Quality Assessment of the A-posteriori Error Estimation in Finite Elements

by

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INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY
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Abstract

The paper addresses results of numerical experimentation with an a-posteriori error estimator which was theoretically analyzed in [1]. The paper concentrates on the problems of the selection of benchmarks which is directed to a numerical verification of basic theoretical features of the estimator.
1 Introduction

During recent years much interest has been focused on the design of a-posteriori error estimates in the finite element method, their experimental verification and their use for adaptive procedures. We refer for example to [1-24] and further citations there.

The a-posteriori error estimators are often derived by purely heuristic reasoning and without mathematical analysis. If a mathematical analysis is made it adresses most often only asymptotic properties and special meshes. Usually the estimators are numerically analyzed (verified) on a set of examples (benchmarks) which are selected at best by heuristic reasoning. Often these benchmark computations are used for the design of a correction factor which is used to improve the quality of the estimators for this set of benchmarks.

This pragmatic approach to test and compare the estimators by benchmark computations has serious shortcomings. The quality of an estimator is usually very sensitive to the structure of the solution and the meshes used. The performance of an estimator can be very different. It may depend on whether the meshes are nearly translation invariant or general, or whether the meshes are general or such that the error indicators are nearly equal on all elements (i.e. the meshes are equilibrated), or whether the solution is smooth, has point singularities or is generally unsmooth, etc. Some of these aspects are adressed in [5]. Hence the benchmark computations could motivate misleading conclusions. Each benchmark computation should be a representative of a more or less precisely defined class of computational problems and the conclusions used for this class of problems.

In this paper we will address various aspects of the relation between theoretical understanding of an estimator and the benchmark selection. We will adress a certain estimator which is used in practice, and discuss its properties. We do not intend to compare it with any other error estimator.

2 The Model Problem

Let \( \Omega \in \mathbb{R}^2 \) be a bounded polygonal domain and let \( \Gamma \) be its boundary. Assume further that \( \overline{\Omega} = \bigcup_{i=1}^{m} \overline{\Omega_i} \), where \( \Omega_i \) is a polygonal domain (with boundary \( \partial \Omega_i \)) and that \( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \). We will be interested in solving the following problem:

\[
- \text{div}(a \nabla u) = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma_D, \quad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma_N, \quad \Gamma_D \cup \Gamma_N = \Gamma, \tag{2.1}
\]

where \( n \) is the outward normal to \( \Omega \). We assume that \( f \in H^1(\Omega), g \in H^1(\Gamma_D), a \geq a_0 \geq 0 \) is constant on \( \Omega_i, i = 1, \ldots, m \). We will assume that either \( \text{meas}(\Gamma_D) \neq 0 \) or, if \( \Gamma_D = \emptyset \), \( \int_{\Omega} f \, dx \, dy + \int_{\Gamma} g \, ds = 0 \) and \( u(A) = 0, A \in \overline{\Omega} \). (Because of the assumptions about \( f \) and \( g \), \( u \in H^2(\Omega), a > 1 \), and so \( u(A) = 0 \) makes sense.)

We will consider the weak solution of (2.1). To this end we denote by \( H^k(\Omega) \) the usual Sobolev space and let

\[
H^1_0 = \{ u \in H^1(\Omega), u = 0 \quad \text{on} \quad \Gamma_D \}
\]

Further we let

\[
B(u, v) = \int_{\Omega} a \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \, dx \, dy \tag{2.2}
\]
be the bilinear form defined on $H^1_0(\Omega) \times H^1_0(\Omega)$ and
\[ F(v) = \int_{\Omega} f v dx dy + \int_{\Gamma_N} g v ds. \] (2.3)
be a linear functional on $H^1_0(\Omega)$. Then the weak solution $u$ of (2.1) in $H^1_0$ exists, is unique and
\[ B(u, v) = F(v) \quad \forall v \in H^1_0. \] (2.4)
If $\Gamma_D = \emptyset$, we consider $H^1(\Omega)$ instead of $H^1_0(\Omega)$ and assume $u(A) = 0$, which is well-defined, as we have in fact higher regularity for $u$.

Let us consider a family $\mathcal{M}$ of triangular meshes $M$ on $\Omega$ satisfying a uniform minimal angle condition. We denote by $T$ the closed triangles of the mesh $M$, $\mathcal{M} = \bigcup T = \bar{\Omega}$. We will assume that if $T \in M$ then interior($T$) $\subset \Omega_i$ for some $i$, i.e. $T$ lies in a domain where $a$ is a constant. To every mesh $M$ we associate a parameter $t(M) \geq 0$, for example $t(M) = N^{-1}$ where $N$ is the number of the nodes that do not belong to $\Gamma_D$ or $t(M) = h$ where $h = \max(d(T))$ and $d(T)$ denotes the diameter of $T$. We will always assume that $h = \max(d(T)) \to 0$ as $t(M) \to 0$.

Further let $S(M)$ be the set of all continuous functions on $\Omega$ whose restriction on a $T \in M$ is a linear function. Obviously $S(M) \subset H^1(\Omega)$. Let $S_0(M) = H^1_0(\Omega) \cap S(M)$ and $S_A(M) = \{u \in S(M), u(A) = 0\}$. Then the finite element solution $u_S \in S_0(M)$ satisfies
\[ B(u_S, v) = B(u, v) \quad \forall v \in S_0(M) \] (2.5)
where $u$ is the exact solution of (2.1). If $\Gamma_D = \emptyset$, we replace $S_0(M)$ by $S_A(M)$. We will consider a family of finite element solutions $u_S$ associated to $M$ and assume that $\|u - u_{S(M)}\|_E = B(u - u_{S(M)}, u - u_{S(M)})^{1/2} \to 0$ as $t(M) \to 0$. By (2.5) we associate to any solution $u$ its finite element solution $u_S = u_S(M, u)$ and an error estimator $\mathcal{E}(u, M)$ that will be defined in the next section. In fact $\mathcal{E}$ depends on $u_S$ and $M$ but because of (2.5) we write $\mathcal{E}(u, M)$. This estimator approximates the energy norm of the error, i.e. $\mathcal{E}(u, M) \approx \|e\|_E = (B(e, e))^{1/2}$, where $e = u - u_S$.

3 The Error Estimator

Let $M \in \mathcal{M}$, $T_i, T_l \in M$ be two triangles with common side $\gamma_{l,i}$ as shown in figure (3.1). By $n_{l,i}$ we denote the outer normal from $T_l$ to $T_i$; we will also write $T_l = T_{in}$ and $T_i = T_{out}$, as $T_{in}$ is the triangle for which the normal $n_{l,i}$ is the outward one.

Let us now define for every side $\gamma_{l,i}$
\[ \left[ \frac{\partial u_S}{\partial n_{l,i}} \right](x) = (a \nabla u_S)_{|\gamma_{out}} - (a \nabla u_S)_{|\gamma_{in}} \cdot n_{l,i} \] (3.1)
the jump of $a \frac{\partial u_S}{\partial n_{l,i}}$ across the side $\gamma_{l,i}$. This value is independent of the choice of $n_{l,i}$, i.e. $\left[ a \frac{\partial u_S}{\partial n_{l,i}} \right](x) = \left[ a \frac{\partial u_S}{\partial n_{i,l}} \right](x)$ and so we will write $\left[ a \frac{\partial u_S}{\partial n} \right](x)$.

For each triangle $T \in M$ let
\[ R_T = -[\text{div}(a \nabla u)]_T + f_T. \] (3.2)
We may write for any $v \in H^1_0$ the residual equation
\[ B(e, v) = \sum_{T \in \mathcal{M}} \int_T R_T v dx dy + \int_{\Gamma_N} (g - a \frac{\partial u_s}{\partial n}) v ds + \sum_{\gamma \in \mathcal{E}} \int_{\gamma} \left[ a \frac{\partial u_s}{\partial n} \right] v ds \] (3.3)
where $G$ stands for the set of all interior edges of $M$ and $n$ for the outward normal to $\Omega$. (If $\Gamma_D = \emptyset$ the equation (3.3) is the same, but $v \in H^1(\Omega)$.)

For any $T \in \mathcal{M}$ we define
\[ \Pi_T = \frac{1}{|T|} \int_T R_T dx dy \] (3.4)
where $|T|$ is the area of $T$. ($\Pi_T$ is the $L^2(T)$ projection of the interior residual onto the set of constants.) Further for each $\gamma \in \Gamma_N$, where $\gamma$ is a side of a triangle $T$, we define
\[ \Pi_\gamma = \frac{1}{|\gamma|} \int_{\gamma} (g - a \frac{\partial u_s}{\partial n}) ds \] (3.5)
where $|\gamma|$ is the length of $\gamma$. ($\Pi_\gamma$ is the $L^2(\gamma)$ projection of the boundary residual onto the set of constants.)

For each edge $\gamma$ of the mesh let
\[ J_\gamma = \begin{cases} \left[ a \frac{\partial u_s}{\partial n} \right] & \text{if } \gamma \in G \\ 2 \Pi_\gamma & \text{if } \gamma \subset \Gamma_N \\ 0 & \text{if } \gamma \subset \Gamma_D \end{cases} \] (3.6)
and for each triangle $T \in \mathcal{M}$ we define the error indicator $\eta_T$:
\[ \eta_T := \left| T \right| \left( \int_T \frac{\Pi_T^2}{a} dx dy + \frac{1}{2} \sum_{\gamma \in \partial T} |\gamma| \left( \int_{\gamma} \frac{J_\gamma^2}{a} ds \right) \right)^{\frac{1}{2}} \] (3.7)
which approximates the energy error on $T$. Further we define the error estimator
\[ E(u, M) := \left( \sum_{T \in \mathcal{M}} \eta_T^2 \right)^{\frac{1}{2}} \kappa \] (3.8)
where $\kappa$ is a suitable constant which will be defined later. It is convenient to write
\[ E^*(u, M) := \left( \sum_{T \in \mathcal{M}} \eta_T^2 \right)^{\frac{1}{2}} \] (3.9)
for the choice $\kappa = 1$. Let us note that the error indicator is meaningful for more general $f$ and $g$ than we have assumed. Nevertheless our generality is sufficient for practical purposes. Obviously the set of problems we are discussing is precisely described.
4 The Quality of the Error Estimator

In Section 3 we defined the error estimator $\mathcal{E}(u, M)$ which approximates $\|e\|_E$. In the following we will discuss the relation between $\mathcal{E}(u, M)$ and $\|e\|_E$, especially the effectivity index of the estimator

$$\xi = \frac{\mathcal{E}(u, M)}{\|e\|_E}.$$

Before addressing this problem we introduce some notation which we will need in the following. Let $S^p(M)$ be the space of all continuous functions on $\Omega$ whose restriction on $T \in M$ is a polynomial of degree $p$. Let $S^p_0(M) = S^p \cap H^1_0(\Omega)$ and $u^p \in S^p_0(M)$ be the finite element solution of problem (2.1), i.e.

$$B(u^p, v) = B(u, v) \forall v \in S^p_0(M).$$

(If $\Gamma_D = \emptyset$, we use $S^p_0(M)$ instead of $S^p_0(M)$.) Obviously $u^p$ is the p-version solution of problem (2.1). For more about the $h$ - $p$-version we refer to [6]. We have for any $k > 1$:

$$\|e\|_E \leq C p^{-k-1} \|u\|_{H^1} h^{\min(p,k-1)}$$

(4.1)

with $C$ dependent on $k$, but not on $p$ and $h$.

Remark: Although in (4.1) the maximal element side is used, an analogous element by element estimate holds.

Now we are able to state the basic theorems about $\mathcal{E}^*(u, M)$ . (See [1] for the detailed proof and Section 5 for an outline of the main ideas.)

Theorem 4.1 For any integer $p$ there is a constant $K^p_2$ depending only on the minimal angle $\alpha_T$ of the triangles used and on $p$, such that

$$\|e\|_E \leq K^p_2 \mathcal{E}^*(u, M) + \|u - u^p\|_E + O(h^{3/2})$$

(4.2)

and the constant $K^p_2$ grows logarithmically with $p$.

Theorem 4.2 There is a constant $K_1 > 0$, depending only on $\alpha_T$ and on the jumps of $a$, such that

$$\mathcal{E}^*(u, M) \leq K_1 \|e\|_E + O(h^{3/2}).$$

(4.3)

The constants $K_1$ and $K^p_2$ are computable. We have

$$K^p_2 = \max_{T \in M} C^p_T$$

(4.4)

and

$$K_1 = \max_{T \in M} C^{'T} Q_a,$$

(4.5)

where

$$Q_a = \max_{T \cap \Gamma_D \neq \emptyset} \left( \frac{a_{TT}^{-1}}{a_{TT}^{-1}} \right).$$

(4.6)

For $2 \leq p \leq 8$ we have the estimates

$$0.548 \log^3 p \sin^{-1/2}(\frac{\alpha_T}{2}) \leq C^p_2 \leq 0.813 \log^3 p \sin^{-1/2}(\frac{\alpha_T}{2})$$

(4.7)
Table 4.1: Constants $C_T$ and $C_T'$ for various angles $\alpha_T$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$C_T$</th>
<th>$C_T'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 2$</td>
<td>$p = 4$</td>
<td>$p = 6$</td>
</tr>
<tr>
<td>7.5</td>
<td>.051</td>
<td>2.390</td>
</tr>
<tr>
<td>15.0</td>
<td>.072</td>
<td>1.682</td>
</tr>
<tr>
<td>22.5</td>
<td>.087</td>
<td>1.363</td>
</tr>
<tr>
<td>30.0</td>
<td>.099</td>
<td>1.169</td>
</tr>
<tr>
<td>37.5</td>
<td>.108</td>
<td>1.035</td>
</tr>
<tr>
<td>45.0</td>
<td>.115</td>
<td>.939</td>
</tr>
<tr>
<td>52.5</td>
<td>.119</td>
<td>.875</td>
</tr>
<tr>
<td>60.0</td>
<td>.121</td>
<td>.850</td>
</tr>
</tbody>
</table>

and

$$3.45 \sin^{-\frac{1}{2}}(\frac{\alpha_T}{2}) \leq C_T' \leq 5.85 \sin^{-\frac{1}{2}}(\frac{\alpha_T}{2})$$ (4.8)

where $\alpha_T$ is the minimal angle of $T$. In Table 4.1 we give the values of $C_T'$ and $C_T'$.

Let us note that for $h$ small only very few elements have a side on the interface $\partial \Omega$; we can in fact use $Q_s = 1$.

Remark: We have dealt only with the Laplace equation. In this case we use the minimal angle $\alpha_T$ of the triangle. For a general differential equation we have to replace it by $\alpha_T$ which is the minimal angle of the triangle after a local transformation of the general equation to Laplace's equation (this is always possible).

The term $O(h^{\frac{3}{2}})$ in (4.2) and (4.3) depends only on $\|f\|_{H^1(\Omega)}$ and $\|g\|_{H^1(\Gamma)}$, but not on $u$. For any $u$ the term $\|u - u_0\|_E$ can be made arbitrarily small by (4.1) using sufficiently large $p$. We note that the smoothness of $u$ is governed by the smoothness of $f$ and $g$, transition of boundary conditions and smoothness of $\Gamma$. It is unessential that the estimates (4.2) and (4.3) have an asymptotic character. For example if $f = \text{const}$ on every $T$ and $g = \text{const}$ on every triangle side of $\Gamma_N$, the term $O(h^{\frac{3}{2}})$ disappears. Hence as $t(M) \to 0$, the term $O(h^{\frac{3}{2}})$ is negligible in general.

Theorems 4.1 and 4.2 indicate the expected theoretical properties of the estimator well. The structure of Theorems 4.1 and 4.2 clearly suggests the set of benchmark experiments that should be performed.
5 The Outline of the Proof of Theorems 4.1 and 4.2

For simplicity assume that $f$ and $g$ are constant on each $T$ resp $l \in \Gamma_N$. (The general case is discussed in [1].) With $e_p := u_S^p - u_S \in S_p^0$, we have

$$
\|e\|_E \leq \|e_p\|_E + \|u - u_S^p\|_E
$$

(5.1)

and

$$
\|e_p\|_E^2 = B(e_p, e_p) = B(u_S^p - u, e_p) + B(u - u_S, e_p) = B(e, e_p)
$$

(5.2)

For any continuous function $v$ let $v'$ denote the linear interpolant of $v$ on $T$. Using (3.3), we have:

$$
\|e_p\|_E^2 = B(e, e_p) = \sum_{T \in \mathcal{M}} \left[ \int_T \Pi_T(e_p - e'_p) dxdy + \frac{1}{2} \sum_{I \in \partial T} \int_I (e_p - e'_p) ds \right]
$$

$$
\leq \sum_{T \in \mathcal{M}} \eta_T \left\{ \left[ \frac{\sqrt{a_T}}{|T|} \int_T (e_p - e'_p) dxdy \right]^2 + \frac{1}{2} \sum_{I \in \partial T} \left[ \frac{\sqrt{a_T}}{|I|} \int_I (e_p - e'_p) ds \right]^2 \right\}^{1/2}
$$

$$
\leq \sum_{T \in \mathcal{M}} \eta_T C_T^2 \|e_p\|_{E,T}
$$

where

$$
C_T^2 = \sup_{v \in P_p \setminus P_0} \left( \frac{\int_T |v - v'| dxdy}{\int_T |\nabla v|^2 dxdy} \right)^{1/2}
$$

Here, $P_p$ and $P_0$ denote the spaces of all polynomials of degree $p$ or $0$ on $T$, respectively.

Hence

$$
\|e_p\|_E \leq K_T^2 \Phi^*(u, M)
$$

with

$$
K_T^2 = \sup_{T \in \mathcal{M}} C_T^2.
$$

Using (5.1) we get (4.2) without the term $O(h^{3/2})$. This is related to the simplification assumption about $f$ and $g$ made in this section.

Let us now address inequality (4.3). For any triangle $T$ as shown in Fig. 5.1, let $\varphi_0$ be the cubic bubble function vanishing on $\partial T$ with $\varphi_0(Q_0) = 1$, where $Q_0$ is the barycenter of $T$. Let

![Figure 5.1: Notation for triangular elements](image)

further $P^2_T(T)$ be the space of quadratic functions vanishing at the vertices of $T$ and let $\{\varphi_i\}_{i=1}^3$
be the canonical basis of this space (i.e. \( \varphi_i(Q_j) = \delta_{ij}, i = 1, 2, 3, j = 1, \ldots, 6 \)) and let \( V_T \) be the space spanned by \( \{\varphi_i\}_{i=0}^6 \). Set \( \omega_T = \sum_{i=0}^6 \omega_i \varphi_i \in V_T \) with coefficients \( \omega_0, \omega_1, \omega_2, \omega_3 \) chosen such that

\[
\int_T \Pi_T \omega_T \, dx \, dy = |T| \int_T \frac{\Pi^2_T}{a} \, dx \, dy \quad (5.3)
\]

\[
\int_{l_i} J_{l_i} \omega_T \, ds = |l_i| \int_{[a]_i} J_{l_i} \, ds, \quad i = 1, 2, 3
\]

where \([a]_i\) is the average of \( a \) over the triangles sharing the edge \( l_i \), i.e.

\[
[a]_i = \begin{cases} 
\frac{1}{2} (a |T| + a |T|) & \text{if } l_i = \partial T_{in} \cap \partial T_{out} \\
\frac{a}{|T|} & \text{if } l_i \subseteq \partial T \cap T
\end{cases}
\]

The coefficients \( \omega_0, \ldots, \omega_3 \) satisfying (5.3) and (5.4) are

\[
\omega_i = \frac{3|l_i|}{2[a]_i}
\]

\[
\omega_0 = \frac{20}{9} \left( \frac{|T|^{1/2} a_{|T|}^{1/2}}{a |T|} - \frac{1}{2} \sum_{i=0}^3 \frac{|l_i|}{[a]_i} J_{l_i} \right)
\]

Since for any interior edge \( l_i \) the values \( \omega_T \) are the same on both triangles \( T \) sharing the edge, the function \( \omega \) composed from \( \omega_T \) on every triangle is continuous and hence \( \omega \in H^1_0(\Omega) \). Using (5.1), (5.2) we get:

\[
(E^*(u, M))^2 = \sum_{T \in M} \eta^2_T
\]

\[
= \sum_{T \in M} \left[ |T| \int_T \frac{\Pi^2_T}{a} \, dx \, dy + \frac{1}{2} \sum_{l \in \partial T} |l| \int_{l} C_T \, ds \right]
\]

\[
\leq C_a \sum_{T \in M} \left[ \int_T \Pi_T \omega_T \, dx \, dy + \frac{1}{2} \sum_{l \in \partial T} |l| \int_{l} \omega_T \, ds \right]
\]

where

\[
C_a = \max_{l \in \partial T, T \in M} \frac{|a|_l}{a |T|}.
\]

Since \( \omega \in H^1_0 \) we get from (3.3)

\[
(E^*(u, M))^2 \leq C_a B(e, \omega) \leq C_a \|e\|_E^2 \|\omega\|_E.
\]

Now let for \( r = \{r_i\}_{i=0}^3 \)

\[
C^2_T = \sup_{\|\omega\|_0^2} \frac{\int_T \|\nabla v_T\|^2 \, dx \, dy}{\|\omega\|_0^2} \geq \frac{1}{2} \sum_{i=1}^3 r_i^2
\]

where \( v_T \in V_T \) is the function satisfying \( \int_T v_T \, dx \, dy = |T|^{1/2} \int_{l_i} v_T \, ds = |l_i| r_i, \ i = 1, 2, 3 \). Therefore \( \omega_T \) is the function \( v_T \) corresponding to

\[
r_0 = \frac{|T|^{1/2}}{a |T|}
\]

\[
r_i = \frac{|l_i| J_{l_i}}{[a]_i}, \quad i = 1, 2, 3
\]
and so

\[ \|\omega_T\|_{E,T}^2 = a_T \left( \int_{\Omega} \|
abla u_T\|^2 \right) \]

\[ \leq C_T^2 a_T \left( \frac{|T|^2 \Pi_T^2}{a_T^2} + \frac{1}{2} \sum_{i=1}^{3} \frac{|I_i|^2 J_{ii}^2}{|a_i|^2} \right) \]

\[ = C_T^2 \left( \frac{|T|^2 \Pi_T^2}{a_T} + \frac{1}{2} \sum_{i=1}^{3} \frac{a_T |a_i|^2}{|a_i|} \frac{|I_i|^2 J_{ii}^2}{a_T} \right) \]

\[ \leq C_T^2 C_a^2 \eta_T^2, \]

where

\[ C_a' = \max_{i \in \Omega, T \in M} \frac{a_T}{|a_i|}. \]

Hence

\[ \|\omega\|_E \leq (\max_{T \in M} C_T') C_a' \mathcal{E}'(u, M) \]

and so, using 5.6, we obtain

\[ \mathcal{E}'(u, M) \leq Q_a (\max_{T \in M} C_T') \|\epsilon\|_E \]

where

\[ Q_a = C_a C_a' = \max_{T \in M} \frac{a_T}{|a_T|} \]

and we get (4.3).

6 The Experimental Results

In this section we will experimentally verify the reliability of various conclusions that can be made from Theorems 4.1, 4.2 for the given estimator.

6.1 The Influence of Mesh Topology

Theorem 4.1 was proven for general meshes without any restrictions on the mesh topology. Hence, it can be expected that particular meshes with different topologies could influence the effectivity index in the same order of magnitude as indicated by the theoretical upper and lower bounds. In addition the influence of the angle has to be visible (and could also depend on the topology). To check the validity of this conclusion for a smooth solution we consider the following problem:

For \( \Omega = (0, 1) \times (0, 1) \), let \( u \) be the solution of

\[ -\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma \]

(6.1)

We choose \( f \) such that \( u = \sin \pi x \sin \pi y \).
6.1.1 Three-direction meshes

Let us consider the three-directional meshes which are obtained by uniform refinement of a basic mesh of the type $\frac{1}{n}, n = 1, 2, 4, 8$ as shown in Fig. 6.1.

In Table 6.1 we report the effectivity index for these meshes. We also report the upper and lower bounds for the effectivity index from Theorems 4.1, 4.2, when $p = 2$ was used.

<table>
<thead>
<tr>
<th>number of elements</th>
<th>mesh</th>
<th>1/1</th>
<th>1/2</th>
<th>1/4</th>
<th>1/8</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.852</td>
<td>0.993</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1.070</td>
<td>1.289</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>1.147</td>
<td>1.503</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>1.256</td>
<td>1.804</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>1.803</td>
<td>2.068</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>1.192</td>
<td>2.145</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>512</td>
<td>1.387</td>
<td>2.169</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1024</td>
<td>1.192</td>
<td>2.169</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2048</td>
<td>1.192</td>
<td>2.169</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4096</td>
<td>1.192</td>
<td>2.169</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1: Effectivity index for problem 6.1 ($u = \sin \pi x \sin \pi y$) and 3-direction meshes

6.1.2 Criss-cross meshes

Let us consider now the ‘criss-cross’ meshes as shown in Fig. 6.2. The effectivity indices for these meshes are given in Table 6.2.

6.1.3 Diamond-shaped meshes

Finally, we consider the ‘diamond-shaped’ meshes as shown in Fig. 6.3. The effectivity indices are given in Table 6.3.

Comparing the tables we can make the following conclusions, which are in agreement with Theorems 4.1, 4.2:
Table 6.2: Effectivity index for problem 6.1 ($u = \sin \pi x \sin \pi y$) and 'criss-cross' meshes
<table>
<thead>
<tr>
<th>number of elements</th>
<th>mesh</th>
<th>1/1</th>
<th>1/2</th>
<th>1/4</th>
<th>1/8</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td></td>
<td>1.465</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>0.750</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>1.017</td>
<td></td>
<td>0.523</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>1.026</td>
<td></td>
<td></td>
<td>0.385</td>
</tr>
<tr>
<td>48</td>
<td></td>
<td>1.107</td>
<td></td>
<td>1.111</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td></td>
<td>1.188</td>
<td></td>
<td></td>
<td>0.959</td>
</tr>
<tr>
<td>80</td>
<td></td>
<td>1.131</td>
<td></td>
<td>1.558</td>
<td></td>
</tr>
<tr>
<td>144</td>
<td></td>
<td>1.234</td>
<td></td>
<td></td>
<td>1.708</td>
</tr>
<tr>
<td>160</td>
<td></td>
<td></td>
<td></td>
<td>1.740</td>
<td></td>
</tr>
<tr>
<td>272</td>
<td></td>
<td></td>
<td></td>
<td>1.246</td>
<td>2.274</td>
</tr>
<tr>
<td>288</td>
<td></td>
<td></td>
<td></td>
<td>1.139</td>
<td></td>
</tr>
<tr>
<td>544</td>
<td></td>
<td></td>
<td></td>
<td>1.793</td>
<td></td>
</tr>
<tr>
<td>576</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.515</td>
</tr>
<tr>
<td>1056</td>
<td></td>
<td>1.138</td>
<td></td>
<td>2.34</td>
<td>2.87</td>
</tr>
<tr>
<td>1088</td>
<td></td>
<td>1.740</td>
<td></td>
<td>3.87</td>
<td>5.38</td>
</tr>
<tr>
<td>2112</td>
<td></td>
<td>1.246</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2176</td>
<td></td>
<td></td>
<td></td>
<td>2.274</td>
<td></td>
</tr>
<tr>
<td>4160</td>
<td></td>
<td></td>
<td></td>
<td>1.139</td>
<td></td>
</tr>
<tr>
<td>4224</td>
<td></td>
<td></td>
<td></td>
<td>1.793</td>
<td></td>
</tr>
<tr>
<td>8448</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.515</td>
</tr>
</tbody>
</table>

Table 6.3: Effectivity index for problem 6.1 \((u = \sin \pi x \sin \pi y)\) and ‘diamond-shaped’ meshes
1. For $h \to 0$ the effectivity index stays in the theoretical bounds. In addition we see that in our concrete case the effectivity index converges to a limiting value. This is not surprising, because the mesh is translation invariant and the solution can be periodically extended. In this case various results about superconvergence can be used (see [21]).

2. The effectivity index depends on the topology. The error estimator is not asymptotically exact.

3. The dependence of the effectivity index on the angle is clearly visible.

4. The theoretical bounds for the effectivity index are relatively pessimistic. This is because these bounds are independent of the topology and the solution. Better bounds in our case could be obtained by exploiting the superconvergence effect (see [5]).

6.2 The Influence of the Solution Structure

Theorems 4.1 and 4.2 have been shown for a general solution. We expect that the effectivity index may depend on the solution structure. To check this conclusion consider the solution of the following problem on $\Omega = (0,1) \times (0,1):$

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ for } x = 0 \text{ and } x = 1, \quad \frac{\partial u}{\partial y} = 0 \text{ for } y = 0 \text{ and } y = 1 \quad (6.2)$$

with $f$ such that $u = \sin \pi x$. We will use meshes as in Fig. 6.1 with the orientation shown there (mesh 1/2 and 1/4) and with the opposite orientation (2/1 and 4/1).

<table>
<thead>
<tr>
<th>number of elements</th>
<th>mesh</th>
<th>1/1</th>
<th>1/2</th>
<th>1/4</th>
<th>2/1</th>
<th>4/1</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td></td>
<td>1.013</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>1.564</td>
<td>0.717</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td></td>
<td>1.107</td>
<td>2.258</td>
<td>0.507</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td></td>
<td>1.598</td>
<td>0.783</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td></td>
<td>1.131</td>
<td>2.273</td>
<td>0.553</td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td></td>
<td>1.608</td>
<td>0.800</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>512</td>
<td></td>
<td>1.137</td>
<td>2.278</td>
<td>0.566</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1024</td>
<td></td>
<td>1.611</td>
<td>0.804</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2048</td>
<td></td>
<td>1.139</td>
<td>2.279</td>
<td>0.569</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4096</td>
<td></td>
<td>1.612</td>
<td>0.806</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>upper bound</td>
<td></td>
<td>2.34</td>
<td>2.87</td>
<td>3.87</td>
<td>2.87</td>
<td>3.87</td>
</tr>
<tr>
<td>lower bound</td>
<td></td>
<td>0.29</td>
<td>0.21</td>
<td>0.15</td>
<td>0.21</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 6.4: Effectivity index for problem 6.2 ($u = \sin \pi x$) and 3-direction meshes

From Table 6.4 we see in fact that the effectivity index depends on the relation between the mesh and the structure of the solution (especially the relation between the second derivatives and the mesh orientation). Hence in general we have to deal with uncertainties in the effectivity index. (No correction factor could improve the situation.) We note that our factor $\kappa = 0.2684\ldots$ could be changed, but in general (for all functions) the range in the effectivity index could not be improved. Again, we see that Theorems 4.1 and 4.2 reliably predict the outcome of the numerical experiments.
6.3 The Influence of the Smoothness of the Solution

Theorems 4.1 and 4.2 contain higher order terms which we neglected in the reported theoretical bounds in Tables 6.1-6.4. When computing the bounds, these terms influence the convergence of the effectivity index as \( h \to 0 \). To check the effect of these terms, we will consider once more problem 6.1, using now the solution

\[
u(x, y) = \sin \alpha x \sin \alpha y.
\]

(6.3)

(We use regular three-direction meshes, as in Fig. 6.1 with mesh ratio 1/1).

\[
\begin{array}{cccccc}
\text{number of elements} & a = 1 & a = 2 & a = 4 & a = 11 \\
8 & 0.852 & 0.833 & 0.438 & \\
32 & 1.070 & 0.876 & 0.932 & 0.379 & \\
128 & 1.147 & 1.080 & 0.887 & 0.530 & \\
512 & 1.180 & 1.161 & 1.090 & 0.780 & \\
2048 & 1.192 & 1.187 & 1.168 & 1.022 & \\
8192 & & & & 1.148 & \\
\hline
\text{upper bound} & 2.34 & 2.34 & 2.34 & 2.34 & \\
\text{lower bound} & 0.29 & 0.29 & 0.29 & 0.29 & \\
\end{array}
\]

Table 6.5: Effectivity index for problem 6.3 \( (u = \sin \alpha x \sin \alpha y) \) and 3-direction meshes

We see that the insufficient smoothness influences the effectivity index and that the effectivity index can behave erratically. Nevertheless, the bounds —where higher order terms are neglected—are still valid. We see in general that the effectivity index decreases with decreasing smoothness of the solution. The lower bound goes down with increasing \( p \), but the upper bound is not influenced. Hence we can expect that the effectivity index will go down (assuming heuristically that the effectivity index moves as the middle of the interval). The same effect will be seen later. Once more we see good agreement with the conclusions stemming from Theorem 4.1 and that we cannot hope for the existence of a universal correction factor.

6.4 Harmonic solutions

The error estimator contains terms related to the residual and the jumps, respectively. To isolate the influence of the jump term, we consider the following problem (again in \( \Omega = (0, 1) \times (0, 1) \)):

\[
-\Delta u = 0 \text{ in } \Omega, \quad u = 0 \text{ for } x = 0, x = 1 \text{ and } y = 0, \quad \frac{\partial u}{\partial n} = g \text{ for } y = 1
\]

(6.4)

and let \( u(x, y) = \sin \pi x \sinh \pi y \). The effectivity index (for a regular three-direction mesh as shown in Fig. 6.1) is given in Table 6.6.

Comparing Table 6.1 and 6.6 we see a similar behaviour influenced only mildly by the structure of the solution. The limiting value for the effectivity index seems to be different for the mesh 1/1. For the other meshes, this effect is less visible. For these particular meshes, this might be explained by a superconvergence phenomenon.
Table 6.6: Effectivity index for problem 6.4 \((u(x, y) = \sin \pi x \sinh \pi y)\) and 3-direction meshes

<table>
<thead>
<tr>
<th>number of elements</th>
<th>mesh 1/1</th>
<th>1/2</th>
<th>1/4</th>
<th>1/8</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.841</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.958</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0.978</td>
<td>1.269</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>1.102</td>
<td>1.795</td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>1.040</td>
<td>1.457</td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>1.166</td>
<td>2.042</td>
<td></td>
<td></td>
</tr>
<tr>
<td>512</td>
<td>1.065</td>
<td>1.534</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1024</td>
<td>1.190</td>
<td>2.136</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2048</td>
<td>1.071</td>
<td>1.560</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4096</td>
<td>1.199</td>
<td>2.166</td>
<td></td>
<td></td>
</tr>
<tr>
<td>upper bound</td>
<td>2.34</td>
<td>2.87</td>
<td>3.87</td>
<td>5.38</td>
</tr>
<tr>
<td>lower bound</td>
<td>0.29</td>
<td>0.21</td>
<td>0.15</td>
<td>0.12</td>
</tr>
</tbody>
</table>

Table 6.7: Effectivity indices for problem 6.4 and nonuniform meshes

<table>
<thead>
<tr>
<th>Sequence A</th>
<th>Sequence B</th>
<th>Sequence C</th>
</tr>
</thead>
<tbody>
<tr>
<td>elements</td>
<td>effectivity index</td>
<td>elements</td>
</tr>
<tr>
<td>8</td>
<td>0.841</td>
<td>8</td>
</tr>
<tr>
<td>19</td>
<td>0.944</td>
<td>19</td>
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<td>47</td>
<td>0.970</td>
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<tr>
<td>94</td>
<td>1.008</td>
<td>67</td>
</tr>
<tr>
<td>181</td>
<td>1.018</td>
<td>107</td>
</tr>
<tr>
<td>323</td>
<td>1.041</td>
<td>201</td>
</tr>
<tr>
<td>593</td>
<td>1.043</td>
<td>355</td>
</tr>
<tr>
<td>1020</td>
<td>1.060</td>
<td>607</td>
</tr>
<tr>
<td>1773</td>
<td>1.051</td>
<td>991</td>
</tr>
</tbody>
</table>

Comparing Tables 6.7 and 6.6 we see that different refinement strategies lead to different effectivity indices, although the difference is not big. Once more we see a range of the effectivity indices. We note that for these general meshes superconvergence results cannot be applied.
Figure 6.4: Nonuniform meshes for problem 6.4
6.6 Effectivity Indices for Problems with Unsmooth Solutions

So far, we have considered problems with a smooth solution. Let us now examine the behaviour of the error estimator for solutions with singularities. Theorems 4.1 and 4.2 indicate a decrease of the effectivity index, when the singularity of the exact solution grows, and also that this effect diminishes when adaptive meshes are used.

To this purpose we consider the domain \( \Omega = (-1, 1) \times (-1, 1) \), partitioned into 4 square subdomains as shown in Fig. 6.5. Let \( u \) be the solution of

\[
- \nabla (a(x,y) \nabla u) = 0 \text{ in } \Omega, \quad a(x,y) \frac{\partial u}{\partial n} = g(x,y) \text{ on } \Gamma \tag{6.5}
\]

where \( a(x,y) \) is constant in each quadrant (\( a(x,y) = 1 \) or \( a(x,y) = \tilde{a} \)), see Fig. 6.5). \( \tilde{a} \) is chosen so that

\[
u(x,y) = r^\alpha (c \cos(\alpha \varphi) + s \sin(\alpha \varphi)) \tag{6.6}
\]

where \((r, \varphi)\) are polar coordinates with center in the origin. For \( \alpha = 0.5 \) and \( \alpha = 0.1 \) we get \( \tilde{a} = 3 + \sqrt{8} \approx 5.828... \) and \( \tilde{a} \approx 166.447... \), respectively.

\[
\begin{array}{|c|c|}
\hline
2 & 1 \\
\hline
a(x,y) = \tilde{a} & a(x,y) = 1 \\
\hline
3 & 4 \\
\hline
a(x,y) = 1 & a(x,y) = \tilde{a} \\
\hline
\end{array}
\]

Figure 6.5: Coefficient \( a \) for problem 6.5

Table 6.8 shows the effectivity index for uniform meshes. (In each quadrant the meshes are uniform three-direction meshes of mesh ratio 1/1 as in Fig. 6.1. The orientation of the meshes is changed in subdomains 2 and 4 to preserve the symmetry of the solution.)

\[
\begin{array}{|c|c|c|}
\hline
\text{number of elements} & \alpha = 0.5 & \alpha = 0.1 \\
\hline
32 & 0.658 & 0.520 \\
128 & 0.694 & 0.512 \\
512 & 0.711 & 0.524 \\
2048 & 0.719 & 0.536 \\
8192 & 0.722 & \\
\hline
\end{array}
\]

Table 6.8: Effectivity index for problem 6.5 (\( u = r^\alpha f(\varphi) \)) and 3-direction meshes

Although the estimator in Theorem 4.1 includes the ratio of \( a \), it influences only elements on the interfaces and hence this factor can be neglected. The decrease of the effectivity index is clearly related to the necessity to use a higher degree \( p \) in 4.2 for the estimate. Hence the
unsmoothness of the solution decreases the effectivity index. This has also been observed in Table 6.5.

We have seen in Table 6.7, that nonuniformity of the mesh does not influence the effectivity index too much. On the other hand an adaptive procedure refines the mesh in the places where the solution is unsmooth and hence a lower $p$ can be used. We have created two sequences of adaptively refined meshes (A and B). The meshes B1 and B2 are less refined than the meshes A1 and A2. The refinement in meshes B is more concentrated around the origin than in meshes A. Examples of each sequences are displayed in Fig. 6.6. If the mesh is properly refined, a lower $p$ can be used and the effectivity index improves. Table 6.9 confirms this conclusion (for $\alpha = 0.5$).

![Meshes A and B](image)

Figure 6.6: Nonuniform meshes for problem 6.5

The effect of the unsmoothness of the solution can also be observed if the singular part of the solution is removed. So we consider problem 6.5 in the domain $\tilde{\Omega} = \Omega - [-0.5, 0.5] \times [-0.5, 0.5]$ and prescribe $\phi = g$ on $\partial \tilde{\Omega}$. Table 6.10 shows the effectivity indices for this problem using uniform meshes as for Table 6.8. Again, $\alpha = 0.5$.

We note that the effectivity index is now very similar to problems 6.1, 6.2.
<table>
<thead>
<tr>
<th>Sequence A</th>
<th></th>
<th>Sequence B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>elements</td>
<td>effectivity index</td>
<td>elements</td>
<td>effectivity index</td>
</tr>
<tr>
<td>32</td>
<td>0.658</td>
<td>32</td>
<td>0.658</td>
</tr>
<tr>
<td>87</td>
<td>0.712</td>
<td>64</td>
<td>0.719</td>
</tr>
<tr>
<td>194</td>
<td>0.738</td>
<td>96</td>
<td>0.787</td>
</tr>
<tr>
<td>370</td>
<td>0.761</td>
<td>168</td>
<td>0.830</td>
</tr>
<tr>
<td>725</td>
<td>0.787</td>
<td>244</td>
<td>0.880</td>
</tr>
<tr>
<td>1297</td>
<td>0.807</td>
<td>363</td>
<td>0.918</td>
</tr>
<tr>
<td>2346</td>
<td>0.828</td>
<td>485</td>
<td>0.960</td>
</tr>
</tbody>
</table>

Table 6.9: Effectivity indices for problem 6.5 \((u = r^α f(φ))\) and adapted meshes

<table>
<thead>
<tr>
<th>number of elements</th>
<th>(\alpha = 0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>0.964</td>
</tr>
<tr>
<td>96</td>
<td>1.027</td>
</tr>
<tr>
<td>384</td>
<td>1.059</td>
</tr>
<tr>
<td>1536</td>
<td>1.070</td>
</tr>
<tr>
<td>6144</td>
<td>1.073</td>
</tr>
</tbody>
</table>

Table 6.10: Effectivity index for problem 6.5 \((u = r^α f(φ)\text{ on } \hat{Ω})\) and 3-direction meshes
6.7 The Oscillation Effect

It is often assumed, that the error of the interpolant is close to the error of the finite element method. To demonstrate that this assumption is in general incorrect, we show that the error of the finite element method could oscillate. To this end we show in Fig. 6.7 the typical error behaviour for the problem discussed in section 6.6 with uniform meshes and \( \alpha = 0.5 \). Displayed are two patches of regular three-direction meshes at the same location—far from the singularity—in \( \Omega \). Fig. 6.7 also shows the error indicators (in parentheses). In Fig. 6.8 we illustrate graphically the error behaviour for the entire mesh where the mesh has 363 elements. (An element is black if the energy norm error in this element exceeds a certain threshold. Here, the threshold is \( 4.0 \times 10^{-4} \).)

<table>
<thead>
<tr>
<th>Small element side: 1/8</th>
<th>Small element side: 1/16</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2(-3) (1.2(-3))</td>
<td>2.5(-3) (1.4(-3))</td>
</tr>
<tr>
<td>1.0(-3) (1.2(-3))</td>
<td>1.1(-3) (1.4(-3))</td>
</tr>
<tr>
<td>2.2(-3) (1.0(-3))</td>
<td>2.4(-3) (1.2(-3))</td>
</tr>
<tr>
<td>9.7(-4) (1.0(-3))</td>
<td>1.1(-3) (1.2(-3))</td>
</tr>
<tr>
<td>5.6(-4) (3.0(-4))</td>
<td>5.9(-4) (3.2(-4))</td>
</tr>
<tr>
<td>2.6(-4) (3.0(-4))</td>
<td>2.7(-4) (3.2(-4))</td>
</tr>
<tr>
<td>5.6(-4) (2.7(-4))</td>
<td>5.9(-4) (3.0(-4))</td>
</tr>
<tr>
<td>2.6(-4) (2.7(-4))</td>
<td>2.6(-4) (3.0(-4))</td>
</tr>
</tbody>
</table>

Figure 6.7: Oscillation of the finite element error

We see that the error indicator shows approximately the average error for adjacent elements and hence does not exhibit the oscillation of the error of the finite element solution. Note that the interpolant does not show the oscillation. This oscillation effect is caused by the pollution of the error stemming from the singularity of strength \( r^{1/4} \). If we enforce a smooth solution by removing the center part of the domain, as in problem 6.5, the results given in Fig. 6.9 show that the oscillation disappears. (We use elements from the same location as in Fig. 6.7; the numbers are therefore directly comparable.)

The oscillation effect is weaker, if \( \alpha \neq 0.5 \) or the mesh is constructed adaptively. We show in Fig. 6.10 the local results for \( \alpha = .8 \) and in Fig. 6.11 the results for \( \alpha = .5 \) and adaptively refined meshes. In Fig. 6.12 we show the error behaviour for an adaptively refined mesh and the threshold \( 2.0 \times 10^{-3} \). In Section 7 we will examine this effect.

7 Analysis of the Oscillation Phenomenon

The oscillation phenomenon we have observed in Section 6.7 occurs also in the case of the following problem on \( \Omega = [-1, 1] \times [0, 1] \)

\[
-\Delta u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D = \{(x, y) \mid 0 \leq x \leq 1, y = 0\}, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma_N = \Gamma - \Gamma_D
\]

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Figure 6.8: Finite element error ($\alpha = 0.5$, uniform meshes)

Figure 6.9: Finite element error and estimators for a problem without singularity
<table>
<thead>
<tr>
<th></th>
<th>6.0(-4)</th>
<th>7.0(-4)</th>
<th>1.4(-4)</th>
<th>1.6(-4)</th>
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<tbody>
<tr>
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</tr>
</tbody>
</table>

Values in (..): error estimator

small element side: 1/8

Figure 6.10: Finite element error and estimators for problem with singularity ($\alpha = .8$, uniform mesh)

<table>
<thead>
<tr>
<th></th>
<th>1.4(-3)</th>
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<th>1.6(-3)</th>
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<td>1.5(-3)</td>
<td>(1.7(-3)</td>
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<td>1.0(-3)</td>
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<td>(1.7(-3)</td>
</tr>
</tbody>
</table>

Values in (..): error estimator

small element side: 1/8

Figure 6.11: Finite element error and estimators for problem with singularity ($\alpha = .5$, adaptively refined mesh)
with \( g \) selected so that

\[
u = r^{1/2} \sin\left(\frac{1}{2} \varphi \right) = \text{Re} z^{1/2}
\]

with \( z = x + iy \).

We consider now a uniform 3-direction mesh of the type shown in Fig. 6.1. Then the finite element solution has essentially the form

\[
u_S(x, y) = u^f(x, y) + G(x, y) + R(x, y),
\]

(7.1)

where \( \|R(\cdot, \cdot)\|_{H^1} \leq o(h) \) and \( G(x, y) \) is a piecewise linear function, which coincides in the nodal points of the mesh with the function

\[
G^*(x, y) = Ar^{1/2} \sin \frac{1}{2} \theta = Ah \text{Re} z^{-1/2}.
\]

(7.2)

The function \( G^*(x, y) \) is the adjoint singularity function. It describes the pollution error, which is of the same order (in \( \| \cdot \|_{E} \)) as the error of the interpolant. The error—measured in the energy norm—is locally (i.e. on single elements) of order \( h^2 \) and is governed by the second derivatives of \( u \). The contributions of \( G \) to the energy error are of the same order, but are governed by the first derivative of \( G \). Now observe that in our case both errors are governed by the second derivative of \( \text{Re} z^{1/2} \). This leads to a cancellation and consequently to the oscillation phenomenon.

To demonstrate this, we consider a patch of two triangles as in Fig. 7.1 with local coordinates \( \xi, \eta \) with the origin in the lower left vertex of the patch.

Using (7.1), (7.2) we can assume that on the patch:

\[
u = a(\xi^2 - \eta^2) + b \xi \eta + \text{linear functions} + \text{h.o.t.}
\]

\[
G = (aa \xi + ab \eta)h + \text{constant term} + \text{h.o.t.}
\]
Figure 7.1: Mesh for local analysis of errors

where $\alpha$ is related to $A$ in (7.2). Neglecting $R$ in 7.1, we get by simple computations

\[
\frac{\|e\|_{L^1}}{\|e\|_{L^2}} = \frac{(a^2 + b^2)(1 + 2\alpha + 3\alpha^2) + a^2}{(a^2 + b^2)(1 - 2\alpha + 3\alpha^2) + a^2}
\]

($\|e\|_{L^1}$ is the energy norm of the error in element $i$) and we see that we have in the case without pollution ($\alpha = 0$)

\[
\frac{\|e\|_{L^1}}{\|e\|_{L^2}} = 1
\]

while the ratio is $\neq 1$ in the presence of the pollution error. This explains our observation in Section 6.7.

Let us note that the said effect occurs when the pollution error and the error of the interpolant have the same strength. This is in agreement with our results in Section 6.7, where the oscillation phenomenon was only visible for the singularity $\alpha = .5$.

8 Conclusions

Summarizing the results that we have presented in the previous sections we draw the following conclusions.

1. The estimator performs as expected from its properties given in Theorems 4.1 and 4.2.

2. The effectivity index depends on the mesh, especially on the angle of the triangles. Hence meshes without small angles are preferable. The notion of the angle depends on the differential operator.

3. The effectivity index is not too sensitive to the topology of the mesh (except for the minimal angle).

4. The effectivity index can be larger or smaller than 1 depending on the character of the solution as predicted in Theorems 4.1 and 4.2.

5. If the solution has singular behaviour, the quality of the error estimator deteriorates. The deterioration can be avoided by adaptive refinement, as follows from Theorem 4.1.
References


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- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

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