REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

AD-A271 050

estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, g and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this g this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

REPORT DATE	3. REPORT TYPE AND DATES COVERED
	Reprints; 1/1/93 - 12/31/93

4. TITLE AND SUBTITLE

Strain Energy Density Bounds for Linear Anisotropic Elastic Materials

5. FUNDING NUMBERS

DAAL03-91-G-0022

(2)

6. AUTHOR(S)

M. M. Mehrabadi, S. C. Cowin, and C. O. Horgan

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Dept. of Applied Mathematics
Thornton Hall
University of Virginia
Charlottesville, VA 22903

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

ARO 28399.3-EG

11. SUPPLEMENTARY NOTES

The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

DTIC
ELECTE
OCT 21 1993
S A D

93-24911
93 10 18 09
420 0n3

13. ABSTRACT *(Maximum 200 words)*

Abstract. We discuss the problem of obtaining upper and lower bounds for the strain-energy density in linear anisotropic elastic materials. One set of bounds is given in terms of the *magnitude of the stress field,* another in terms of the *magnitude of the strain field.* Results of this kind play a major role in the analysis of Saint-Venant's Principle for anisotropic materials and structures. They are also useful in estimating global quantities such as total energies, buckling loads, and natural frequencies. For several classes of elastic symmetry (e.g., cubic, transversely isotropic, hexagonal, and tetragonal symmetry) the *optimal constants* appearing in these bounds are given *explicitly* in terms of the elastic constants. This makes the results directly accessible to the design engineer. Such explicit results are rare in the filed of anisotropic elasticity. For more elaborate symmetries (e.g., orthotropic, monoclinic, and triclinic) the optimal constants depend on the solution of cubic and sextic equations, respectively.

14. SUBJECT TERMS

Anisotropic; Linear Elasticity; Strain Energy Density; Optimal Upper and Lower Bounds

15. NUMBER OF PAGES

6

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

Journal of Elasticity 30: 191–196, 1993.

© 1993 *Kluwer Academic Publishers. Printed in the Netherlands.*

Strain energy density bounds for linear anisotropic elastic materials

M.M. MEHRABADI,[1] S.C. COWIN,[2,*] and C.O. HORGAN[3]
[1]*Department of Mechanical Engineering, Tulane University, New Orleans, LA 70118, USA;* [2]*Department of Mechanical Engineering, City College of CUNY, 138th and Convent, New York, NY 10031, USA (*author for correspondence);* [3]*Department of Applied Mathematics, University of Virginia, Charlottesville, VA 22903, USA*

Received 2 August 1990; in revised form 5 August 1992

Abstract. Upper and lower bounds are presented for the magnitude of the strain energy density in linear anisotropic elastic materials. One set of bounds is given in terms of the magnitude of the stress field, another in terms of the magnitude of the strain field. Explicit algebraic formulas are given for the bounds in the case of cubic, transversely isotropic, hexagonal and tetragonal symmetry. In the case of orthotropic symmetry the explicit bounds depend upon the solution of a cubic equation, and in the case of the monoclinic and triclinic symmetries, on the solution of sixth order equations.

Bounds on the magnitude of the strain energy density in linear anisotropic elastic materials are needed in proofs of Saint-Venant's principle for these materials (see, for example, Toupin [1], Horgan [2], and the review of Horgan and Knowles [3]). In this note we extend the method of bounding the strain energy density employed by Horgan [4] to include specific anisotropic elastic symmetries. We do this using a result of Mehrabadi and Cowin [5] in which the coefficients of elasticity are expressed as a second rank tensor in a six-dimensional space rather than as a fourth rank tensor in a three-dimensional space. The results of Mehrabadi and Cowin [5] are a development of ideas due to Kelvin [6]; see also Rychlewski [7]. The eigenvalues of the six-dimensional, second rank elasticity tensor are the numerical coefficients in the bounds obtained.

The anisotropic form of Hooke's law is often written in indicial notation as

$$T_{ij} = C_{ijkm} E_{km}, \tag{1}$$

where the C_{ijkm} are the components of the elasticity tensor. There are three important symmetry restrictions on the tensor C_{ijkm}. These restrictions, which require that components with the subscripts *ijkm*, *jikm*, and *kmij* be equal, follow from the symmetry of the stress tensor, the symmetry of the strain tensor, and the requirement that no work be produced by the elastic material

in a closed loading cycle, respectively. Written as a linear transformation in six dimensions, Hooke's law (1) has the representation, $\mathbf{T} = \mathbf{c}\mathbf{E}$, or

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}, \tag{2}$$

in the notation of Voigt. The relationship of the components of C_{ijkm} to the components of the symmetric matrix $\mathbf{c}$ is easily seen.

Introducing new notation, (2) can be rewritten in the form,

$$\hat{\mathbf{T}} = \hat{\mathbf{c}}\hat{\mathbf{E}}, \tag{3}$$

where the shearing components of these new six-dimensional stress and strain vectors, denoted by $\hat{\mathbf{T}}$ and $\hat{\mathbf{E}}$, respectively, are multiplied by $\sqrt{2}$, and $\hat{\mathbf{c}}$ is a new six-by-six matrix. Thus the matrix form of (3) is given by

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ \sqrt{2}T_{23} \\ \sqrt{2}T_{13} \\ \sqrt{2}T_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & \sqrt{2}c_{15} & \sqrt{2}c_{16} \\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & \sqrt{2}c_{25} & \sqrt{2}c_{26} \\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & \sqrt{2}c_{35} & \sqrt{2}c_{36} \\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & 2c_{44} & 2c_{45} & 2c_{46} \\ \sqrt{2}c_{15} & \sqrt{2}c_{25} & \sqrt{2}c_{35} & 2c_{45} & 2c_{55} & 2c_{56} \\ \sqrt{2}c_{16} & \sqrt{2}c_{26} & \sqrt{2}c_{36} & 2c_{46} & 2c_{56} & 2c_{66} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \sqrt{2}E_{23} \\ \sqrt{2}E_{13} \\ \sqrt{2}E_{12} \end{bmatrix}. \tag{4}$$

The symmetric matrix $\hat{\mathbf{c}}$ can be shown, Mehrabadi and Cowin [5], to represent the components of a second rank tensor in a six-dimensional space, whereas the components of the matrix $\mathbf{c}$ appearing in (2) do not form a tensor. The inverse of the elasticity tensor $\hat{\mathbf{c}}$ is the compliance tensor $\hat{\mathbf{s}}$ where $\hat{\mathbf{s}} = \hat{\mathbf{c}}^{-1}$, thus

$$\hat{\mathbf{E}} = \hat{\mathbf{s}}\hat{\mathbf{T}}, \tag{5}$$

where $\hat{\mathbf{s}}$ is also a second rank tensor in a six-dimensional space. The eigenvalues of the matrix $\hat{\mathbf{c}}$ are the six numbers Λ satisfying the equation

$$(\hat{\mathbf{c}} - \Lambda\mathbf{1})\hat{\mathbf{N}} = \mathbf{0}, \tag{6}$$

and their inverses are the eigenvalues of the matrix $\hat{\mathbf{s}}$

$$(\hat{\mathbf{s}} - (1/\Lambda)\mathbf{1})\hat{\mathbf{N}} = \mathbf{0}, \tag{7}$$

where the vectors $\hat{\mathbf{N}}$ represent the eigenvectors of $\hat{\mathbf{c}}$ or $\hat{\mathbf{s}}$. The linear transformation (3) defined by $\hat{\mathbf{c}}$ is a six-dimensional symmetric transformation, assumed to be positive definite, which has, of course, six positive eigenvalues. These eigenvalues will be denoted by Λ_i, $i = 1, \ldots, 6$, and ordered by the inequalities $\Lambda_1 \geqslant \cdots \geqslant \Lambda_6 > 0$. It follows that

$$\Lambda_6|\hat{\mathbf{E}}| \leqslant |\hat{\mathbf{T}}| \leqslant \Lambda_1|\hat{\mathbf{E}}|, \quad \frac{1}{\Lambda_1}|\hat{\mathbf{T}}| \leqslant |\hat{\mathbf{E}}| \leqslant \frac{1}{\Lambda_6}|\hat{\mathbf{T}}|, \tag{8}$$

where the vertical bars on either side of a vector indicate the norm of the vector, e.g.

$$|\hat{\mathbf{T}}| = \sqrt{\hat{\mathbf{T}} \cdot \hat{\mathbf{T}}}. \tag{9}$$

The strain energy density is denoted by Σ where

$$2\Sigma = \hat{\mathbf{T}} \cdot \hat{\mathbf{E}} = (\hat{\mathbf{c}}\hat{\mathbf{E}}) \cdot \hat{\mathbf{E}} = \hat{\mathbf{T}} \cdot (\hat{\mathbf{s}}\hat{\mathbf{T}}), \tag{10}$$

and it can be expressed in terms of the strain eigenvector $|\hat{\mathbf{E}}|\hat{\mathbf{N}}$ using the eigenvalues of (6) as

$$2\Sigma = |\hat{\mathbf{E}}|^2\{\Lambda_1[\hat{N}_1]^2 + \Lambda_2[\hat{N}_2]^2 + \Lambda_3[\hat{N}_3]^2 + \Lambda_4[\hat{N}_4]^2 + \Lambda_5[\hat{N}_5]^2 + \Lambda_6[\hat{N}_6]^2\}, \tag{11}$$

or in terms of the stress eigenvector $|\hat{\mathbf{T}}|\hat{\mathbf{N}}$ using the eigenvalues of (7) as

$$2\Sigma = |\hat{\mathbf{T}}|^2\left\{\frac{1}{\Lambda_1}[\hat{N}_1]^2 + \frac{1}{\Lambda_2}[\hat{N}_2]^2 + \frac{1}{\Lambda_3}[\hat{N}_3]^2 + \frac{1}{\Lambda_4}[\hat{N}_4]^2 + \frac{1}{\Lambda_5}[\hat{N}_5]^2 + \frac{1}{\Lambda_6}[\hat{N}_6]^2\right\}. \tag{12}$$

Recalling that the vector $\hat{\mathbf{N}}$ is a unit vector ($\hat{\mathbf{N}} \cdot \hat{\mathbf{N}} = 1$) in six dimensions, recalling also the ordering of the eigenvalues by the inequalities $\Lambda_1 \geqslant \cdots \geqslant \Lambda_6 > 0$, the results (11) and (12) yield the inequalities

$$\Lambda_6|\hat{\mathbf{E}}|^2 \leqslant 2\Sigma \leqslant \Lambda_1|\hat{\mathbf{E}}|^2, \quad \text{and} \quad \frac{1}{\Lambda_1}|\hat{\mathbf{T}}|^2 \leqslant 2\Sigma \leqslant \frac{1}{\Lambda_6}|\hat{\mathbf{T}}|^2, \tag{13}$$

respectively. These inequalities represent the bounds of interest. For a particular elastic symmetry the bounds (13) are employed with the values of Λ_1 and Λ_6 taken to be the numerically largest and smallest, respectively, of the eigenvalues listed in Table 1 for that particular symmetry. The second of the inequalities (13) has been previously derived in [4] (p. 232) and illustrated

Table 1. The sets of distinct eigenvalues Λ for each of the ten distinct elastic symmetries. The multiplicity of these eigenvalues is discussed in [5].

Isotropic symmetry

$c_{11} + 2c_{12}, 2c_{44}$ (i.e., $3\lambda + 2\mu, 2\mu$)

Cubic symmetry

$c_{11} + 2c_{12}, c_{11} - c_{12}, 2c_{44}$

Transversely isotropic symmetry

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) + \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) - \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$c_{11} - c_{12}, 2c_{44}$$

Hexagonal (7) symmetry

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) + \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) - \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$\tfrac{1}{2}[(c_{11} - c_{12} + 2c_{44}) + \sqrt{16(c_{14}^2 + c_{15}^2) + (c_{11} - c_{12} - 2c_{44})^2}],$$

$$\tfrac{1}{2}[(c_{11} - c_{12} + 2c_{44}) - \sqrt{16(c_{14}^2 + c_{15}^2) + (c_{11} - c_{12} - 2c_{44})^2}],$$

Hexagonal (6) symmetry

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) + \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) - \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$\tfrac{1}{2}[(c_{11} - c_{12} + 2c_{44}) + \sqrt{16c_{14}^2 + (c_{11} - c_{12} - 2c_{44})^2}],$$

$$\tfrac{1}{2}[(c_{11} - c_{12} + 2c_{44}) - \sqrt{16c_{14}^2 + (c_{11} - c_{12} - 2c_{44})^2}],$$

Tetragonal (7) symmetry

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) + \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) - \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$\tfrac{1}{2}[(c_{11} - c_{12} + 2c_{66}) + \sqrt{16c_{16}^2 + (c_{11} - c_{12} - 2c_{66})^2}],$$

$$\tfrac{1}{2}[(c_{11} - c_{12} + 2c_{66} - \sqrt{16c_{16}^2 + (c_{11} - c_{12} - 2c_{66})^2}], 2c_{44}$$

Tetragonal (6) symmetry

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) + \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$\tfrac{1}{2}[(c_{11} + c_{12} + c_{33}) - \sqrt{8c_{13}^2 + (c_{11} + c_{12} - c_{33})^2}],$$

$$c_{11} - c_{12}, 2c_{66}, 2c_{44}$$

Table 1 (Continued)

Orthotropic symmetry

$2c_{44}$, $2c_{55}$, $2c_{66}$, and the eigenvalues of the matrix $\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$

Monoclinic symmetry

Eigenvalues of the matrix $\begin{bmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & 0 & 0 \\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & 0 & 0 \\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & 2c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c_{55} & 2c_{56} \\ 0 & 0 & 0 & 0 & 2c_{56} & 2c_{66} \end{bmatrix}$

Triclinic symmetry

Eigenvalues of the matrix $\begin{bmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & \sqrt{2}c_{15} & \sqrt{2}c_{16} \\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & \sqrt{2}c_{25} & \sqrt{2}c_{26} \\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & \sqrt{2}c_{35} & \sqrt{2}c_{36} \\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & 2c_{44} & 2c_{45} & 2c_{46} \\ \sqrt{2}c_{15} & \sqrt{2}c_{25} & \sqrt{2}c_{35} & 2c_{45} & 2c_{55} & 2c_{56} \\ \sqrt{2}c_{16} & \sqrt{2}c_{26} & \sqrt{2}c_{36} & 2c_{46} & 2c_{56} & 2c_{66} \end{bmatrix}$

there for the case of isotropic symmetry. The first of (13) has been given in Gurtin [8] (p. 85) and, in the case of isotropic symmetry, by Villaggio [9] (p. 46). Toupin [1] and Gurtin [8] employ the terminology *maximum elastic modulus* for Λ_1 and *minimum elastic modulus* for Λ_6; see also [3] (p. 240). Kelvin [6] called the Λ_i, $i = 1, \ldots, 6$, the six *principal elasticities* of the material; Pipkin [10] uses the term "principal compliance" for the inverse of the same quantities; and Rychlewski [7] suggests, with persuasive historical justification (but contrary to the contemporary trend to avoid eponyms), that they be called the *Kelvin moduli*. Applications of the inequality represented by the first of (13), in the isotropic case, to obtain bounds for total energies and related quantities are described by Villaggio [9] (see, e.g., p. 400, 418).

In closing, we remark that the positivity of Λ_i, $i = 1, \ldots, 6$, is equivalent to the positive definiteness of Σ, expressed as a quadratic form in either $\hat{\mathbf{T}}$ or $\hat{\mathbf{E}}$. The explicitness of the eigenvalues tabulated in Table 1 here thus enables one to write down necessary and sufficient conditions for positive definiteness of Σ directly in terms of the symmetric Voigt matrix $\mathbf{c}$. The resulting conditions are

simpler than the expressions involving principal minors of **c** that are usually employed in the literature.

Acknowledgements

The research of S.C.C. was supported by NSF Grant No. BSC-8822401 and by grant number 669301 from the PSC-CUNY Research Award Program of the City University of New York, while that of C.O.H. was supported by NSF Grant No. MSS-91-02155 and ARO Grant No. DAAL 03-91-G-0022. The authors appreciate the correction of an error by Don Carlson.

References

1. R.A. Toupin, Saint-Venant's principle. *Arch. Rat. Mech. Anal.* 18 (1965) 83.
2. C.O. Horgan, On Saint-Venant's principle in plane anisotropic elasticity. *J. Elasticity* 2 (1972) 169.
3. C.O. Horgan and J.K. Knowles, Recent developments concerning Saint-Venant's principle. *Adv. Appl. Mech.* 23 (1983) 179.
4. C.O. Horgan, On the strain energy density in linear elasticity. *J. Engng. Math.* 7 (1973) 231.
5. M.M. Mehrabadi and S.C. Cowin, Eigentensors of linear anisotropic elastic materials. *Quart. J. Mech. Appl. Math.* 43 (1990) 15.
6. W. Thomson (Lord Kelvin), Elements of a mathematical theory of elasticity. *Phil. Trans. R. Soc.* 166 (1856) 481.
7. J. Rychlewski, On Hooke's law. *Prikl. Matem. Mekhan.* 48 (1984) 303.
8. M.E. Gurtin, The linear theory of elasticity. In: S. Flugge (ed.), *Handbuch der Physik*, vol. VI a/2. Berlin: Springer Verlag (1972) pp. 1–295.
9. P. Villaggio, *Qualitative Methods in Elasticity*. Leyden: Noordhoff (1977).
10. A.C. Pipkin, Constraints in linear elastic materials. *J. Elasticity* 6 (1976) 179.

Accesion For
NTIS CRA&I ☑
DTIC TAB ☐
Unannounced ☐
Justification
By
Distribution /
Availability Codes
Dist | Avail and / or Special
A-1 | 20

DTIC QUALITY INSPECTED 2