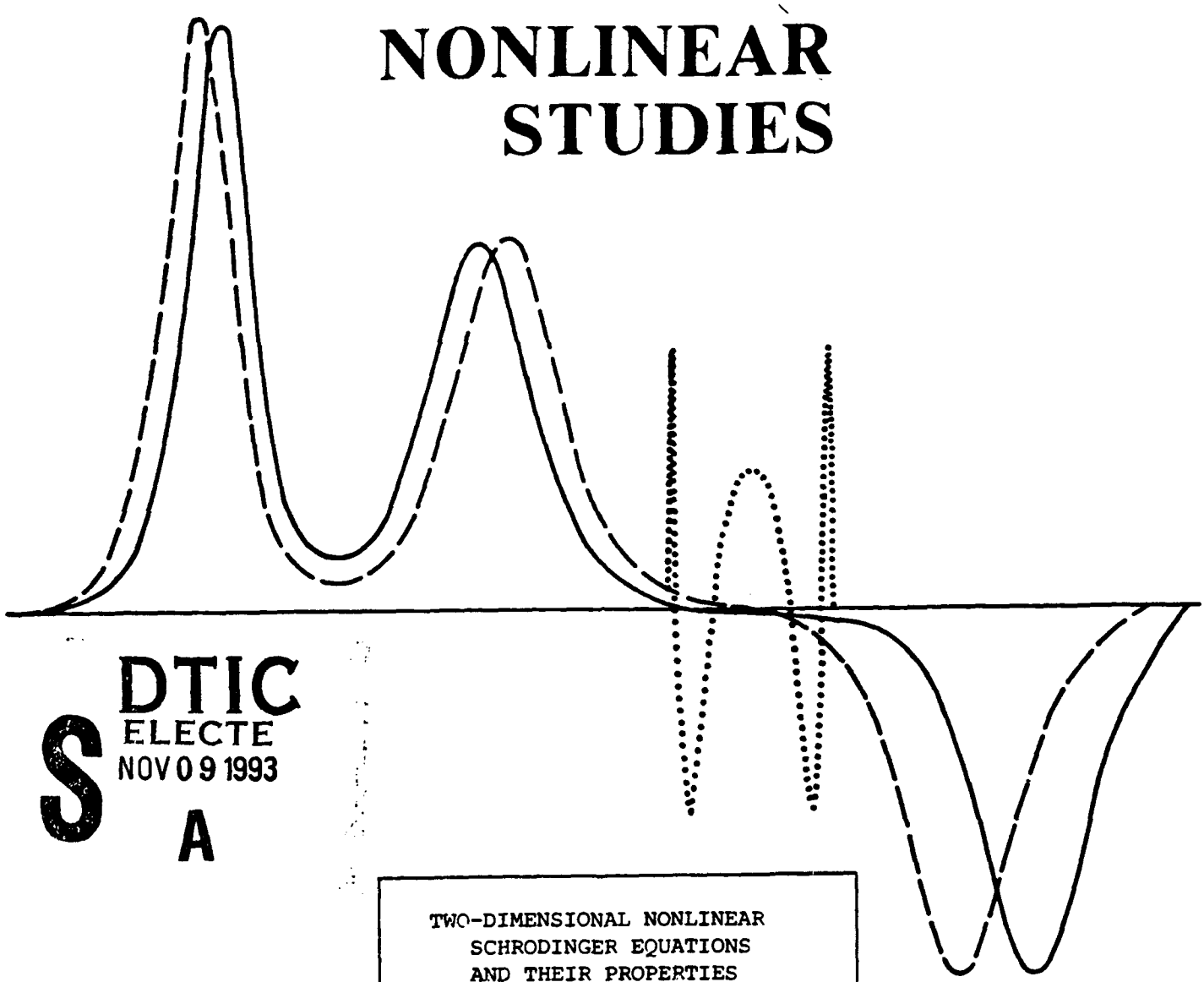
AD-A272 184

INS 241

(2)

INSTITUTE FOR NONLINEAR STUDIES



DTIC
ELECTE
NOV 09 1993
S A

TWO-DIMENSIONAL NONLINEAR
SCHRODINGER EQUATIONS
AND THEIR PROPERTIES

by

Ronald E. Kates

D.J. Kaup

October 1993

This document has been approved for public release and sale; its distribution is unlimited

CLARKSON UNIVERSITY
Potsdam, New York 13699-5815, U.S.A.

93-27278

93 11 5 083

TWO-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATIONS AND THEIR PROPERTIES

Ronald E. Kates
D.J. Kaup

Clarkson University
Potsdam, NY 13699-5815
USA

Abstract

This paper studies a general class of two-dimensional systems of the cubic nonlinear Schrödinger type (2DNLS), defined by $i\partial_t q + O_1 q = pq$ and $O_2 p = O_3(q^* q)$, where each $O_n \equiv D_{ij}^{(n)} \partial_i \partial_j$, $n = 1, 2, 3$, is a linear, second-order, operator with constant coefficients. This class generalizes the Djordjevic-Redekopp (DR) system, which has previously been encountered in the context of water waves. Integrability is characterized simply in terms of covariant conditions on the O_n. We obtain all integrable cases, including the known cases Davey-Stewartson I and II as well as other known integrable cases. The 2DNLS is modulationally stable if $\mathcal{D}^{(1)}\mathcal{D}^{(2)}\mathcal{D}^{(3)} > 0 \quad \forall \vec{k}$, where $\mathcal{D}^{(n)} = k_i k_j D_{ij}^{(n)}$. All other regimes are modulationally unstable and have projections satisfying the ordinary (1D) NLS with soliton solutions, though in all known cases these 1D solitons are unstable with respect to transverse perturbations. The self-focusing regime is characterized by the eigenvalues of the $D_{ij}^{(n)}$: O_1 and O_2 must both be elliptic, and for that choice of variables for which $D_{ij}^{(1)}$ and $D_{ij}^{(2)}$ both have positive signature, $D_{ij}^{(3)}$ must have at least one negative eigenvalue. The self-focusing regime is distinct from the modulationally stable regime and also from the integrable regime, while the integrable cases may be modulationally stable or unstable. There are no soliton solutions known in those integrable cases that are modulationally stable, whereas those integrable cases in which 2D solitons are known correspond to the modulational instability regime.

Accesion For
NTIS CRA&I ✓
DTIC TAB
Unannounced
Justification
By
Distribution /
Availability Codes
Dist | Avail and / or Special
A-1

1. Motivation

Reduction of nonlinear physical problems to systems of evolution equations with known properties has proven to be quite fruitful in recent years. Though a comprehensive bibliography is beyond the scope of the present paper, the flavor of such computations may be seen in our own work on (1+1)-dimensional "intense propagation" problems as they arise in plasma physics (see Kates & Kaup 1989a,b; 1991; 1992a; 1992b and references cited therein) and cosmology (Kates & Kaup 1988). Useful examples may also be found in a series of papers on the "reductive perturbation method" (Ichikawa et al., 1976; Taniuti, 1974). Considering the wide variety of applications, even modest improvement in our understanding of the resulting nonlinear evolution equations could be of enormous value.

If nonlinear effects are not too strong, and if symmetry permits the assumption of dependence on only one spatial coordinate, then the essential features of a nonlinear pulse propagation problem may often be deduced from the well-known properties of the one-dimensional nonlinear Schrödinger equation (NLS)

$$iq_t + q_{xx} \pm |q|^2 q = 0 \quad . \tag{1}$$

In the reduction of a complicated 1D system to the NLS, the most important computation is the dependence of the *sign* in (1) on the parameters of the problem: The minus sign indicates modulational stability, while the plus sign indicates modulational instability and soliton solutions.

In this paper, we investigate the qualitatively much richer range of phenomena associated with "2D" instabilities arising in nonlinear pulse propagation. That is, we consider the evolution of nonlinear instabilities having functional dependence on (2+1) dimensions (two spatial dimensions and time).

An important area of application concerns 2D modulations of electromagnetic pulses. Since Eq. (1) governs 1D modulations of weakly relativistic EM pulses, one might have supposed that 2D modulations of EM waves could be described simply by replacing the 1D dispersion term q_{xx} in the NLS (1) with a 2D dispersion term:

$$iq_t + [D_{xx}\partial_x^2 + 2D_{xy}\partial_x\partial_y + D_{yy}\partial_y^2]q \pm |q|^2 q = 0 \quad , \tag{2a}$$

where

$$D_{ij} \equiv \frac{\partial^2 \omega}{\partial k_i \partial k_j} \quad . \tag{2b}$$

A system of the form (2) is indeed obtained for a circularly polarized beam if longitudinal perturbations are negligible (Spatschek, 1977). This case corresponds to the "thin-beam" approximation. However in many applications both longitudinal and transverse perturbations may occur. Careful analysis of fluid modes associated with longitudinal and transverse ponderomotive forces (Kates & Kaup, 1993) shows that a proper description of

2D modulations of EM pulses then involves two independent, *coupled* potential functions. This point was recognized by Karpman (1990).

Some hints as to *generic* 2D behavior are provided by water waves (Djordjevic & Redekopp, 1977 (DR in what follows); Ablowitz & Segur, 1979 ("A&S" in what follows)): Capillary-gravity waves are described by the "DR" system, which (for historical reasons) is expressed in the form

$$iA_\tau + \lambda A_{\xi\xi} + \mu A_{\eta\eta} = \chi A^* A^2 + \chi_1 A \Phi_\xi \tag{3a}$$

$$\alpha \Phi_{\xi\xi} + \Phi_{\eta\eta} = -\beta (A^* A)_\xi \quad . \tag{3b}$$

Both (1) and (2) are special cases of (3): If the initial data depend only on ξ or η or some linear combination thereof, the DR system (3) may be reduced to the 1D nonlinear Schrödinger equation (1). By properly choosing the coordinates, one may write (2) in the DR form (3), in which case $\chi_1 = 0$, so that the potentials are decoupled. Eq. (2) or equivalently (3a) with $\chi_1 = 0$ will be refered to below as the "decoupled 2DNLS."

As written, the DR system (3) contains six "arbitrary" constants: α, β, λ μ, χ, and χ_1. (However, by suitable coordinate transformations, one can reduce the dimensionality of this parameter space to two parameters and two signs, as is occasionally convenient.) Since a rather large class of 2D systems with cubic nonlinearity can be reduced to DR form, it represents an important advance in generalizing the NLS to two dimensions. However, as discussed below, the DR system still does not include all possible 2D generalizations of the NLS. Indeed, it does not contain all possible 2DNLS integrable cases. Moreover, in practice, reduction of a given system to the DR form (3) may require a tedious search for appropriate coordinate transformations. These considerations already suffice to motivate our introduction of a more general, covariant form (Section 2), which seems to deserve the designation "canonical." This canonical form includes a much wider class of 2D generalizations of the NLS and also facilitates a comprehensive treatment of all integrable forms of the 2DNLS. The importance of this form was first noted by Zakharov and Schulman (1990).

In Section 3, we compute the regimes of modulational instability for the generic 2DNLS. More precisely, we study perturbations about spatially constant, but nonlinear solutions. Modulational instability of the 2DNLS is always associated with the existence of one-dimensional NLS soliton solutions. Whether such 1D solitons truly arise in a system depending on two spatial coordinates is of course a separate question.

The canonical form turns out to be of great benefit in understanding self-focusing. Self-focusing has been studied for those regions of the DR parameter space accessible to water wave systems: A&S showed that a singularity in moment of inertia occurs in their "region F," characterized by χ_1, β, α, μ, and λ positive, χ, and $\nu \equiv \chi - \chi_1 \beta/\alpha$ negative in Eqs. (3). They also studied quasi-self-similar solutions in this region, obtaining growing solutions consistent with a self-focusing interpretation. The question of self-focusing in their "Region E" (χ_1, β, α, μ, λ, χ, and ν all positive) was left open. (Incidentally, the A&S paper contains a few minor misprints: The quantity λ^2 given in their coordinate

transformations should read $|\lambda|$ throughout. In their Eq. (4.1d), χ_1^3 should be replaced by χ_1.)

However, several intended applications to experiments in plasma physics and nonlinear optics require at the very least a knowledge of self-focusing conditions for the entire range of DR parameters (not just the region of DR parameter space accessible to water waves). For example, the interesting question of self-focusing for positive χ and *negative* ν did not arise in the A&S paper, because for water waves ν and χ have the same sign. Other applications may require self-focusing results for systems that cannot be reduced to DR form but that can be put into our canonical 2DNLS form: for example, (Kates & Kaup, 1992b) the Zakharov equations (Zakharov, 1972).

In Section 4, we derive self-focusing conditions for the general 2DNLS in terms of the signatures of three operators. As a consequence we will be able to answer the open question referred to by A&S concerning the possibility of self-focusing for their "region E" in the negative. On the other hand, self-focusing *is* possible for positive χ and *negative* ν (assuming the other parameters are positive).

2. Canonical Form for the 2DNLS and Integrability

Of particular interest for applications such as multi-dimensional plasma phenomena is the question of whether parameters can be chosen such that the system is *integrable*. In this case the system may have nontrivial soliton solutions. In this section, we discuss a general canonical form for two-dimensional generalizations of the NLS ("2DNLS"). Conditions for integrability are most simply and conveniently expressed in terms of this canonical form instead of the DR form.

We begin our discussion by observing that if both eigenvalues of D_{ij} have the same sign as the nonlinear term, then we may rescale the decoupled NLS (2) to

$$i\partial_t q + \nabla^2 q + |q|^2 q = 0 \quad . \tag{4}$$

In two dimensions, Eq. (4) is of considerable interest in its own right, because it exhibits self-focusing behavior (Synakh, 1975; Spatschek, 1977). Eq. (4) would appear to be the "simplest" possible multi-dimensional generalization of (1) in any number of dimensions; if the 1D version of a physical system obeys the NLS (1), then it *sometimes* happens that the multi-dimensional version can be reduced to (4). However, its properties are quite different: In more than one dimension, Eq. (4) is *not integrable.*

All known *integrable* forms of the multidimensional NLS involve an auxiliary function. Such a function arises naturally in the physics of water waves (DR), in the Zakharov system (see below) and in electromagnetic pulses propagation (Kates and Kaup, 1992b).

Recently, Boiti, Pimpinelli and Sabatier (1992) demonstrated that there are only three distinct integrable forms of a 2DNLS. Below, we will show that all three may be expressed compactly in terms of the "canonical" form (Zakharov and Schulman, 1990)

$$i\partial_t q + O_1 q = pq \tag{5a}$$

$$O_2 p = O_3(q^* q) \quad , \tag{5b}$$

where each of the operators O_n in (5) is a second-order linear, symmetric (dispersion) operator of the form

$$O_n = D^{(n)}_{xx}\partial_x^2 + 2D^{(n)}_{xy}\partial_x\partial_y + D^{(n)}_{yy}\partial_y^2 \quad . \tag{5c}$$

The criteria for integrability of (5) now become simply that at least one of O_2 and O_3 not be elliptic,

$$D^{(1)}_{ij} = \kappa D^{(3)}_{ij} \tag{6a}$$

be satisfied for some constant κ,

$$\sum_{ij} D^{(2)}_{ij}[(D^{(3)})^{-1}]_{ji}) = 0 \quad , \tag{6b}$$

and

$$\det(D^{(n)}) \neq 0 \tag{6c}$$

for each of $n = 1, 2, 3$. Thus, expressing the system in the canonical form (5) offers the advantage of allowing a straightforward, coordinate-independent test of integrability.

Note that whenever $\det(D^{(n)}) = 0$, then $D^{(n)}$ is simply a projection. Thus condition (6c) ensures that each O_n remains two-dimensional. Condition (6b) means that in some sense, the operator O_2 must be orthogonal to O_3 (and naturally also to O_1).

We now relate conditions (6) to the integrability conditions of Boiti, Pimpinelli and Sabatier (1992): Using powerful analytical techniques, they showed that, in multiple dimensions, the only possible integrable equations having at most two derivatives are obtainable from a Lax pair of the following form:

$$\partial_2 v_1 = q v_2 \tag{7a}$$

$$\partial_1 v_2 = r v_1 \tag{7b}$$

$$i\partial_t v_1 = \sigma_1 \partial_1^2 v_1 + i b_1 \partial_1 v_1 + c_1 \partial_2 v_1 + d_1 v_1 + e_1 v_2 \tag{8a}$$

$$i\partial_t v_2 = \sigma_2 \partial_2^2 v_2 + i b_2 \partial_2 v_2 + c_2 \partial_1 v_1 + d_2 v_1 + e_2 v_2 \quad , \tag{8b}$$

where $\partial_1 = \partial/\partial w_1; \partial_2 = \partial/\partial w_2$; w_1 and w_2 are independent spatial variables; v_1 and v_2 are eigenfunctions; σ_1 and σ_2 are constants; and $b_1, b_2, c_1, c_2, d_1, d_2, e_1$ and e_2 are functions to be determined. Eqs. (7) constitute a scattering problem which was first studied by Ablowitz and Haberman (1975). For given q and r, particular scattering data are determined. Conversely, given the scattering data, one can reconstruct the potentials, q and r.

The Lax pair (7) and (8) can only have a common solution when certain integrability conditions are satisfied. These integrability conditions yield precisely the integrable forms of the 2DNLS detailed by Boiti, Pimpinelli and Sabatier (1992). One obtains them by cross-differentiating (7) and (8), using (7) and (8) to simplify the resulting conditions. One finds

$$c_1 = \sigma_2 q \tag{9a}$$
$$c_2 = \sigma_1 r \tag{9b}$$
$$\partial_2 b_1 = 0 \tag{9c}$$
$$\partial_1 b_2 = 0 \tag{9d}$$
$$e_1 = iqb_2 - \sigma_2 \partial_2 q \tag{9e}$$
$$d_2 = irb_1 - \sigma_1 \partial_1 r \tag{9f}$$

and the evolution equations

$$i(\partial_t - b_1\partial_1 - b_2\partial_2)q = (\sigma_1\partial_1^2 - \sigma_2\partial_2^2)q + (d_1 - e_2 + i\partial_2 b_2)q \tag{10a}$$

$$-i(\partial_t - b_1\partial_1 - b_2\partial_2)r = (\sigma_1\partial_1^2 - \sigma_2\partial_2^2)r + (d_1 - e_2 - i\partial_1 b_1)r \quad , \tag{10b}$$

where the potentials d_1 and e_2 are determined by

$$\partial_2 d_1 = -2\sigma_1\partial_1(rq) \tag{11a}$$
$$\partial_1 e_2 = -2\sigma_2\partial_2(rq) \quad . \tag{11b}$$

The quantities b_1 and b_2 are trivial, being simply a group velocity for the q and r fields. Thus we take them to be zero. Now, if we identify

$$r = \sigma_0 q^* \quad , \tag{12}$$

where $\sigma_0 = \pm 1$, then the two equations (10) are obviously equivalent. Defining

$$p = d_1 - e_2 \quad , \tag{13}$$

we then have from (11)

$$\partial_1\partial_2 p = -2\sigma_0(\sigma_1\partial_1^2 - \sigma_2\partial_2^2)(q^* q) \tag{14a}$$

and (10a) becomes

$$i\partial_t q = (\sigma_1\partial_1^2 - \sigma_2\partial_2^2)q + pq \quad . \tag{14b}$$

Lastly, if we define

$$O_2 = \partial_1 \partial_2 \tag{15a}$$
$$O_3 = -2\sigma_0(\sigma_1 \partial_1^2 - \sigma_2 \partial_2^2) \tag{15b}$$
$$O_1 = (\sigma_1 \partial_1^2 - \sigma_2 \partial_2^2) \quad , \tag{15c}$$

the system takes the canonical form (5) and satisfies the conditions given in (6). Conversely, conditions (6) are invariant under general linear coordinate transformations as well as under constant rescalings of the dependent variables. This invariance simplifies considerably the problem of determining when an equation of the form (5) is integrable. One does not have to search for special coordinate scalings. One simply checks whether or not conditions (6) hold in *some* coordinate system. When (6) are satisfied, then one can always find coordinates such that the O_n are given by (15).

Let us now make some remarks on this canonical form. First, there is an arbitrariness in the choice of signs of w_1 and/or w_2. By using this, we could absorb the sign σ_0 into either w_1 or w_2. Note that (7) would be transformed if the sign of σ_0 is so absorbed. However, it is convenient to carry σ_0 in the computation and leave the coordinates fixed. Furthermore, the dispersion operator O_2 is not necessarily restricted to be hyperbolic, as it may appear at first glance. When the dispersion operator O_2 is hyperbolic, then it is only necessary to choose the coordinates w_1 and w_2 to be real. Then in order to obtain real operators O_3 and O_1, one would choose σ_1 and σ_2 real. In this case, O_3 and O_1 may be either elliptic or hyperbolic, depending on the signs chosen for σ_1 and σ_2.

However, suppose the actual problem demands O_2 to be elliptic. One may also treat this case by allowing w_1 and w_2 to be complex. (In the above derivation, ∂_1 and ∂_2 were only required to be linear operators. They quite easily could have been a linear combination of derivatives of real coordinates with complex coefficients as below.) The simplest example is to take

$$\partial_1 = \partial_x + i\partial_y \tag{16a}$$
$$\partial_2 = \partial_x - i\partial_y \quad , \tag{16b}$$

where x and y are real coordinates. Now the dispersion operators all have to be real for physical reasons, so care has to be taken to ensure that this remains so when w_1 and w_2 are complex. The condition that O_3 (and O_1) be real requires $\sigma_2 = -\sigma_1^*$, so that only one of σ_1 and σ_2 may be chosen arbitrarily. One may show that for O_2 elliptic, O_3 (and O_1) must be hyperbolic. This is in contrast to the case O_2 hyperbolic, in which O_3 (and O_1) could be either elliptic or hyperbolic. Also, one may verify that this does not affect the choice (12), which remains a valid identification.

This case of O_2 elliptic contains the case of the classical Davey-Stewartson equation (which has been called the "Davey-Stewartson II") in hydrodynamics. We refer the reader to the literature [Kaup (1980), Fokas and Ablowitz (1983), Fokas and Ablowitz (1984), Boiti, Leon, Martina and Pempinelli (1988), Boiti, Leon and Pempinelli (1989), Boiti,

Pempinelli and Sabatier (1992), Arkadiev, Pogrebkov and Polivanov (1989)] for the solution of the initial value problems and various specific solutions.

Now, an important point in regard to this canonical system is that any system which can be expressed in the DR form (3) can also be expressed in the canonical form (5). (The converse is not always true.) The transformation can be easily obtained by differentiating (3b) with respect to ξ and making the identifications

$$O_2 = \frac{1}{\chi_1}\partial_\eta^2 + \frac{\alpha}{\chi_1}\partial_\xi^2 \tag{17a}$$

$$O_3 = \frac{\chi}{\chi_1}\partial_\eta^2 + (\frac{\alpha\chi}{\chi_1} - \beta)\partial_\xi^2 \tag{17b}$$

$$O_1 = \mu\partial_\eta^2 + \lambda\partial_\xi^2 \tag{17c}$$

$$q = A \tag{17d}$$

$$p = \chi(A^*A) + \chi_1\partial_\xi\Phi \quad . \tag{17e}$$

(Note that x corresponds to η and y to ξ. The "decoupled" case of vanishing χ_1 is excluded.)

The so-called Davey-Stewartson I equations (DS-I)

$$-iQ_\tau = \frac{1}{2}(Q_{xx} + Q_{yy}) + Q(\phi_y \pm Q^*Q) \tag{18a}$$

$$(\phi_{xx} - \phi_{yy}) \pm 2(QQ^*)_y = 0 \tag{18b}$$

and DS-II equations (for the upper sign)

$$-iQ_\tau = \frac{1}{2}(Q_{xx} - Q_{yy}) - Q(\phi_y \pm Q^*Q) \tag{19a}$$

$$(\phi_{xx} + \phi_{yy}) \pm 2(QQ^*)_y = 0 \tag{19b}$$

are the integrable cases of DR.

Conditions for integrability of a system expressed in the DR form (3) may be obtained by the following tedious procedure: one introduces constant transformations to normalize all but two of the coefficients in (3). Inspection of the resulting equations reveals that reduction to Eqs. (18) (DS-I) requires positive λ and satisfaction of two additional conditions:

$$\beta\mu\chi_1 = -2\chi\lambda \tag{20a}$$

and

$$\alpha\mu = -\lambda \quad . \tag{20b}$$

If λ is negative, the DR system (3) reduces to Eqs. (19) (DS-II) if χ is positive and (20) hold. Applying (6) to the operators O_n defined in Eqs. (17a-c), one sees that the conditions for integrability are consistent with those given in (20) for the DR form.

The "standard" forms (18) and (19) for DS-I and DS-II appear to distinguish a direction of propagation (longitudinal coordinate) from a transverse coordinate, whereas there is no such distinction in (5). Now, of course such a distinction may be defined in the underlying problem, but it is not inherent in the DS equations themselves, since one can easily transform either DS system to (14).

In some cases, we can reverse the procedure and convert (14) to DR form. However, we emphasize that *not all integrable cases can be expressed in DR form*: Suppose that a transformation exists such that p can be decomposed into

$$p = \chi q^* q + (n_1\partial_1 + n_2\partial_2)\Phi \quad , \tag{21a}$$

where χ is a coupling constant and $(n_1\partial_1 + n_2\partial_2)$ is the gradient in some "transverse" direction. From (14a), one then must obtain the equation for Φ in the form

$$\partial_1\partial_2\Phi = 2\gamma(n_1\partial_1 + n_2\partial_2)(q^* q) \quad , \tag{21b}$$

where γ is simply a sign. One finds that this can be done whenever the following conditions can be satisfied:

$$\sigma_1 = -\gamma {n_1}^2 \tag{22a}$$
$$\sigma_2 = \gamma {n_2}^2 \quad , \tag{22b}$$

where χ will then be given by

$$\chi = -4\gamma n_1 n_2 \quad . \tag{23}$$

When O_2 is elliptic, then since $\sigma_2 = -{\sigma_1}^*$, a solution will always exist for the n_i's where $n_2 = {n_1}^*$, leading to a real transverse gradient operator, $(n_1\partial_1 + n_2\partial_2)$ (recall here that the partials are complex operators). However, in the case where O_2 is hyperbolic and the coordinates w_1 and w_2 are real, from (22) it is clear that a solution exists only if O_3 is elliptic (DS-I). When in addition O_3 is hyperbolic, then no real solution can exist for n_1 and n_2, i.e., at least one of the direction cosines would have to be complex. *Consequently the DR form is not a canonical form covering all possible integrable cases.* In particular, it cannot cover the case where both O_2 and O_3 are hyperbolic. For this reason, the form (5) is preferable, because it does cover all integrable cases.

3. Modulational Instability of the 2DNLS

As in the one-dimensional NLS (1), important hints to the asymptotic behavior of solutions of the *generic* 2DNLS (5) can be obtained by studying modulational stability following the method of Paper III. We first observe that (5) admits spatially constant solutions of the form $p = p_0 = \text{const}$, $q = q_0(t)$, where p_0 and q_0 are any solutions of

$$i\partial_t q_0 = p_0 q_0 \quad , \tag{24}$$

for example,

$$q_0 = \exp(ip_0 t) \quad . \tag{25}$$

We next write

$$p = p_0(1 + p_1) \tag{26a}$$

$$q = q_0(1 + q_1) \tag{26b}$$

(where q_1 and p_1 depend on all variables) and linearize Eqs. (5) in p_1 and q_1, obtaining

$$i\partial_t q_1 + O_1 q_1 - p_0 p_1 = 0 \tag{27a}$$

$$-i\partial_t q_1^* + O_1 q_1^* - p_0 p_1 = 0 \tag{27b}$$

$$O_2 p_1 = \frac{q_0^* q_0}{p_0} O_3(q_0 + q_0^*) \quad . \tag{27c}$$

Expressing the dependent variables q_1, q_1^*, and p_1 as Fourier integrals and being careful to include the possibility that the time dependence of the phase may be complex, we then obtain the dispersion relation

$$\omega^2 = \mathcal{D}^{(1)}[\mathcal{D}^{(1)} + 2\frac{\mathcal{D}^{(3)}}{\mathcal{D}^{(2)}} q_0^* q_0] \tag{28}$$

where $\mathcal{D}^{(n)}$ is defined by

$$\mathcal{D}^{(n)} \exp(i\vec{k} \cdot \vec{x}) = -O_n \exp(i\vec{k} \cdot \vec{x}) \quad , \tag{29a}$$

that is,

$$\mathcal{D}^{(n)} = k_i k_j D_{ij}^{(n)} \quad . \tag{29b}$$

Let us assume that $\mathcal{D}^{(2)}$ is nonvanishing. Since it can be shown that higher wavenumbers are stabilizing, we obtain the condition

$$\mathcal{D}^{(1)}\mathcal{D}^{(2)}\mathcal{D}^{(3)} > 0 \quad \forall \vec{k} \tag{30}$$

for modulational stability of the general canonical system (5).

One important application of (30) concerns one-dimensional subspaces (see also Discussion): For any *fixed* $\hat{k}$, we can define $y = \hat{k} \cdot \vec{x}$ and substitute $p = p(t, y)$ and $q = q(t, y)$ into (5). Assuming nonvanishing $\mathcal{D}^{(2)}$, (5b) can easily be integrated twice, and p can be eliminated form (5a), resulting in a different one-dimensional NLS (1) for each $\hat{k}$. It is then easy to verify that (30) yields the correct NLS modulational stability condition. Additional consequences will be discussed in the final section.

It is especially interesting to combine (30) with the integrability conditions (6): these imply

$$\mathcal{D}^{(1)} = \kappa \mathcal{D}^{(3)} \quad , \tag{31}$$

and thus

$$\kappa \mathcal{D}^{(2)} > 0 \tag{32}$$

is the condition for modulational stability of an *integrable* 2DNLS system. Consequences will be discussed in the final section.

Although the above derivation already covers the DR form (3) in principle (via Eqs. (17)), for future reference we repeat the argument explicitly for the DR form (3): The spatially constant solution for A is

$$A_0(\tau) \equiv a_0 \exp\left(-i\chi\tau a_0^2\right) \tag{33}$$

with $\Phi_0 = 0$. We take

$$a = A_0(1 + a_1(\tau, \xi, \eta)) \quad , \tag{34}$$

and linearize (3) in a_1 and Φ. The dispersion relation is then

$$\omega^2 = k^2(\lambda \sin^2(\theta) + \mu \cos^2(\theta))[(\lambda \sin^2(\theta) + \mu \cos^2(\theta))k^2 + 2a_0^2\chi - \frac{2a_0^2\beta\chi_1 \sin^2(\theta)}{\cos^2(\theta) + \alpha \sin^2(\theta)}] \quad , \tag{35}$$

where $k_\eta = k\cos(\theta)$, $k_\xi = k\sin(\theta)$, i.e., $k^2 = k_\xi^2 + k_\eta^2$. Since evidently higher wavenumbers k are stabilizing, the maximal instability region may be determined by considering the limit $k \to 0$ of γ^2/k^2 . The resulting inequality reduces properly in the limiting cases $\theta = 0$ and $\theta = \pi/2$ to the known conditions for modulational instability of the "longitudinal" and "transverse" NLS. It is also is of great utility in understanding the changes in the qualitative behavior of solutions of the DR system for limiting values of the coefficients.

Finally, we note that the zeroth-order solution (33), which was assumed spatially constant, may be generalized to an arbitrary plane wave by Galilean invariance without changing the regimes of modulational stability.

4. Self-focusing

Probably the most dramatic difference between 1D and 2D modulations is the possibility of a violent "self-focusing" singularity (Zakharov & Synakh, 1975; Spatschek, 1977). Self-focusing is also a well-known phenomenon in nonlinear optics (Kelley, 1965). For the general 2DNLS (5), which contains two independent potentials, the situation is a quite a bit more complicated than for the decoupled 2DNLS (4), which involves only one potential. Below, we derive restrictions on the operators O_n in (5) and give conditions on the initial conditions leading to self-focusing.

We are interested in the time evolution of the generalized "moment of inertia" integral

$$J \equiv \int\int d^2x \sum_{ij} [(D^{(1)})^{-1}]_{ij} \quad x_i x_j q^* q \quad . \tag{35}$$

We assume that O_1 and O_2 are elliptic and that all fields vanish at infinity sufficiently rapidly in what follows (see end of this section for discussion). Without further loss of generality, we take the matrix $D_{ij}^{(1)}$ to have two positive eigenvalues. Note that in the absence of singularities J is a positive definite quantity. (This would not be true for hyperbolic O_1.) For our purposes, the question of self-focusing amounts to asking under what circumstances J develops a singularity in finite time.

It may be verified (see Appendix A) under the above assumptions that

$$\frac{d^2 \quad J}{d\tau^2} = 8I_1 \quad , \tag{36}$$

where

$$I_1 \equiv \int\int d\xi d\eta \quad H_1 \tag{37a}$$

and

$$H_1 \equiv \sum_{ij} D_{ij}^{(1)} q_i q_j^* + \frac{1}{2} p q^* q \quad . \tag{37b}$$

If I_1 exists, is conserved, and is initially negative, then integration of (36) implies that J will develop a singularity in finite time. Now, for arbitrary O_n, I_1 is not generally conserved, even for diagonal O_1 and O_2 (i.e., the DR system). Nevertheless, as we shall now show, I_1 will be constant in time under the assumptions stated above.

Consider a ("generalized DR") system of the form

$$iA_\tau + O_1 A = \chi A^* A^2 + \chi_1 A \Phi_\xi \tag{38a}$$

$$O_2 \Phi = -\beta (A^* A)_\xi \quad , \tag{38b}$$

where O_1 and O_2 are as in (5). The DR form (3) corresponds to the special case of (38) in which O_1 and O_2 are both diagonal. Our procedure will now be to write the 2DNLS (5) in the form (38) and obtain conservation laws for I_1 in this form.

Let us write

$$q = A \tag{39}$$

and

$$p = \chi A^* A + P \quad . \tag{40}$$

Eq. (5b) becomes

$$O_2 P = (O_3 - \chi O_2)(A^* A) \quad . \tag{41}$$

Since O_2 is elliptic, it can be shown (see following paragraph) that there exists a linear transformation to coordinates η, ξ such that

$$O_3 - \chi O_2 = -b \partial_\xi^2 \tag{42}$$

for some constant b, where

$$\partial_\xi = \sum_j n_j \partial_j \quad , \tag{43}$$

and where without loss of generality $\vec{n}$ is a unit vector. Identifying

$$P = \Phi_\xi \quad , \tag{44}$$

we can then integrate (41).

Now let us prove the above statement leading to (42): Since by assumption both eigenvalues of $D^{(2)}$ are positive, we first transform it into the identity matrix by means of a constant linear coordinate transformation. In the coordinate system (x_1, x_2), we thus seek a solution of the matrix equation

$$D^{(3)}_{ij} - \chi\delta_{ij} + bn_i n_j = 0 \quad . \tag{45}$$

Eq. (45) is equivalent to three scalar equations. One of these, the trace of (45), yields an expression for b:

$$b = -(D^{(3)}_{11} + D^{(3)}_{22}) + 2\chi \quad . \tag{46}$$

Let $n_1 = \cos\theta$ and $n_2 = \sin\theta$ for some θ. The remaining two scalar equations then yield

$$\chi = \frac{1}{2}\left(D^{(3)}_{11} + D^{(3)}_{22} \pm [(D^{(3)}_{11} - D^{(3)}_{22})^2 + 4(D^{(3)}_{12})^2]^{1/2}\right) \tag{47}$$

and

$$\cos^2\theta = \frac{D^{(3)}_{11} - \chi}{D^{(3)}_{11} + D^{(3)}_{22} - 2\chi} \leq 1 \quad . \tag{48}$$

Therefore, under the stated assumptions we may always write the 2DNLS (5) in the form (38).

In order to prove conservation of I_1, it is convenient to define a Lagrangian density for (38):

$$L = \frac{i}{2}(A_\tau A^* - A^*_\tau A) - H(A, A^*, \Phi) \tag{49}$$

where

$$H \equiv \sum_{ij} D^{(1)}_{ij} A_i A^*_j + \frac{\chi}{2}(AA^*)^2 + \frac{\chi_1}{2\beta}\sum_{ij} D^{(2)}_{ij}\Phi_i\Phi_j + \chi_1 AA^*\Phi_\xi \quad . \tag{50}$$

Equations (38) are then equivalent to

$$\delta\left[\int d\tau \int\int d\xi d\eta \quad L\right] = 0 \quad , \tag{51}$$

where independent variations are to be performed with respect to A^*, A, and Φ. In particular, (51) implies

$$\frac{d \quad I_0}{d\tau} = 0 \quad , \tag{52a}$$

where

$$I_0 \equiv \int\int d\xi d\eta \quad H \quad . \tag{52b}$$

¿From (38b) one obtains

$$I_2 \equiv \frac{\chi_1}{\beta} \int\int d\xi d\eta \quad (\sum_{ij} D_{ij}^{(2)} \Phi_i \Phi_j) + \chi_1 \int\int d\xi d\eta \quad A^* A \Phi_\xi = 0 \quad . \tag{53}$$

Combining (52) and (53), we obtain the additional conservation law

$$\frac{d \quad I_3}{d\tau} = 0 \quad , \tag{54a}$$

where

$$I_3 \equiv \int\int d\xi d\eta \quad H_3 \quad , \tag{54b}$$

$$H_3 \equiv \sum_{ij} D_{ij}^{(1)} A_i A_j^* + \frac{\chi}{2}(AA^*)^2 + \frac{\chi_1}{2} AA^* \Phi_\xi \quad . \tag{54c}$$

On the other hand, substituting (40) and (44) into (37) and taking the special case $\chi_1 = 1$, we find that $H_1 = H_3$. Therefore, I_1 is conserved.

Let us now study conditions for negative I_1: Since the first term in H_1 is positive definite, self-focusing requires :

$$I_4 < 0 \quad , \tag{55a}$$

where

$$I_4 \equiv \int\int d^2x \quad pq^*q. \tag{55b}$$

A sufficient condition for (55a) is that O_2 and O_3 have opposite signatures: We first express p and q^*q as 2D Fourier integrals

$$p = \int\int d^2k \quad \psi_{\vec{k}} u_{\vec{k}} \tag{56a}$$

$$q^*q = \int\int d^2k \quad \rho_{\vec{k}} u_{\vec{k}} \quad , \tag{56b}$$

where

$$u_{\vec{k}} \equiv \frac{1}{2\pi} \exp(i\vec{k} \cdot \vec{x}) \quad . \tag{57}$$

We then have

$$I_4 = \int\int d^2k \quad \psi_{\vec{k}}^* \rho_{\vec{k}} \quad . \tag{58}$$

Observing that (5b) is a linear equation in p and q^*q, we can now solve for $\psi_{\vec{k}}$ in terms of $\rho_{\vec{k}}$ using

$$\sum_{ij} D_{ij}^{(2)} k_i k_j \psi_{\vec{k}} = \sum_{ij} D_{ij}^{(3)} k_i k_j \rho_{\vec{k}} \quad . \tag{59}$$

The integrand in (58) is therefore negative definite if the signatures of O_2 and O_3 are opposite, and therefore (55a) is satisfied as claimed.

For self-focusing, we are interested in initial data such that not just I_4, but also I_1 is negative. It is intuitively clear that this can always be accomplished if q is chosen to have sufficiently large power at small $\vec{k}$.

We observe that if the signatures of O_2 and O_3 are the same, then I_4 is positive definite. In this case, the system is clearly outside of the self-focusing regime as defined by the prediction of a singularity in J. This is the case of "Region E" defined by A&S for the DR system (3).

In the case of $D^{(3)}$ having one negative and one positive eigenvalue (hyperbolic O_3), Eq. (59) no longer guarantees opposite signs of $\psi_{\vec{k}}$ and $\rho_{\vec{k}}$ for arbitrary $\vec{k}$. An example is the case of positive χ and negative ν in (3) (other parameters positive) mentioned in the introduction. Nevertheless, self-focusing can still occur if the initial conditions have enough power at low wave number and also most of the power in q^*q is concentrated in modes with wave numbers $\vec{k}$ intersecting a hyperbola $D_{ij}^{(3)} k_i k_j = -1$ (i.e., in the notation of (29a), modes satisfying $\mathcal{D}^{(3)} < 0$).

At the beginning of this section, we assumed that O_1 and O_2 are elliptic. Now, the condition O_1 elliptic obviously is needed to argue for a singularity in J (Eq. (35)). The condition that O_2 must be elliptic was derived for the DR system (3) by A&S, and their arguments are simply repeated here for the 2DNLS (5): First, suppose one has initial data for q which decays sufficiently rapidly at infinity or has compact support. Then partial integration of global integrals involving only q and its derivatives leave no boundary terms. If O_2 is elliptic, then appropriate boundary conditions for (5b) are that p vanish at infinity, so integration by parts of global integrals involving p also leaves no boundary terms. This argument breaks down if O_2 is hyperbolic, since boundary terms involving p do not in general vanish at infinity.

Summarizing, for the self-focusing regime, the operators O_2 and O_1 in (6) must be elliptic, while $D^{(3)}$ must have at least one negative eigenvalue (taking $D^{(1)}$ and $D^{(2)}$ positive definite).

5. Discussion

The canonical form (5) of the 2DNLS has proven to be quite useful in understanding integrability, modulational instability and self-focusing. An overview is given in Table 1. [For example, the entry "DSI" under "Integrability" in the sixth row of the table means that for (O_1, O_2, O_3) = (elliptic, hyperbolic, elliptic), conditions (6) are equivalent to DSI.]

We have obtained all of the integrable cases (see conditions (6) which also contain some degenerate cases [Zakharov, 1980]). Eq. (32) is the condition for modulational stability of an integrable system. Observe that the DS-I system, which is known to admit solitons, is modulationally unstable, as one would have expected, while the DS-II system, which is not known to admit solitons, is in fact modulationally stable (for $\kappa > 0$). It thus seems plausible that all integrable cases that do not give rise to solitons are in the modulationally stable regime.

Recall that, for the self-focusing regime as we understand it here, the operators O_2 and O_1 in (6) must be elliptic, while $D^{(3)}$ must have at least one negative eigenvalue (taking $D^{(2)}$ and $D^{(1)}$ to have positive signature without loss of generality). Hence, the regimes of *self-focusing* and *modulational stability* (30) are *mutually exclusive*. For applications to practical experiments, it is especially interesting to compare (28), the dispersion relation for modulational instability (arbitrary $\vec{k}$), with the conditions for self-focusing (negativity of I_1; see (37)) in light of Eq. (59): In (28), higher wavenumbers are stabilizing, and in I_1, too much power at high wavenumbers prevents self-focusing, even if I_4 (see Eqs. (55) and (58)) is negative.

In particular, for hyperbolic O_3, self-focusing requires not only sufficient power at low wavenumbers, but also that most of the power in q^*q be concentrated in modes $\vec{k}$ such that $\mathcal{D}^{(3)} < 0$ (in the notation of (29a)). These are just the unstable modes according to (28).

Next, we observe that the conditions for *integrability* and those for *self-focusing* are *mutually exclusive*. Suppose the contrary: As stated above Eq. (6a), for integrability at least one of O_2 and O_3 must not be elliptic. For self-focusing, O_2 must be elliptic, so O_3 would have to be hyperbolic. But then by (6a) O_1 would also be hyperbolic, which violates our conditions for self-focusing.

In Section 3, we saw that for any 1D subspace defined by $y = \hat{k} \cdot \vec{x}$ (where $\hat{k}$ is any direction such that $\mathcal{D}^{(2)}$ does not vanish), the 2DNLS (5) may be reduced to Eq. (1), the (1D) NLS. Modulational instabilities of the 2DNLS predicted by (30) are therefore associated with ordinary NLS solitons. However, from the work of A&S we know at least for the DR equations that all 1D solitons are *unstable* with respect to long-wavelength perturbations in the remaining spatial coordinate if these perturbations are compatible with the boundary conditions. An intriguing question for future work is thus whether all 1D solitons are unstable for the *full* 2DNLS (5) and, if so, what they evolve into. Of special interest is the regime where all three operators are elliptic and $D^{(3)}$ has two negative eigenvalues (taking $D^{(1)}$ and $D^{(2)}$ positive definite). This regime corresponds to "Region F" of A&S in the DR case and is the most favorable for self-focusing. Begging the reader's tolerance for pure speculation, we suggest that if "nearly 1D" solitons do form and persist for some time in this regime, they may decay due to growth of long-wavelength "transverse" perturbations and eventually develop self-focusing singularities.

It is also evident from (30) that a system which reduces to a modulationally stable NLS in a 1D subspace defined by $y = \hat{k} \cdot \vec{x}$ may still be unstable in 2D. In particular, it may self-focus or, if it happens to be integrable, form 2D solitons. This simple observation,

together with the instability of 1D solitons discussed in the preceeding paragraph, poses a severe limitation on the validity of conclusions drawn from a one-dimensional analysis of any intrinsically multi-dimensional, NLS-type system.

Table 1: Self-Focussing, Modulational Stability, and Integrability of 2DNLS (Eqs. (5))

O_1	O_2	O_3	Self-focussing	Modulational stability	Integra-bility
H	H	H	?	U	YES/NO
H	H	E	?	U/S	NO
H	E	H	?	U/S	DSII/DSII'/NO
H	E	E	?	U	NO
E	H	H	?	U/S	NO
E	H	E	?	U	DSI/NO
E	E	H	YES	U	NO
E	E	E	YES/NO	U/S	NO

H = hyperbolic, E=elliptic, U = unstable, S = stable

Where there is a choice, which case will occur will depend on the exact values of the coefficients (see text).

DSI, DSII, etc.: (see text)

Appendix A: Derivation of Eq. (42)

Consider the moment of inertia integral

$$J \equiv \int\int d^2x \sum_{ij} [(D^{(1)})^{-1}]_{ij} \quad x_i x_j q^* q \quad . \tag{A1}$$

We assume that O_1 and O_2 are elliptic, all fields vanish at infinity sufficiently rapidly, and that q^*q and p can be expressed as 2D Fourier integrals.

Eq. (5a) may be expressed in the form

$$i\partial_t q = \mathcal{H} \tag{A2a} \quad ,$$

where

$$\mathcal{H} \equiv -O_1 + p \tag{A2b}$$

is Hermitian. Using (A2), one obtains

$$\frac{d \quad J}{dt} = 2i \int\int d^2x [q\vec{x} \cdot \nabla q^* - q^* \vec{x} \cdot \nabla q] \quad . \tag{A3}$$

Differentiating, one obtains

$$\frac{d^2 \quad J}{dt^2} = 2 \int\int d^2x \, (-q^* [\mathcal{H} \quad , \vec{x} \cdot \nabla] q + \text{c.c.}) \quad , \tag{A4}$$

where [,] indicates the commutator of two operators. We note the commutation relations

$$[-O_1 \quad , \vec{x} \cdot \nabla] = -2O_1 \tag{A5}$$

and

$$[p \quad , \vec{x} \cdot \nabla] = -\vec{x} \cdot \nabla \quad . \tag{A6}$$

Partially integrating, one obtains

$$\frac{d^2 \quad J}{d\tau^2} = 8 \int\int d^2x \sum_{ij} D_{ij}^{(1)} q_i q_j^* - 4I_5 \quad , \tag{A7a}$$

where

$$I_5 \equiv \int\int d^2x q^* q (\vec{x} \cdot \nabla p) \quad . \tag{A7b}$$

Let us write p, q^*q, and $\vec{x} \cdot \nabla p$ as 2D Fourier integrals

$$p = \int\int d^2k \quad \psi_{\vec{k}} u_{\vec{k}} \tag{A8}$$

$$\vec{x} \cdot \nabla p = \int\int d^2k \quad \Psi_{\vec{k}} u_{\vec{k}} \tag{A9}$$

$$q^* q = \int\int d^2k \quad \rho_{\vec{k}} u_{\vec{k}} \quad , \tag{A10}$$

where

$$u_{\vec{k}} \equiv \frac{1}{2\pi} \exp(i\vec{k} \cdot \vec{x}) \quad . \tag{A11}$$

I_5 then reduces to

$$I_5 = \int\int d^2k \quad \Psi^*_{\vec{k}} \rho_{\vec{k}} \quad . \tag{A12}$$

For simplicity, we drop the subscripts on u, Ψ, ψ, and ρ in what follows. From (A9) - (A10) one obtains

$$\Psi = -2\psi - \sum_i k_i \frac{\partial \psi}{\partial k_i} \quad . \tag{A13}$$

Now, since O_2 is elliptic, Eq. (5b) implies

$$\psi = R\rho \quad , \tag{A14a}$$

where

$$R \equiv \frac{\mathcal{D}^{(3)}}{\mathcal{D}^{(2)}} \tag{A14b}$$

and

$$\mathcal{D}^{(n)} \equiv k_i k_j D^{(n)}_{ij} \quad . \tag{A14c}$$

Using (A13) and (A14) and vigorously integrating by parts in k-space, one obtains

$$I_5 = -\int\int d^2k \quad \psi^*_{\vec{k}} \rho_{\vec{k}} \quad . \tag{A15}$$

Inverting the Fourier transforms, one then has

$$I_5 = -\int\int d^2x \quad pq^* q \quad . \tag{A16}$$

Combining (A16) and (A7), one then obtains Eq. (42).

Note that ellipticity of O_2 was essential to the argument. Otherwise, the quantity R defined in (A14b) would have been singular.

Acknowledgements

This research was supported in part by the Office of Naval Research and the Air Force Office of Scientific Research.

REFERENCES

Ablowitz, M.J. & Haberman, R. 1975 *Phys. Rev. Letters* **35**, 1185.

Ablowitz, M.J. & Segur, H. 1979 *J. Fluid Mechanics* **92**, 691.

Arkadiev, V.A., Pogrebkov, A.K. & Polivanov, M.C. 1989 *Physica* **D36**, 189.

Boiti, M., Leon, J.J.-P., Martina, L. & Pempinelli, F. 1988, *Physics Letters A* **132**, 432.

Boiti, M., Leon, J.J.-P. & Pempinelli, F. 1989, *Physics Letters A* **141**, 101.

Boiti, M., Pempinelli, F. & Sabatier, P.C. 1992 *First and Second Order Nonlinear Evolution Equations from an Inverse Scattering Problem,* (preprint).

Borisov, A. et al. 1992a *Phys. Rev. Lett.* **65**, 1753.

Borisov, A. et al. 1992b *Phys. Rev. A* **45**, 5830.

Djordjevic, V. & Redekopp, L. 1977 *J. Fluid Mechanics* **79**, 703.

Fokas, A.S. & Ablowitz, M.J. 1983 *Phys. Rev. Letters* **51**, 7.

Fokas, A.S. & Ablowitz, M.J. 1984 *J. Math. Phys.* **25**, 2494.

Ichikawa, Y., Mituhashi, T., Konno, K. 1976 *J. Phys. Soc. Japan* **41**, 1382.

Karpman, V. 1990 "Nonlinear Interaction between Short and Long Waves," in Compte & Boccara, "Partially Integrable Evolution Equations in Physics," 83.

Karpman, V. & Washimi, H. 1977 *J. Plasma Physics* **18**, 173.

Kates, R. & Kaup, D. 1988 *Astron. Astrophys.* **206**, 9.

Kates, R. & Kaup, D. 1989 *J. Plasma Physics* **42**, 507 (Paper I).

Kates, R. & Kaup, D. 1989 *J. Plasma Physics* **42**, 521 (Paper II).

Kates, R. & Kaup, D. 1991 *J. Plasma Physics* **46**, 85 (Paper III).

Kates, R. & Kaup, D. 1992a *J. Plasma Physics* **48**, 119 (Paper IV).

Kates, R. & Kaup, D. 1992b *J. Plasma Physics* **48**, 397 (Paper V).

Kates, R. & Kaup, D. 1993, to be published.

Kaup, D.J. 1980 *Physica* **1D**, 45.

Kelley, P. 1965 *Phys. Rev. Lett.* **15**, 1005.

Shukla, P. & Stenflo, L. 1984 *Phys. Rev.* **A30**, 2140.

Spatschek, K 1977 *J. Plasma Physics* **18**, 293.

Taniuti, T. 1974 *Suppl. Progr. Theor. Phys.* **55**.

Zakharov, V.E. 1980, Article in "Solitons", R.K. Bullough and P.J. Caudrey (eds) [Springer-Verlag, NY], (see page 250).

Zakharov, V.E. 1972 Soviet Physics JETP **35**, 908 (Zh. Eksp. Teor. Fiz. **62** 1745).

Zakharov, V.E. & Synakh, V.S. 1976 Soviet Physics JETP **41**, 465 (Zh. Eksp. Teor. Fiz. **68** 940).

Zakharov, V.E. & Schulman, E.I. 1990, Article in "What is Integrability?", V.E. Zakharov (ed.) [Springer-Verlag, NY], (see page 230).