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CONTINUUM MODELING OF MATERIALS THAT CAN UNDERGO MARTENSITIC PHASE TRANSITIONS

by

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1. Introduction

In this paper, a continuum model for materials that can undergo martensitic phase transformations is developed and applied to the study of several problems that involve such transformations. Among other things, the continuum model that is developed provides the correct material symmetry group for each phase of the material, and results in the corresponding boundary value problem being in a form that permit direct linearization, while still retaining finite shape deformations for the martensite phases. All of the problems that this continuum model is applied to in this paper deal with the issue of which phase or which variant of martensite is preferred during the growth process when a boundary traction is applied.[1] Among these problems are the case of a uniaxial tensile traction applied to a cylindrical body, and the case of a hydrostatic pressure applied to a material that has a finite shape deformation with an infinitesimal dilatation.

The term martensitic phase transformation was originally given to the diffusionless phase transformation that occurs when the high temperature austenite phase of a steel is quenched, and the term martensite was originally given to the phase that is created from this solid-solid phase transformation. The term martensitic phase transformation has since been given to almost all solid-solid phase transformations that proceed by a diffusionless cooperative movement of atoms at the phase boundary, involve a change in crystal structure, have a distinct orientation relation between the crystal lattices of the parent phase and the product phase, have a deformation associated with the product phase, and have continuity of displacements, with possible discontinuity of strain, at the phase boundary.[2] Similarly, the phase that is created

[1] This continuum model was also applied to the study of the temperature at the interface and the quasi-static motions of a two-phase thermoelastic bar [9], [10], and to the study of the longitudinal free vibrations of a finite fixed-free two-phase elastic bar [9], [11].

[2] See [8] and [15] for more detailed and comprehensive disucussions about martensitic phase transformations.

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by a transformation that is considered to be martensitic is sometimes referred to as the martensite phase of the given material. In this paper, when a martensitic phase transformation between a high temperature phase and a low temperature phase of a material is being considered, the high temperature phase will usually be referred to as the austenite phase, and the low temperature phase will usually be referred to as the martensite phase, regardless of which phase is being created by the martensitic phase transformation. Additionally, the martensitic phase transformation that creates the austenite phase may sometimes be referred to as the austenite phase transformation, and the martensitic phase transformation that creates the martensite phase may sometimes be referred to as the martensite phase transformation.

As mentioned above, the martensite phase is characterized by having a deformation relative to the undeformed parent phase. This deformation consists mostly of the deformation that occurs solely from the mechanisms of the martensitic phase transformation that occur at the phase boundary. This portion of the total deformation corresponds to an unstressed state of the martensite and will henceforth be referred to as the shape deformation of the martensite.[3] Additionally, the strain corresponding to the shape deformation will sometimes be referred to as the transformation strain. The shape deformation is in general a finite deformation, and consists primarily of the deformation that would be necessary to deform the austenite crystal lattice into the martensite crystal lattice and, for some materials, the deformation that is necessary to maintain continuity of displacements at the phase boundary. The remaining portion of the total deformation corresponding to the martensite is due to the surrounding matrix material constraining the formation of the shape deformation and/or by any applied boundary tractions. These deformations are

[3] Because the shape deformation corresponds to an unstressed state of the martensite, it can be considered to represent the undeformed martensite.

usually infinitesimal deformations, even though they may be greater than the yield strain of the martensite, and in this paper they will be considered to be superimposed upon the finite shape deformation. If the martensite is stressed, it is due solely to these superimposed deformations.

The change in crystal structure that occurs during a martensitic phase transformation may involve a change in crystal symmetry, as is the case with austenite-martensite phase transformations in most materials that can undergo such transformations, or it may result in the product phase having the same type of crystal lattice as the parent phase, but with a different orientation, as is the case when one variant of martensite is transformed into another variant of the same martensite. The symmetry of the crystal lattice of a material is reflected in the material symmetry group of the material. The material symmetry group of a material restricts the functional form of the constitutive equations of the material (see [13]). Thus, when a particle of material in its austenite phase is transformed to its martensite phase, the material symmetry group of that particle of material should change accordingly, and this should be reflected in the constitutive equations of that particle of material both before and after the transformation. This change in material symmetry group should not only represent the change in the type of crystal lattice, it should also represent the *orientation relation* between the crystal lattices of the austenite and the martensite.

In the next few sections, the continuum model for materials that can undergo martensitic phase transformations is developed and the corresponding field equations and jump conditions are derived for a purely mechanical process. A similar continuum model can be developed and the corresponding field equations and jump conditions can be derived for a process that is thermo-mechanical in a similar manner (see [9]).

2. The Eulerian Global Form of the Balance Laws

It is assumed that the process under consideration takes place at a constant, uniform temperature and with no heat conduction. Such a process is an isothermal and adiabatic process and is also known as a purely mechanical process. It is assumed that this process takes place in a time interval $\Gamma = [t_0, t_1]$.[4] The body B that is considered is assumed to occupy a regular region R_t at time $t \in \Gamma$ and it is assumed that R_t is a subset of the three-dimensional Euclidean space E_3. A point or the position vector of a point in R_t is denoted by $\mathbf{y}$. The traction on the surface with unit normal $\mathbf{n}(\mathbf{y}, t)$ is denoted by $\mathbf{t}(\mathbf{y}, \mathbf{n}, t)$, the body force per unit volume by $\mathbf{b}(\mathbf{y}, t)$, and the mass per unit volume by $\bar{\rho}(\mathbf{y}, t)$. Also, the Eulerian form of the velocity of the particle of material at $\mathbf{y} \in R_t$ at time $t \in \Gamma$ is denoted by $\bar{\mathbf{v}}(\mathbf{y}, t)$. The familiar Eulerian global form of the balance of mass, linear momentum, and angular momentum are

$$\frac{d}{dt}\int_{D_t} \bar{\rho} dV = 0, \tag{2.1}$$

$$\int_{\partial D_t} \mathbf{t} dA + \int_{D_t} \mathbf{b} dV = \frac{d}{dt}\int_{D_t} \bar{\rho}\bar{\mathbf{v}} dV, \tag{2.2}$$

$$\int_{\partial D_t} \mathbf{y} \times \mathbf{t} dA + \int_{D_t} \mathbf{y} \times \mathbf{b} dV = \frac{d}{dt}\int_{D_t} \mathbf{y} \times \bar{\rho}\bar{\mathbf{v}} dV, \tag{2.3}$$

respectively, $\forall\, D_t \subset R_t$ and $\forall\, t \in \Gamma$.[5] Equations (2.1), (2.2), and (2.3) can be used to derive the rate of work-energy equation given by

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[4] In this paper, math-italic quantities denote scalars and bold-faced quantities denote tensors, including vectors.

[5] In this paper, unless otherwise stated, whenever a subset of a regular region is considered, it is assumed that the subset is a regular subregion. Similarly, all subregions of regular regions are assumed to be regular.

$$\int_{\partial D_t} \mathbf{t} \cdot \bar{\mathbf{v}} dA + \int_{D_t} \mathbf{b} \cdot \bar{\mathbf{v}} dV = \int_{D_t} \tau \cdot \nabla_{\mathbf{y}} \bar{\mathbf{v}} dV + \frac{d}{dt} \int_{D_t} \tfrac{1}{2} \bar{\rho} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} dV, \tag{2.4}$$

$\forall\, D_t \subset R_t$ and $\forall\, t \in \Gamma$, where $\tau(\mathbf{y}, t)$ is the true (or Cauchy) stress tensor and $\nabla_{\mathbf{y}} \bar{\mathbf{v}}$ denotes the gradient of $\bar{\mathbf{v}}(\mathbf{y}, t)$ with respect to $\mathbf{y}$.

3. Multiple Reference Configurations and the Continuum Model

Consider a region R in E_3 that the body B can occupy, in the sense that there exists a suitably smooth and invertible mapping that maps R into R_t. Note that R is not such that the body B has to occupy it at some time $t \in \Gamma$. Such a region R can be used as a reference configuration for the body B.

In the continuum model that is developed in this paper, each phase has its own constitutive relation. These constitutive relations, however, are not all defined with respect to the same fixed reference configuration. Instead, each phase has its own reference configuration for the definition of its constitutive equations, and for the expression of its field equations. More specifically, for the continuum model that is develped, it is assumed that each phase has a configuration corresponding to an unstressed undeformed state, and that each phase behaves elastically for some range of deformations about its unstressed undeformed configuration. The reference configuration for each phase of these materials is taken to coincide with the unstressed undeformed configuration of that phase. With respect to the undeformed austenite phase, these reference configurations coincide the shape deformations of the martensites.

For simplicity, in the following sections, the continuum model will be developed and the corresponding field equations and jump conditions will be derived for a

two-phase material. The corresponding results for a material that consists of more than two phases can be obtained in a similar manner.

4. The Kinematics Using Multiple Reference Configurations

Consider the body B that was described in Section 2. Additionally, assume that this body consists of two phases, which will henceforth be referred to as phase 1 and phase 2. Assume that at each $t \in \Gamma$ phase 1 occupies a subregion R_t^- of R_t, and phase 2 occupies a subregion R_t^+ of R_t, where $R_t^- \cup R_t^+ = R_t$ and $R_t^- \cap R_t^+ = \emptyset$ (Figure 1). These two subregions of R_t are separated by an interface S_t which can pass over particles of material in R_t. If this occurs, R_t^- will increase in size while R_t^+ decreases in size, or vise-versa, depending on the direction of motion of S_t.

It is assumed that there exists a configuration R of phase 1 that corresponds to an unstressed undeformed state of that phase. This configuration R will be used as a stationary reference configuration for R_t (Figure 1). Let $\mathbf{x}$ denote a point or the position vector of a point in R. Let $\hat{\mathbf{y}}(\mathbf{x},t)$ be the suitably smooth and invertible mapping which maps R into R_t at each $t \in \Gamma$, with $\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x},t) = \mathbf{x}+\hat{\mathbf{u}}(\mathbf{x},t)\,\forall(\mathbf{x},t)\in R\times\Gamma$, where $\mathbf{u} = \hat{\mathbf{u}}(\mathbf{x},t)$ is the displacement of the point $\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x},t)$ from the point $\mathbf{x}$ at time $t \in \Gamma$. The deformation gradient of $\hat{\mathbf{y}}$ is defined as $\mathbf{F}(\mathbf{x},t) = \nabla\hat{\mathbf{y}}(\mathbf{x},t)$, and the Jacobian of $\mathbf{F}$ is defined as $J(\mathbf{x},t) = \det\mathbf{F}(\mathbf{x},t)$, where it is required that $J > 0$ to exclude reflections.[6] The velocity of the particle of material at $\mathbf{y} = \hat{\mathbf{y}}(\mathbf{x},t)$ is defined as $\mathbf{v}(\mathbf{x},t) = \frac{\partial}{\partial t}\hat{\mathbf{y}}(\mathbf{x},t)$, and the Eulerian form of the velocity used in Section 2 can be defined as $\bar{\mathbf{v}}(\mathbf{y},t) = \mathbf{v}(\hat{\mathbf{x}}(\mathbf{y},t),t)$, where $\hat{\mathbf{x}}(\cdot,t): R_t \mapsto R$ at each $t \in \Gamma$ is the inverse of $\hat{\mathbf{y}}$. Let $R^- = \hat{\mathbf{x}}(R_t^-,t)$, $R^+ = \hat{\mathbf{x}}(R_t^+,t)$, and $S = \hat{\mathbf{x}}(S_t,t)$. Note that $R^- \cup R^+ = R$, $R^- \cap R^+ = \emptyset$, S is the surface separating R^- from R^+, and S moves within R as S_t moves within R_t.

[6] ∇ denotes the gradient operator with respect to $\mathbf{x} \in R$.

It is assumed that there exists a shape deformation for phase 2 with respect to R which corresponds to an unstressed undeformed configuration of that phase. Let R_1^+ be the reference configuration coinciding with this shape deformation of phase 2 for all $t \in \Gamma$ (Figure 1). We note that because R has been assumed to be stationary and there must be continuity of displacements at the phase boundary for all $t \in \Gamma$, R_1^+ will most likely be moving if a problem other than a static problem is considered.[7] Let $\mathbf{x}_1$ denote a point or the position vector of a point in R_1^+, and let $\tilde{\mathbf{x}}_1(\mathbf{x},t)$ be the suitably smooth and invertible mapping that maps R^+ into R_1^+ at each $t \in \Gamma$, with $\mathbf{x}_1 = \tilde{\mathbf{x}}_1(\mathbf{x},t)\ \forall\, \mathbf{x} \in R^+$ at each $t \in \Gamma$. This mapping $\tilde{\mathbf{x}}_1$ is assumed to be given in a problem. Let $\tilde{\mathbf{F}}(\mathbf{x},t) = \nabla\tilde{\mathbf{x}}_1(\mathbf{x},t)$ and $\tilde{J}(\mathbf{x},t) = \det\tilde{\mathbf{F}}(\mathbf{x},t)$, with $\tilde{J} > 0$. Let $\hat{\mathbf{y}}_1(\mathbf{x}_1,t)$ be the suitably smooth and invertible mapping that maps R_1^+ into R_t^+ at each $t \in \Gamma$, with $\mathbf{y} = \hat{\mathbf{y}}_1(\mathbf{x}_1,t) = \mathbf{x}_1 + \hat{\mathbf{u}}_1(\mathbf{x}_1,t)\ \ \forall\, \mathbf{x}_1 \in R_1^+$ at each $t \in \Gamma$. Note also that $\mathbf{y} = \hat{\mathbf{y}}_1(\tilde{\mathbf{x}}_1(\mathbf{x},t),t) = \hat{\mathbf{y}}(\mathbf{x},t)\ \forall\, \mathbf{x} \in R^+$ at each $t \in \Gamma$. Let $\mathbf{F}_1(\mathbf{x}_1,t) = \nabla_1\hat{\mathbf{y}}_1(\mathbf{x}_1,t)$ and $J_1(\mathbf{x}_1,t) = \det\mathbf{F}_1(\mathbf{x}_1,t)$, with $J_1 > 0$.[8] The velocity field of the particles of material in phase 2 as a function of $\mathbf{x}_1 \in R_1^+$ is given by $\bar{\mathbf{v}}_1(\mathbf{x}_1,t) = \bar{\mathbf{v}}(\hat{\mathbf{y}}_1(\mathbf{x}_1,t),t)\ \forall\, \mathbf{x}_1 \in R_1^+$ at each $t \in \Gamma$, or equivalently as $\bar{\mathbf{v}}_1(\mathbf{x}_1,t) = \mathbf{v}(\tilde{\mathbf{x}}(\mathbf{x}_1,t),t)\ \forall\, \mathbf{x}_1 \in R_1^+$ at each $t \in \Gamma$, where $\tilde{\mathbf{x}}(\cdot,t): R_1^+ \mapsto R^+$ at each $t \in \Gamma$ is the inverse of $\tilde{\mathbf{x}}_1$. Also, $\hat{\mathbf{x}}_1(\cdot,t): R_t^+ \mapsto R_1^+$ at each $t \in \Gamma$ represents the inverse of $\hat{\mathbf{y}}_1$.

Let $\mathbf{N}(\mathbf{x},t)$ represent a unit vector normal to S that points into R^+, and let $\mathbf{L}(\mathbf{x},t)$ represent a vector tangent to S, both at a point on S coinciding with the point $\mathbf{x} \in R$ at time $t \in \Gamma$. Also, let $\mathbf{V}(\mathbf{x},t)$ represent the velocity of the point on S coinciding with the point $\mathbf{x} \in R$ at time $t \in \Gamma$.[9] In the following, if $g(\mathbf{x},t)$ represents

[7] See Section 6, Equation (6.9), for an example of this.

[8] ∇_1 denotes the gradient operator with respect to $\mathbf{x}_1 \in R_1^+$.

[9] In the rest of this thesis, a point on the surface S that coincides with the point $\mathbf{x} \in R$ at time $t \in \Gamma$ will simply be referred to as the point $\mathbf{x} \in S$. Also, note that V represents a nominal-type velocity and not the velocity of a point on S_t.

a generic field quantity that is discontinuous at S, then $g^-(\mathbf{x},t)$ and $g^+(\mathbf{x},t)$ denote the limiting values of g at $\mathbf{x} \in \mathrm{S}$ as this point $\mathbf{x}$ is approached from negative and positive sides of S, respectively, in directions parallel to $\mathrm{N}(\mathbf{x},t)$.[10]

Because the displacements at the interface separating two phases involved in a martensitic phase transformation are continuous while the strains at the interface may be discontinuous, we require that $\hat{\mathbf{y}}(\mathbf{x},t)$ be continuous on $\mathrm{R} \times \Gamma$, and allow the first and second derivatives of $\hat{\mathbf{y}}$ to be piecewise continuous on $\mathrm{R} \times \Gamma$, with discontinuities occurring only at points on S. As a result of this, we have

$$\hat{\mathbf{y}}_1^+(\tilde{\mathbf{x}}_1(\mathbf{x},t),t) = \hat{\mathbf{y}}^-(\mathbf{x},t), \tag{4.1}$$

$\forall\, \mathbf{x} \in \mathrm{S}$ at each $t \in \Gamma$. Taking the differential of both sides of (4.1) while keeping time fixed (and recalling the continuity conditions on $\hat{\mathbf{y}}$) yields

$$\left(\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-\right)\mathbf{L} = 0, \tag{4.2}$$

$\forall\, \mathbf{x} \in \mathrm{S}$ at each $t \in \Gamma$, and for every vector $\mathbf{L}$ tangent to S at $\mathbf{x} \in \mathrm{S}$, where $\mathbf{x}_1 = \tilde{\mathbf{x}}_1(\mathbf{x},t)$ in $\mathbf{F}_1^+$. Differentiating (4.1) with respect to time yields

$$\bar{\mathbf{v}}_1^+ - \mathbf{v}^- + \left(\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-\right)\mathbf{V} = 0, \tag{4.3}$$

$\forall\, \mathbf{x} \in \mathrm{S}$ at each $t \in \Gamma$, where $\mathbf{x}_1 = \tilde{\mathbf{x}}_1(\mathbf{x},t)$ in $\bar{\mathbf{v}}_1^+$ and $\mathbf{F}_1^+$.[11]

It can be shown that if given an $\mathbf{F}^-, \mathbf{F}_1^+$, and an $\tilde{\mathbf{F}}^+$ such that Equation (4.2) is satisfied at a point on some surface, there exists vectors $\hat{\mathbf{a}}$ and $\mathbf{N}$ defined at that point such that

$$\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^- = \hat{\mathbf{a}} \otimes \mathbf{N}, \tag{4.4}$$

[10] If a piecewise continuous field quantity is discontinuous at S, it is sometimes said that the quantity *jumps* across S, and an equation relating g^- and g^+ is sometimes referred to as a *jump condition*.

[11] The corresponding forms of Equations (4.2) and (4.3) involving only one reference configuration are well known and can be found in, e.g., [2].

where N is normal to the surface at that point.[12] Also, note that $\hat{\mathbf{a}} \otimes \mathbf{N}$ in this equation is a rank-one two-tensor.

5. The Nominal Form of the Field Equations Using Multiple Reference Configurations

In addition to the continuity requirements on $\hat{\mathbf{y}}$ and $\mathbf{u}$ discussed in the previous section, it is assumed that $\mathbf{b}$ is continuous $\forall\, \mathbf{y} \in \mathrm{R}_t$ and $\forall\, t \in \Gamma$, and that $\boldsymbol{\tau}$ and its gradient are piecewise continuous $\forall\, \mathbf{y} \in \mathrm{R}_t$ and $\forall\, t \in \Gamma$, with discontinuities occurring only at points on S_t.

The global form of the balance of mass given by Equation (2.1) can be expressed with respect to R as

$$\frac{d}{dt}\int_D \rho dV = 0, \tag{5.1}$$

where $\mathrm{D} = \hat{\mathbf{x}}(\mathrm{D}_t, t)$, and $\rho = J\bar{\rho}$ represents the mass per unit volume of R. Note that since (2.1) is valid $\forall\, \mathrm{D}_t \subset \mathrm{R}_t$, (5.1) is valid $\forall\, \mathrm{D} \subset \mathrm{R}$. We require that ρ be continuous $\forall\, \mathbf{x} \in \mathrm{R}$. Localizing Equation (5.1) yields the familiar result that ρ must be independent of time $\forall\, \mathbf{x} \in \mathrm{R}$. Thus,

$$\rho(\mathbf{x}) = J(\mathbf{x}, t)\bar{\rho}(\hat{\mathbf{y}}(\mathbf{x}, t), t), \tag{5.2}$$

$\forall (\mathbf{x}, t) \in \mathrm{R} \times \Gamma$. Note that the global field equations given by (2.1)-(2.3) are still valid in the regions of space and time indicated there, for the case where ρ, $\boldsymbol{\tau}$, and $\mathbf{b}$ have the continuity conditions specified in this section and $\hat{\mathbf{y}}$ has the continuity conditions specified in the previous section. However, for these continuity conditions, the work-energy equation given by (2.4) is valid only for subregions of R_t not containing a portion of S_t, as will be discussed further in Section 8.

[12] The corresponding form of this equation with $\mathbf{F}_1^+\bar{\mathbf{F}}^+$ replaced with $\mathbf{F}^+$, where $\mathbf{F}^+ = \mathbf{F}_1^+\bar{\mathbf{F}}^+$, is well known and can be found in, e.g., [14].

For subregions D_t^- of R_t^-, the global forms of the balance of linear and angular momentum for phase 1 given by (2.2) and (2.3), respectively, can be expressed with respect to R^- as

$$\int_{\partial D^-} \sigma \mathbf{n} dA + \int_{D^-} \mathbf{f} dV = \frac{d}{dt}\int_{D^-} \rho \mathbf{v} dV, \tag{5.3}$$

$$\int_{\partial D^-} \hat{\mathbf{y}} \times \sigma \mathbf{n} dA + \int_{D^-} \hat{\mathbf{y}} \times \mathbf{f} dV = \frac{d}{dt}\int_{D^-} \hat{\mathbf{y}} \times \rho \mathbf{v} dV, \tag{5.4}$$

respectively, where $D^- = \hat{\mathbf{x}}(D_t^-, t)$, $\sigma(\mathbf{x}, t) = J(\mathbf{x}, t)\tau(\hat{\mathbf{y}}(\mathbf{x}, t), t)\mathbf{F}^{-T}(\mathbf{x}, t)$ is the nominal stress tensor with respect to R^-, $\mathbf{f}(\mathbf{x}, t) = J(\mathbf{x}, t)\mathbf{b}(\hat{\mathbf{y}}(\mathbf{x}, t), t)$ is the nominal body force per unit volume of R^-, and $\mathbf{n}(\mathbf{x})$ is the outward unit normal vector field on the boundary of D^-. These nominal field equations can be obtained in the usual way from Equations (2.2) and (2.3), respectively (see, e.g., [13]).

For subregions D_t^+ of R_t^+, the global forms of the balance of linear and angular momentum for phase 2 given by (2.2) and (2.3), respectively, can be expressed with respect to R_1^+ as

$$\int_{\partial D_1^+} \sigma_1 \mathbf{n}_1 dA + \int_{D_1^+} \mathbf{f}_1 dV = \frac{d}{dt}\int_{D_1^+} \bar{\rho}_1 \bar{\mathbf{v}}_1 dV, \tag{5.5}$$

$$\int_{\partial D_1^+} \hat{\mathbf{y}}_1 \times \sigma_1 \mathbf{n}_1 dA + \int_{D_1^+} \hat{\mathbf{y}}_1 \times \mathbf{f}_1 dV = \frac{d}{dt}\int_{D_1^+} \hat{\mathbf{y}}_1 \times \bar{\rho}_1 \bar{\mathbf{v}}_1 dV, \tag{5.6}$$

respectively, where $D_1^+ = \hat{\mathbf{x}}_1(D_t^+, t)$, $\bar{\rho}_1(\mathbf{x}_1, t) = J_1(\mathbf{x}_1, t)\bar{\rho}(\hat{\mathbf{y}}_1(\mathbf{x}_1, t), t)$ is the mass per unit volume of R_1^+, $\sigma_1(\mathbf{x}_1, t) = J_1(\mathbf{x}_1, t)\tau_1(\hat{\mathbf{y}}_1(\mathbf{x}_1, t), t)\mathbf{F}_1^{-T}(\mathbf{x}_1, t)$ represents the stress tensor with respect to R_1^+, $\mathbf{f}_1(\mathbf{x}_1, t) = J_1(\mathbf{x}_1, t)\mathbf{b}(\hat{\mathbf{y}}_1(\mathbf{x}_1, t), t)$ represents the

body force per unit volume of R_1^+, and $\mathbf{n}_1(\mathbf{x}_1, t)$ is the outward unit normal vector field on the boundary of D_1^+. Also, $\bar{\rho}_1$ is related to ρ by

$$\rho(\mathbf{x}) = \tilde{J}(\mathbf{x}, t)\bar{\rho}_1(\tilde{\mathbf{x}}_1(\mathbf{x}, t), t), \tag{5.7}$$

$\forall\, \mathbf{x} \in R^+$ at each $t \in \Gamma$. Note that $\boldsymbol{\sigma}_1$, $\mathbf{f}_1$, and $\bar{\rho}_1$ do not have a true nominal form since $\tilde{\mathbf{x}}_1$ is a function of time. In fact, because of this time dependence these quantities are closer to having an Eulerian form. Note also that Equations (5.5) and (5.6) can be obtained from Equations (2.2) and (2.3) in a manner completely analogous to that used to obtain Equations (5.3) and (5.4), regardless of whether R_1^+ is stationary or whether points in R_1^+ are moving, which is the case if $\tilde{\mathbf{x}}_1$ is a function of time.

Localization of the global nominal balance laws given by (5.3) and (5.4) using (5.2) yields

$$\begin{aligned} \mathrm{div}\boldsymbol{\sigma} + \mathbf{f} &= \rho\mathbf{a}, \\ \boldsymbol{\sigma}\mathbf{F}^{\mathrm{T}} &= \mathbf{F}\boldsymbol{\sigma}^{\mathrm{T}}, \end{aligned} \tag{5.8}$$ [13]

respectively, $\forall\, \mathbf{x} \in R^-$ at each $t \in \Gamma$, where $\mathbf{a}(\mathbf{x}, t) = \frac{\partial}{\partial t}\mathbf{v}(\mathbf{x}, t)$ is the acceleration of the particle of material in R_t^- that corresponds to the point $\mathbf{x} \in R^-$.

Localization of the global nominal balance laws given by (5.5) and (5.6) using (5.7) yields

$$\begin{aligned} \mathrm{div}_1\boldsymbol{\sigma}_1 + \mathbf{f}_1 &= \bar{\rho}_1\bar{\mathbf{a}}_1, \\ \boldsymbol{\sigma}_1\mathbf{F}_1^{\mathrm{T}} &= \mathbf{F}_1\boldsymbol{\sigma}_1^{\mathrm{T}}, \end{aligned} \tag{5.9}$$ [14]

respectively, $\forall\, \mathbf{x}_1 \in R_1^+$ at each $t \in \Gamma$, where $\bar{\mathbf{a}}_1(\mathbf{x}_1, t) = \left[\frac{\partial}{\partial t}\bar{\mathbf{v}}_1(\tilde{\mathbf{x}}_1(\mathbf{x}, t), t)\right]_{\tilde{\mathbf{x}}(\mathbf{x}_1, t)}$

[13] div denotes the divergence operator with respect to $\mathbf{x} \in R$.

[14] div_1 denotes the divergence operator with respect to $\mathbf{x}_1 \in R_1^+$.

is the acceleration of the particle of material in R_t^+ that corresponds to the point $\mathbf{x}_1 = \tilde{\mathbf{x}}_1(\mathbf{x}, t)$ in R_1^+.

For subregions D_t of R_t which contain a portion of S_t, the global form of the balance of linear momentum given by Equation (2.2) can be expressed with respect to R in terms of the field quantities defined on R^- and R_1^+. Localizing this equation at points on S would then yield

$$\left(\left(\tilde{J}\sigma_1\tilde{\mathbf{F}}^{-\mathrm{T}}\right)^+ - \sigma^-\right)\mathbf{N} + \rho(\bar{\mathbf{v}}_1^+ - \mathbf{v}^-)(\mathbf{V}\cdot\mathbf{N}) = 0, \tag{5.10}$$

$\forall\, \mathbf{x} \in S$ at each $t \in \Gamma$, where $\mathbf{x}_1 = \tilde{\mathbf{x}}_1(\mathbf{x}, t)$ in σ_1^+ and $\bar{\mathbf{v}}_1^+$.[15]

The balance of angular momentum given by Equation (2.3) for a subregion D_t of R_t containing a portion of S_t is automatically satisfied if the jump conditions (4.2), (4.3), and (5.10) are satisfied at the points on S corresponding to the points on the portion of S_t contained in D_t. Also, the local field equations given by (5.2) and (5.7)-(5.9), the kinematic jump condition (4.1), or (4.2) and (4.3), and the linear momentum jump condition (5.10) are all together equivalent to the global field equations given by (2.1)-(2.3).

6. Elastic Materials and Multiple Reference Configurations

In the following, it is assumed that phase 1 behaves elastically for some range of deformations about R^- and that phase 2 behaves elastically for some range of deformations about R_1^+. In particular, it is assumed that phase 1 possesses an elastic

[15] The corresponding form of this jump condition involving only one reference configuration is well known and can be found in, e.g., [2]. In fact, the jump conditions (4.2), (4.3), and (5.10) can formally be obtained from the jump conditions presented in [2] by replacing $\mathbf{F}^+$, $\mathbf{v}^+$, and σ^+ with $\mathbf{F}_1^+\tilde{\mathbf{F}}^+$, $\bar{\mathbf{v}}_1^+$, and $\left(\tilde{J}\sigma^+\tilde{\mathbf{F}}^{-\mathrm{T}}\right)^+$, respectively.

potential

$$W = W(\mathbf{F}(\mathbf{x},t), \mathbf{x}), \tag{6.1}$$

$\forall \mathbf{x} \in \mathrm{R}^-$ and $\forall \mathbf{F} \in \widehat{\mathcal{L}}^+$, such that the nominal stress tensor for this phase is given by

$$\sigma(\mathbf{x}, t) = W_{\mathbf{F}}(\mathbf{F}(\mathbf{x}, t), \mathbf{x}), \tag{6.2}$$

$\forall \mathbf{x} \in \mathrm{R}^-$ and $\forall \mathbf{F} \in \widehat{\mathcal{L}}^+$, where $\widehat{\mathcal{L}}^+$ is a subset of $\mathcal{L}^+$, $\mathcal{L}^+$ is the set of all two-tensors with positive determinants, and $\widehat{\mathcal{L}}^+$ represents the range of deformations about R^- for which phase 1 behaves elastically. Equations (6.1) and (6.2) correspond to the standard definition of an elastic (or hyperelastic) material defined with respect to a single fixed reference configuration, that can be found in almost any textbook on finite elasticity (see, e.g., [13]). The constitutive equations (6.1) and (6.2) can have these standard forms because the reference configuration R^- is stationary. In fact, as a result of R^- being stationary and (6.1) and (6.2), the general field equations in terms of the displacements for phase 1, and consequently the linearized constitutive equations and the linearized field equations in terms of the displacements for phase 1, will all have the standard well known forms corresponding to those for elastic materials where single fixed reference configurations are used.

For phase 2, we assume that there exists an elastic potential W_1 *defined with respect to* R_1^+. It is assumed that W_1 is a function of $\mathbf{F}_1$. Additionally, as one might expect, it is required that the inhomogeneity of W_1 remain the same for each particle of material in phase 2 as time progresses. If $\tilde{\mathbf{x}}_1$ is a function of time, the reference point $\mathbf{x}_1 \in \mathrm{R}_1^+$ for a given particle of material in phase 2 is changing as time progresses, and a different particle of material occupies a given point $\mathbf{x}_1 \in \mathrm{R}_1^+$ at different times $t \in \Gamma$. Because of this, the inhomogeneity of W_1 cannot be

expressed with respect to points $\mathbf{x}_1 \in \mathrm{R}_1^+$ if $\tilde{\mathbf{x}}_1$ is a function of time.[16] Instead, the inhomogeneity of W_1 must be represented with respect to points $\mathbf{x}$ in the stationary reference configuration R^+, so that the effect of the inhomogeneity of W_1 on a given particle of material in phase 2 follows that particle for all $t \in \Gamma$. Thus, we assume that W_1 has the form

$$W_1 = W_1(\mathbf{F}_1(\mathbf{x}_1,t), \tilde{\mathbf{x}}(\mathbf{x}_1,t)), \tag{6.3}$$

$\forall\, \mathbf{x}_1 \in \mathrm{R}_1^+$, $\forall\, t \in \Gamma$, and $\forall\, \mathbf{F}_1 \in \widehat{\mathcal{L}}_1^+$, where $\widehat{\mathcal{L}}_1^+$ is a subset of $\mathcal{L}^+$ and represents the range of deformations about R_1^+ for which phase 2 behaves elastically. It is further assumed that W_1 is such that the stress tensor for phase 2 with respect to R_1^+ is given by

$$\sigma_1(\mathbf{x}_1,t) = W_{1_{\mathbf{F}_1}}(\mathbf{F}_1(\mathbf{x}_1,t), \tilde{\mathbf{x}}(\mathbf{x}_1,t)) \tag{6.4}$$

$\forall\, \mathbf{x}_1 \in \mathrm{R}_1^+$, $\forall\, t \in \Gamma$, and $\forall\, \mathbf{F}_1 \in \widehat{\mathcal{L}}_1^+$.

As discussed in the Introduction, the symmetry of the crystal lattice of a material is represented in the material symmetry group of that material (see [13]). It is required that the elements of a material symmetry group of a solid be unimodular, so that they preserve volume.[17] The material symmetry group for phase 1 is defined with respect to R^-, and the material symmetry group for phase 2 is defined with respect to R_1^+. These material symmetry groups restrict the functional forms of the elastic potentials of their respective phases. More specifically, if $\mathcal{G}(\mathbf{x})$ is the material symmetry group for phase 1, W is required to be such that

$$W(\mathbf{FH}, \mathbf{x}) = W(\mathbf{F}, \mathbf{x}), \tag{6.5}$$

[16] However, if $\tilde{\mathbf{x}}_1$ is not a function of time, which would be appropriate for the static case, the reference configuration R_1^+ is stationary, and consequently the inhomogeneity of W_1 can be expressed explicitly with respect to points $\mathbf{x}_1 \in \mathrm{R}_1^+$.

[17] By definition, a unimodular two-tensor has a determinant equal to one.

$\forall\ \mathbf{H} \in \mathcal{G}(\mathbf{x})$ and $\forall\ \mathbf{F} \in \mathcal{L}^+$ at each $\mathbf{x} \in \mathrm{R}^-$. Similarly, if $\mathcal{G}_1(\mathbf{x})$ is the material symmetry group for phase 2, W_1 is required to be such that

$$W_1(\mathbf{F}_1\mathbf{H}_1, \mathbf{x}) = W_1(\mathbf{F}_1, \mathbf{x}), \tag{6.6}$$

$\forall\ \mathbf{H}_1 \in \mathcal{G}_1(\mathbf{x})$ and $\forall\ \mathbf{F}_1 \in \mathcal{L}^+$ at each $\mathbf{x} \in \mathrm{R}^+$. We note that because a different elastic potential is used for each phase, the material symmetry groups for phase 1 and phase 2 can be chosen independently of each other and arbitrarily, with the exception that they must be subsets of the unimodular group. Consequently, the material symmetry group of each phase can be chosen to reflect any type of crystal symmetry with any orientation. Therefore, the change in crystal structure that takes place during a martensitic phase transformation and the orientation relation between the crystal lattices of the austenite and the martensite can be accurately represented. In fact, because the elastic potential for each phase is defined with respect to the undeformed configuration of that phase, the material symmetry group for that elastic potential can be chosen to be the crystallographic point group corresponding to the crystal symmetry of that phase. This certainly makes constructing the elastic potential for each phase much more feasible.

We next require that $\tilde{\mathbf{x}}_1$ has the form

$$\tilde{\mathbf{x}}_1(\mathbf{x}, t) = \Psi(\mathbf{x}) + \Theta(t), \tag{6.7}$$

$\forall\, \mathbf{x} \in \mathrm{R}^+$ at each $t \in \Gamma$. If $\tilde{\mathbf{x}}_1$ has this form, the constitutive behavior for phase 2 given by (6.3) and (6.4) results in the stress-power for any subregion of phase 2 being equal to the time rate of change of the integral of W_1 over that subregion; i.e. if $\tilde{\mathbf{x}}_1$ has the form given by (6.7),

$$\int_{D_1^+} \boldsymbol{\sigma}_1 \cdot \nabla_1 \bar{\mathbf{v}}_1 dV = \frac{d}{dt} \int_{D_1^+} W_1 dV, \tag{6.8}$$

$\forall D_1^+ \subset R_1^+$ and $\forall t \in \Gamma$ (see [9]). For this case, it can easily be shown that the change in the total strain energy and kinetic energy of any subregion of phase 2 in a complete cycle is zero. This is certainly the most important property of an elastic material, and any choice of $\tilde{\mathbf{x}}_1$ and the forms of W_1 and the corresponding stress tensor σ_1 that did not result in this would not be consistent with the standard definition of an elastic material.[18] For the important special case where $\tilde{\mathbf{x}}_1$ is a homogeneous deformation and there is continuity of displacements at the phase boundary when $\mathbf{F} = \mathbf{1} \forall \mathbf{x} \in R^-$ and $\mathbf{F}_1 = \mathbf{1} \forall \mathbf{x}_1 \in R_1^+$, at each $t \in \Gamma$, $\tilde{\mathbf{x}}_1$ must have the form

$$\tilde{\mathbf{x}}_1(\mathbf{x},t) = \mathbf{x} + \left(\left[\mathbf{x} - \tilde{\mathbf{x}}(t)\right] \cdot \tilde{\mathbf{n}}\right)\tilde{\mathbf{a}}, \tag{6.9}$$

where $\tilde{\mathbf{x}}(t)$ is any point on the interface S, and $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{n}}$ are constant vectors. We note that $\tilde{\mathbf{x}}_1$ in (6.9) has the form of an invariant plane strain [15]. Additionally, for the $\tilde{\mathbf{x}}_1$ given by (6.9) and when $\mathbf{F} = \mathbf{1} \forall \mathbf{x} \in R^-$ and $\mathbf{F}_1 = \mathbf{1} \forall \mathbf{x}_1 \in R_1^+$, at each $t \in \Gamma$, the interface S is a plane with unit normal $\tilde{\mathbf{n}}$ and translates with velocity $\mathbf{V}(t) = \frac{d}{dt}\bar{\mathbf{x}}(t)$, and all points in phase 2 translate with velocity $\tilde{\mathbf{v}}(t) = -\left(\mathbf{V}(t) \cdot \tilde{\mathbf{n}}\right)\tilde{\mathbf{a}}$.

7. The Domains of the Elastic Potentials

As indicated in the previous section, the elastic potentials given by (6.1) and (6.3) are defined for finite deformations in $\hat{\mathcal{L}}^+ \subset \mathcal{L}^+$ and $\hat{\mathcal{L}}_1^+ \subset \mathcal{L}^+$, respectively. Additionally, $\hat{\mathcal{L}}_1^+$ is with respect to R_1^+ in the sense that the two-tensors in $\hat{\mathcal{L}}_1^+$

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[18] Note also that if $\tilde{\mathbf{x}}_1$ has the form given by (6.7), the transformation strain for phase 2 is independent of time, which is probably most appropriate for the case considered here (i.e., for the case where phase 2 represents a phase that is in a purely mechanical process and behaves elastically).

represent deformations $\mathbf{F}_1$ with respect to R_1^+. The corresponding subset of $\mathcal{L}^+$ that represents $\widehat{\mathcal{L}}_1^+$ with respect to R can be defined as

$$\bar{\mathcal{L}}_1^+ = \left\{ \bar{\mathbf{F}} / \, \bar{\mathbf{F}} = \mathbf{F}_1 \tilde{\mathbf{F}}, \text{ where } \mathbf{F}_1 \in \widehat{\mathcal{L}}_1^+ \right\}. \tag{7.1}$$

Let $\mathcal{S}^+$ denote the set of all symmetric positive definite two-tensors. In the space $\mathcal{S}^+$, the deformations with respect to R corresponding to the undeformed configurations of phase 1 and phase 2 are $\mathbf{C} = \mathbf{1}$ and $\tilde{\mathbf{C}} = \tilde{\mathbf{F}}^{\mathrm{T}}\tilde{\mathbf{F}}$, respectively. Considering this, the sets

$$\begin{aligned} \widehat{\mathcal{S}}^+ &= \left\{ \mathbf{C} / \, \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \text{ where } \mathbf{F} \in \widehat{\mathcal{L}}^+ \right\}, \\ \bar{\mathcal{S}}_1^+ &= \left\{ \bar{\mathbf{C}} / \, \bar{\mathbf{C}} = \bar{\mathbf{F}}^{\mathrm{T}}\bar{\mathbf{F}}, \text{ where } \bar{\mathbf{F}} \in \bar{\mathcal{L}}_1^+ \right\} \end{aligned} \tag{7.2}$$

represent the domains with respect to R about $\mathbf{C} = \mathbf{1}$ and $\tilde{\mathbf{C}} = \tilde{\mathbf{F}}^{\mathrm{T}}\tilde{\mathbf{F}}$ in $\mathcal{S}^+$ that phase 1 and phase 2, respectively, behave elastically for. We note that if $\widehat{\mathcal{S}}^+$ and $\bar{\mathcal{S}}_1^+$ are relatively large and $\mathbf{1}$ and $\tilde{\mathbf{C}}$ are relatively close, these two phase domains may intersect in $\mathcal{S}^+$. In this case, some might consider this constitutive model to be multivalued. Either way, an intersection of these two phase domains will not present any difficulties in setting up or solving a boundary value problem.

Most metals, however, have phases that are not elastic for finite deformations about their undeformed configurations. They instead have phases that have elastic-plastic constitutive behaviors with yield stresses corresponding to infinitesimal deformations. If the material under consideration represents such a material and if the stresses within phase 1 and phase 2 are less than the yield stresses of these phases, respectively, for all $t \in \Gamma$, both of these phases will behave elastically in this time interval. Additionally, in this case, because the deformations are infinitesimal for all $t \in \Gamma$, the linearized forms of the constitutive equations given by (6.1)

and (6.2) about $\mathbf{F} = \mathbf{1}$ and (6.3) and (6.4) about $\mathbf{F}_1 = \mathbf{1}$ can be used. This is probably most appropriate in the temperature interval containing the martensitic start temperature, where the stress necessary to induce nucleation or growth of a variant of martensite is less than the yield stresses of both phases. In this case, the material will deform by a martensitic phase transformation before it will deform by plastic deformation.[19] This martensitic phase transformation may be associated with the austenite phase transforming into a martensite phase, or with one variant of martensite transforming into another variant of the same martensite, which is also known as *reorientation*. Also, for this case where $\widehat{\mathcal{L}}^+$ and $\widehat{\mathcal{L}}_1^+$ contain only deformations such that $|\nabla \hat{\mathbf{u}}| << 1$ and $|\nabla_1 \hat{\mathbf{u}}_1| << 1$, respectively, if $\bar{\mathbf{x}}_1$ is a finite deformation, there is no chance that $\widehat{\mathcal{S}}^+$ and $\bar{\mathcal{S}}_1^+$ will intersect.

8. The Driving Traction

As mentioned in Section 5, the global form of the rate of work-energy equation given by (2.4) is not valid for subregions of R_t containing a portion of S_t since the continuity assumptions that are necessary for the derivation of that equation from Equations (2.1)-(2.3) do not exist for these subregions. For the two-phase elastic material under consideration, the global form of the rate of work-energy equation for subregions D_t of R_t containing a portion S_t^* of S_t can be expressed as

$$\begin{aligned} &\int_{\partial D_t} \mathbf{t} \cdot \bar{\mathbf{v}} dA + \int_{D_t} \mathbf{b} \cdot \bar{\mathbf{v}} dV - \int_{S^*} f \mathbf{N} \cdot \mathbf{V} dA \\ &= \frac{d}{dt} \int_{D^-} W dV + \frac{d}{dt} \int_{D_1^+} W_1 dV + \frac{d}{dt} \int_{D_t} \bar{\rho} \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} dV, \end{aligned} \tag{8.1}$$

[19] A material deforms by a martensitic phase transformation in the sense that the creation of the martensite phase produces a deformation due to the shape deformation of that martensite.

where $\mathrm{D}^- = \hat{\mathbf{x}}(\mathrm{D}_t^-, t)$, $\mathrm{D}_1^+ = \hat{\mathbf{x}}_1(\mathrm{D}_t^+, t)$, $\mathrm{S}^* = \hat{\mathbf{x}}(\mathrm{S}_t^*, t)$, $\mathrm{D}_t^- \cup \mathrm{D}_t^+ = \mathrm{D}_t$, and

$$f = \left(\tilde{J}^+ W_1^+ - W^-\right) - \tfrac{1}{2}\left(\left(\tilde{J}\sigma_1 \tilde{\mathbf{F}}^{-\mathrm{T}}\right)^+ + \sigma^-\right) \cdot \left(\mathbf{F}_1^+ \tilde{\mathbf{F}}^+ - \mathbf{F}^-\right). \tag{8.2}$$

The quantity $f\mathbf{N}$ is referred to as the driving traction, and f as the scalar driving traction.[20] The integral

$$\int_{S^*} f\mathbf{N} \cdot \mathbf{V} dA \tag{8.3}$$

can be interpreted as representing the rate of work done on the interface S^* by the traction $f\mathbf{N}$ exerted by the body on the interface. We require that this integral be positive so that it represents a dissipation of energy. Localizing this equation at points on the interface S then yields

$$f\mathbf{N} \cdot \mathbf{V} \geq 0 \tag{8.4}$$

$\forall\, \mathbf{x} \in \mathrm{S}$ at each $t \in \Gamma$.[21] Note that energy can be dissipated only at the phase boundary, and that if $f\mathbf{N} \cdot \mathbf{V} = 0\ \forall\, \mathbf{x} \in \mathrm{S}$ at each $t \in \Gamma$, energy is conserved. Thus, by allowing $f\mathbf{N} \cdot \mathbf{V} > 0$ we are in effect considering a nonconservative system even though both phases behave elastically for points not on the interface.

We can also postulate a constitutive relation relating the scalar driving traction to the normal component of the phase boundary velocity at each $\mathbf{x} \in \mathrm{S}$ (see [2]).

[20] Refer to [2] and [5] for more extensive discussions about the driving traction. Also, the scalar driving traction given by (8.2) can formally be obtained from the scalar driving traction derived in [2] by replacing W^+, σ^+, and $\mathbf{F}^+$ with $\tilde{J}^+ W_1^+$, $\left(\tilde{J}\sigma^+ \tilde{\mathbf{F}}^{-\mathrm{T}}\right)^+$, and $\mathbf{F}_1^+ \tilde{\mathbf{F}}^+$, respectively.

[21] This requirement is also equivalent to the second law of thermodynamics for the type of process under consideration (see [2]).

More specifically, we can postulate that

$$V_n = \Phi(f, \mathbf{x}), \tag{8.5}$$

at each $\mathbf{x} \in S$, where $\Phi(f, \mathbf{x})$ is given and depends on the material, and $V_n = \mathbf{V} \cdot \mathbf{N}$. This may be done primarily to provide an extra equation for the extra unknowns — the variables describing the location of the phase boundary.

9. The Linearized Problem

As mentioned in the Introduction, one of the main advantages of using the continuum model that was developed in this paper is that the field quantities and equations are in forms that permit direct linearization, while retaining finite shape deformations for the martensite phases. This is the case since the displacements for each phase are measured from the reference configuration coinciding with an unstressed undeformed configuration of that phase (i.e. the shape deformation of that phase), and consequently, for the appropriate boundary and initial conditions, the displacement gradients can be considered infinitesimal.[22] Another advantage of using this continuum model is that for the linear case the nominal stress for each phase is approximately equal to the true stress for that phase. This is very convenient for solving certain types of boundary value problems, as will become more apparent in Sections 13 and 14 where several of these types of problems are considered.

In the next few sections, the linearized field equations and jump conditions in terms of the displacements for the two-phase elastic material under consideration are considered. The general equations in terms of the unspecified elastic potentials given by (6.1) and (6.2) for finite deformations can be obtained in a similar manner (see [9]).

[22] It is also assumed here that the unstressed undeformed configuration of each phase corresponds to a local minimum of the elastic potential for that phase.

10. The Linearized Constitutive Equations and Field Equations

In the following, it is assumed that the undeformed configurations of phase 1 and phase 2 correspond to local minima of their respective elastic potentials. For the linear problem, we assume that $|\nabla\hat{\mathbf{u}}| << 1\ \forall\,\mathbf{x} \in \mathrm{R}^-$ and $|\nabla\hat{\mathbf{u}}_1| << 1\ \forall\,\mathbf{x}_1 \in \mathrm{R}_1^+$ at each $t \in \Gamma$. In this case, we have for phase 1

$$\begin{aligned} W &= W^* + \tfrac{1}{2}\nabla\hat{\mathbf{u}}\cdot(\mathbf{C}\nabla\hat{\mathbf{u}}) + \mathrm{O}\big(|\nabla\hat{\mathbf{u}}|^3\big) \\ &= W^* + \tfrac{1}{2}\boldsymbol{\varepsilon}\cdot(\mathbf{C}\boldsymbol{\varepsilon}) + \mathrm{O}\big(|\boldsymbol{\varepsilon}|^3\big) \\ \boldsymbol{\sigma} &= \mathbf{C}\nabla\hat{\mathbf{u}} + \mathrm{O}\big(|\nabla\hat{\mathbf{u}}|^2\big) \\ &= \mathbf{C}\boldsymbol{\varepsilon} + \mathrm{O}\big(|\boldsymbol{\varepsilon}|^2\big) \end{aligned} \tag{10.1}$$

$$\boldsymbol{\tau}(\hat{\mathbf{y}}(\mathbf{x},t),t) = \boldsymbol{\sigma}(\mathbf{x},t) + \mathrm{O}\big(|\nabla\hat{\mathbf{u}}|^2\big)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} + \mathrm{O}\big(|\nabla\hat{\mathbf{u}}|^2\big)$$

$\forall\,\mathbf{x} \in \mathrm{R}^-$ at each $t \in \Gamma$, where $\boldsymbol{\varepsilon} = \frac{1}{2}\Big(\nabla\hat{\mathbf{u}} + (\nabla\hat{\mathbf{u}})^{\mathrm{T}}\Big)$ is the infinitesimal strain tensor for phase 1, $W^*(\mathbf{x}) = W(\mathbf{1},\mathbf{x})$, and $\mathbf{C}(\mathbf{x})$ with $C_{ijkl} = \left.\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}\right|_{\mathbf{F}=\mathbf{1}}$ is the elasticity four-tensor for phase 1. For phase 2, we have

$$\begin{aligned} W_1 &= W_1^* + \tfrac{1}{2}\nabla_1\hat{\mathbf{u}}_1\cdot(\mathbf{C}_1\nabla_1\hat{\mathbf{u}}_1) + \mathrm{O}\big(|\nabla_1\hat{\mathbf{u}}_1|^3\big) \\ &= W_1^* + \tfrac{1}{2}\boldsymbol{\varepsilon}_1\cdot(\mathbf{C}_1\boldsymbol{\varepsilon}_1) + \mathrm{O}\big(|\boldsymbol{\varepsilon}_1|^3\big) \\ \boldsymbol{\sigma}_1 &= \mathbf{C}_1\nabla_1\hat{\mathbf{u}}_1 + \mathrm{O}\big(|\nabla_1\hat{\mathbf{u}}_1|^2\big) \\ &= \mathbf{C}_1\boldsymbol{\varepsilon}_1 + \mathrm{O}\big(|\boldsymbol{\varepsilon}_1|^2\big) \end{aligned} \tag{10.2}$$

$$\boldsymbol{\tau}_1(\hat{\mathbf{y}}_1(\mathbf{x}_1,t),t) = \boldsymbol{\sigma}_1(\mathbf{x}_1,t) + \mathrm{O}\big(|\nabla_1\hat{\mathbf{u}}_1|^2\big)$$

$$\sigma_1 = \sigma_1^{\mathrm{T}} + O(|\nabla_1 \hat{\mathbf{u}}_1|^2)$$

$\forall \mathbf{x}_1 \in \mathrm{R}_1^+$ at each $t \in \Gamma$, where $\boldsymbol{\varepsilon}_1 = \frac{1}{2}\left(\nabla_1 \hat{\mathbf{u}}_1 + (\nabla_1 \hat{\mathbf{u}}_1)^{\mathrm{T}}\right)$ is the infinitesimal strain tensor for phase 2, $W_1^*(\tilde{\mathbf{x}}(\mathbf{x}_1, t)) = W_1(\mathbf{1}, \tilde{\mathbf{x}}(\mathbf{x}_1, t))$, and $\mathbf{C}_1(\tilde{\mathbf{x}}(\mathbf{x}_1, t))$ with $C_{1_{ijkl}} = \left.\frac{\partial^2 W_1}{\partial F_{1_{ij}} \partial F_{1_{kl}}}\right|_{\mathbf{F}_1 = \mathbf{1}}$ is the elasticity four-tensor for phase 2. The elasticity tensors $\mathbf{C}$ and $\mathbf{C}_1$ are both four-tensors which contain the material coefficients and reflect the material symmetry of phase 1 and phase 2, respectively. Additionally, both $\mathbf{C}$ and $\mathbf{C}_1$ are positive-definite, since it has been assumed that the undeformed configuration of each phase corresponds to a relative minimum of the elastic potential for that phase. These four-tensors also have the usual major and minor symmetries.

Substituting the constitutive equation $(10.1)_2$ into the equation representing the balance of linear momentum for phase 1 given by $(5.8)_1$ and expressing $\mathbf{a}$ in that equation in terms of $\hat{\mathbf{u}}$ yields, in a given coordinate frame,

$$\frac{\partial C_{ijkl}}{\partial x_j} \frac{\partial \hat{u}_k}{\partial x_l} + C_{ijkl} \frac{\partial^2 \hat{u}_k}{\partial x_l \partial x_j} + f_i = \rho \frac{\partial^2 \hat{u}_i}{\partial t^2}, \tag{10.3}$$

$\forall \mathbf{x} \in \mathrm{R}^-$ at each $t \in \Gamma$, where the second-order terms have been neglected.

Similarly, substituting the constitutive equation $(10.2)_2$ into the equation representing the balance of linear momentum for phase 2 given by $(5.9)_1$ and expressing $\bar{\mathbf{a}}_1$ in that equation in terms of $\tilde{\mathbf{x}}_1$ and $\hat{\mathbf{u}}_1$ yields

$$\mathrm{div}_1(\mathbf{C}_1 \nabla_1 \hat{\mathbf{u}}_1) + \mathbf{f}_1 = \bar{\rho}_1 \left(\ddot{\tilde{\mathbf{x}}}_1 + \ddot{\hat{\mathbf{u}}}_1\right), \tag{10.4}$$

$\forall \mathbf{x}_1 \in \mathrm{R}_1^+$ at each $t \in \Gamma$, where $\ddot{\tilde{\mathbf{x}}}_1 = \frac{\partial^2}{\partial t^2} \tilde{\mathbf{x}}_1(\mathbf{x}, t) = \frac{d^2}{dt^2} \Theta(t)$, $\ddot{\hat{\mathbf{u}}}_1 = \left[\frac{\partial^2}{\partial t^2} \hat{\mathbf{u}}_1(\tilde{\mathbf{x}}_1(\mathbf{x}, t), t)\right]_{\tilde{\mathbf{x}}(\mathbf{x}_1, t)}$, and the second-order terms have been neglected. In a given frame, the term $\ddot{\hat{\mathbf{u}}}_1$ has components

$$\frac{\partial^2}{\partial t^2}\hat{u}_{1_i}(\tilde{\mathbf{x}}_1(\mathbf{x},t),t) = \frac{\partial}{\partial t}\left[\left(\frac{\partial \hat{u}_{1_i}}{\partial x_{1_k}}\tilde{v}_k + \frac{\partial \hat{u}_{1_i}}{\partial t}\right)\right]_{\tilde{\mathbf{x}}_1(\mathbf{x},t)}$$

$$= \left(\frac{\partial^2 \hat{u}_{1_i}}{\partial x_{1_k}\partial x_{1_n}}\tilde{v}_k\tilde{v}_n + 2\frac{\partial^2 \hat{u}_{1_i}}{\partial t \partial x_{1_k}}\tilde{v}_k + \frac{\partial \hat{u}_{1_i}}{\partial x_{1_k}}\frac{d\tilde{v}_k}{dt} + \frac{\partial^2 \hat{u}_{1_i}}{\partial t^2}\right), \tag{10.5}$$

where $\tilde{\mathbf{v}}(\mathbf{x},t) = \frac{\partial}{\partial t}\tilde{\mathbf{x}}_1(\mathbf{x},t) = \frac{d}{dt}\Theta(t)$. The inertial-type terms in (10.5) obviously occur because points in R_1^+ are moving. This also results in the boundary conditions on the boundary of $\mathrm{R}_1^+/\mathrm{S}_1$, where $\mathrm{S}_1 = \hat{\mathbf{x}}_1(\mathrm{S}_t,t)$, being with respect to a moving boundary. Both of these issues complicate solving the corresponding boundary value problem, whether it be by using analytical methods or by constructing a finite difference or a finite element computer program.

Fortunately, the boundary value problem with the balance of linear momentum for phase 2 in the form given by Equation (10.4) does not have to be solved. Instead, Equation (10.4) can be transformed into a more tractable equation. In particular, we can use the mapping $\tilde{\mathbf{x}}_1$ and define the function $\bar{\mathbf{u}}_1$ as

$$\bar{\mathbf{u}}_1(\mathbf{x},t) = \hat{\mathbf{u}}_1(\tilde{\mathbf{x}}_1(\mathbf{x},t),t), \tag{10.6}$$

$\forall\, \mathbf{x} \in \mathrm{R}^+$ and $\forall\, t \in \Gamma$, and then solve for $\mathbf{u}_1 = \bar{\mathbf{u}}_1(\mathbf{x},t)$ instead of $\mathbf{u}_1 = \hat{\mathbf{u}}_1(\mathbf{x}_1,t)$ in the boundary value problem. Also, note that $\hat{\mathbf{u}}_1(\mathbf{x}_1,t) = \bar{\mathbf{u}}_1(\tilde{\mathbf{x}}(\mathbf{x}_1,t),t)\ \forall\, \mathbf{x}_1 \in \mathrm{R}_1^+$ at each $t \in \Gamma$.

In terms of $\bar{\mathbf{u}}_1$, the elastic potential and stress tensor for phase 2 given by Equations $(10.2)_1$ and $(10.2)_2$, respectively, become

$$W_1 = W_1^* + \tfrac{1}{2}(\nabla\bar{\mathbf{u}}_1\nabla_1\tilde{\mathbf{x}})\cdot[\mathbf{C}_1(\nabla\bar{\mathbf{u}}_1\nabla_1\tilde{\mathbf{x}})] + \mathrm{O}\big(|\nabla_1\bar{\mathbf{u}}_1|^3\big) \tag{10.7}$$

$$\boldsymbol{\sigma}_1 = \mathbf{C}_1(\nabla\bar{\mathbf{u}}_1\nabla_1\tilde{\mathbf{x}}) + \mathrm{O}\big(|\nabla\bar{\mathbf{u}}_1|^2\big)$$

respectively, $\forall\, \mathbf{x} \in \mathrm{R}^+$ at each $t \in \Gamma$, where $\mathbf{x}_1 = \tilde{\mathbf{x}}_1(\mathbf{x}, t)$ in $\nabla_1 \tilde{\mathbf{x}}_1$.

Substituting $(10.7)_2$ into the equation representing the balance of linear momentum for phase 2 given by $(5.9)_1$ and expressing the acceleration term in that equation in terms of $\tilde{\mathbf{x}}_1$ and $\bar{\mathbf{u}}_1$ yields, in a given coordinate frame,

$$\begin{aligned}
&\frac{\partial C_{1_{ijkl}}}{\partial x_n} \frac{\partial \tilde{x}_n}{\partial x_{1_j}} \frac{\partial \bar{u}_{1_k}}{\partial x_m} \frac{\partial \tilde{x}_m}{\partial x_{1_l}} \\
&+ C_{1_{ijkl}} \left\{ \frac{\partial^2 \bar{u}_{1_k}}{\partial x_m \partial x_n} \frac{\partial \tilde{x}_m}{\partial x_{1_l}} \frac{\partial \tilde{x}_n}{\partial x_{1_j}} + \frac{\partial \bar{u}_{1_k}}{\partial x_m} \frac{\partial^2 \tilde{x}_m}{\partial x_{1_l} \partial x_{1_j}} \right\} + f_{1_i} \\
&= \frac{\rho}{\tilde{J}} \left(\frac{\partial^2 \tilde{x}_{1_i}}{\partial t^2} + \frac{\partial^2 \bar{u}_{1_i}}{\partial t^2} \right),
\end{aligned} \tag{10.8}$$

$\forall \mathbf{x} \in \mathrm{R}^+$ at each $t \in \Gamma$, where $\mathbf{x}_1 = \tilde{\mathbf{x}}_1(\mathbf{x}, t)$ in $\nabla_1 \tilde{\mathbf{x}}$. From Equation (10.8), we can observe that not only have most of the inertial-type terms been eliminated, the resulting boundary value problem is in a completely Lagrangian description; i.e. it is completely in terms of the coordinates of the fixed reference configuration. This includes the boundary conditions for phase 2 being specified with respect to a fixed boundary. The only penalty that is paid for this coordinate transformation are the additional terms in Equation (10.8). For the important special case where $\tilde{\mathbf{x}}_1$ is a homogeneous deformation, phase 2 is a homogeneous material (i.e. $\mathbf{C}_1$ is independent of $\mathbf{x}$), and $\mathbf{f}_1 = 0$, Equation (10.8) given above reduces to

$$C_{1_{ijkl}} \frac{\partial^2 \bar{u}_{1_k}}{\partial x_m \partial x_n} \frac{\partial \tilde{x}_m}{\partial x_{1_l}} \frac{\partial \tilde{x}_n}{\partial x_{1_j}} = \frac{\rho}{\tilde{J}} \left(\frac{\partial^2 \tilde{x}_{1_i}}{\partial t^2} + \frac{\partial^2 \bar{u}_{1_i}}{\partial t^2} \right), \tag{10.9}$$

where $\mathbf{x}_1 = \tilde{\mathbf{x}}_1(\mathbf{x}, t)$ in $\nabla_1 \tilde{\mathbf{x}}$. For this case, the additional terms in (10.9) contribute only to the coefficients of the terms involving second spatial derivatives of $\bar{\mathbf{u}}_1$. Thus,

in *at least* this case, the advantages of using multiple reference configurations along with this coordinate transformation far outweigh their disadvantages.

For the static problem, points in R_1^+ are stationary and it is probably much more convenient to work in terms of $\mathbf{u}_1 = \hat{\mathbf{u}}_1(\mathbf{x}_1)$ instead of $\mathbf{u}_1 = \bar{\mathbf{u}}_1(\mathbf{x})$ in the boundary value problem. In this case the inhomogeneity of W_1 can be expressed explicitly in terms of points $\mathbf{x}_1 \in R_1^+$, and the equation representing the balance of linear momentum for phase 2 would have the same form as the equation representing the balance of linear momentum for phase 1 given by (10.3), except that all of the quantities would have 1 subscripts and the inertial terms would, of course, be equal to zero.

For the linear problem where it is assumed that $|\nabla\hat{\mathbf{u}}| << 1 \ \forall\, \mathbf{x} \in R^-$ and $|\nabla\hat{\mathbf{u}}_1| << 1 \ \forall\, \mathbf{x}_1 \in R_1^+$ at each $t \in \Gamma$, we can consider a process where the first and second time derivatives of $\hat{\mathbf{u}}$ and $\bar{\mathbf{u}}_1$ are negligible $\forall\, \mathbf{x} \in R^-$ *and* $\forall\, \mathbf{x} \in R^+$, respectively, $|\mathbf{V}| << 1$, and $\frac{d}{dt}\mathbf{V}$ is negligible, for all $t \in \Gamma$. Such a process corresponds to a process where the strains of each phase are infinitesimal, the motions of each particle of each phase relative to the undeformed configuration of that phase are negligible, the magnitude of the phase boundary velocity is infinitesimal, and the acceleration of the phase boundary is negligible, for all $t \in \Gamma$. In the following, this type of process will be referred to as a quasi-static process, even though it does not conform to the exact definition of such a process. More specifically, this type of process is not a true quasi-static process since time is not just a parameter in *all* variables of the problem, and consequently the set of all solutions as time is varied does not consist of only static equilibrium solutions. Static equilibrium occurs only when $\mathbf{V} = 0$. For this quasi-static process, it is probably much more convenient to solve for $\mathbf{u}_1 = \hat{\mathbf{u}}_1(\mathbf{x}_1)$ instead of $\mathbf{u}_1 = \bar{\mathbf{u}}_1(\mathbf{x})$ in the boundary value

problem. However, unlike the static problem, points in R_1^+ are moving. Therefore, the inhomogeneity of W_1 still needs to be expressed with respect to points $\mathbf{x} \in R^+$ (see Section 6). Thus, for a quasi-static process the balance of linear momentum for phase 2 would have the same form as it would have for the static case, except that $\partial C_{1_{ijkl}}/\partial x_{1_j}$ would be replaced by $\left(\partial C_{1_{ijkl}}/\partial x_n\right)\left(\partial \tilde{x}_n/\partial x_{1_j}\right)$.

11. The Linearized Jump Conditions

The most direct form of the conitinuity of displacements condition is given by (4.1). If we work with this condition in this form, there is nothing to linearize. However, the jump conditions (4.2) and (4.3), which together represent the continuity of displacements, are such that they or equations equivalent to them can be linearized. In particular, as was mentioned in Section 4, Equation (4.2) is equivalent to Equation (4.4). Since both $\hat{\mathbf{a}}$ and $\mathbf{N}$ are real quantities in that equation, we can conclude that the two-tensor $\hat{\mathbf{a}} \otimes \mathbf{N}$, and hence $\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-$, has two zero eigenvalues and one real not necessarily zero eigenvalue.[23] Thus, we can write the characteristic equation for $\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-$ as

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0 \tag{11.1}$$

where

$$\begin{aligned} I_1 &= \mathrm{tr}\left(\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-\right), \\ I_2 &= \tfrac{1}{2}\left\{\left[\mathrm{tr}\left(\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-\right)\right]^2 - \mathrm{tr}\left(\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-\right)^2\right\}, \\ I_3 &= \det\left(\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-\right), \end{aligned} \tag{11.2}$$

and

$$\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^- = (\nabla\tilde{\mathbf{u}}_1)^+ + (\nabla\tilde{\mathbf{u}}_1\nabla_1\tilde{\mathbf{x}}\nabla\tilde{\mathbf{u}}_1)^+ + (\nabla\tilde{\mathbf{u}}_1\nabla_1\tilde{\mathbf{x}})^+ - (\nabla\hat{\mathbf{u}})^-. \tag{11.3}$$

[23] This can easily be seen by working in the coordinate frame where either $\mathbf{N}$ or $\hat{\mathbf{a}}$ coincides with a basis vector.

A necessary condition for the existence of two zero eigenvalues of $\mathbf{F}_1^+\tilde{\mathbf{F}}^+ - \mathbf{F}^-$ is that

$$I_2 = 0, \qquad I_3 = 0. \tag{11.4}$$

Equations (11.4) can both be linearized with respect to $\nabla\hat{\mathbf{u}}$ and $\nabla\bar{\mathbf{u}}_1$. The set of all vectors $\mathbf{L}$ that satisfy the linearized form of (4.2) can then be determined. Once this is done, $\mathbf{N}(\mathbf{x}, t)$ can be determined such that $\mathbf{N}\cdot\mathbf{L} = 0\ \forall\,\mathbf{x} \in \mathrm{S}$ at each $t \in \Gamma$. $\mathbf{N}$ can then be substituted into the linear momentum jump condition given by (5.10) yielding three scalar equations which can be linearized with respect to $\nabla\hat{\mathbf{u}}$ and $\nabla\bar{\mathbf{u}}_1$.

Consider the path of the phase boundary during a motion in the time interval Γ. Next consider the special case where we assume that at each location of the phase boundary in this path there is continuity of displacements at all points on the phase boundary for $\hat{\mathbf{u}} = 0\ \forall\,\mathbf{x} \in \mathrm{R}^-$ and $\hat{\mathbf{u}}_1 = 0\ \forall\,\mathbf{x}_1 \in \mathrm{R}_1^+$. In this case, as can be seen by substituting $\mathbf{F}^- = \mathbf{F}_1^+ = \mathbf{1}$ into (4.4), $\tilde{\mathbf{F}}^+$ must have the form

$$\tilde{\mathbf{F}}^+ = \mathbf{1} + \tilde{\mathbf{a}} \otimes \tilde{\mathbf{n}} \tag{11.5}$$

at each point on the phase boundary at each $t \in \Gamma$, where the vectors $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{n}}$ may both be functions of $\mathbf{x} \in \mathrm{S}$.[24] In this case, we can conclude from substituting (11.5) into (4.2) that the first-order approximation of $\mathbf{L}$ is $\tilde{\mathbf{L}}$, where $\tilde{\mathbf{L}}\cdot\tilde{\mathbf{n}} = 0$. Additionally, in this case, the first-order approximation of the linear momentum jump condition given by (5.10) is

$$(\boldsymbol{\sigma}_1^+ - \boldsymbol{\sigma}^-)\tilde{\mathbf{n}} + \rho(\bar{\mathbf{v}}_1^+ - \mathbf{v}^-)V_n = \mathbf{0}, \tag{11.6}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_1$ are given by their respective linear constitutive relations.

For the static case, jump conditions (4.1) and (4.2) are the same, jump condition (4.3) is trivially satisfied, and the linear momentum jump condition given by (5.10)

[24] This may be a good assumption for a material that has continuity of displacements at the phase boundary and is unstressed when it is in static equilibrium with no applied boundary tractions.

reduces to

$$\left(\left(\tilde{J}\sigma_1\tilde{\mathbf{F}}^{-\mathrm{T}}\right)^+ - \sigma^-\right)\mathbf{N} = 0, \tag{11.7}$$

which is equivalent to the continuity of traction across the interface. For the case where $\tilde{\mathbf{F}}^+ = \mathbf{1} + \tilde{\mathbf{a}} \otimes \tilde{\mathbf{n}}$ (11.7) reduces to

$$\left(\sigma_1^+ - \sigma^-\right)\tilde{\mathbf{n}} = 0. \tag{11.8}$$

For a quasi-static process, jump conditions (4.1) and (4.2) are the same. However, for this type of process, the first-order approximation of jump condition (4.3) is

$$\left(\frac{\partial \tilde{\mathbf{x}}_1}{\partial t}\right)^+ = -(\nabla \tilde{\mathbf{u}}_1)^+\mathbf{V}. \tag{11.9}$$

If the displacements are to be continuous at the phase boundary for all $t \in \Gamma$ with respect to a first-order approximation, the mapping $\tilde{\mathbf{x}}_1$ should be chosen such that (11.9) is satisfied. For the case where the shape deformation is a homogeneous invariant plane strain for all $t \in \Gamma$, $\tilde{\mathbf{x}}_1$ must have the form given by (6.9) in order for (11.9) to be satisfied at each $t \in \Gamma$ (see Section 6). Also, for a quasi-static process, the first-order approximation of the linear momentum jump condition is the same as that for the static case.

12. The Linearized Driving Traction and Kinetic Relation

The driving traction acting on the phase boundary was presented and discussed in Section 8. For the linear case where we assume that $\tilde{\mathbf{F}}^+ = \mathbf{1} + \tilde{\mathbf{a}} \otimes \tilde{\mathbf{n}}$ at each point on the phase boundary at each $t \in \Gamma$, the driving traction given by (8.2) becomes

$$f = \left(\left(\tilde{J} W_1^* \right)^+ - (W^*)^- \right) - \tfrac{1}{2} \left[(\sigma_1^+ + \sigma^-) \tilde{\mathbf{n}} \right] \cdot \tilde{\mathbf{a}} + O(|\nabla \hat{\mathbf{u}}|^2, |\nabla \tilde{\mathbf{u}}_1|^2) \tag{12.1}$$

For the quasi-static case, using (4.4) and (11.7), the driving traction given by (8.2) can be written as

$$f = \left(\tilde{J}^+ W_1^+ - W^- \right) - \sigma^- \cdot \left(\mathbf{F}_1^+ \tilde{\mathbf{F}}^+ - \mathbf{F}^- \right). \tag{12.2}$$

For the linear case and when $\tilde{\mathbf{F}}^+ = \mathbf{1} + \tilde{\mathbf{a}} \otimes \tilde{\mathbf{n}}$, this driving traction becomes

$$f = \left(\left(\tilde{J} W_1^* \right)^+ - (W^*)^- \right) - (\sigma^- \tilde{\mathbf{n}}) \cdot \tilde{\mathbf{a}} \tag{12.3}$$

where the second-order terms have been neglected. The implications of this form of the driving traction for the quasi-static case will be discussed further in Section 13.

As mentioned in Section 8, we can postulate a kinetic relation at the phase boundary which relates the driving traction to the normal component of the phase boundary velocity. It is required that this kinetic relation satisfy (8.4) so that energy is dissipated (or conserved if the equality sign holds), instead of being created, during a martensitic phase transformation. We next consider a kinetic relation which has the general form

$$\Phi(f) \begin{cases} > 0, & & f > f_2 \\ = 0, & \text{for} & f_1 \le f \le f_2 \\ < 0, & & f < f_1 \end{cases} \tag{12.4}$$

where the constants f_1 and f_2 are material-dependent and are such that $f_1 \le$

0 and $f_2 \geq 0$.[25] We note that such a kinetic relation satisfies (8.4). For the linear case, a kinetic relation which has the form given by (12.4) is

$$\Phi(f) = \begin{cases} \frac{1}{\nu_2}(f - f_2), & f > f_2 \\ 0, \quad \text{for} & f_1 \leq f \leq f_2 \\ \frac{1}{\nu_1}(f - f_1), & f < f_1 \end{cases} \tag{12.5}$$

where f is the linearized driving traction and ν_1 and ν_2 are constants which depend on the given material. We note that as ν_1 and $\nu_2 \to \infty$, $V_n \to 0$ and the phase boundary moves with the particles of material at the interface but does not pass over them converting them from one phase to the other.

13. Reorientation

The phenomena where a boundary traction is applied resulting in the phase boundary separating two variants of the same martensite moving and transforming one variant into the other is known as reorientation. In a material that is fully martensitic, reorientation takes place until all of the variants of martensite in the material are the same variant of the same martensite,[26] or the boundary traction is removed.

The issue of which variant of martensite is preferred during the growth process, whether the process is reorientation or simply an austenite-martensite phase transformation where several variants of martensite nucleate at different points in the material, is an issue that has received much attention. For the case where a uniaxial tensile traction is applied, it has been observed from experiments in [12] that for

[25] Kinetic relations of this form have been studied in [1].

[26] If this occurs in a single crystal of austenite, it will transform into a single crystal of one variant of martensite (see [12]).

18R martensitic alloys the variant of martensite that is preferred during the growth process is the variant which yields the largest amount of extension due to its shape deformation in the direction of the uniaxial tensile traction. The consistency of this experimental observation with a minimum energy criterion is shown and discussed in [4]. There have also been proposed criteria involving the shear traction on the interface of the martensite,[27] and criteria based on the shear stress on the plane of slip for internally slipped martensites.

In the rest of this section, the issue of which variant of martensite is preferred for the case of a stress-induced austenite-martensite phase transformation and for the case of reorientation for a general state of stress at the interface are considered. The issue of which variant is preferred for the special case of a uniaxial tensile traction applied to a cylindrical body is then investigated. For this problem, a result is obtained that corresponds with observations made from experiments.

13.1. The General Case

Consider the quasi-static case where the shape deformation of phase 2 is a homogeneous invariant plane strain. In this case, the driving traction is a special case of the driving traction given by (12.3), where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{n}}$ are constant vectors. We next assume that all of the variants of martensite of the material under consideration have the same value for their elastic potentials in their undeformed states,[28] and we assume that the material has a kinetic relation of the form (12.4).

To consider the case of an austenite-martensite phase transformation, we let phase 1 represent the austenite and phase 2 represent a variant of martensite. From the

[27] Such a criteria is discussed in, e.g., [6].

[28] For this comparison of the elastic potentials, they are all considered to be with respect to the same reference configuration.

assumptions given above, we can conclude that $\left(\tilde{J}\, W_1^*\right)^+ - (W^*)^-$ has the same value regardless of which variant of martensite phase 2 represents. We next note that $(\sigma^- \tilde{\mathbf{n}})$ is the traction on the interface and that $(\sigma^- \tilde{\mathbf{n}}) \cdot \tilde{\mathbf{a}}$ is the component of this traction in the direction of the amplitude vector of the variant of martensite that phase 2 represents multiplied by the magnitude of this amplitude vector. We also note that the variant that has the largest negative value for its driving traction will grow at the fastest rate. From the discussion above, we can conclude that this variant is the variant with the largest value of $(\sigma^- \tilde{\mathbf{n}}) \cdot \tilde{\mathbf{a}}$. Thus, for the case under consideration, we can conclude that the variant of martensite with the largest value of the component of traction on its interface in the direction of its amplitude vector multiplied by the magnitude of its amplitude vector will be the variant that is preferred during the growth process in a stress-induced austenite to martensite phase transformation.

To consider the case of reorientation, we let phase 1 and phase 2 represent two different variants of the same martensite. From the assumptions given previously, we can conclude that $\left(\tilde{J}\, W_1^*\right)^+ - (W^*)^- = 0$ for every combination of variants that phase 1 and phase 2 can represent. Therefore, the variant that has an amplitude vector $\tilde{\mathbf{a}}$ and an interface normal $\tilde{\mathbf{n}}$ that result in $(\sigma^- \tilde{\mathbf{n}}) \cdot \tilde{\mathbf{a}}$ having the largest value will grow at the fastest rate. We note, however, that in this case $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{n}}$ for the variant under consideration are measured with respect to the neighboring variant across the interface with unit normal $\tilde{\mathbf{n}}$.

13.2. The Case of a Uniaxial Tensile Traction

Consider a cylindrical body parallel to the unit vector $\mathbf{e}_1$. Assume that a tensile traction $\mathbf{t} = -\sigma_0 \mathbf{e}_1$ is applied at the end with unit normal $-\mathbf{e}_1$, a tensile traction $\mathbf{t} = \sigma_0 \mathbf{e}_1$ is applied at the end with unit normal $\mathbf{e}_1$, and the remaining surface of the

cylindrical body is traction free. For this case, the stress tensor

$$\tau = \sigma_0 \mathbf{e}_1 \otimes \mathbf{e}_1 \tag{13.1}$$

is a solution of the field equations, the linear momentum jump condition, and the boundary conditions for the quasi-static process under consideration.[29] Considering $(10.1)_3$, the first-order approximation of $\boldsymbol{\sigma}^-$ is

$$\boldsymbol{\sigma}^- = \sigma_0 \mathbf{e}_1 \otimes \mathbf{e}_1 \tag{13.2}$$

For this special case, the linearized driving traction given by (12.3) becomes

$$f = f_{cr} - \sigma_0 a \cos\phi \cos\alpha, \tag{13.3}$$

where $f_{cr} = \left(\tilde{J} W_1^*\right)^+ - (W^*)^-$, $a = |\tilde{\mathbf{a}}|$, $a\cos\phi = \tilde{\mathbf{a}} \cdot \mathbf{e}_1$, and $\cos\alpha = \tilde{\mathbf{n}} \cdot \mathbf{e}_1$.

We next note that the extension of a unit fibre of material originally parallel to $\mathbf{e}_1$ due only to the shape deformation is

$$\delta = \left|\tilde{\mathbf{F}}\mathbf{e}_1\right| - |\mathbf{e}_1| = \sqrt{\mathbf{e}_1 \cdot \tilde{\mathbf{C}}\mathbf{e}_1} - 1, \tag{13.4}$$

where $\tilde{\mathbf{C}} = \tilde{\mathbf{F}}^{\mathrm{T}}\tilde{\mathbf{F}}$. The component of extension in the direction of the tensile traction is

$$\delta' = \mathbf{e}_1 \cdot \tilde{\mathbf{F}}\mathbf{e}_1 - \mathbf{e}_1 \cdot \mathbf{e}_1 = a\cos\phi\cos\alpha. \tag{13.5}$$

This is also equal to the extension given by (13.4) if rotations are neglected or do not occur. From this, we can write the driving traction given by (13.3) as

$$f = f_{cr} - \sigma_0 \delta'. \tag{13.6}$$

Thus, for a given σ_0, the variant of martensite that yields the largest component of extension in the direction of the uniaxial tensile traction will be preferred during

[29] This can readily be seen by expressing these equations in there Eulerian form.

the growth process. This corresponds with the experimental observations presented in [12].

Let $\tilde{\mathbf{e}}$ be a unit vector in the $(\tilde{\mathbf{a}}, \tilde{\mathbf{n}})$ plane such that $\tilde{\mathbf{e}} \cdot \tilde{\mathbf{n}} = 0$ and $\tilde{\mathbf{e}} \cdot \tilde{\mathbf{a}} > 0$. For the case of a uniaxial tensile traction described above, the (shear) component of traction on the interface in the direction of $\tilde{\mathbf{e}}$ is

$$\tilde{S} = \tilde{\mathbf{e}} \cdot [(\sigma_0 \mathbf{e}_1 \otimes \mathbf{e}_1)\tilde{\mathbf{n}}] = \sigma_0 \cos\lambda \cos\alpha \tag{13.7}$$

where $\tilde{\mathbf{e}} \cdot \mathbf{e}_1 = \cos\lambda$. For the case where $\tilde{\mathbf{F}}$ is a simple shear, $\tilde{\mathbf{e}}$ is parallel to $\tilde{\mathbf{a}}$, which results in $\lambda = \phi$, and (13.7) becomes

$$\tilde{S} = \sigma_0 \cos\phi \cos\alpha. \tag{13.8}$$

Thus, for this case, the driving traction given by (13.3) can be written as

$$f = f_{cr} - a\tilde{S}. \tag{13.9}$$

From this, we can conclude that a shear stress criterion which states that the variant that is preferred is the variant with the largest value of $\tilde{S}$ will correspond to the variant with the largest negative driving traction for a given σ_0 only if each variant has a simple shear shape deformation and has the same value for $|\tilde{\mathbf{a}}|$.

14. An Applied Uniform Hydrostatic Pressure

In this section, the effect that an applied hydrostatic pressure has on a martensitic phase transformation in a given material is studied.[30] The two-phase elastic material that was described in the previous sections is considered. If a given loading results in an increase in the driving traction, that loading is considered to favor the growth of phase 1, and if a given loading results in a decrease in the driving traction, that

[30] See [8] for a discussion of this type of problem and for a list of some references where such problems are considered.

loading is considered to favor the growth of phase 2. If, on the other hand, the given loading results in no change in the value of the driving traction, neither phase is favored by that loading. For the problem under consideration it is assumed that a hydrostatic pressure exists such that

$$\tau = -p\mathbf{1}, \tag{14.1}$$

at every point of the deformed body, where $p > 0$. A quasi-static process is considered where $|\nabla\hat{\mathbf{u}}| << 1 \forall \mathbf{x} \in R^-$ and $|\nabla\hat{\mathbf{u}}_1| << 1 \forall \mathbf{x}_1 \in R_1^+$. For this case, we can conclude from $(10.1)_3$ that the first-order approximation of σ^- is

$$\sigma^- = -p\mathbf{1} \tag{14.2}$$

It is assumed that the shape deformation of phase 2 is homogeneous and has the form $\tilde{\mathbf{F}} = \mathbf{1} + \tilde{\mathbf{a}} \otimes \tilde{\mathbf{n}}$. For the following, let $(\pi/2) - \delta$ denote the angle between $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{n}}$. Note that δ is a measure of the dilatation (or volume expansion) of the shape deformation of phase 2. For the case under consideration and when δ is not infinitesimal, the linearized driving traction given by (12.3) becomes

$$\begin{aligned} f &= \left(\left(\tilde{J}W_1^*\right)^+ - \left(W^*\right)^-\right) + p\tilde{\mathbf{a}} \cdot \tilde{\mathbf{n}} + O\left(|\nabla\tilde{\mathbf{u}}|^2\right) \\ &= f_{cr} + pa\sin\delta + O\left(|\nabla\tilde{\mathbf{u}}|^2, |\nabla_1\tilde{\mathbf{u}}_1|^2\right), \end{aligned} \tag{14.3}$$

Thus, if δ is not infinitesimal, we can conclude from (14.3) that a hydrostatic pressure favors the austenite phase transformation in a material where the martensite phase is such that $\sin\delta > 0$,[31] and a hydrostatic pressure favors a martensite phase transformation in a material where the martensite phase is such that $\sin\delta > 0$. Additionally, as can be observed from (14.3), these results are independent of the

[31] This corresponds with the well known result that a hydrostatic pressure stabilizes the austenite phase in many iron alloys [8]. The martensite phases in these iron alloys have small, but not infinitesimal, volume expansions.

material coefficients of both the austenite and martensite. However, for the case where $|\delta| << 1$, the term $pa\sin\delta$ becomes a second-order term.[32] Therefore, in this case, the lowest-order approximation of $f - f_{cr}$ is a second-order approximation, and the second-order terms that have been neglected in (14.3) must be retained to obtain a lowest-order approximation of $f - f_{cr}$. We note that these second-order terms contain the material coefficients of both phases.

In the rest of this section, the case where $|\delta| << 1$ is considered. The strains in each phase corresponding to the hydrostatic pressure are calculated, and the continuity of displacements condition (the only nontrivial portion of the boundary value problem) is enforced. The driving traction is then calculated to determine what effect the hydrostatic pressure, the shape deformation, the material coefficients, and the orientation of the phase boundary have on the sign of $f - f_{cr}$; i.e. on the *martensitic phase transformation*. The main result of this section is for the case where phase 1 and phase 2 represent two different variants of the same martensite that are twin related. For this case, a result that is expected from physical considerations is obtained. These problems demonstrate both the convenience and accuracy of using this continuum model.

14.1 The Assumptions

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form an orthonormal basis for vectors in E_3. For the following problem, a state of plane strain is assumed with $\mathbf{e}_3$ normal to the plane of plane strain, and a quasi-static process is considered. It is assumed that both phase 1 and phase 2 are homogeneous and have tetragonal symmetry. The case of cubic symmetry for phase 1 and/or phase 2 can be considered as a special case, once the general

[32] Note that because p in (14.3) is related to the infinitesimal strains through the constitutive equations, it is considered to be a first-order term in the equation representing the driving traction.

results are obtained. It is assumed that each phase has preferred directions [13]. For the case considered here where the tetragonal symmetry reflects the symmetry of a tetragonal crystal lattice, the three preferred directions of each phase can be considered to be in the [1 0 0], [0 1 0], [0 0 1] directions of a tetragonal unit cell of each phase, respectively. Let $\{\mathbf{h}_1^+, \mathbf{h}_2^+, \mathbf{h}_3^+\}$ represent the preferred directions of phase 2 and assume that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are such that they are aligned with $\{\mathbf{h}_1^+, \mathbf{h}_2^+, \mathbf{h}_3^+\}$, respectively. Let $\{\mathbf{h}_1^-, \mathbf{h}_2^-, \mathbf{h}_3^-\}$ represent the preferred directions of phase 1, with $\mathbf{h}_3^-$ in the direction of $\mathbf{e}_3$ and the angle between $\mathbf{h}_1^-$ and $\mathbf{e}_1$ denoted by α(Figure 3). For this plane strain problem, it is assumed that the shape deformation is homogeneous with $\tilde{\mathbf{F}} = \mathbf{1} + \tilde{\mathbf{a}} \otimes \tilde{\mathbf{n}}$, where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{n}}$ are in the plane of plane strain; i.e., in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ plane.[33] Let ϕ denote the angle between $\tilde{\mathbf{n}}$ and $\mathbf{e}_1$, and let δ be defined as it was in the beginning of this section. As mentioned above, it is assumed that $|\delta| << 1$ in the following. It is also assumed that a plane strain hydrostatic pressure exists such that

$$\boldsymbol{\tau} = -p\mathbf{1}, \tag{14.4}$$

at every point of the deformed body. Additionally, in the following analysis, we will decompose the displacement gradients into their symmetric and skew-symmetric parts as follows

$$\begin{aligned} \nabla \hat{\mathbf{u}} &= \boldsymbol{\varepsilon} + \boldsymbol{\omega}, \\ \nabla_1 \hat{\mathbf{u}}_1 &= \boldsymbol{\varepsilon}_1 + \boldsymbol{\omega}_1, \end{aligned} \tag{14.5}$$

where $\boldsymbol{\omega} = \text{skew}\nabla\hat{\mathbf{u}}$, $\boldsymbol{\omega}_1 = \text{skew}\nabla_1\hat{\mathbf{u}}_1$.

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[33] Unless otherwise indicated, for the following plane strain problem, it is assumed that all tensors, including vectors, correspond to the two-dimensional Euclidean space containing $\{\mathbf{e}_1, \mathbf{e}_2\}$.

14.2. The Stress and Infinitesimal Strain Tensors

Considering (14.4), $(10.1)_3$, and $(10.2)_3$, we can conclude that the first-order approximations of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_1$ are

$$\begin{aligned} \boldsymbol{\sigma} &= -p\mathbf{1}, \\ \boldsymbol{\sigma}_1 &= -p\mathbf{1}, \end{aligned} \tag{14.6}$$

respectively.[34]

For phase 2, because $\{\mathbf{h}_1^+, \mathbf{h}_2^+\}$ coincide with $\{\mathbf{e}_1, \mathbf{e}_2\}$, the inplane components of infinitesimal strain tensor $\boldsymbol{\varepsilon}_1$ in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ frame as functions of the components of stress in that frame are

$$\begin{Bmatrix} \varepsilon_{11}^+ \\ \varepsilon_{22}^+ \\ \varepsilon_{12}^+ \end{Bmatrix} = \begin{bmatrix} K_{1111}^+ & K_{1122}^+ & 0 \\ K_{1122}^+ & K_{2222}^+ & 0 \\ 0 & 0 & 2K_{1212}^+ \end{bmatrix} \begin{Bmatrix} \sigma_{11}^+ \\ \sigma_{22}^+ \\ \sigma_{12}^+ \end{Bmatrix}, \tag{14.7}$$

where $K_{\alpha\beta\gamma\delta}^+$ denote the components of the compliance four-tensor for phase 2 in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ frame (see [3], [7]).[35] Substituting $(14.6)_2$ into (14.7) yields

$$\begin{Bmatrix} \varepsilon_{11}^+ \\ \varepsilon_{22}^+ \\ \varepsilon_{12}^+ \end{Bmatrix} = -p \begin{Bmatrix} K_{1111}^+ + K_{1122}^+ \\ K_{1122}^+ + K_{2222}^+ \\ 0 \end{Bmatrix}, \tag{14.8}$$

For phase 1, the inplane components of infinitesimal strain $\boldsymbol{\varepsilon}$ in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ frame in terms of the stress given by $(14.6)_1$ are

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[34] Note that since $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\varepsilon}$, and $\boldsymbol{\varepsilon}_1$ are homogeneous, $\boldsymbol{\sigma}^- = \boldsymbol{\sigma}$, $\boldsymbol{\sigma}_1^+ = \boldsymbol{\sigma}_1$, $\boldsymbol{\varepsilon}^- = \boldsymbol{\varepsilon}$, and $\boldsymbol{\varepsilon}_1^+ = \boldsymbol{\varepsilon}_1$.

[35] Note that the compliance four-tensor is the inverse of the elasticity four-tensor. Also, in [3] and [7], orthotropic symmetry is considered. Orthotropic symmetry and the type of tetragonal symmetry considered here are equivalent.

$$\begin{Bmatrix} \varepsilon_{11}^{-} \\ \varepsilon_{22}^{-} \\ \varepsilon_{12}^{-} \end{Bmatrix} = -p \begin{Bmatrix} \{K_{1111}^{-}(\cos\alpha)^2 + K_{2222}^{-}(\sin\alpha)^2 + K_{1122}^{-}\} \\ \{K_{1111}^{-}(\sin\alpha)^2 + K_{2222}^{-}(\cos\alpha)^2 + K_{1122}^{-}\} \\ \frac{1}{2}(K_{1111}^{-} - K_{2222}^{-})\sin 2\alpha \end{Bmatrix}, \tag{14.9}$$

where $K_{\alpha\beta\gamma\delta}^{-}$ denote the components of the compliance four-tensor for phase 1 *in the* $\{\mathbf{h}_1^{-}, \mathbf{h}_2^{-}\}$ *frame.*

14.3. The Continuity of Displacement Condition

Because the true stress is assumed to be uniform and constant throughout the body, the linear momentum field equations and the traction jump condition are trivially satisfied. A necessary condition for the continuity of displacements at the phase boundary is that

$$\det\left(\mathbf{F}_1^{+}\tilde{\mathbf{F}}^{+} - \mathbf{F}^{-}\right) = 0. \tag{14.10}$$

For the linear case that is considered here, the first-order approximation of $\mathbf{L}$ is $\tilde{\mathbf{L}}$, where $\tilde{\mathbf{L}}\cdot\tilde{\mathbf{n}} = 0$. In terms of the displacement gradients, the above equation becomes

$$\det\left(\tilde{\mathbf{a}}\otimes\tilde{\mathbf{n}} + \nabla_1\hat{\mathbf{u}}_1^{+}(\tilde{\mathbf{a}}\otimes\tilde{\mathbf{n}}) + \nabla_1\hat{\mathbf{u}}_1^{+} - \nabla\hat{\mathbf{u}}^{-}\right) = 0. \tag{14.11}$$

Using (14.5), the first-order approximation of the above equation can be written as

$$\frac{1}{2}(\varepsilon_{11}^{+} - \varepsilon_{22}^{+} + \varepsilon_{22}^{-} - \varepsilon_{11}^{-})\sin 2\phi + \varepsilon_{12}^{-}\cos 2\phi + \omega^{-} - \omega^{+} = 0\,, \tag{14.12}$$

where ω^{-} and ω^{+} denote the ω_{12}^{-} and ω_{12}^{+} components of ω and ω_1 in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ frame, respectively, and ε_{12}^{+} has been taken to be equal to zero because of (14.8). We note that for the linear case under consideration, the symmetric parts of the displacement gradients of $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}_1$ are completely determined by the stress distribution and are given by (14.8) and (14.9), respectively, and Equation (14.12)

is the only equation that restricts the skew-symmetric parts of these displacement gradients. Substituting (14.8) and (14.9) into (14.12) and solving for $\omega^+ - \omega^-$ yields

$$\omega^+ - \omega^- = \frac{p}{2}(K^+_{2222} - K^+_{1111})\sin 2\phi + \frac{p}{2}(K^-_{2222} - K^-_{1111})\sin(2\alpha - 2\phi)\,. \qquad (14.13)$$

14.4. The Driving Traction

The second-order approximation of the driving traction corresponding to the problem under consideration is as follows:

$$f = f_{cr} + p\left\{a\delta + \tilde{\mathbf{n}} \cdot [(\nabla_1 \hat{\mathbf{u}}_1)\tilde{\mathbf{a}}] + \tfrac{1}{2}\,\mathrm{tr}\boldsymbol{\varepsilon}_1 - \tfrac{1}{2}\,\mathrm{tr}\boldsymbol{\varepsilon}\right\}, \qquad (14.14)$$

where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{n}}$ are given by their lowest-order approximations. Using (14.5), (14.8), and (14.9), the second-order approximation of the driving traction given above can be written as

$$f = f_{cr} + p\left\{a\delta + a\omega^+ + \tfrac{1}{2}\,ap(K^+_{1111} - K^+_{2222})\sin 2\phi + \tfrac{1}{2}p[\mathrm{tr}(\mathbf{K1}) - \mathrm{tr}(\mathbf{K}_1\mathbf{1})]\right\}. \qquad (14.15)$$

14.5. The Nondimensional Form of the Driving Traction and Infinitesimal Rotations

Equations (14.13) and (14.15) can be written in nondimensional form by using the following nondimensional quantities

$$\tilde{\omega}^- = \frac{2a\omega^-}{pK^+_{1111}}, \quad \tilde{\omega}^+ = \frac{2a\omega^+}{pK^+_{1111}}, \quad \tilde{K}^+_2 = \frac{K^+_{2222}}{K^+_{1111}}, \qquad (14.16)$$

$$\tilde{K}^-_1 = \frac{K^-_{1111}}{K^+_{1111}}, \quad \tilde{K}^-_2 = \frac{K^-_{2222}}{K^+_{1111}}, \quad \tilde{f} = \frac{2f}{p^2K^+_{1111}},$$

$$\tilde{f}_{cr} = \frac{2f_{cr}}{p^2 K^+_{1111}}, \quad \tilde{\delta} = \frac{2a\delta}{pK^+_{1111}}, \quad \tilde{\beta} = \frac{1}{K^+_{1111}}\left[\mathrm{tr}(\mathbf{K1}) - (\mathbf{K}_1\mathbf{1})\right].$$

The nondimensional form of Equation (14.13) is

$$\tilde{\omega}^+ - \tilde{\omega}^- = a\left(\tilde{K}^+_2 - 1\right)\sin 2\phi + a\left(\tilde{K}^-_2 - \tilde{K}^-_1\right)\sin(2\alpha - 2\phi). \tag{14.17}$$

The nondimensional driving traction $\tilde{f}$ is

$$\tilde{f} = \tilde{f}_{cr} + \tilde{\delta} + \tilde{\omega}^+ + \tilde{\beta} + a\left(1 - \tilde{K}^+_2\right)\sin 2\phi. \tag{14.18}$$

Note that $\tilde{\omega}^+$ is a parameter in the above equation for $\tilde{f}$. We can obtain $\tilde{f}$ as a function of $\tilde{\omega}^-$ by using (14.17). The resulting equation is

$$\tilde{f} = \tilde{f}_{cr} + \tilde{\delta} + \tilde{\omega}^- + \tilde{\beta} + a\left(\tilde{K}^-_2 - \tilde{K}^-_1\right)\sin(2\alpha - 2\phi). \tag{14.19}$$

14.6. An Austenite-Martensite Phase Transformation

To investigate the effect of the hydrostatic pressure on an austenite-martensite phase transformation, we assume that phase 1 represents the austenite and phase 2 represents the martensite. From the discussion in the beginning of this section, a hydrostatic pressure will favor the austenite phase transformation if it results in $\tilde{f} - \tilde{f}_{cr} > 0$, and a hydrostatic pressure will favor the martensite phase transformation if it results in $\tilde{f} - \tilde{f}_{cr} < 0$. We note that a value for $\tilde{\omega}^-$ and $\tilde{\omega}^+$ in (14.18) and (14.19), respectively, might be prescribed by some displacement boundary condition that is applied to the material in addition to the hydrostatic pressure and that is consistent with the assumed state of stress within the material. Thus, from (14.18), (14.19), and the nondimensional variables given by (14.16), one can observe what

values for the shape deformation of the martensite, the material coefficients of both phases, the orientation of the phase boundary, and the additional boundary conditions will result in the hydrostatic pressure favoring the martensite phase transformation and what values of these quantities will result in the hydrostatic pressure favoring the austenite phase transformation. Also, we note that as δ increases in magnitude, the results that were obtained above for the second-order case approach the results that were obtained in the beginning of this section for the first-order case, as, of course, should be expected.

14.7. The Case of Twin Related Variants

Consider the case where phase 1 and phase 2 represent two variants of the same martensite which are twin related and the isothermal process is at a temperature near the transformation temperature. In this case we have: (1) $\delta = 0$, since one twin has a simple shear shape deformation relative to the other twin; (2) $f_{cr} = 0$, since $\tilde{J}\,W_1(1) = W(1)$ for two variants of the same martensite near the transformation temperature; (3) $K^+_{ijkl} = K^-_{ijkl}$, since both phases represent the same material; (4) $\tilde{\beta} = 0$, as a result of (3); and (5) $\alpha = 2\phi$, as can be seen from Figure 3.2. In this case, the driving traction given by (14.18) becomes

$$\tilde{f} = \tilde{\omega}^+ + a\left(1 - \tilde{K}_2^+\right)\sin 2\phi\,, \tag{14.20}$$

and Equation (14.17) becomes

$$\tilde{\omega}^+ - \tilde{\omega}^- = -2a\left(1 - \tilde{K}_2^+\right)\sin 2\phi\,. \tag{14.21}$$

If we now require that the deformation of one twin be symmetric with respect to the deformation of the other twin (due to the *symmetric* stress distribution), we must require that

$$\omega^- = -\omega^+. \tag{14.22}$$

In this case, substituting (14.22) into (14.21), solving for $\tilde{\omega}^+$ in the resulting equation, and then substituting the resulting equation for $\tilde{\omega}^+$ into (14.20), yields

$$f = 0. \tag{14.23}$$

This is what should be expected, since everything else in the problem is symmetric, and consequently one twin should not be preferred over the other.

15. Summary and Concluding Remarks

In this paper, a continuum model for materials that can undergo martensitic phase transformations was developed. The continuum model was then used to study several problems that deal with which phase or which variant of martensite is preferred during the application of a mechanical loading.

In the continuum model that was developed in this paper, each phase has its own constitutive equations which are defined with respect to a reference configuration that coincides with an unstressed undeformed configuration of that phase. This reference configuration is also used for the expression of the field equations for that phase. With respect to the undeformed austenite phase, these reference configurations coincide the shape deformations of the martensites. The field equations and jump conditions for a two-phase material in a purely mechanical process were derived and discussed. The general form of the constitutive equations that are most appropriate for the case where each phase of the material behaves elastically were presented and discussed. It was pointed out that the mapping representing the reference configuration for each phase should be such that the stress-power for that phase is equal to the time rate of change of the integral of the elastic potential for that phase, where the elastic potential and stress tensor for that phase are defined with respect to the reference configuration for that phase. This is the case if the mapping for each phase has the form given

by (6.7). Additionally, one can argue that for a purely mechanical process and a material with phases that behave elastically $\tilde{\mathbf{x}}_1$ should always have the form given by (6.7) since this is the form that $\tilde{\mathbf{x}}_1$ must have in order for $\tilde{\mathbf{F}}$, and consequently the transformation strain, to be independent of time. It was also pointed out that if the reference configuration for a given phase is moving, the quantities that are defined with respect to that reference configuration (e.g. the stress) do not have a true nominal form. Because different elastic potentials are used for the different phases of the material, the material symmetry groups of the different phases can be chosen independently of each other and can be chosen to reflect any type of crystal symmetry with any orientation. Thus, the change in crystal structure that takes place during a martensitic phase transformation and the orientation relation between the crystal lattices of the different phases of the material can be accurately represented. In fact, because the elastic potential for each phase is defined with respect to the undeformed configuration of that phase, the material symmetry group for that elastic potential can be chosen to be the crystallographic point group corresponding to the crystal symmetry of that phase. The driving traction and kinetic relation corresponding to the two-phase material were also considered.

One of the main advantages of working with this continuum model is that the field equations and jump conditions are in forms that permit direct linearization, while still retaining finite shape deformations for the martensites. This is the case since the field quantities for each phase are in terms of displacements that are measured from the reference configuration corresponding to an unstressed undeformed configuration of that phase, and consequently, for the appropriate initial and boundary conditions, the displacement gradients can be considered infinitesimal. The linearized field equations, jump conditions, driving traction, and kinetic relation were then

considered. It was pointed out that working with displacements that are functions of points on a moving reference configuration results in the accelerations in terms of these displacements containing many inertial-type terms and results in the boundary conditions for the phases having to be expressed with respect to moving boundaries. A coordinate transformation that eliminates most of these inertial-type terms and results in a completely Lagrangian description of the boundary value problem while still working with field quantities that are defined with respect to the different reference configurations was discussed. Another advantage of using this continuum model is that for the linear problem the true stress for each phase is approximately equal to the nominal stress for that phase. This is very useful for solving certain types of boundary value problems, including the linear problems that were considered in the rest of this paper. All of these problems dealt with the issue of which phase or which variant of martensite is preferred during the growth process during the application of a mechanical loading. After the general problem was discussed, the special case of an applied uniaxial tensile traction was considered. A result that corresponds with observations that were made from experiments was derived. The last problem that was considered in this paper involves the application of a hydrostatic pressure to a two-phase material with a martensite phase that has a finite shape deformation with an infinitesimal dilatation. The case where the two phases represent two different variants of the same martensite was then considered as a special case. It was shown that for this case the hydrostatic pressure favors the growth of neither variant, which is what should be expected since everything else in the problem is completely symmetric. All of the problems that were considered in this paper demonstrate the convenience and accuracy of using the continuum model that was developed in this paper.

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