where \( \rho \) is associated to some type context \( \Gamma \) such that \( \Gamma \vdash t : s \rightarrow t \) respectively.

Variables, Errors, Constants, Arithmetic

\[
\begin{align*}
x = C, & \quad \rho \cdot x \Downarrow C \\
\rho \cdot n \Downarrow n & \\
\rho \cdot M \Downarrow m & \quad \rho \cdot N \Downarrow n \\
\rho \cdot M + N \Downarrow m + n & \quad (\text{similar for all } op \in \Sigma \text{ and } =)
\end{align*}
\]

Type products

\[
\begin{align*}
\frac{\rho \cdot M \Downarrow C}{\rho \cdot \bot \Downarrow \bot} & \quad \frac{\rho \cdot N \Downarrow D}{\rho \cdot (M, N) \Downarrow (C, D)} & \quad \frac{\rho \cdot M \Downarrow (C, D)}{\pi_1(M) \Downarrow C} & \quad \frac{\rho \cdot M \Downarrow (C, D)}{\pi_2(M) \Downarrow D}
\end{align*}
\]

Type sums

\[
\begin{align*}
\frac{\rho \cdot M \Downarrow C}{\rho \cdot \text{in}_1(M) \Downarrow \text{in}_1(C)} & \quad \frac{\rho \cdot M \Downarrow C}{\rho \cdot \text{in}_2(M) \Downarrow \text{in}_2(C)} & \quad \frac{\rho \cdot \text{case } M \text{ of } \text{in}_1(x) \Rightarrow N | \text{in}_2(x) \Rightarrow P \Downarrow D}{\rho \cdot M \Downarrow C}
\end{align*}
\]

Functions

\[
\begin{align*}
\frac{\rho \cdot N \Downarrow C}{\rho \cdot F(N) \Downarrow D} & \quad \frac{\rho \cdot F(C) \Downarrow D}{\rho \cdot (\lambda x. M)(C) \Downarrow D}
\end{align*}
\]

Iteration

\[
\begin{align*}
\frac{\rho \cdot P(C) \Downarrow \text{false}}{\rho \cdot \text{while}(P, F)(C) \Downarrow C} & \quad \frac{\rho \cdot P(C) \Downarrow \text{true}}{\rho \cdot (P, F)(C) \Downarrow D}
\end{align*}
\]

Collections

\[
\begin{align*}
\frac{\rho \cdot M \Downarrow C}{\rho \cdot [M] \Downarrow [C]} & \quad \frac{\rho \cdot M \Downarrow [C_0, \ldots, C_{m-1}]}{\rho \cdot N \Downarrow [D_0, \ldots, D_{n-1}]} & \quad \frac{\rho \cdot M \Downarrow [C_0, \ldots, C_{m-1}, D_0, \ldots, D_{m-1}]}{\rho \cdot N \Downarrow [C_0, \ldots, C_{m-1}]} & \quad \frac{\rho \cdot M \Downarrow [C_0, \ldots, C_{m-1}]\{C_{m+1}, \ldots, C_n\}}{\rho \cdot \text{length}(M) \Downarrow n} & \quad \frac{\rho \cdot M \Downarrow [C_0, \ldots, C_{m-1}]}{\rho \cdot \text{flatten}(M) \Downarrow [C_0, C_1]} & \quad \frac{\rho \cdot F(C) \Downarrow D_0}{\rho \cdot \text{map}(F)([C_0, \ldots, C_n]) \Downarrow [D_0, \ldots, D_{n-1}]}
\end{align*}
\]

Sequences

\[
\begin{align*}
\frac{\rho \cdot M \Downarrow [C_0, \ldots, C_{n-1}]}{\rho \cdot \text{zip}(M, N) \Downarrow [(C_0, D_0), \ldots, (C_{n-1}, D_{n-1})]} & \quad \frac{\rho \cdot M \Downarrow [C_0, \ldots, C_{n-1}]}{\rho \cdot \text{enumerate}(M) \Downarrow [0, \ldots, n-1]} & \quad \frac{\rho \cdot \text{split}(M, N) \Downarrow [(C_0, \ldots, C_{m-1}), (\ldots, \ldots)]}{\rho \cdot N \Downarrow [n_0, \ldots, n_{m-1}]}
\end{align*}
\]

Weakening

\[
\begin{align*}
\frac{\rho \cdot M \Downarrow C}{x = C', \rho \cdot M \Downarrow C} & \quad \frac{\rho \cdot F(C) \Downarrow D}{x = C', \rho \cdot F(C) \Downarrow D}
\end{align*}
\]
C  The Nested Sequence Algebra N\textsc{sa}

Errors, Constants, Arithmetic

\[
\begin{array}{llll}
\Omega^n : unit \rightarrow t & n \in \mathbb{N} & op \in \Sigma & op : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \\
\end{array}
\]

Function identity and composition

\[
\begin{array}{ll}
id_t : t \rightarrow t & f : \tau \rightarrow s \quad g : s \rightarrow t \\
\end{array}
\]

Type products

\[
\begin{array}{llll}
t_1 : t \rightarrow unit & f_1 : s \rightarrow t_1 & f_2 : s \rightarrow t_2 & (f_1, f_2) : s \rightarrow t_1 \times t_2 \\
\end{array}
\]

Type sums

\[
\begin{array}{llll}
im_1 : t_1 \rightarrow t_1 + t_2 & im_2 : t_2 \rightarrow t_1 + t_2 & f_3 : s_1 - t & f_2 : s_1 + s_2 - t \\
\end{array}
\]

Iteration

\[
p : t \rightarrow \mathbb{B} \quad f : t \rightarrow t \\
\text{while}(p, f) : t \rightarrow t
\]

Collections

\[
\begin{array}{llll}
[] : unit \rightarrow [t] & \text{singleton} : t \rightarrow [t] & \@ : [t] \times [t] \rightarrow [t] & \text{flatten} : [[t]] \rightarrow [t] \\
\text{length} : [t] \rightarrow \mathbb{N} & \text{get} : [t] \rightarrow t & \text{map}(f) : [s] \rightarrow [t] & f : s \rightarrow t \\
\end{array}
\]

Sequences

\[
\begin{array}{llll}
\text{zip} : [s] \times [t] \rightarrow [s \times t] & \text{enumerate} : [t] \rightarrow \mathbb{N} & \text{split} : [t] \times [\mathbb{N}] \rightarrow [[t]] & \\
\end{array}
\]

Broadcast  This replaces the “free variables” present in N\textsc{sc}.

\[
\rho_2 : s \times [t] \rightarrow [s \times t]
\]

The evaluation relation \( f(C) \Downarrow C' \), for \( f \) some function in N\textsc{sa} of type \( s \rightarrow t \) and \( C, C' \) S-objects of type \( s \) and \( t \) respectively, is defined in a way similar to the definition for N\textsc{sc}, but simpler because functions in N\textsc{sa} do not have free variables, hence there is no need for an environment. The time and work complexity \( T(f, C) \) and \( W(f, C) \) are defined accordingly.

**Proposition C.1** Any closed function \( f \in \text{Nsc} \) with time and work complexity \( T, W \) is expressible in N\textsc{sa} by some function \( f' \) with time and work complexity \( O(T), O(W) \), and vice versa. Thus, N\textsc{sc} and N\textsc{sa} have the same expressive power.
Scalar types are: \( s := \text{unit} \mid \mathbb{N} \mid s \times s \mid s + s \). Scalar functions \( \varphi : s \to s' \) are given by:

<table>
<thead>
<tr>
<th>Constants, Arithmetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \in \mathbb{N} )</td>
</tr>
<tr>
<td>( n : \text{unit} \to \mathbb{N} )</td>
</tr>
<tr>
<td>( op \in \Sigma )</td>
</tr>
<tr>
<td>( \mathbb{N} \times \mathbb{N} \to \mathbb{N} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Function identity and composition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( id_s : s \to s )</td>
</tr>
<tr>
<td>( \varphi : s \to s' )</td>
</tr>
<tr>
<td>( \psi : s' \to s'' )</td>
</tr>
<tr>
<td>( \psi \circ \varphi : s \to s'' )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scalar type products</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bot : \text{unit} \to \bot )</td>
</tr>
<tr>
<td>( \pi_1 : s_1 \times s_2 \to s_1 )</td>
</tr>
<tr>
<td>( \pi_2 : s_1 \times s_2 \to s_2 )</td>
</tr>
<tr>
<td>( \varphi_1 : s \to s_1 )</td>
</tr>
<tr>
<td>( \varphi_2 : s \to s_2 )</td>
</tr>
<tr>
<td>( (\varphi_1, \varphi_2) : s \to s_1 \times s_2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scalar type sums</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 : s_1 \to s_1 + s_2 )</td>
</tr>
<tr>
<td>( m_2 : s_2 \to s_1 + s_2 )</td>
</tr>
<tr>
<td>( \varphi_1 : s_1 \to s )</td>
</tr>
<tr>
<td>( \varphi_2 : s_2 \to s )</td>
</tr>
<tr>
<td>( \varphi_1 + \varphi_2 : s_1 + s_2 \to s )</td>
</tr>
<tr>
<td>( \delta : (s_1 + s_2) \times s \to s_1 \times s + s_2 \times t )</td>
</tr>
</tbody>
</table>

(Cont'd next page)
Flat types are: $\text{unit} \mid [s] \mid t \times t \mid t + t$. Functions in SA $f : t \rightarrow t'$ are given by:

### Errors and Scalar operations

- **$\Omega' : \text{unit} \rightarrow t$**
- **$\varphi : s \rightarrow s'$ a scalar function**
- **$\text{map}(\varphi) : [s] \rightarrow [s']$**

### Function identity and composition

- **$f : t \rightarrow t'$**
- **$g : t' \rightarrow t''$**
- **$g \circ f : t \rightarrow t''$**

### Flat type products

- **$! : t \rightarrow \text{unit}$**
- **$\pi_1 : t_1 \times t_2 \rightarrow t_1$**
- **$\pi_2 : t_1 \times t_2 \rightarrow t_2$**
- **$f_1 : t \rightarrow t_1$**
- **$f_2 : t \rightarrow t_2$**
- **$(f_1, f_2) : t \rightarrow (t_1, t_2)$**

### Flat type sums

- **$\text{in}_1 : t_1 \rightarrow t_1 + t_2$**
- **$\text{in}_2 : t_2 \rightarrow t_1 + t_2$**
- **$f_1 : t_1 \rightarrow t$**
- **$f_2 : t_2 \rightarrow t$**
- **$f_1 + f_2 : t_1 + t_2 \rightarrow t$**
- **$\delta : (t_1 + t_2) \times t \rightarrow t_1 \times t + t_2 \times t$**

### Iterations

- **$p : t \rightarrow B$**
- **$f : t \rightarrow t$**
- **$\text{while}(p, f) : t \rightarrow t$**

### Collections

- **$[] : \text{unit} \rightarrow [s]$**
- **$\text{singleton} : \text{unit} \rightarrow [\text{unit}]$**
- **$\oplus : [s] \times [s] \rightarrow [s]$**
- **$\text{length} : [s] \rightarrow [\mathbb{N}]$**
- **$\text{empty?} : [s] \rightarrow B$**
- **$\sigma_1 : [s_1 + s_2] \rightarrow [s_1]$**
- **$\sigma_2 : [s_1 + s_2] \rightarrow [s_2]$**

### Sequences

- **$\text{zip} : [s] \times [s'] \rightarrow [s \times s']$**
- **$\text{enumerate} : [s] \rightarrow [\mathbb{N}]$**
- **$\text{bm\_route} : ([s] \times [\mathbb{N}]) \times [s'] \rightarrow [s']$**
- **$\text{sbm\_route} : ([s] \times [\mathbb{N}]) \times ([s'] \times [\mathbb{N}]) \rightarrow [s']$**

### Example D.1

Informally we show how to compute combine : $[B] \times [s] \times [s] \rightarrow [s]$, where combine($f, x, y$) combines the lists $x$ and $y$, according to the flags given by $f$. The resulting list will have the same length as $f$, and will contain some $x_i$ on those positions where $f$ is true, and some $y_j$ where $f$ is false. E.g. when $f = [\text{true}, \text{false}, \text{false}, \text{true}, \text{false}, \text{true}, \text{true}]$ and $x = [x_0, x_1, x_2, x_3]$, $y = [y_0, y_1, y_2]$, then combine($f, x, y$) must be $[x_0, y_0, x_1, y_1, y_2, x_2, x_3]$. To compute combine in SA, start by enumerating $f$, to get $[0, 1, 2, 3, 4, 5, 6]$, and by transforming the booleans into 0 and 1, to get $[0, 0, 1, 0, 1, 1, 1]$. Now apply $\text{bm\_route}$ to select from the first list those elements having a 1 in the second list, and obtain $[0, 3, 5, 6]$. Similarly, we obtain $[1, 2, 4]$. These two lists tell us on which position each element of $x$ and $y$ must end up. Next, we subtract each number in this list from its right neighbor (by considering $7 = \text{length}(f)$ to be the right neighbor of the last element), with the exception of the first position, where we also add the number itself. I.e., we get: $[0 + (3 - 0), 5 - 3, 6 - 5, 7 - 6] = [3, 2, 1, 1]$ and $[1 + (2 - 1), 4 - 2, 7 - 4] = [2, 2, 3]$. Now we $\text{bm\_route}$ $x$ and $y$, using these two lists as replication sequences, and get $[x_0, x_0, x_1, x_1, x_2, x_2, x_3]$ and $[y_0, y_0, y_1, y_1, y_2, y_2, y_2]$ respectively (both have the length of $f$). Finally, we zip them together with $f$, and map some scalar function which selects $x_i$ or $y_i$ according to the flag.