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A Piezothermoelastic Shell Theory Applied to Active Structures

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A PIEZOTHERMOELASTIC SHELL THEORY
APPLIED TO ACTIVE STRUCTURES

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ABSTRACT

"Smart" structures with integrated sensors, actuators, and control electronics are of importance to next-generation high-performance structural systems. Piezoelectric materials possess unique electromechanical properties, the direct and converse effects which can be used in sensor and actuator applications. In this study, piezothermoelastic characteristics of piezoelectric shell continua are studied and applications of the theory to active structures in sensing and control are discussed. A generic piezothermoelastic shell theory for thin piezoelectric shells is derived using the linear piezoelectric theory and Kirchhoff-Love assumptions. It shows that the dynamic equations, in three principal directions, include thermal induced loads as well as conventional electric and mechanical loads. The electric membrane forces and moments induced by the inverse effect can be used to control the thermal and mechanical loads. A simplification procedure, based on Lame's parameters and radii of curvatures, is proposed and applications of the theory to 1) a piezoelectric cylindrical shell and 2) a piezoelectric beam are demonstrated.

INTRODUCTION

Development of "smart" structures with integrated sensors, actuators, and control electronics are crucial to next-generation structural systems. New sensor/actuator materials are investigated and new technologies are developed in recent years. Among those commonly used sensor/actuator materials (e.g., piezoelectric materials, shape-memory alloys, electrorheological fluids, electrostrictive materials, magnetorheological fluids, etc.), piezoelectric materials possess unique electromechanical properties (the direct and converse piezoelectric effects) which can be respectively used in sensor and actuator applications (Tzou & Fukuda, 1991; Tzou & Anderson, 1992). General theories derived from a generic shell continuum can be applied to a broad class shell and non-shell structures (Soedel, 1981). Chau (1986) proposed a variational formulation to describe the electromechanical equilibrium of completely anisotropic piezoelectric shells. Rogacheva (1982, 1984a, 1984b, 1986) studied state equations and boundary conditions of piezoelectric shells polarized along coordinate directions. Senik and Kudriavtsev (1980) formulated the equations of motion for piezoelectric shells transversely polarized. Dökmeci (1978) derived a theory for coated thermopiezoelectric laminae. Tzou and Gadre (1989) proposed a general theory for multi-layered piezoelectric shell actuators based on equivalent induced strains. Tzou (1991) derived a general distributed sensing and control theory for a generic shell continuum using piezoelectric thin layers. A thin piezoelectric solid finite element with three internal degrees of freedom was formulated and applied to distributed sensing and control of continua (Tzou & Tseng, 1991). Tzou and Zhong (1990) derived a piezoelectric vibration theory for a hexagonal symmetrical piezoelectric thick shell with three effective principal axes, and this theory was applied to distributed shell convolving sensors (Tzou & Zhong, 1991a) and active structural control (Tzou & Zhong, 1991b). In this study, the piezoelectric shell vibration theory is extended to include thermal induced effects due to temperature variations. Piezothermoelastic behaviors of piezoelectric shell continua are investigated.

Based on the linear piezoelectric theory and Kirchhoff-Love assumptions, a generic piezothermoelastic shell theory for thin piezoelectric shells is derived first. A simplification procedure, based on Lame's parameters and radii of curvatures, is proposed and applications of the theory to a number of piezoelectric continua (a piezoelectric cylindrical shell and a piezoelectric beam) are demonstrated. Thermal effects to sensing and control are discussed.

DEFINITIONS

It is assumed that a generic piezoelectric shell continuum is defined in a curvilinear tri-orthogonal coordinate system in

205
where the \( \alpha_1 \) and \( \alpha_2 \) define the neutral surface and \( \alpha_3 \) defines the normal. Figure 1. Since the shell is thin, the electric field \( E_3 \) is considered across the shell thickness and the external electric charge \( Q_3 \) is on the top and bottom surfaces only. In this section, assumptions and constitutive equations are defined. (Note that this shell is generic, which can be simplified to a broad class of shell and non-shell geometries. Examples are demonstrated in case studies.)

It is assumed that the piezoelectric shell is thin as compared with the other two in-plane dimensions. The transverse shear deformations and rotary in-plane dimensions. Thus, the displacement \( (U_3) \) of any given point in the shell continuum can be represented as a summation of the component due to contraction/expansion of the neutral surface and the component due to bending:

\[
U_3(\alpha_1, \alpha_2, \alpha_3) = U_0(\alpha_1, \alpha_2) + \alpha_3/2, \quad \alpha_3 \geq 0
\]

where \( J_1 \) denotes the bending angle and \( J_2 = \dot{\alpha}_3 \) defines the distance measured from the neutral surface. Based on Kirchhoff–Love assumptions, the transverse shear strain \( S_22 \) and \( S_23 \) are negligible, i.e., \( S_13 = 0 \) and \( S_{23} = 0 \). Thus, the two bending angles can be derived as:

\[
\begin{align*}
J_1 &= -\frac{u_1}{R_1} - \frac{1}{A_1} \frac{\partial^2 S_{11}}{\partial \alpha^2}, \\
J_2 &= -\frac{u_2}{R_2} - \frac{1}{A_2} \frac{\partial^2 S_{22}}{\partial \alpha^2}
\end{align*}
\]

Fig. 1. A piezoelectric shell continuum.

The constitutive equation of piezothermoelasticity is defined as:

\[
\begin{align*}
\{ \Sigma \} &= [c] \{ S \} - \{ e \} \{ E \} - \{ \lambda \} \Delta T_0, \\
\{ \partial \} &= \{ e \} \{ S \} + \{ e \} \{ E \} + \{ \gamma \} \Delta T_0
\end{align*}
\]

where \( \{ \Sigma \} \) is a stress vector; \([c]\) is the elastic moduli matrix; \([e]\) is the piezoelectric constant; \([\lambda]\) is the coefficient of thermal expansion; \( \{ \partial \} \) is the electric displacement vector; \( \{ S \} \) is the mechanical strain vector; \( \{ e \} \) is the dielectric constant matrix; \( \{ E \} \) is the electric field vector; \( \{ \gamma \} \) is the pyroelectric constant; and \( \Delta T_0 \) is the temperature change. It is assumed that the piezothermoelastic behaviors are instantly balanced in the temperature change. It is assumed that the piezoelectric shell is thin as compared with the other two in-plane dimensions. The transverse shear deformations and rotary in-plane dimensions. Thus, the displacement \( (U_3) \) of any given point in the shell continuum can be represented as a summation of the component due to contraction/expansion of the neutral surface and the component due to bending:

\[
U_3(\alpha_1, \alpha_2, \alpha_3) = U_0(\alpha_1, \alpha_2) + \alpha_3/2, \quad \alpha_3 \geq 0
\]

where \( J_1 \) denotes the bending angle and \( J_2 = \dot{\alpha}_3 \) defines the distance measured from the neutral surface. Based on Kirchhoff–Love assumptions, the transverse shear strain \( S_22 \) and \( S_23 \) are negligible, i.e., \( S_13 = 0 \) and \( S_{23} = 0 \). Thus, the two bending angles can be derived as:

\[
\begin{align*}
J_1 &= -\frac{u_1}{R_1} - \frac{1}{A_1} \frac{\partial^2 S_{11}}{\partial \alpha^2}, \\
J_2 &= -\frac{u_2}{R_2} - \frac{1}{A_2} \frac{\partial^2 S_{22}}{\partial \alpha^2}
\end{align*}
\]

Note that the transverse displacement \( U_3 \) is independent of thickness, i.e., \( U_3 = u_3(\alpha_1, \alpha_2) \) and the transverse strain \( S_{23} \) can be neglected, except where a concentrated load is applied. The mechanical strains of the thin shell consist of an in-plane membrane strain component \( S_{11} \), and an out-of-plane bending component \( k_{ij} \).

The membrane and bending strains, \( S_{ij} \) and \( k_{ij} \) are defined as follows:

\[
\begin{align*}
S_{11} &= \frac{1}{A_1} \frac{\partial^2 \alpha_1}{\partial \alpha^2} + \frac{u_1}{R_1} + \frac{1}{A_1} \frac{\partial^2 \alpha_1}{\partial \alpha^2} + \frac{R_1}{A_1} \\
S_{22} &= \frac{1}{A_2} \frac{\partial^2 \alpha_2}{\partial \alpha^2} + \frac{u_2}{R_2} \frac{1}{A_2} \frac{\partial^2 \alpha_2}{\partial \alpha^2} + \frac{R_2}{A_2} \\
S_{12} &= \frac{1}{A_2} \frac{\partial^2 \alpha_1}{\partial \alpha^2} + \frac{u_1}{R_1} \frac{1}{A_2} \frac{\partial^2 \alpha_1}{\partial \alpha^2} + \frac{R_1}{A_2} \\
k_{11} &= \frac{1}{A_1} \frac{\partial^2 \alpha_1}{\partial \alpha^2} + \frac{u_1}{R_1} \frac{1}{A_1} \frac{\partial^2 \alpha_1}{\partial \alpha^2} + \frac{R_1}{A_1} \\
k_{12} &= \frac{1}{A_2} \frac{\partial^2 \alpha_1}{\partial \alpha^2} + \frac{u_2}{R_2} \frac{1}{A_2} \frac{\partial^2 \alpha_2}{\partial \alpha^2} + \frac{R_2}{A_2}
\end{align*}
\]

where \( \partial \)s are defined in Eqs. (7) and (8). Note that there is no shear strain on the \( \alpha_1 \) face such that there is not an induced electric field in the \( \alpha_1 \) and the \( \alpha_2 \) directions. Considering the piezothermoelastic constitutive equations and the stress–strain relations of thin shells, one can define the mechanical stress \( T_{ij} \) induced by the mechanical strains, the electric displacement \( S_{ij} \) induced by strains, and the stress \( E_{ij} \) induced by electric fields.

\[
\begin{align*}
T_{11} &= c_{11} S_{11} - c_{12} S_{12} \\
T_{12} &= c_{21} S_{11} - c_{22} S_{12} \\
T_{13} &= 0 \\
T_{13} &= c_{44} S_{12}
\end{align*}
\]
These terms will be used in conjunctions with the energy expressions and the variational equations.

**FORCES AND MOMENTS**

In this section, all forces and moments introduced by mechanical, electric, and thermal effects are defined. These force and moment components will be used in Hamilton's equation when deriving the shell piezothermoelastic equations. The mechanical membrane forces are

\[ N_{ij}^m = \int_{a_3} T_{ij} da_3 = K(S_{ij} + \mu S_{ij}) . \]  

where \( K = Yh/(1-\nu^2) \) is the membrane stiffness and \( N_{ij}^m \) is the total force acting on the \( i \)th face in the \( j \)th direction due to mechanical effects. The mechanical bending moments are

\[ M_{ij}^m = \int_{a_3} T_{ij} da_3 = D(k_{ij} + \mu k_{ij}) . \]

where \( D = Yh^3/(1-\nu^2) \) is the bending stiffness and \( M_{ij}^m \) is the total bending moment on the \( i \)th face in the \( j \)th direction due to mechanical effects. The mechanical transverse shear forces \( Q_{ij}^m \) are

\[ Q_{ij}^m = \int_{a_3} T_{ij} da_3 . \]

Using Eq.(2), one can derive the electric membrane forces:

\[ N_{ij}^e = \int_{a_3} e_{ij} E da_3 \]

\[ = -\epsilon_{ij12} h_3 \rho \partial_t^3 S_{ij} - \epsilon_{ij13} h_2 \partial_t^2 S_{ij} - \epsilon_{ij21} h_1 \partial_t^3 S_{ij} - \epsilon_{ij23} h_1 \partial_t^2 S_{ij} + \epsilon_{ij12} \nu \frac{k_{ij} + \mu k_{ij}}{h^3} \]  

where the first term is contributed by the converse effect, the second term by the pyroelectric effect (temperature), the third term by the elastic strains via the direct effect. The electric bending moments are

\[ M_{ij}^e = \int_{a_3} e_{ij} E da_3 \]

\[ = -\frac{h^3}{12} \epsilon_{ij12} (k_{ij} + \mu k_{ij}) . \]

The electric transverse shear forces \( Q_{ij}^e \) are

\[ Q_{ij}^e = \int_{a_3} e_{ij} E da_3 . \]

where there is no shear forces in the \( a_3 \) direction due to the electric effects. The thermal membrane forces \( N_{ij}^t \) are defined by

\[ N_{ij}^t = \int_{a_3} \lambda_i \Delta t da_3 = -h \lambda_i \Delta t . \]

The thermal bending moments are

\[ M_{ij}^t = \int_{a_3} \lambda_i \Delta t^2 da_3 = 0 . \]

It is noted that the piezoelectric continua experience only in-plane thermal expansion/contraction and no bending moments in a uniformly distributed thermal field.

**HAMILTON'S PRINCIPLE AND PIEZOTHERMOELASTIC EQUATIONS**

In this section, piezothermoelastic equations in three principal directions will be derived using Hamilton's principle and the variational procedures. Hamilton's principle gives (Tzou & Zhong, 1990)

\[ \int_{x_0}^{x_f} \left[ \int_S \left( -\rho \partial_t^2 U_j - H(S_k, E_j) \right) dV dt + \int_S \left( \frac{\partial \mathcal{L}}{\partial \partial_t U_j} - \mathcal{L} \right) dS dt \right] \]

where \( H \) is the electric enthalpy; \( \rho \) is the mass density; \( U_j \) is the displacement; \( E_j \) is the electric field; \( \tilde{t}_j \) is the surface traction in the \( a_j \) direction; \( S_k \) is the strain on the \( k \)th face and in the \( j \)th direction; \( \partial \mathcal{L}/\partial \partial_t U_j \) is the surface charge; and \( \partial \mathcal{L}/\partial \partial_t U_j \) is the electric potential. The electric fields in the curvilinear coordinate system are:

\[ E_1 = -\frac{1}{A_1} \frac{\partial \phi}{\partial a_1} \]

\[ E_2 = -\frac{1}{A_2} \frac{\partial \phi}{\partial a_2} \]

\[ E_3 = -\frac{1}{A_3} \frac{\partial \phi}{\partial a_3} \]

where \( A_1 \) and \( A_3 \) are the radii of curvatures, and \( \lambda_1 \) and \( \lambda_3 \) are Lame's parameters. These define the electric fields as the gradient of the electric potential. The electric enthalpy is
defined as
\[ H = \{S\}^T \{T\} - \{E\}^T \{D\} \] (Tzou & Zhong, 1990).

Using the piezothermoelastic constitutive equations, one can derive

\[ H = \{S\}^T \{S\} - \{E\}^T \{E\} - \{S\}^T \{\lambda\} \Delta p - \{E\}^T \{p\} \Delta p. \] (23)

Substituting the strain–stress expressions into the electric

\[ H = \{S\}^T \{S\} - \{E\}^T \{E\} = \{S\}^T \{\lambda\} \Delta p - \{E\}^T \{p\} \Delta p. \] (24)

Substituting the electric enthalpy and all other energy

\[ \frac{\partial (N^T - N_1 - N_2)}{\partial \alpha} + \frac{\partial (N^T - N_1)}{\partial \alpha} = \frac{\partial (N^T - N_1 - N_2)}{\partial \alpha} \frac{\partial \Delta A}{\partial \alpha} \\
+ Q_{1A} A_1 A_2 + N_1 \frac{\partial \Delta A}{\partial \alpha} = \rho A_1 A_2 \frac{\partial \Delta A}{\partial \alpha} \] (25–a)

\[ \frac{\partial (N^T - N_1 - N_2)}{\partial \alpha} + \frac{\partial (N^T - N_1 - N_2)}{\partial \alpha} = \frac{\partial (N^T - N_1 - N_2)}{\partial \alpha} \frac{\partial \Delta A}{\partial \alpha} \\
+ Q_{1A} A_1 A_2 + N_1 \frac{\partial \Delta A}{\partial \alpha} = \rho A_1 A_2 \frac{\partial \Delta A}{\partial \alpha} \] (26–a)

\[ \frac{\partial (Q_{1A})}{\partial \alpha} + \frac{\partial (Q_{2A})}{\partial \alpha} = \frac{\partial (Q_{1A})}{\partial \alpha} \frac{\partial \Delta A}{\partial \alpha} \\
+ N_1 \frac{\partial \Delta A}{\partial \alpha} = \rho A_1 A_2 \frac{\partial \Delta A}{\partial \alpha} \] (27–a)

where \( h \) is the thickness of piezoelectric shell. The superscripts m, e, and t respectively denote mechanical, electric, and thermal components. \( Q_{1A} \) and \( Q_{2A} \) in Eqs. (25)–(27) are defined by

\[ Q_{1A} A_2 = \frac{\partial (M^T - M_{1T})}{\partial \alpha} + \frac{\partial (M^T - M_{2T})}{\partial \alpha} \]

\[ + \frac{\partial (M^T - M_{3T})}{\partial \alpha} + \frac{\partial (M^T - M_{4T})}{\partial \alpha} \] (28)

\[ Q_{2A} A_1 = \frac{\partial (M^T - M_{1T})}{\partial \alpha} + \frac{\partial (M^T - M_{2T})}{\partial \alpha} \]

\[ + \frac{\partial (M^T - M_{3T})}{\partial \alpha} + \frac{\partial (M^T - M_{4T})}{\partial \alpha} \] (29)

Note that the equation of motions include the mechanical

\[ \frac{\partial (e_2S_{11} + e_3S_{22} + e_4S_{33})}{\partial \alpha} + \frac{\partial (e_1S_{11} + e_2S_{22} + e_3S_{33} + p_3 \Delta p)}{\partial \alpha} = 0. \] (30)

thermal induced terms (\( N^T_{ij} / M^T_{ij} \)), and external mechanical

which implies that the quantity \( e_2S_{11} + e_3S_{22} + e_4S_{33} + p_3 \Delta p \) is equal to a constant and the thickness variation is equal
to zero. Note that this equation can be used to estimate an external electric output as functions of induced mechanical strains and

\[ \frac{\partial (N^T - N_1 - N_2)}{\partial \alpha} + \frac{\partial (N^T - N_1 - N_2)}{\partial \alpha} = \frac{\partial (N^T - N_1 - N_2)}{\partial \alpha} \frac{\partial \Delta A}{\partial \alpha} \\
+ \frac{\partial (M^T - M_{1T})}{\partial \alpha} + \frac{\partial (M^T - M_{2T})}{\partial \alpha} \] (25–b)

\[ + \frac{\partial (M^T - M_{3T})}{\partial \alpha} + \frac{\partial (M^T - M_{4T})}{\partial \alpha} \] (26–b)

\[ + \frac{\partial (M^T - M_{1T})}{\partial \alpha} + \frac{\partial (M^T - M_{2T})}{\partial \alpha} \] (27–b)

temperature variation, i.e., \( E_3 = - (1/\epsilon_{13})[e_2(S_{11} + S_{22}) + p_3 \Delta p] \).


BOUNDARY CONDITIONS

Boundary conditions are directly derived from the variational equation (Appendix). The boundary conditions are defined by the surface traction forces and the surface charge.

Mechanical Boundary Conditions

Mechanical boundary conditions defined by either

force/moment or displacement/rotation are summarized in

Table 1 in which terms with a superscript * denote external

boundary components.
The mechanical shear stress resultants are defined as

\[ \tau_k = \frac{M_k}{A} \]

It is observed that the total surface charge including the thermal induced membrane force only occurs in the principal direction (i.e., \( k = 1 \) and \( k = 2 \)). The subscript \( t \) denotes the tangential direction (i.e., \( t = 1 \)).

For a totally fixed edge at \( \alpha_1 = \alpha_2 \) (i.e., no motion allowed), the boundary conditions are: \( u_1 = 0, J_1 = 0, u_3 = 0 \), and \( u_2 \) = 0.

For a totally free edge at \( \alpha_2 = \alpha_2' \), i.e., no external forces and moments, the boundary conditions at \( \alpha_2 = \alpha_2' \) are: \( N_{12} = -N_{12}^e = 0, M_{12} = M_{12}^e = 0, V_{12} = 0 \), and \( T_{12} = 0 \). In the case where the surface traction forces \( t_{ij} \) are defined, the boundary membrane forces are

\[ N_{11} = \int t_{11} \, da_1, \quad (33-a) \]
\[ N_{22} = \int t_{22} \, da_2, \quad (33-b) \]
\[ N_{12} = \int t_{12} \, da_1, \quad (33-c) \]
\[ N_{21} = \int t_{21} \, da_2 \quad (33-d) \]

where \( N_{1j} \) is the total force on the \( j \)th face in the \( j \)th direction due to the surface tractions. The induced boundary bending moments \( M_{ij} \) are

\[ M_{11} = \int t_{12} \, da_2, \quad (34-a) \]
\[ M_{22} = \int t_{21} \, da_1, \quad (34-b) \]
\[ M_{12} = \int t_{12} \, da_1, \quad (34-c) \]
\[ M_{21} = \int t_{21} \, da_2 \quad (34-d) \]

Accordingly, the boundary transverse shear forces \( Q_{ij} \) are

\[ Q_{11} = Q_{11}^t = \int t_{12} \, da_2, \quad (35-a) \]
\[ Q_{22} = Q_{22}^t = \int t_{21} \, da_1 \quad (35-b) \]

where \( Q_{ij}^t \) is the shear force on the \( i \)th face in the \( j \)th direction.

Electric Boundary Condition

The electric boundary condition is defined as

\[ e_1 S_{11} + e_2 S_{22} = \epsilon_0 \frac{\partial \phi}{\partial a_1} + \tau_{ij} \frac{\partial \psi}{\partial a_j} - Q_{ij} = 0 \quad (36) \]

It is observed that the total surface charge including the mechanical, electric, and temperature effects is equal to the external surface charge \( Q_3 \).

Note that the piezothermoelastic equations for the thin shell continuum and the boundary conditions can be reduced to conventional elastic shell equations by neglecting all electric and thermal coupling terms (Soedel, 1981). Again, transverse shear deformation and rotary inertia effects were not considered.

### PIEZOTHERMEOELECTRICITY OF SIMPLIFIED GEOMETRIES

The piezothermoelastic theory derived above is for a generic piezoelectric shell continuum exposed to mechanical, thermal, and electric fields. The generic shell was defined in a curvilinear tri-orthogonal coordinate system defined by \( a_1 \), \( a_2 \), and \( a_3 \) axes. The in-plane two axes define the neutral surface experiencing only membrane effects. Each of the in-plane axes is defined by its radius of curvature, e.g., \( R_1 \) for \( a_1 \) and \( R_2 \) for \( a_2 \). In addition, there are two Lamé’s parameters \( A_1 \) and \( A_2 \) defined by a fundamental form: \( (da)^2 = A_1 (da_1)^2 + A_2 (da_2)^2 \). For a given geometry, \( R_1 \) and \( R_2 \) can usually be directly observed from the coordinate system and \( A_1 \) and \( A_2 \) can be derived from the fundamental form. Substituting the four parameters into the generic shell equation and simplifying them accordingly, one can derive the corresponding piezothermoelastic equations and boundary conditions for the geometry. In this section, these procedures are used to derive the piezothermoelastic equations for 1) a piezoelectric cylindrical shell and 2) a piezoelectric beam.

#### Example-1: Piezoelectric Cylindrical Shell

It is assumed that the cylindrical shell is defined in a cylindrical coordinate system in which \( x \) axis \( (a_1) \) is aligned with the height and its radius of curvature \( R_1 = a \). The second axis \( \theta \) \( (a_2) \) defines the circumferential direction which has a radius of curvature \( R_2 = \theta \). Note that the \( x \) and \( \theta \) axes constitute the neutral surface. The third axis \( a_3 \) is normal to the neutral surface. Figure 2 illustrates the piezoelectric cylinder and its coordinate system. Piezothermoelastic effects of the cylindrical shell will be discussed.
Note that the superscripts m, e, and t are for the mechanical, electric, and thermal effects respectively. Substituting the force and moment terms into the $Q_1^1$ and $Q_1^3$ equations, one can derive

$$Q_1^1 = \left[ - D - e_{\text{m}1}^1 \frac{h^1}{12} \frac{\partial^2 u_1^1}{\partial x^2} - \frac{D}{2} \frac{\partial^2 u_1^1}{\partial x^2} \right] \frac{T_1}{T_1^i}$$

$$\frac{\partial^2 u_1^1}{\partial x^2} = \left[ - \frac{D}{2} - e_{\text{m}1}^1 \frac{h^1}{12} \frac{\partial^2 u_1^1}{\partial x^2} \right] \frac{T_1}{T_1^i}$$

$$Q_1^3 = \left[ - D - e_{\text{m}1}^1 \frac{h^1}{12} \frac{\partial^2 u_1^1}{\partial x^2} - \frac{D}{2} \frac{\partial^2 u_1^1}{\partial x^2} \right] \frac{T_1}{T_1^i} + \frac{D}{2} \frac{\partial^2 u_1^1}{\partial x^2}$$

Thus, the piezothermoelastic equations in three principal directions for the piezoelectric cylindrical shell are derived

$$\frac{\partial (N_{11}^1 - N_{11}^3 - N_{11}^2)}{\partial x} + \frac{\partial (N_{11}^2)}{\partial y} = \frac{\partial (N_{11}^3)}{\partial x}$$

$$\frac{\partial (Q_{11}^1)}{\partial x} + \frac{\partial (Q_{11}^3)}{\partial y} = \frac{Q_{11}^2}{|\xi|}$$

It is observed that the thermal effects only contribute to the membrane forces. Removing the electric and thermal related terms, one can simplify the system equations to those corresponding to an elastic cylindrical shell.

Using the charge boundary condition, one can define the electric field strength $E_3$ at the location $\xi$ above/below the neutral surface as a function of the mechanical strains, temperature effect, and charge effect.

$$E_3 = \frac{e_{33}^1}{\epsilon_3^2} \left[ \frac{\partial u_0}{\partial x} + \frac{1}{\epsilon_3^2} \left( \frac{\partial u_0}{\partial y} + u_3 \right) - \frac{\partial^2 u_1^1}{\partial x^2} \right]$$

The electric field strength is contributed by the direct piezoelectric effect (the first term), the pyroelectric effect (the second term), and the external surface charge (the third term) as defined in the constitutive equation. Note that the resulting voltage is $V_3 = \int_0^1 E_3 \, dx$ in an open-circuit condition. The bending components, with $\alpha_3$ terms, vanish after the integration. It is also observed that the output signal has a temperature related term induced by the pyroelectric effect in sensor applications. Note that it is assumed that the external charge is zero in sensor applications (Zhong et al., 1991a).

Example 2: Piezoelectric Beam

A beam is a special case of an open ring with zero curvature, $R = \infty$. In this case, the $a_1$ axis is aligned with the longitudinal direction of the cantilever beam, i.e., $a_1 = x$. The second axis is in the width direction, $a_2 = y$. Figure 3 shows the piezoelectric beam. It is assumed that the beam only experiences transverse oscillations, $a_3 = z$. Governing equation and piezothermoelastic behaviors of the beam are discussed.

The fundamental form of the beam is

$$\frac{(ds)^2}{(dx)^2} + (dy)^2 = 0$$

210
where $dx$ and $dy$ are infinitesimal distances in the $x$ and $y$ directions respectively. Thus, $A_x = 1$, $A_y = 1$, $R_1 = x$, $R_2 = y$. Since only the bending oscillation is considered, the membrane strains are zeros, i.e., $S_{11} = 0$, $S_{22} = 0$, and $S_{12} = 0$. The bending strain at the $a_1$ location is defined by $k_{11} = -\frac{\partial^2 u_1}{\partial x^2}$ and $k_{12} = 0$. The total strains at $a_1$ location are defined as

$$S_{11} = -\alpha_1 \frac{\partial^2 u_1}{\partial x^2} \quad S_{22} = 0 \quad S_{12} = 0 \quad (51)$$

Again, the beam experiences only transverse oscillation. The membrane (longitudinal) force components are all zeros, i.e., $N_{11} = 0$, $N_{22} = 0$, $N_{12} = 0$. The resultant moments are

$$M_{11} = D(k_{11} + \mu k_{22}) = -D \frac{\partial^2 u_1}{\partial x^2} \quad (52-a)$$
$$M_{22} = D(k_{22} + \mu k_{11}) = -\mu D \frac{\partial^2 u_1}{\partial x^2} \quad (52-b)$$
$$M_{12} = \frac{D}{2} k_{12} = 0 \quad (52-c)$$

Note that the moment $M_{12}$ is primarily introduced by Poisson's effect. The electric force and moment resultants due to the external charge and temperature are

$$V_{11} = -h \frac{\epsilon_{11}}{\epsilon_{33}} Q_2 - h \frac{\epsilon_{11}}{\epsilon_{33}} \frac{\partial^2 u_1}{\partial x^2} \quad (53-a)$$
$$V_{22} = -h \frac{\epsilon_{11}}{\epsilon_{33}} Q_3 - h \frac{\epsilon_{11}}{\epsilon_{33}} \frac{\partial^2 u_1}{\partial x^2} \quad (53-b)$$
$$M_{11} = \frac{h}{2} \frac{\epsilon_{11}}{\epsilon_{33}} \frac{\partial^2 u_1}{\partial x^2} \quad (54-a)$$
$$M_{22} = \frac{h}{2} \frac{\epsilon_{11}}{\epsilon_{33}} \frac{\partial^2 u_1}{\partial x^2} \quad (54-b)$$
$$N_{11} = h \lambda_1 \Delta t_p \quad (55-a)$$
$$N_{22} = h \lambda_1 \Delta t_p \quad (55-b)$$

Substituting the system parameters and force/moment resultants into the original shell equation, one can derive the transverse piezothermoelastic equation

$$-\left[ D \frac{\epsilon_{11}}{\epsilon_{33}} \frac{\partial^2 u_1}{\partial x^2} \right] = \alpha h \frac{\partial^2 u_1}{\partial x^2} \quad (56)$$

For a beam with a rectangular cross-section (width $b$ and thickness $h$), the transverse equation of motion is

$$\left[ 1 + \frac{E_b h^2}{12 t_0} \right] \frac{\partial^2 u_1}{\partial x^2} = \frac{q_1}{h} \quad (57)$$

where $I = \frac{bh^3}{12}$ Note that the elasticity part has one more term contributed by the piezoelectricity. The piezoelectricity contributed elasticity is very small, about $1\%$ for piezoelectric polyvinylidene fluoride polymer [Tzou & Zhong, 1993]. However, the temperature has no contribution to the transverse oscillation because the thermal forces are primarily in the neutral surface neutral axis in this case. This piezoelectric effect will contribute to the longitudinal oscillation.

The electric field strength at the location $a_1$ above below the neutral axis is defined by the external charge, temperature induced pyroelectric effect and bending strain

$$E_1 = \frac{Q_1}{\epsilon_{33}} - \frac{i}{\epsilon_{33}} \frac{\partial^2 u_1}{\partial x^2} \quad (58)$$

However, the resultant open-circuit voltage $V_1$ is in fact only contributed by the pyroelectric effect $Q_3 = 0$ and

$$\frac{a_1}{\epsilon_{33}} \frac{\partial^2 u_1}{\partial x^2}$$

SUMMARY AND CONCLUSIONS

A linear piezothermoelastic theory of piezoelectric shell continua was proposed and piezothermoelastic phenomena were evaluated. It was assumed that the electric, thermal, and elastic fields are instantaneously balanced and a quasistatic condition is used in the piezothermoelastic constitutive equations. A generic theory for a piezoelectric thin shell continuum was derived using Hamilton's principle and Kirchhoff's assumptions. The governing equations show close coupling effects among electric, thermal, and elastic fields. Both mechanical and electric effects contribute to the resultant forces/moments for the shell continuum. However, it was observed that the thermal effect only contributes to the membrane force resultants, not the bending resultants due to a uniform temperature assumption. Thermal induced bending could appear if there is a non-uniform temperature distribution. Note that the electric force/moment resultants in the piezothermoelastic equations can be used to control the shell continuum.

The derived piezothermoelastic equations are general, which can be simplified to a variety of piezoelectric continua if two radii of curvatures and two Lamé's parameters are defined. This simplification was demonstrated in three examples: 1) a cylindrical shell and 2) a beam. Detailed piezothermoelastic phenomena of each geometry were discussed along with the derived governing equations. The same procedure can be applied to a variety of other piezoelectric continua and so the piezothermoelasticity evaluated. Note that the theory was derived based on linear assumptions and the material nonlinearity was not considered. However, these material constants (e.g., piezoelectric constants, elastic constants, etc.) could vary when temperature variation is significant. Thus, extending the present theory to encompass the material nonlinearity would further enhance the theoretical development and understand more about the complicated behaviors of piezoelectric sensors/actuators operating in non-ideal environments.
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