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Nicholas D. Blackaby

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TOLLMIEN-SCHLICHTING/VORTEX INTERACTIONS IN COMPRESSIBLE BOUNDARY LAYER FLOWS

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ABSTRACT

The weakly nonlinear interaction of oblique Tollmien–Schlichting waves and longitudinal vortices in compressible, high Reynolds number, boundary–layer flow over a flat plate is considered for all ranges of the Mach number. The interaction equations comprise of equations for the vortex which is indirectly forced by the waves via a boundary condition, whereas a vortex term appears in the amplitude equation for the wave pressure. The downstream solution properties of interaction equations are found to depend on the sign of an interaction coefficient. Compressibility is found to have a significant effect on the interaction properties; principally through its impact on the waves and their governing mechanism, the triple–deck structure. It is found that, in general, the flow quantities will grow slowly with increasing downstream co–ordinate; i.e. in general, solutions do not terminate in abrupt, finite–distance 'break–ups'.

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1. INTRODUCTION

The nonlinear interaction between two oblique three-dimensional Tollmien–Schlichting (TS) waves and their induced streamwise (longitudinal)-vortex flow is considered theoretically for a compressible boundary–layer flow; this study is an extension of the paper by Hall & Smith (1989) who considered an incompressible boundary–layer flow. The same theory applies to destabilisation of an incident vortex motion by sub-harmonic TS waves, followed by interaction. The interaction is considered for all ranges of the Mach number in order to investigate the effect of flow–compressibility.

The motivation for such a study is essentially the same as expressed by Hall & Smith in the introduction to their paper; namely that often in experimental studies of laminar–to–turbulent transition on a *flat* plate (eg. Aihara & Koyama, 1981; Aihara *et al*, 1985), there appear to be longitudinal vortices co–existing, and interacting, with the viscous TS modes. As there is no concave curvature of the surface, these longitudinal vortices are not Taylor–Görtler vortices driven by surface–curvature (see Hall, 1982a,b and subsequent studies); instead one could postulate that they are in fact being driven by, and/or interacting with TS modes. The reader is referred to the paper by Hall & Smith (1989) for a fuller account of relevant experimental findings, as well as supporting computational work (see, for example, Spalart & Yang, 1986). These experimental studies are all for incompressible flow; the author is unaware of any experimental work specifically relevant to this compressible study. We note in passing that, for compressible flow over a heated plate, buoyancy–driven longitudinal vortices may also be possible (see Hall & Morris, 1992).

Recently, the origin of streamwise vortices in a turbulent boundary layer has been investigated theoretically by Jang *et al* (1986). The Reynolds number is taken to be finite and their formulation is of the Orr–Sommerfeld–type. They show that two oblique travelling waves can combine nonlinearly to produce a stationary, streamwise vortex — this is essentially the theoretical idea later used by Hall & Smith in their independent work. However the latter's approach, that adopted in this paper, takes advantage of the feature that the Reynolds numbers of interest in reality are large and so the Reynolds number is taken as a large parameter throughout. The nonlinear interaction is powerful, starting

at quite low amplitudes with a triple-deck structure for the TS waves but a large-scale structure for the induced vortex, after which strong nonlinear amplification can occur. Non-parallelism is accommodated within the scales involved.

The nonlinear interaction is governed by a partial-differential system for the vortex flow coupled with an ordinary-differential equation for the TS–waves' pressure. The solution of this coupled system depends crucially upon so–called interaction coefficients which are functions of the Mach number; additionally, the TS waves are significantly affected by the inclusion of compressibility. It is found that the interaction coefficients, for subsonic flow, do not differ significantly in nature from the incompressible ones, but as the flow becomes supersonic the restriction (for high Reynolds numbers) that the TS waves must be directed outside the local Mach–wave cone (Ryzhov, 1984; Zhuk & Ryzhov, 1981) excludes a particular flow solution which is only possible for less oblique modes. The flow properties point to the second stages of interaction associated with higher amplitudes.

It is found that the present formulation breaks down as the Mach number becomes large: for then, *even when* the presence of shock/boundary layer interaction is neglected, the viscous sublayers coalesce to form a single boundary-layer. The structure applicable in this hypersonic limit has been considered by Smith (1989) and Blackaby (1991).

The theoretical idea is basically that, if two low–amplitude TS waves are present, proportional to $E_{1,2} = \exp[i(\alpha X \pm \frac{\beta}{2} Z - \Omega \hat{t})]$ say; then nonlinear inertial effects produce the combination $E_1 E_2^{-1} = \exp[i\beta], = E_3$ say, at second order, among other contributions, i.e. a standing–wave or longitudinal–vortex flow is induced. Here α, β and Ω are real–valued scaled wavenumbers and frequency, whilst X, Z and $\hat{t}$ are scaled length– and time–scales (see later). Equally, the combination of the vortex and one TS wave provokes the other TS wave.

Since the Reynolds number is assumed to be large, the TS waves are supported by the triple–deck structure (Smith, 1979,89); however an extra sub–boundary layer and a further streamwise length–scale are necessary to capture their interaction with the longitudinal vortices. The present vortex/wave interaction mechanism is very similar to that of Hall & Smith (1989); the difference is caused by an error in the latter, uncovered by Smith & Blennerhassett (1992). The amended interaction still has the induced vortices lying

at the top of the lower deck but now the forcing from the TS waves is solely from an inner boundary condition. The wall–shear of the induced vortices modifies the wall–shear of the basic flow at the same order as the latter's leading–order nonparallel correction. These corrections to the wall shear force secondary TS waves in the lower–deck, whilst the amplitude of the primary TS waves here is governed by an amplitude equation involving these corrections to the wall shear. The behaviour of the primary TS quantities at the top of the lower deck then leads to longitudinal–vortex activity being forced there. Thus the system is truly interactive: the longitudinal vortices are driven by the TS waves, the amplitude of which is determined by an amplitude equation involving a vortex–term.

We consider the interaction for the case of compressible laminar flow over a semi–infinite plate. In the next section the underlying boundary–layer flow is outlined and the triple–deck structure, for such compressible flows, is reviewed. In §3 the interaction equations are derived, and a few special limiting cases of these equations are considered. In §4 numerical results are reported and in the last section some conclusions are drawn.

2. FORMULATION

We consider the boundary layer due to high-speed uniform flow of a compressible over a flat plate. Suppose that L is the distance from the leading edge, and u^*_∞, a^*_∞, ρ^*_∞ and μ^*_∞, are the velocity, speed of sound, density and shear viscosity of the free stream flow, then we assume that the Reynolds number, $Re = \rho^*_\infty u^*_\infty L/\mu^*_\infty$, is large. This is not unreasonable as one is already assuming the presence of a boundary layer. The second important parameter is the Mach number, $M_\infty = u^*_\infty/a^*_\infty$, which we take to be $O(1)$ for the time being.

A nondimensionalisation based on coordinates $L\underline{x}$ (where x is in the direction of flow and y is normal to plate), velocities $u^*_\infty \underline{u}$, time Lt/u^*_∞, pressure $\rho^*_\infty u^{*2}_\infty p$, density $\rho^*_\infty \rho$, temperature $T^*_\infty T$ and shear viscosity $\mu^*_\infty \mu$ is adopted, where the subscript ∞ denotes the value of the quantity in the free-stream. We assume that the fluid's viscosity and temperature are related by Sutherland's formula. Full details of the Navier-Stokes equations equations for compressible flow; the resulting boundary–layer equations and associated

similarity solutions can be found in several books and articles (eg. Stewartson, 1964). Note that only the 'wall-values' of the steady boundary-layer flow solution occur within scales considered in this paper; however, these quantities depend on the choice of viscosity-temperature relation as well as other factors such as whether the plate is cooled and the nature of the external pressure gradient.

2.1 The 3-D compressible triple-deck equations.

The underlying structure, of the vortex-wave interaction to be considered later, is that of the three-dimensional, compressible Tollmien-Schlichting (TS) waves at large values of the Reynolds number, namely the three-dimensional 'compressible triple-deck'. This structure has been studied by, in particular, Zhuk & Ryzhov (1981), Ryzhov (1984) and Smith (1989); the two-dimensional, compressible triple-deck theory was first considered by Stewartson & Williams (1969). Recently, Cowley & Hall (1990) and Duck & Hall (1990) have shown that the theory can be adapted to include the effects of a shock for flow over a wedge, and cylindrical geometry, respectively; whilst Seddougui, Bowles & Smith (1991) have considered the effects of wall-cooling. For definiteness, we assume that the flow is supersonic ($M_\infty > 1$) during the formulation of the interaction equations; the subsonic and other cases follow very similarly and in §4 results are also presented for these cases.

In the following scalings, the Reynolds number is assumed to be large whilst the other factors are taken be be $O(1)$. The latter are introduced to normalise the resulting governing equations as far as possible; however, the Mach number still remains in the upper-deck's pressure-disturbance equation and hence it appears in the TS-eigenrelation.

The streamwise, spanwise length-scales and the time-scale, for $M_\infty > 1$, are

$$(x - x_0\ , z - z_0) = Re^{-\frac{3}{8}} K_1 (X\ , Z), \quad t = Re^{-\frac{1}{4}} \lambda_w^{-\frac{3}{2}} \mu_w^{-\frac{1}{2}}\, T_w^{\frac{3}{2}} (M_\infty^2 - 1)^{-\frac{1}{4}} \hat{t},$$

$$K_1 = \lambda_w^{-\frac{5}{4}} \mu_w^{-\frac{1}{4}}\, T_w^{\frac{7}{4}} (M_\infty^2 - 1)^{-\frac{3}{8}}; \qquad (2.1a - c)$$

here (x_0, z_0) corresponds to the location of the initial disturbance of the laminar base-flow. In the viscous sublayer, or lower deck,

$$y = Re^{-\frac{5}{8}} \lambda_w{}^{-\frac{3}{4}} \mu_w^{\frac{1}{4}}\, T_w^{\frac{5}{4}} (M_\infty^2 - 1)^{-\frac{1}{8}} Y,$$

$$(u,v,w) = Re^{-\frac{1}{8}}\lambda_w^{\frac{1}{4}}\mu_w^{\frac{1}{4}}T_w^{\frac{1}{2}}(M_\infty^2-1)^{-\frac{1}{8}}(U,\ \ Re^{-\frac{1}{2}}\lambda_w^{\frac{1}{2}}\mu_w^{\frac{1}{2}}T_w^{\frac{-3}{4}}V,\ \ W),$$

and

$$p - p_\infty = Re^{-\frac{1}{4}}\lambda_w^{\frac{1}{2}}\mu_w^{\frac{1}{2}}T_w^{-\frac{1}{2}}(M_\infty^2-1)^{-\frac{1}{4}}P. \qquad (2.2a-e)$$

The resulting lower deck equations are

$$U_X + V_Y + W_Z = 0,$$

$$U_{\hat{t}} + UU_X + VU_Y + WU_Z = -P_X + U_{YY},$$

$$W_{\hat{t}} + UW_X + VW_Y + WW_Z = -P_Z + W_{YY},$$

$$P_Y = 0, \qquad (2.3a-d)$$

to be solved subject to

$$U = V = W = 0, \quad \text{on} \quad Y = 0; \quad U \to (Y + A(X,Z,\hat{t})), \quad \text{as} \quad Y \to \infty. \qquad (2.3e-f)$$

The main deck has

$$y = Re^{-\frac{1}{2}}\mu_w^{\frac{1}{2}}T_w^{\frac{1}{2}}\tilde{y}, \qquad (2.4)$$

and merely transmits the small displacement effect, A, across the boundary layer as well as smoothing out an induced spanwise velocity. The displacement, A, is related to the pressure, P, via a pressure–displacement law stemming from matching solutions across the three decks (see Smith, 1989).

The upper deck occurs where

$$y = Re^{-\frac{3}{8}}K_1(M_\infty^2-1)^{-\frac{1}{4}}\bar{y}; \qquad (2.5)$$

note that the Mach number can be scaled out of all but the upper deck equations.

2.2 Linear Tollmien–Schlichting modes.

The eigenrelation, for linear supersonic TS–modes, is easily derived using the triple–deck scales and equations discussed in the previous subsection. It can be written in the form

$$(i\alpha)^{\frac{1}{3}}(\alpha^2 + \frac{\beta^2}{4}) = (Ai'/\kappa)(\xi_0)\{\frac{\beta^2}{4(M_\infty^2-1)} - \alpha^2\}^{\frac{1}{2}}, \qquad (2.6a)$$

where Ai signifies the Airy function,

$$\kappa = \int_{\xi_0}^{\infty} Ai(q)dq \quad \text{and} \quad \xi_0 = -i^{\frac{1}{3}}\Omega/\alpha^{\frac{2}{3}}. \tag{2.6b, c}$$

Here α, $\beta/2$ and Ω are the scaled wavenumbers and frequency of the mode. The vortex–wave interaction to be described concerns only neutral modes, so α, β and Ω are all real.

In Figure 1 we present the 'neutral' solutions of the eigenrelation (2.6a), and its subsonic counterpart, for a few (illustrative) choices of the Mach number M_∞. Here (and hereinafter) the 'wave [obliqueness] angle' is defined by

$$0 \leq \theta = \tan^{-1}(\beta/2\alpha) \leq 90°. \tag{2.7a}$$

We see for subsonic values of the Mach number ($M_\infty < 1$) that neutral modes are possible for *all* wave–angles. However, for increasing supersonic Mach number values ($M_\infty > 1$) the solution properties start to differ noticeably, with only an ever decreasing range of very oblique TS–wave propagation angles, θ, being possible. Thus the restriction (2.7a), which can be re–written

$$\theta \geq \tan^{-1}\left[(M_\infty^2 - 1)^{\frac{1}{2}}\right], \tag{2.7b}$$

is clearly evident in this figure. We shall see later, once the interaction has been formulated and numerical values have been calculated for the important interaction coefficients, that this restriction proves to be a more significant 'compressibility–effect' on the interaction than the 'direct' effect due to the Mach number appearing in the interaction coefficients.

Smith (1989) gives a comprehensive account of the consequences of the eigenrelation (2.6a) on the stability of the flow to linear TS-modes (note the factor of 2 difference between the definition of β here and that used by Smith, 1989 and Blackaby, 1991) Our concern in this paper is with a vortex–wave interaction based on these length- and time–scales. In the next subsection we deduce the size of additional x- and y- scales necessary to capture this interaction.

2.3 The interaction scales.

In deriving the interaction scales, the same argument as Hall & Smith (1989) is followed but based on the compressible triple-deck scales quoted in §2.1. However, we take the coupled lower- and upper-deck equations as our starting point, rather than returning to the compressible Navier-Stokes equations; this approach appears simpler.

We know that TS-waves are governed by the triple-deck structure, and in particular by the unsteady interactive boundary-layer equations holding in the lower-deck coupled with the upper-deck equations via a pressure-displacement law. If the three-dimensional (3D) TS-wave amplitudes are comparatively small, say of order $h \ll 1$ relative to fully nonlinear sizes, then nonlinear inertial effects force a vortex motion at relative order h^2; the TS-modes are taken to be proportional to

$$E_1 = \exp[i(\alpha X + \frac{\beta}{2}Z - \Omega\hat{t})], \quad E_2 = \exp[i(\alpha X - \frac{\beta}{2}Z - \Omega\hat{t})] \tag{2.8a, b}$$

and we see that combinations yield, in particular, induced longitudinal-vortex terms proportional to

$$E_3 = \exp[i(\beta Z)], \tag{2.8c}$$

having only spanwise dependence.

It can be easily shown that spanwise inertial effects (such as the 'UW_X' term of the $z-$momentum equation) decay slowly like $1/Y^2$ resulting in the spanwise velocity component of the induced vortex to grow logarithmically like $\ln Y$ (Hall & Smith, 1984,89). Hall & Smith (1989) introduced the concept of a new sub-layer ('the buffer layer') situated within, and at the top of, the lower-deck, along with a longer length-scale (for amplitude modulation) to dampen down this logarithmic growth. They showed that the main vortex activity was confined to this region.

Before deriving sizes for the modulation length-scale and the thickness of the buffer-layer, we briefly mention the link between the $x-$scales present and nonparallel effects. The triple-deck is a local structure located at nondimensionalised distance $x = x_0$ from the leading edge. It is short, its length being $O(Re^{-\frac{3}{8}}K_1)$ compared to the $O(1)$ development of the underlying boundary layer, and all the $X-$dependence of the TS-modes is taken to

be in the E_1 and E_2 factors. The modulation of the modes is assumed to be on a longer x-scale, leaving the eigenrelation (2.6a) unaffected. We define this modulation x-scale, $\overline{X}$ say, by

$$x - x_0 = \delta_2 \overline{X} + Re^{-\frac{3}{8}} K_1 X, \quad Re^{-\frac{3}{8}} K_1 \ll \delta_2 \ll 1 \ , \tag{2.9a}$$

where δ_2 is to be determined. Thus we have multiple-scales in x; formally we should make the replacement

$$\frac{\partial}{\partial X} \to \frac{\partial}{\partial X} + \frac{Re^{-\frac{3}{8}} K_1}{\delta_2} \frac{\partial}{\partial \overline{X}} \tag{2.9b}$$

in the triple deck equations (2.3).

At leading order the wall shear $\lambda_w \equiv \lambda_w(x)$ is constant (with respect to the $X-$ and $\overline{X}-$ scales) but here we wish to balance the next order term into the interaction equations; in fact, for later convenience we have scaled the leading order value, $\lambda_w(x_0)$, out of the triple-deck equations. As a Taylor expansion about the local station $x = x_0$,

$$\lambda_w(x) = \lambda_w(x_0)(1 + \delta_2 \overline{X} \lambda_b(x_0) + \cdots); \tag{2.10}$$

here $\lambda_b = \lambda_w^{-1} d\lambda_w/dx$ is $O(1)$ and represents the first influence of nonparallelism (streamwise boundary-layer growth).

In the buffer layer, where $Y = \delta \overline{Y}$ say ($\delta \gg 1$), the size of the spanwise velocity of the induced vortex in the buffer layer is $O(h^2 \ln Y) \sim h^2 \ln \delta$, leading to an induced streamwise velocity of order $\dfrac{\delta_2 Re^{\frac{3}{8}}}{K_1} h^2 \ln \delta$, by continuity (and noting that the modulation is on $\overline{X}$), which alters the basic shear λ_w by a relative amount of order $\dfrac{\delta_2 Re^{\frac{3}{8}}}{K_1} \dfrac{h^2}{\delta} \ln \delta$ and this is the same order as the 'non-parallel' λ_b-term if

$$\delta_2 \sim \frac{\delta_2 Re^{\frac{3}{8}}}{K_1} \frac{h^2}{\delta} \ln \delta \ . \tag{2.11a}$$

The $\overline{X}-$modulation has been introduced to damp the induced-vortex velocity components in the buffer layer, and this requires the inertial operator, $Y\partial/\partial X$, to balance the viscous one, $\partial^2/\partial Y^2$, i.e.

$$\delta^3 \sim \frac{\delta_2 Re^{\frac{3}{8}}}{K_1} \ . \tag{2.11b}$$

One further relation (between the unknowns h, δ and δ_2) is required and results from balancing the slower $\overline{X}$ modulation with the nonparallel effects too (i.e. in the x-momentum equation, balancing the λ_b term with $P_{\overline{X}}$), yielding

$$\delta_2 = Re^{-\frac{3}{16}} K_1^{\frac{1}{2}} \ . \tag{2.11c}$$

The other two sizes now follow immediately from (2.11a-c):

$$\delta \sim Re^{\frac{1}{16}} K_1^{-\frac{1}{6}} \qquad \text{and} \qquad h \sim \left(Re^{-\frac{15}{16}} K_1^{\frac{5}{6}} / \ln \delta \right)^{\frac{1}{2}} \ . \tag{2.11d, e}$$

The logarithmic factor in (2.11a,e) is important (Smith & Blennerhassett, 1992); it is wrong to dismiss it like Hall & Smith (1989). We note that δ is large, whilst h and δ_2 are small, as required.

3. THE INTERACTION EQUATIONS

The method of deriving the interaction equations is identical to that used by Hall & Smith (1989); essentially, it involves a standard weakly-nonlinear triple-deck analysis but slightly complicated due to (i) the wall shear being weakly z-dependent, and (ii), the extra (buffer) layer. Thus here we present only the briefest outline of the derivation; our main concern in this paper being the effect of compressibility on their solution properties. Fuller details can be found in Blackaby (1991) (see also Hall & Smith, 1984,89; and Smith, 1989).

3.1 The equations and their derivation.

The interaction equations comprise of a set of equations for the vortex-terms in the buffer-layer coupled with an equation for the TS-wave pressures resulting from matching the solutions for the waves between the lower-deck and the upper-deck. The vortex equations are forced by the wave via a boundary condition whereas a vortex term explicitly enters the wave-equation.

In the buffer-layer, the vortex terms, $\hat{u}_{33}(\hat{X},\hat{Y}), \hat{v}_{33}(\hat{X},\hat{Y})$ and $\hat{w}_{33}(\hat{X},\hat{Y})$, have the following sizes relative to the lower-deck scales (2.2)

$$(U,V,W)_{\text{vortex}} = h^2 \ln \delta \ (Re^{\frac{3}{8}} \delta_2 \hat{u}_{33}/K_1, \ \delta \hat{v}_{33}, \ \hat{w}_{33}) E_3 + \text{c.c.}; \tag{3.1}$$

here the notation of Hall & Smith (1989) has been adhered to and c.c. denotes complex conjugate. These scales, together with the wall shear term $\delta\overline{Y}$ in the U expansion, lead to the vortex equations

$$\hat{u}_{33\overline{YYY}} - \overline{Y}\hat{u}_{33\overline{XY}} = -i\beta\hat{w}_{33},$$

$$\hat{w}_{33\overline{YY}} - \overline{Y}\hat{w}_{33\overline{X}} = 0, \qquad (3.2a, b)$$

which must be solved subject to the boundary conditions:

$$\hat{w}_{33}(\overline{X}, \infty) = \hat{u}_{33\overline{Y}}(\overline{X}, \infty) = \hat{u}_{33\overline{YY}}(\overline{X}, 0) = 0 \quad \text{and} \quad \hat{w}_{33}(\overline{X}, 0) = -i\beta K|\tilde{p}_{11}|^2; \qquad (3.2c - f)$$

here,

$$K = 1 - (\beta^2/4\alpha^2) \qquad (3.3)$$

and $\tilde{p}_{11}$ is the amplitude of the TS-waves which we choose to be of equal amplitude. Thus, we see that the vortex equations are only forced by the TS-waves via a boundary-condition which matches the solution in the buffer layer with that found in the lower-deck.

The desired equation for the pressure amplitude $|\tilde{p}_{11}|$ of the primary TS-waves can be derived by solving the triple-deck equations for the primary and some forced TS-waves; after some manipulation we find that it takes the form

$$a\,|\tilde{p}_{11}(\overline{X})|_{\overline{X}} + b\,\lambda_b\overline{X}|\tilde{p}_{11}(\overline{X})| + c\,\hat{u}_{33\overline{Y}}(\overline{X}, 0)|\tilde{p}_{11}(\overline{X})| = 0. \qquad (3.4)$$

This equation was first derived by Hall & Smith (1989); however, here the so-called compatibility coefficients a, b and c are functions of the Mach number. The presence of M_∞ in these coefficients is one of the reasons for the solution properties for compressible flow differing from those for the incompressible case. In fact, the for supersonic case

$$a = \frac{2r_1\gamma D\xi_0\Delta^{-2}}{3\alpha} - iB^{-\frac{1}{2}}\left[\frac{2B}{\gamma} + \frac{B}{3\alpha^2} + 1\right], \quad b = -\frac{2r_1\gamma D\xi_0\alpha\Delta^{-5}}{3} - \frac{5B^{\frac{1}{2}}}{3\alpha}$$

$$\text{and} \quad c = iD\xi_0\Delta^{-2}\left(\frac{2r_1\gamma}{3} + \frac{\beta^2 r_2}{4}\right) - \alpha^{-1}B^{-\frac{1}{2}}\left(\frac{5B}{3} - \frac{3\beta^2 B}{4\gamma}\right), \qquad (3.5a - c)$$

where

$$r_1 = \frac{Ai(\xi_0)}{Ai'(\xi_0)}, \quad r_2 = \frac{\kappa(\xi_0)}{Ai(\xi_0)}, \quad D = 1 + \frac{\xi_0\kappa(\xi_0)}{Ai'(\xi_0)}, \quad \gamma = \alpha^2 + \frac{\beta^2}{4},$$

$$B = \frac{\beta^2}{4(M_\infty^2 - 1)} - \alpha^2 \quad \text{and} \quad \Delta = (i\alpha)^{\frac{1}{3}}. \qquad (3.6a - f)$$

The coefficients for other flows can be derived very similarly and, in fact, numerical results for the subsonic and incompressible cases are presented in the next section. An alternative, less physically motivated, derivation of the interaction coefficients is outlined by Blackaby (1991) who considers a generalised TS eigenrelation. The quantitative values of these coefficients, and their resulting effect on the interaction properties are considered later. The (normalised) interaction coefficients

$$c_{1r} = Real(b/a) \quad \text{and} \quad c_{2r} = Real(c/a), \qquad (3.7a, b)$$

are crucial to the solution properties; especially the large–$\overline{X}$ behaviour.

It is possible to derive a nonlinear 'integro–differential' equation,

$$\frac{d|\tilde{p}_{11}|}{d\overline{X}} + \left(c_{1r}\lambda_b \overline{X} + \frac{c_{2r}\beta^2 K Ai'(0)}{\Gamma\left(\frac{2}{3}\right) Ai(0)} \int_0^{\overline{X}} |\tilde{p}_{11}|^2(\psi)(\overline{X} - \psi)^{-\frac{1}{3}} d\psi \right) |\tilde{p}_{11}| = 0, \qquad (3.8)$$

for the pressure amplitude $|\tilde{p}_{11}|$ from the previous equations. A similar equation has been found by Smith & Walton (1989), in their study of vortex–wave interactions.

3.2 Possible limiting forms for large–$\overline{X}$.

Let us consider analytically the possible flow solutions for large–$\overline{X}$. Hall & Smith (1989) found four such options for their system of equations; however, the necessary amendments to their work render one of these options is no longer feasible, namely that of exponential growth. Moreover, there is a swap in the signature required for the crucial quantity Kc_{2r} for the finite–distance–blow–up and the algebraic–growth–to–infinity eventualities to be possible. Thus, the conclusions, drawn later, for the case of zero Mach number are quite different from those found in Hall & Smith (1989). In §4, numerical solutions of the interaction equations will be presented and compared with the large–$\overline{X}$ asymptotic predictions that follow.

(i) Option I: Finite distance break up.

Hall & Smith (1989) showed that a possible, ultimate behaviour of the nonlinear interactive flow, as $\overline{X}$ increases, was that of an algebraic singularity arising at a finite position, say as $\overline{X} \to \overline{X}_0^-$; this option is, in fact, still possible for the corrected system of interaction equations but with some changes in the details. The similarity forms they proposed are appropriate, apart from that for the pressure. As $\overline{X} \to \overline{X}_0^-$, the behaviours for the interaction quantities must have the forms:

$$|\tilde{p}_{11}| \sim (\overline{X}_0 - \overline{X})^{-\frac{5}{6}} \tilde{P}(\check{\eta}), \quad \hat{w}_{33} \sim (\overline{X}_0 - \overline{X})^{-\frac{5}{3}} \tilde{W}(\check{\eta}),$$

$$\hat{u}_{33\overline{Y}} \sim (\overline{X}_0 - \overline{X})^{-1} \tilde{\tau}(\check{\eta}), \quad \text{where} \quad \check{\eta} = \overline{Y}(\overline{X}_0 - \overline{X})^{-\frac{1}{3}}. \qquad (3.9a - d)$$

When these forms are substituted into the interaction equations, the resulting similarity equations can be solved and we deduce that we require the quantity

$$Kc_{2r} < 0, \qquad (3.10)$$

for this option of finite–distance break–up to be a possible large–$\overline{X}$ state of the vortex–wave interaction. Note the change of sign necessary for this option to be possible; this change is due to the modifications found necessary by Smith & Blennerhassett (1992).

The next option that we consider is less 'catastrophic', as far as the laminar flow is concerned, with the solution continuing to downstream infinity.

(ii) Option II: Algebraic response at infinity.

This option is still also possible with the corrected equations. The flow quantities must have the following forms

$$|\tilde{p}_{11}| \sim \overline{X}^{\frac{1}{6}} \tilde{\check{P}}(\check{\check{\eta}}), \quad \hat{w}_{33} \sim \overline{X}^{\frac{1}{3}} \tilde{\check{W}}(\check{\check{\eta}}), \quad \hat{u}_{33\overline{Y}} \sim \overline{X} \tilde{\check{\tau}}(\check{\check{\eta}}), \quad \check{\check{\eta}} = \overline{Y}\overline{X}^{-\frac{1}{3}}, \qquad (3.11a - d)$$

as $\overline{X} \to \infty$.

It is easy to show, from the integro–differential equation (3.8), that we require

$$Kc_{2r} > 0 \qquad (3.12)$$

for this option to be possible; note that this is also a different result than Hall & Smith (1989) found.

The third large–$\overline{X}$ option proposed by Hall & Smith (see also Smith & Walton, 1989) is that of an exponential growth as $\overline{X} \to \infty$. This option is no longer possible as it relies on a forcing term in the $\hat{w}_{33}$–equation that is not present in the corrected equations. A further option, mentioned by Hall & Smith, that is still feasible; is that of decoupling due to linearisation. Here the TS pressure disturbance $|\tilde{p}_{11}|$ becomes very small, and the vortex flow then grows slowly on its own with downstream variable $\overline{X}$ from its initial upstream state. However, this option is ultimately unstable to the TS–waves since the nonparallel–growth term, proportional to λ_b, will dominate the vortex skin friction $\hat{u}_{33\overline{Y}}(\overline{X}, 0)$.

3.3 The transonic and hypersonic limits.

There are three obvious limiting cases to consider for the value of the Mach number; below we consider the transonic and hypersonic limits, $M_\infty \to 1$ and $M_\infty \to \infty$ respectively, whilst the incompressible case $M_\infty = 0$ is considered in the next section.

(i) The Mach number tending to unity.

In his study of the eigenrelation (2.6a), Smith (1989) investigated various limiting cases, including those of $M_\infty \to 1$ and $M_\infty \to \infty$; Here the 'transonic limit' will be considered; without loss of generality, we suppose that the flow is (just) supersonic and define

$$\hat{m} = (M_\infty^2 - 1)^{\frac{1}{2}}, \tag{3.13}$$

so that $\hat{m}$ is small. Smith (1989) showed that, in this case, the TS–wave quantities behave like

$$(\alpha, \beta, \Omega) \sim (\hat{m}^{-\frac{3}{4}}\alpha^*, \hat{m}^{-\frac{3}{4}}\beta^*, \hat{m}^{-\frac{3}{4}}\Omega^*) + \cdots. \tag{3.14}$$

Substituting these into the formula for the interaction coefficients, we find that

$$(c_{1r}, c_{2r}) \sim \hat{m}^{-\frac{3}{4}}(c_{1r}^*, c_{2r}^*) + \cdots, \quad c_{1r}^*, c_{2r}^* \sim O(1), \tag{3.15}$$

so that the interaction scales $\overline{X}, \overline{Y}$, the vortex disturbances $\hat{u}_{33\overline{Y}}, \hat{w}_{33}$ and the TS- pressure amplitude $|\tilde{p}_{11}|$ also need to be scaled with $\hat{m}$:

$$(\overline{X}, \overline{Y}, \hat{u}_{33\overline{Y}}, \hat{w}_{33}, |\tilde{p}_{11}|) = (\hat{m}^{\frac{3}{8}}\overline{X}^*, \hat{m}^{\frac{1}{8}}\overline{Y}^*, \hat{m}^{\frac{3}{8}}, \hat{u}^*_{33\overline{Y}^*}, \hat{m}^{\frac{7}{8}}\hat{w}^*_{33}, \hat{m}^{\frac{13}{16}}|\tilde{p}_{11}|^*) + \cdots. \quad (3.16)$$

These scales and the resulting set of equations can be used to check numerical results, for the general supersonic case, by providing a 'transonic' asymptote. We do not consider the transonic limit any further here; Bowles (1990) has considered transonic boundary layer transition but the author is unaware of any vortex–wave formulations for the transonic regime.

(ii) The large Mach number limit.

Another limiting case that Smith (1989) went on to investigate was the so-called *hypersonic limit* when $M_\infty \gg 1$; this limiting case leads to some interesting consequences for the whole triple–deck structure. Thus it would be most instructive to consider the same limit here, as our interaction structure is, of course, dependent to a very great extent on the underlying triple–deck scales.

First, let us recap those results of Smith (1989) which are relevant here, before going on to investigate the result of increasingly large Mach number on the interaction–coefficients, equations and length-scales. For $M_\infty \gg 1$ the main features revolve around the small regime

$$(\alpha, \beta, \Omega) \sim (M_\infty^{-\frac{3}{2}}\tilde{\alpha}, M_\infty^{-\frac{1}{2}}\tilde{\beta}, M_\infty^{-1}\tilde{\Omega}) + \cdots, \quad (3.17)$$

where $\tilde{\alpha}, \beta$ and $\tilde{\Omega}$ are $O(1)$. Since α, β and Ω appear in the interaction coefficients (3.7), it is necessary to rescale the coefficients as follows:

$$(c_{1r}, c_{2r}) = M_\infty^{-\frac{3}{2}}(\tilde{c}_{1r}, \tilde{c}_{2r}) + \cdots, \quad \text{where } \tilde{c}_1, \tilde{c}_2 \sim O(1). \quad (3.18a, b)$$

It also is necessary to rescale the quantities appearing in the interaction equations,

$$(\overline{X}, \overline{Y}, \hat{u}_{33\overline{Y}}, \hat{w}_{33}, |\tilde{p}_{11}|) = (M_\infty^{3/4}\tilde{X}, M_\infty^{1/4}\tilde{Y}, M_\infty^{3/4}\tilde{\hat{u}}_{33\tilde{Y}}, M_\infty^{3/4}\tilde{\hat{w}}_{33}, \ M_\infty^{-3/8}|\tilde{\tilde{p}}_{11}|). \quad (3.18c-g)$$

A set of interaction equations for this hypersonic–limiting case can be derived (Blackaby, 1991). These appear exactly the same as the general interaction equations which obscures the fact that the whole multi–layered boundary–layer structure is radically altered as the Mach number increases. It was shown by Blackaby (1991), based on the theory of Smith (1989), that as $M_\infty \nearrow Re^{\frac{1}{9}}$, the triple–deck streamwise length–scale, $Re^{-\frac{3}{8}}K_1X$, rises to become $O(1)$ in size; implying that a normal–mode decomposition is no longer rational because the nonparallelism of the underlying, growing boundary–layer is now a leading–order effect. Further, in this limit it was shown that the lower–deck thickens to coalesce with the main-deck.

This collapse of the underlying compressible–triple–deck structure, as the Mach number increases, will obviously occur for the large Mach number behaviour of the vortex–wave interaction being considered. However the large Mach number destiny of buffer–layer (in particular, its thickness) and the amplitude–modulation scale remain to be established. Intuitively, as the buffer–region is 'sandwiched' between the lower– and main–decks which merge into a single viscous layer in this limit, we would also expect the buffer-region to collapse into the same viscous layer. Similarly, as the modulation–scale is 'sandwiched' between the triple–deck's streamwise length–scale (which emerges as $O(1)$ in this Mach number limit) and the $O(1)$–length–scale of the underlying flow, we would expect that the modulation–scale also lengthens to that of the underlying base flow (as $M_\infty \nearrow Re^{\frac{1}{9}}$). We now show that these suspicions are correct, by formally considering the large Mach number properties of the scales involved.

Recall that, in the streamwise direction, we have the multiple scales,

$$\partial_x \longrightarrow \partial_x + \delta_2^{-1}\partial_{\overline{X}} + Re^{\frac{3}{8}}K_1^{-1}\partial_X;$$

necessary to capture the vortex–wave interaction. The quantities K_1 and δ_2 are as defined by (2.1c) and (2.11c), respectively. In the large Mach number limit, we have seen that

$$\partial_{\overline{X}} \sim M_\infty^{-\frac{3}{4}} \quad \text{whilst} \quad \partial_X \sim \alpha \sim M_\infty^{-\frac{3}{2}},$$

so that the unscaled length-scales, L_w and L_v say, of the TS-waves and the modulation of the induced vortices, respectively, are

$$L_w \sim Re^{-\frac{3}{8}} K_1 M_\infty^{\frac{3}{2}} \ll 1 \quad \text{and} \quad L_v \sim \delta_2 M_\infty^{\frac{3}{4}} \ll 1.$$

The Sutherland temperature–viscosity relation leads to,

$$K_1 \sim M_\infty^{\frac{15}{8}}, \tag{3.19a}$$

and so

$$L_w \sim Re^{-\frac{3}{8}} M_\infty^{\frac{27}{8}}, \nearrow O(1), \quad \text{as} \quad M_\infty \nearrow Re^{\frac{1}{9}}.$$

As far as the amplitude–modulation scale is concerned, we find that

$$L_v \sim \delta_2 M_\infty^{\frac{3}{4}} \sim Re^{-\frac{3}{16}} K_1^{\frac{1}{2}} M_\infty^{\frac{3}{4}} \sim Re^{-\frac{3}{16}} M_\infty^{\frac{27}{16}} \nearrow O(1), \quad \text{as} \quad M_\infty \nearrow Re^{\frac{1}{9}}. \tag{3.19b}$$

Thus, as predicted earlier, this modulation scale does indeed rise to $O(1)$–size in this limit of the Mach number.

Now, let us consider the buffer-region; it lies at the top of the lower–deck, where the lower–deck normal–variable $Y = \delta \overline{Y}$ and δ is defined by (2.11d). For large Mach numbers, we have found, (3.18d), that the buffer–region is characterised by the location where $\tilde{Y} = M_\infty^{-\frac{1}{4}} \overline{Y} \sim O(1)$. Thus the buffer-region lies where

$$Y \sim \delta M_\infty^{\frac{1}{4}} \sim Re^{\frac{1}{16}} K_1^{-\frac{1}{6}} M_\infty^{\frac{1}{4}} \sim Re^{\frac{1}{16}} M_\infty^{-\frac{1}{16}}$$

$$\rightarrow O(Re^{\frac{1}{18}}) \sim O(M_\infty^{\frac{1}{2}}), \quad \text{as} \quad M_\infty \nearrow Re^{\frac{1}{9}}. \tag{3.20}$$

Note that for large Mach number, the lower–deck variable, Y, also scales on M_∞; in fact $Y \sim M_\infty^{\frac{1}{2}}$ — hence from this and (3.20) we deduce that the buffer–layer merges with the lower–deck, which in turn coalesces with the main–deck. Thus the three sub–boundary–layers, present for $M_\infty \sim O(1)$, have all merged into one single viscous layer.

Summarising, when $M_\infty \rightarrow Re^{\frac{1}{9}}$ the four–layered, short–scaled structure underlying the vortex–wave interaction collapses into the two–tiered, long structure found by Smith (1989) and considered by Blackaby (1991).

4 RESULTS AND DISCUSSION.

This study was motivated by the desire to find out what changes to the theory, predictions and conclusions of the original work by Hall & Smith, are brought about by the inclusion of compressibility–effects. However, ironically, the changes brought about by the correction of the former turn out to be more significant. For this reason, and for later comparison, the new results for incompressible flow will also be presented in §4.2. In the following subsection, we show how the interaction equations can be 'normalised' so that their solution depends merely on initial conditions imposed and the sign of Kc_{2r}. Numerical solutions of the normalised interaction equations are presented for both choices of the sign of Kc_{2r}.

4.1 The interaction equations renormalised.

In §3.2 we considered possible limiting–forms, for solutions to the interaction equations as $\overline{X} \to \infty$, and found that the sign of the quantity Kc_{2r} was crucial in deciding whether particular limiting forms were, in fact, possible. This suggests that the interaction equations, (3.2,3.4), can be renormalised. This being desirable, we investigated further and found this was, indeed, the case.

Writing

$$\overline{X} = |c_{1r}\lambda_b|^{-\frac{1}{2}} X^*, \quad \overline{Y} = |c_{1r}\lambda_b|^{-\frac{1}{6}} Y^*, \quad \hat{w}_{33} = -i\beta K |c_3|^{-1} W^*,$$

$$|\tilde{p}_{11}| = |c_3|^{-\frac{1}{2}} P^* \quad \text{and} \quad \hat{u}_{33\overline{Y}} = -\beta^2 K |c_{1r}\lambda_b|^{-\frac{1}{3}} |c_3|^{-1} \tau^*, \tag{4.1a – e}$$

where

$$c_3 = -Kc_{2r}\beta^2 / |c_{1r}\lambda_b|^{\frac{5}{2}}, \tag{4.2}$$

leads to the normalised system

$$W^*_{Y^*Y^*} - Y^* W^*_{X^*} = 0, \quad \tau^*_{Y^*Y^*} - Y^* \tau^*_{X^*} = W^*$$

$$\text{and} \quad P^*_{X^*} + [\text{sgn}(c_{1r}\lambda_b) X^* - \text{sgn}(Kc_{2r}) \tau^*(X^*, 0)] \, P^* = 0, \tag{4.3a – c}$$

which must be solved subject to initial conditions (at $X^* = 0$), together with the boundary conditions

$$W^*(X^*, \infty) = \tau^*(X^*, \infty) = \tau^*_{Y^*}(X^*, 0) = 0 \text{ and } W^*(X^*, 0) = P^{*2}(X^*). \qquad (4.3d - g)$$

Thus, the interaction equations (and hence their solutions) are dependent only on the initial conditions imposed; $\mathrm{sgn}(c_{1r}\lambda_b)$ and $\mathrm{sgn}(Kc_{2r})$. In all the numerical calculations carried out, it was found that $c_{1r} > 0$, whilst $\lambda_b < 0$ for a growing 'Blasius–similarity–variable–type' boundary layer — appropriate to the present study, if we assume that there is no significant wall–cooling or pressure–gradient effects. We therefore set $\mathrm{sgn}(c_{1r}\lambda_b) = -1$ and, apart from consistent initial conditions, the only parameter remaining is $\mathrm{sgn}(Kc_{2r})$. Thus, with hindsight, it is not surprising that the (predicted) solution properties for, large–$\overline{X}$, depend crucially on the value of $\mathrm{sgn}(Kc_{2r})$. Recall that earlier, in §3.2, we noted the following predictions:

$$\mathrm{sgn}(Kc_{2r}) \begin{cases} > 0 & : \text{Algebraic response, as } \overline{X} \to \infty \\ < 0 & : \text{Finite-distance break-up, as } \overline{X} \to \overline{X}_0^-, \end{cases} \qquad (4.4)$$

for the behaviour of the solutions to the interaction equations.

To check these predictions, the normalised system, (4.3), was solved numerically; for both possible values of $\mathrm{sgn}(Kc_{2r})$, and for different (consistent) initial conditions. The large–$\overline{X}$ ($X^* \gg 1$) properties of the solutions were found to depend solely on $\mathrm{sgn}(Kc_{2r})$; the initial conditions were found to affect only the initial development of the imposed disturbances. The equations were solved by taking 'central differences' in Y^* and 'forward differences' in X^* (following the method of Hall & Smith, 1989); the appropriate numerical checks were performed.

In Figures 2a,b, we present typical results for both values of $\mathrm{sgn}(Kc_{2r})$. In both of these computations the system was initialised at $X^* = -1$ (upstream of the neutral TS point) using

$$P^* = P_0^*, \; W^* = P_0^{*2}(1 + Y^{*2})\exp[-Y^{*2}], \; \tau^* = (1 - \frac{P_0^{*2}}{2} + Y^{*2})\exp[-Y^{*2}], \quad (4.5a - c)$$

with $P_0^* = 0.1$. Note that this initial state, which is consistent with the interaction equations plus boundary conditions, corresponds to a 'mixed' wave/vortex state. Moreover,

we see from the 'forcing' boundary condition (eg. (4.3g)) that admissible initial states *cannot* consist of just TS waves alone; the longitudinal vortices must initially be present. It appears to the author that the initial states used by Hall & Smith (1989) (see their section 5; particularly figures 2–5) are inconsistent with their system of interaction-equations plus boundary conditions; they do not appear to satisfy the boundary conditions. In their study of vortex/wave interactions, Smith & Walton (1989) do not comment on the initial conditions they choose.

Returning to Figures 2a,b, we see that these numerical results are in full agreement with the theoretical large-X^* predictions, (4.4). Thus, in the following subsections, it is sufficient to merely calculate values of $\mathrm{sgn}(Kc_{2r})$ in order to determine the solution properties for large-X^*; these being of principal interest.

4.2 The incompressible case ($M_\infty = 0$).

In their study, Hall & Smith (1989) considered 'this' vortex–wave interaction for incompressible boundary–layer flow. They cleverly deduced the scales and formulated the interaction; unfortunately, they made two unrelated errors in their analysis, both of which have a significant effect on the results and conclusions. The first of these errors concerning the missing logarithmic term in the interaction scales (most kindly pointed out to the current author by Dr. P. Blennerhassett and Prof. F.T. Smith), leads to a simpler system of interaction–equations, as well as leading to changes in the possible large–$\overline{X}$ states and the necessary parameter values for them to be possible. The second, the term $\tilde{w}^{(1)}\lambda_{3Z}Y$ missing from the left–hand side of Hall & Smith's (1989) equation (3.9b), was spotted by the current author and leads to a corrected form for c, and hence, a corrected value for the crucial quantity c_{2r}.

The interaction coefficients a, b, are as given by Hall & Smith (1989); whilst the corrected form for c was given by Blackaby (1991) (see also the Appendix A of Smith & Blennerhassett, 1992). In Figure 3 the new numerical values for the important interaction quantities, c_{1r} and c_{2r} are plotted, versus TS–wave obliqueness angle θ; recall that, for

incompressible flow, all such wave angles are possible. Note that $c_{1r} > 0$ for all θ; whilst c_{2r} has one zero, at $\theta \simeq 32.21°$. Recalling the definition of K,

$$K = 1 - (\beta^2/4\alpha^2) = 1 - \tan^2(\theta) \begin{cases} > 0 & : \text{if } \theta < 45° \\ < 0 & : \text{otherwise,} \end{cases}$$

we see that, when $M_\infty = 0$,

$$\text{sgn}(Kc_{2r}) = \begin{cases} -1 & : \text{if } 32.21° \leq \theta < 45° \\ +1 & : \text{otherwise.} \end{cases}$$

Thus, from this last result and the numerical calculations described in §4.1, we deduce the following: (i) if $32.21° \leq \theta < 45°$ then the solution to the interaction equations will 'blow-up' in a finite-distance; otherwise (ii) the solutions will grow slowly (far slower than the linear TS-solutions if there were no vortices present), with amplitudes proportional to algebraic powers of X^*, as $X^* \to \infty$. Note that these conclusions are quite different from those of Hall & Smith (who concluded that the 'finite-distance break-up' option was most likely, apart from the small range $45° \leq \theta <\sim 50°$ where an 'exponential-growth' option was favoured). Thus, the theoretically-exciting 'finite-distance break-up' option is now the exception, rather than the rule.

4.3 The subsonic, supersonic and hypersonic cases.

For subsonic ($M_\infty < 1$) and some supersonic ($1 < M_\infty <\sim 1.15$) flows, the properties of the interaction-coefficients were remarkably similar to those found for the incompressible case i.e. graphs of c_{1r}, c_{2r} against θ appear very similar to Figure 3. However, the TS-wave angle restriction (2.7b) is found to have a far more significant effect for 'more' supersónic flows — essentially it can be regarded as preventing wave-angles that would allow $\text{sgn}(Kc_{2r}) < 0$, corresponding to the finite-distance break-up option. This is illustrated more clearly in Figure 4 where the results are summarised; we see that the θ - M_∞ plane splits into four regions (labelled I - IV, as shown). Region IV corresponds to the 'barred' area, where no neutral TS-modes are possible. We see how the border of this region acts as an 'abrupt cut-off' to the larger-M_∞ extent of Region II (finite-distance break-up option). This is so much so that, for Mach numbers above $\sqrt{2}$, the possibility

of finite distance break up has gone. Thus summarising, in the subsonic case the results are almost identical to the incompressible case; whereas, in general, the finite distance break up eventuality is not possible for supersonic flows, mainly due to the severe cut off restriction for large Reynolds numbers. To illustrate the last point, in Figure 5 we have plotted c_{2r} versus β and θ for $M_\infty = 3$ — note (i) that there is no zero for c_{2r}, and (ii), the very oblique wave angles encountered (so that K is always negative and, hence, Kc_{2r} is always positive).

The last set of results that we present are for hypersonic flow over a wedge, as considered by Cowley & Hall (1990), in which a shock is fitted into the upper-deck (at $\overline{y} = \overline{y}_s$, where $\overline{y}$ is the normal-variable of the upper-deck), leading to a modified form of Smith's hypersonic TS-eigenrelation (the reader is referred to the paper by Cowley & Hall for all details of the formulation). In Figure 6, we present results for the first (lowest) neutral curve for the case $\overline{y}_s = 1$; here α_{CH}, $\beta_{CH} \sim O(1)$ correspond to α,β in the notation of that paper. It is sufficient to note that, in *our* notation,

$$\frac{\beta}{2\alpha} \sim M_\infty \frac{\beta_{CH}}{\alpha_{CH}} \gg 1,$$

and so the waves they consider are, in general, very oblique. Of particular interest here is the (small) interval where $c_{2r} > 0$, so that $Kc_{2r} < 0$, corresponding to the finite-distance break-up option; this is an effect of the shock. No such interval is found for the 'higher' neutral curves; this interval appears to be a feature of the 'lowest' neutral curves only (for each choice of $\overline{y}_s$) and corresponds to 'crossing' the 'divide' $\alpha_{CH} = \beta_{CH}$.

Finally, we report that for the 'hypersonic and transonic' limiting cases mentioned in §3.3, the numerical results and the predicted asymptotic behaviours (for the interaction coefficients) were in extremely good agreement.

5 CONCLUSIONS

Many of the conclusions of Hall & Smith (1989) carry over to the present study and so we concentrate on compressibility-related aspects here. In this paper it has been demonstrated that, within the triple-deck framework ($Re \gg 1$), pairs of small-amplitude Tollmien-Schlichting waves and longitudinal vortices can interact, leading to

mutual growth. We have seen that two possible 'eventualities', for the downstream evolution of the interaction, exist; one in which the solutions grow relatively slowly as $\overline{X} \to \infty$; whilst the other terminates at a finite-distance in a 'break-up'. Further, we have seen that the latter is no longer possible, in general, for supersonic flows ($Re \gg 1$).

The interaction has been considered for all ranges of the Mach number; corrected results for the incompressible case have been presented and the main effect of compressibility is through its impact on the TS waves via their governing mechanism, the triple-deck structure. In the transonic and hypersonic limiting-cases the interaction modulation scale $\overline{X}$ must be rescaled; in the transonic limit this modulation scale shortens, whilst in the hypersonic limit the opposite is true. The investigation of such vortex/wave interactions in transonic and hypersonic flows (not their 'limits') should prove interesting; note that the former flow has been studied by Bowles (1990), whereas the latter flow regime has been considered by Blackaby (1991).

Other effects which could be incorporated into the present theory include pressure-gradient effects; wall-cooling effects (see Seddougui, Bowles & Smith, 1991); cylindrical geometry (see Duck & Hall, 1990) and spanwise-modulation (cf Smith & Walton, 1989). Finally, we note that Hall & Smith (1991) and Walton & Smith (1992) consider the properties of 'strongly nonlinear' TS-wave/vortex interactions corresponding to larger wave amplitudes than those considered in this paper.

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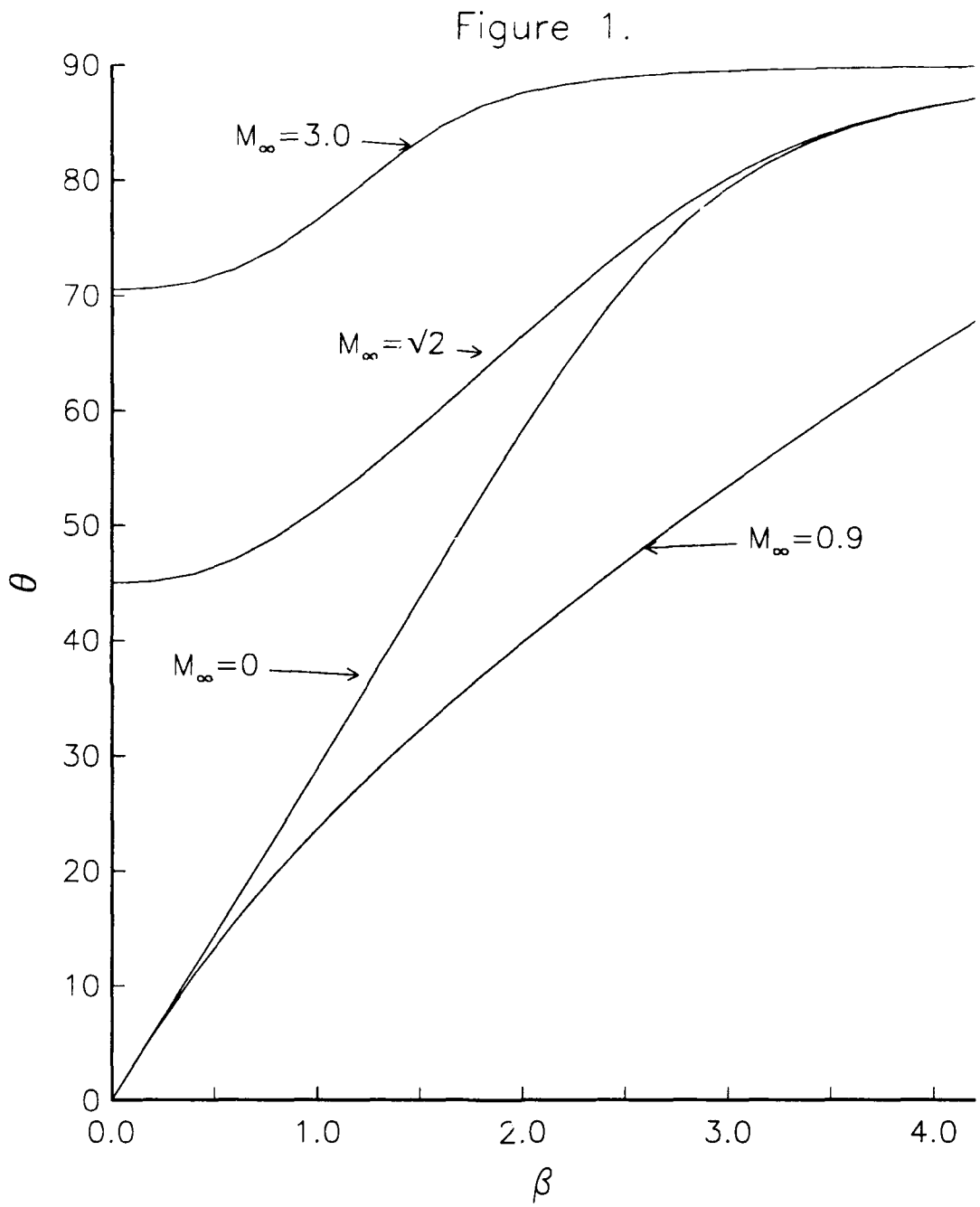

Figure 1. TS wave obliqueness angles θ versus scaled spanwise wavenumber for neutral modes.

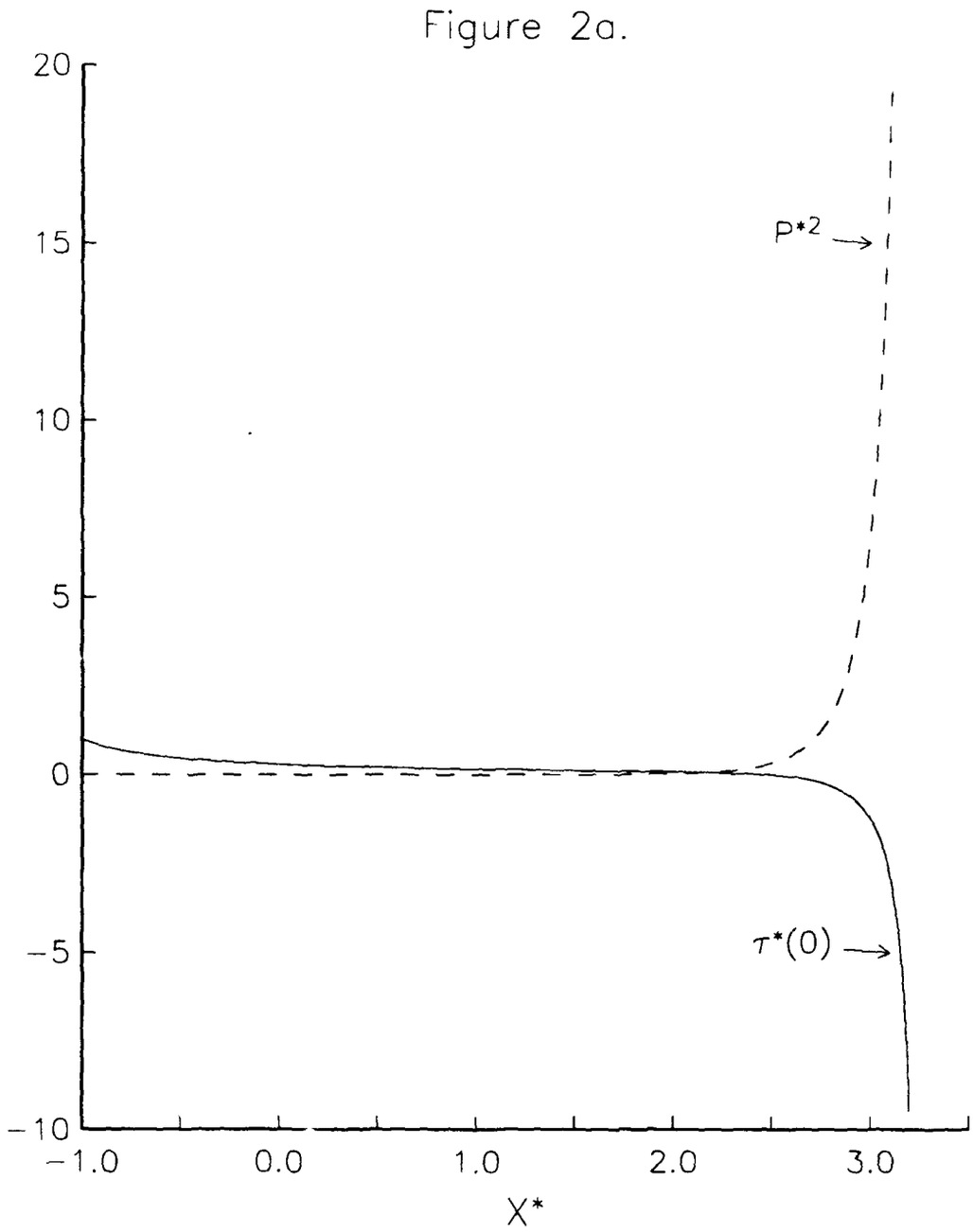

Figure 2a. Numerical solution of the interaction equations (4.3) with $Kc_{2r} < 0$.

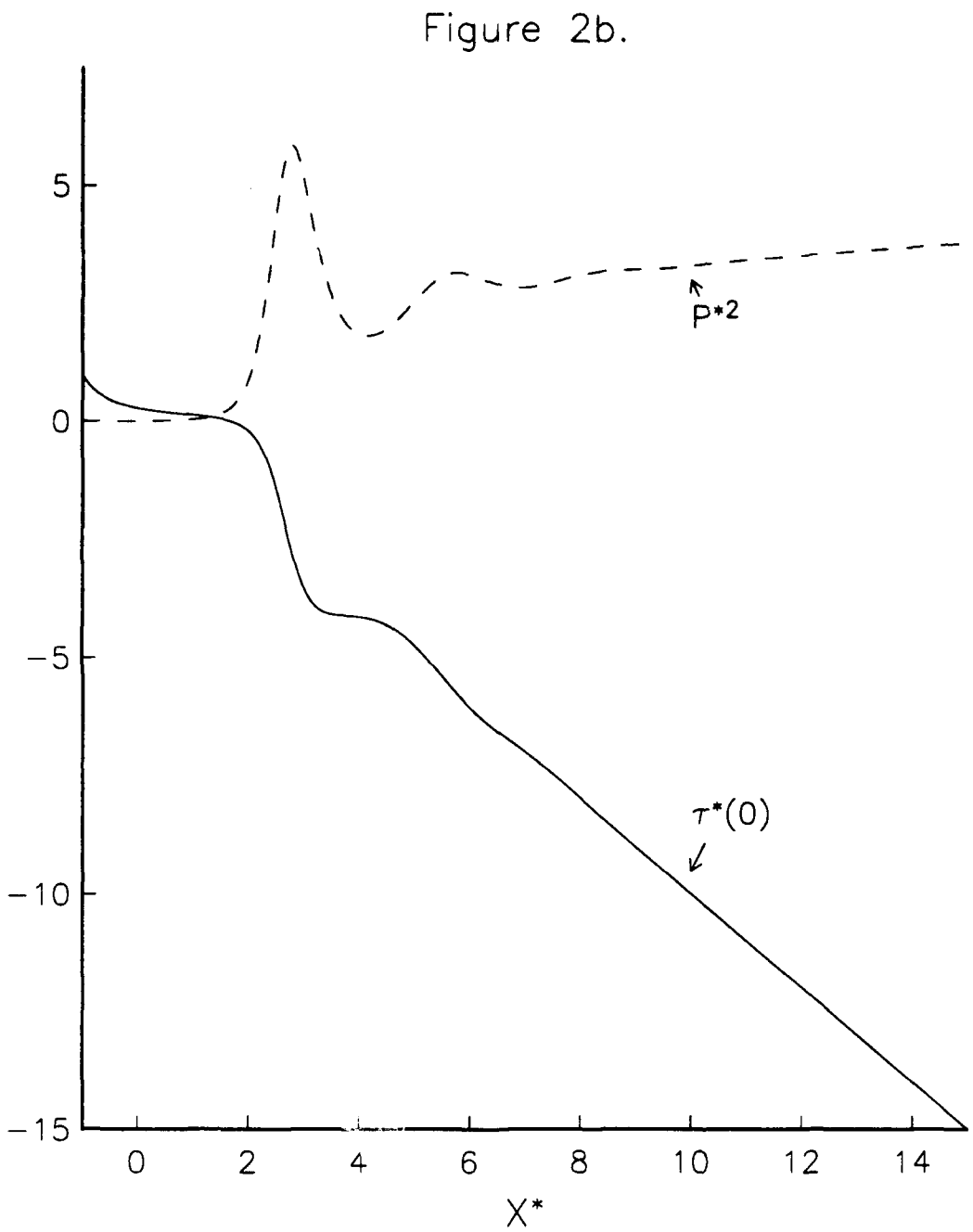

Figure 2b. Numerical solution of the interaction equations (4.3) with $Kc_{2r} > 0$.

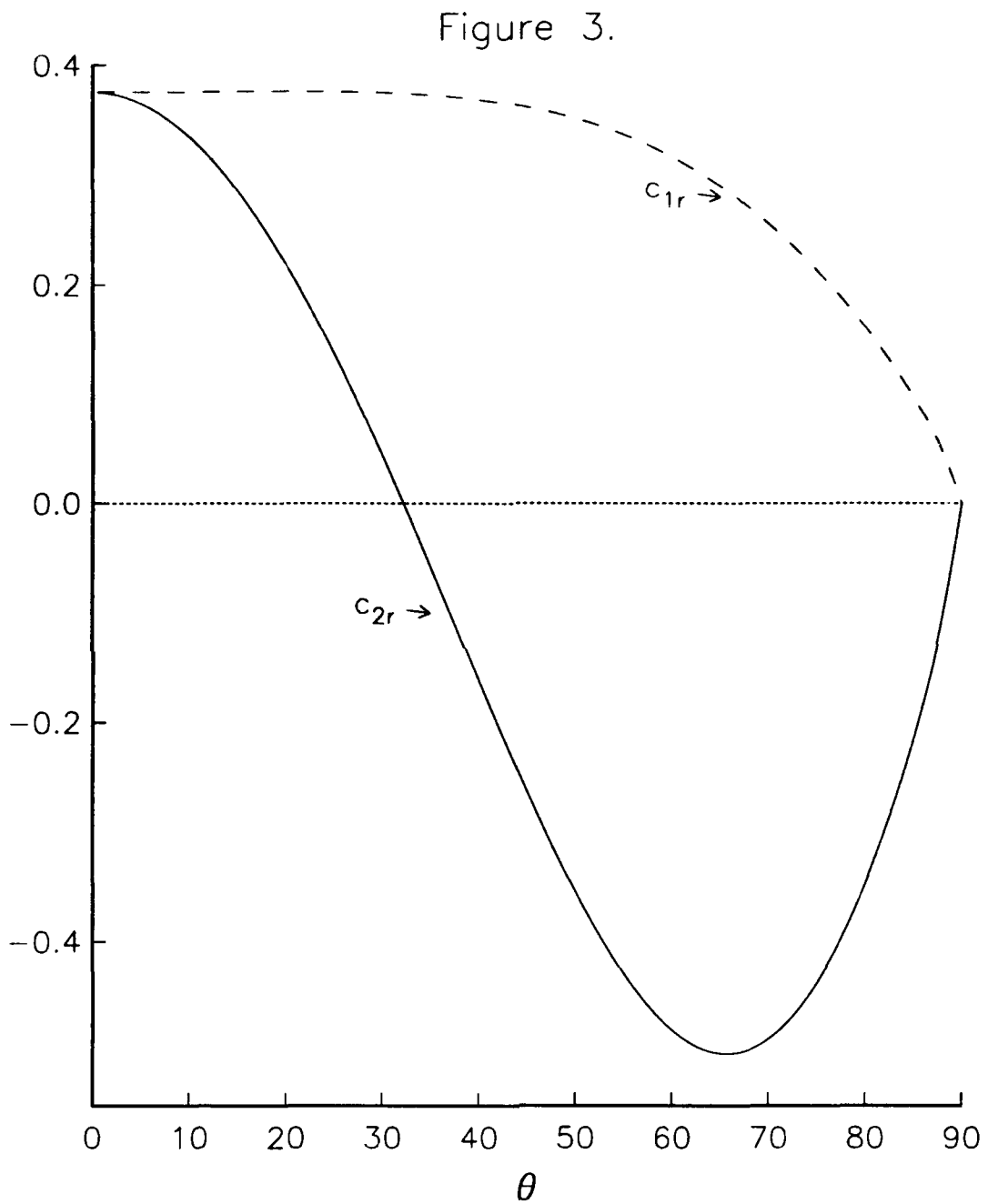

Figure 3. The interaction coefficients, c_{1r} and c_{2r}, versus TS wave angle θ for the incompressible case.

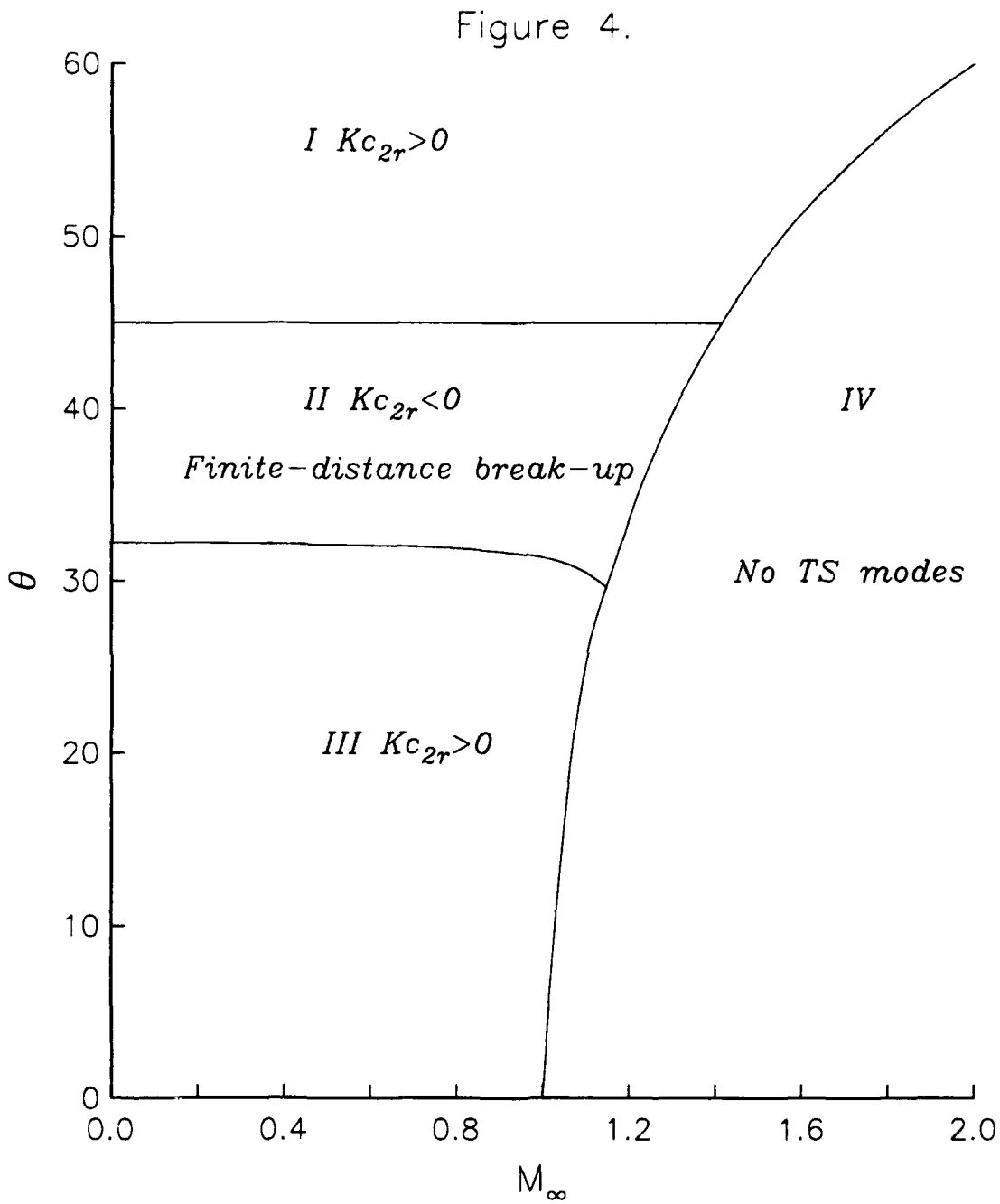

Figure 4. The regions of the $\theta - M_\infty$.

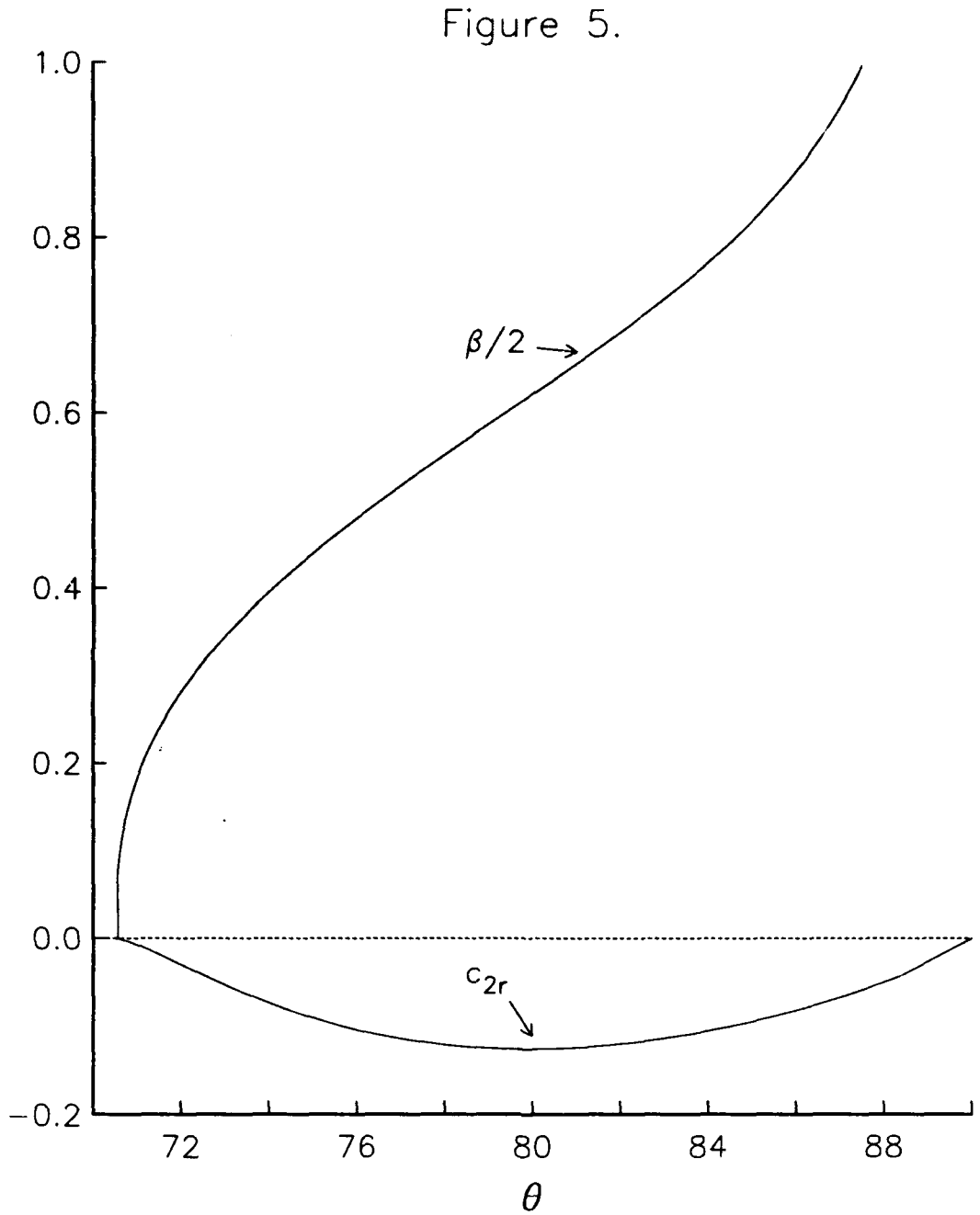

Figure 5. The wavenumber $\beta/2$ and the interaction coefficient c_{2r}, versus TS wave angle θ, for $M_\infty = 3$.

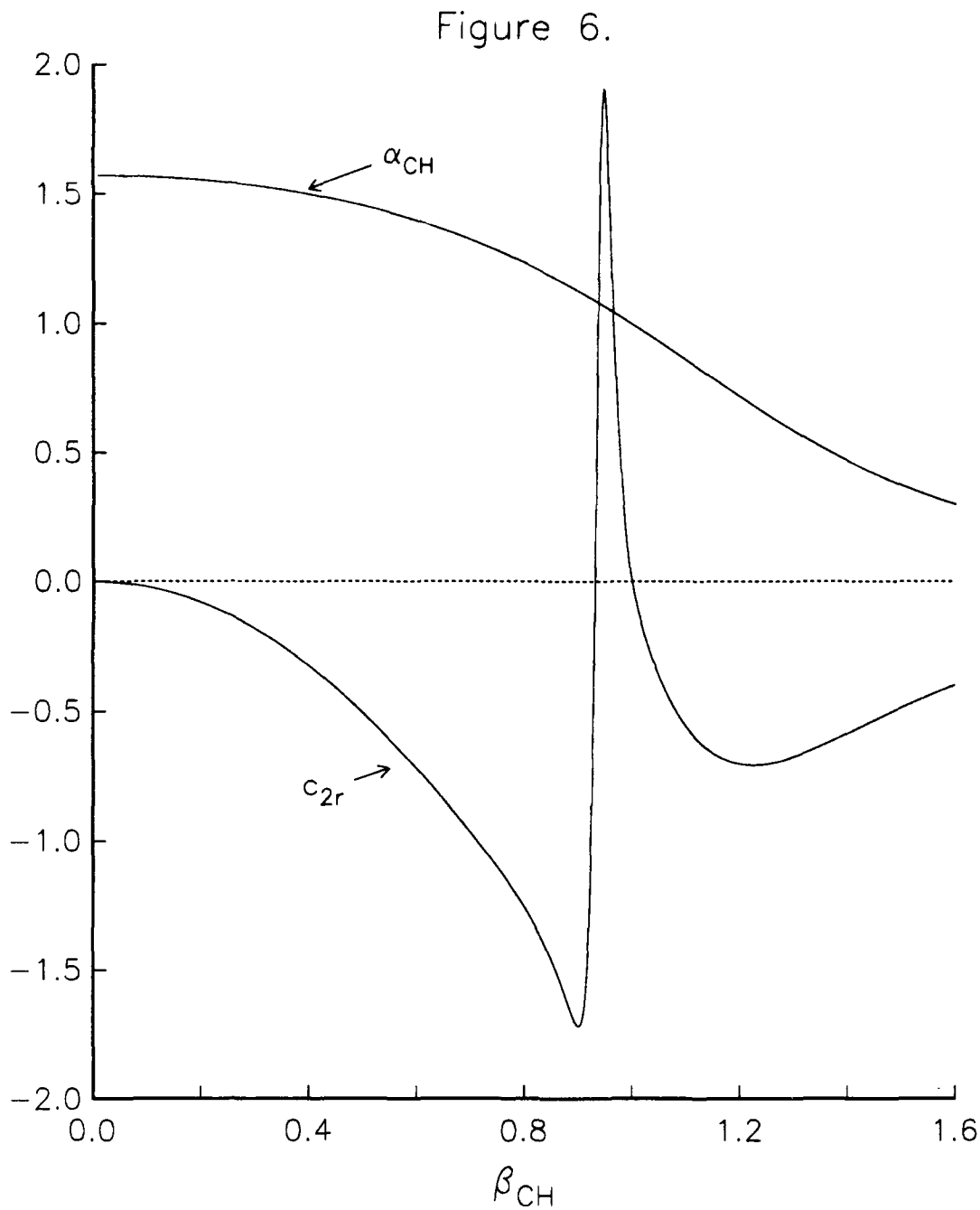

Figure 6. The quantities α_{CH} and c_{2r}, versus β_{CH}, for hypersonic flow over a wedge: $\overline{y}_s = 1$, lowest neutral curve (the lower curve in figure 2a of Cowley & Hall, 1990).

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The weakly nonlinear interaction of oblique Tollmien-Schlichting waves and longitudinal vortices in compressible, high Reynolds number, boundary-layer flow over a flat plate is considered for all ranges of the Mach number. The interaction equations comprise of equations for the vortex which is indirectly forced by the waves via a boundary condition, whereas a vortex term appears in the amplitude equation for the wave pressure. The downstream solution properties of interaction equations are found to depend on the sign of an interaction coefficient. Compressibility is found to have a significant effect on the interaction properties; principally through its impact on the waves and their governing mechanism, the triple-deck structure. It is found that, in general, the flow quantities will grow slowly with increasing downstream co-ordinate; i.e. in general, solutions do not terminate in abrupt, finite-distance 'break-ups'.

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