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The Probability of Multiple Correct Packet Receptions in a Multireceiver Frequency-Hopped Spread-Spectrum System

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In this report we develop exact expressions and approximations for the probability of exactly l correct packet receptions in a group of m receivers, given that k packets are transmitted simultaneously from users employing frequency-hopping spread-spectrum (FH/SS) and Reed-Solomon (RS) coding. This quantity is essential for the design and performance evaluation of multiple-access protocols in spread-spectrum packet radio networks. The evaluations are carried out for MFSK modulation with noncoherent demodulation and various types of RS minimum-distance decoding with erasures and/or errors decoding. Packet synchronous FH/SSMA systems that are hop-synchronous or hop-asynchronous are considered.

The exact expressions developed have computational complexity that grows exponentially with m, thus making their evaluation for large m prohibitive. In light of such difficulties, two approximations are considered: one based on Gaussian multivariate distributions with linear computational requirements in m and k, and another based on the assumption of independent receiver operation (IROA), which has minimum computational complexity. Extensive comparisons of the approximations with the exact results establish their validity over different ranges of the system parameters.

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THE PROBABILITY OF MULTIPLE CORRECT PACKET RECEPTIONS IN A MULTIRECEIVER FREQUENCY-HOPPED SPREAD-SPECTRUM SYSTEM

1. INTRODUCTION AND PROBLEM FORMULATION

An important quantity in spread-spectrum radio networks is the probability that exactly l out of m packet transmissions are successful, given that k users attempt to transmit their packets simultaneously; this quantity is denoted by $P(l, m-l|k)$. This quantity is essential for the integration of the first three layers (physical, data link, and network layer) of the ISO network model for packet radio networks using spread-spectrum signaling and forward-error-control and, as such, it enables the design and performance evaluation of multiple-access protocols for such networks.

The integer m in $P(l, m-l|k)$ denotes the number of receivers of interest; in most practical situations, $m \leq k$. Specifically, in problems involving multireception with a bank of m receivers at a single location, the probability mass function (pmf)

$$P(l, m-l|k) \text{ for } l = 0, 1, \cdots, m \text{ and } m \leq k$$

describes the multireceiver performance. Moreover, in problems in which the evaluation of the throughput or delay of various packet radio network protocols is desirable,

$$P_c(l|k) = P_T(l, k-l|k) = \binom{k}{l} P(l, k-l|k) \text{ for } l = 0, 1, \cdots, k$$

is required, where $P_c(l|k)$ denotes the probability of any l correct packet receptions out of k simultaneous transmissions (see [1],[2]).

Consequently, in practical spread-spectrum packet radio networks, there is an undisputed need to evaluate the probabilities $P(l, m-l|k)$ and $P_T(l, m-l|k) = \binom{m}{l} P(l, m-l|k)$ ($l = 0, 1, \cdots, m$ and $m \leq k$) for different spreading signaling formats, data modulation schemes, and error-control coding schemes.

In this report, we evaluate these quantities for frequency-hopped (FH) spread-spectrum multiple-access (SSMA) networks. Specifically, we develop an exact expression for $P(l, m-$

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$l|k$) and an approximation based on Gaussian multivariate densities. The **exact expression** is difficult to compute and its computational complexity grows exponentially with m. By contrast, the **Gaussian approximation** is computationally efficient and its complexity grows linearly with m. Numerical results obtained from these two methods are compared with those obtained via the **independence assumption** method commonly used in the literature [1],[2]. This method assumes that packet errors among different receivers are mutually independent, which greatly simplifies the computation. We further establish that the independence assumption can be trusted in most cases and derive the range of parameters in which each of the two approximations (Gaussian and independence) is preferable.

Derivations and comparisons are carried out for FH/SS systems employing MFSK modulation with noncoherent demodulation and Reed-Solomon (RS) (n, k_c) forward error-control coding with erasures-only, errors-only, and errors /erasures minimum-distance [3] decoding. It is assumed that each RS symbol carries one M-ary symbol (i.e., $n = M$), that each FH dwell time (hop) carries one RS symbol, and that one RS codeword per packet is transmitted. The frequency-hopping patterns of the different users are modeled as random memoryless hopping patterns [4]. Thus, each of q available frequencies are visited with equal probability and independently of each other during any dwell time (hop) by each user and mutually independent hopping patterns are assigned to distinct users. The various users are packet-synchronous but may be hop-asynchronous; in this context, both hop-synchronous and hop-asynchronous FH/SSMA systems are considered. Also thermal noise modeled as additive white Gaussian noise (AWGN) is incorporated in the analysis.

This report is organized as follows: In Section 2 exact expressions for $P(l, m-l|k)$ are derived for all cases of interest enumerated above. In Section 3, the corresponding expressions based on the Gaussian approximation technique are derived. In Section 4, the approximation based on the independence assumption is cited. Numerical results and comparisons of the three approaches are presented provided in Section 5. In Section 6, conclusions are drawn.

2. DERIVATION OF EXACT MULTIRECEPTION PROBABILITIES

We are interested in finding an exact expression for the probability $P(l, m-l|k)$ of l receivers receiving correctly and $m-l$ ones receiving erroneously, for a specific set of receivers. It assumed that all users employ similar MFSK modulation (with M frequency tones) with noncoherent demodulation and identical extended RS (n, k_c) codes with codeword length n symbols ($n = M$) and k_c information symbols per codeword. (Refer to the additional assumptions in the previous section.) Due to the symmetry in the system, we can equivalently find the probability of the first l receivers decoding correctly, while the remaining $m-l$ receivers decode in error. In the sequel, we implicitly assume that $m \geq 2$. For $m = 1$, the model reduces to the single receiver model analyzed in [4].

For FH/SSMA communications, the probability of a coded symbol error is upper-bounded by the probability of a hit, which is a function of the available frequency slots q and the number of contending users k, where $k \geq 1$. Subsequently, we denote by $P_h(q, k)$ the probability of a hit. Recall from [4] that for hop-synchronous FH/SSMA systems

$$P_h(q, k) = 1 - (1 - 1/q)^{k-1} \tag{2.1a}$$

and for hop-asynchronous systems

$$P_h(q, k) = 1 - (1 - 2/q)^{k-1}. \tag{2.1b}$$

It is assumed that $k \geq m \geq 2$ and $q \geq m \geq 2$ so that the above quantities are guaranteed to be nonnegative. For asynchronous systems the above upper bound is valid for any number of symbols per hop (dwell-time) equal or larger than 1 (slow frequency-hopping).

2.1. Errors-Only or Erasures-Only RS Decoding

The ith receiver receives correctly the transmitted packet, if the number of hits $h(i)$, for $1 \leq i \leq m$, satisfies

$$0 \leq h(i) \leq t,$$

where t denotes the correction capability of the (n, k_c) block code (k_c is the number of information symbols and n the total number of symbols per codeword) For pure error-correction,

$$t = \left\lfloor \frac{d_{min} - 1}{2} \right\rfloor = \frac{n - k_c}{2} \tag{2.2a}$$

while, for erasure correction,

$$t = d_{min} - 1 = n - k_c. \tag{2.2b}$$

In the synchronous case, each receiver output depends only on other receivers outputs during the same dwell time. This applies also to the asynchronous case, if proper interleaving takes place. Then the total system operation becomes memoryless.

As amplified in Appendix A,

$$P(l, m-l|k) =$$

$$= P(0 \le h(1) \le t, \ldots, 0 \le h(l) \le t, t+1 \le h(l+1) \le n, \ldots, t+1 \le h(m) \le n)$$

$$= \sum_{\ell_1} \sum_{\ell_2} \cdots \sum_{\ell_{2^m-2}} \sum_{\ell_{2^m-1}} \binom{n}{\ell_1} \binom{n-\ell_1}{\ell_2} \cdots \binom{n - \sum_{nn=1}^{2^m-2} \ell_{nn}}{\ell_{2^m-1}}$$

$$\cdot P_1^{\ell_1} \cdot P_2^{\ell_2} \ldots P_{2^m-1}^{\ell_{2^m-1}} \cdot (P_0)^{n - \sum_{nn=1}^{2^m-1} \ell_{nn}} \tag{2.3}$$

where $P_{nn} = P(E_{nn})$ denotes the probability of the event E_{nn}, under which the m demodulator outputs during a particular symbol of the codeword (packet) correspond to the binary representation of nn (recall 0 and 1 denote correct and incorrect reception, respectively), and ℓ_{nn} is the number of times the event E_{nn} occurs in one codeword. Of course, any other correspondence of the above events and the natural numbers would work as well. Note that in (2.3), all events having same weight have equal probabilities, although this does not simplify the expression.

The range of ℓ_{nn} for the sums in (2.3) is to be obtained from a Diofantine analysis of the inequalities:

$$0 \le \sum_{nn=1}^{2^m-1} a_{nn}^{(i)} \cdot \ell_{nn} \le t \quad , \quad i = 1, 2, \ldots l \tag{2.4a}$$

and

$$t+1 \le \sum_{nn=1}^{2^m-1} a_{nn}^{(i)} \cdot \ell_{nn} \quad , \quad i = l+1, l+2, \ldots m \tag{2.4b}$$

where $a_{nn}^{(i)} = 1$ or 0, according to whether the ith component of the vector event E_{nn} is 1 or 0, that is, it takes part in the ith receiver error count or not. We have to add another constraint to the m constraints posed by (2.4a,2.4b), namely that

$$0 \leq \sum_{nn=1}^{2^m-1} \ell_{nn} \leq n \quad . \tag{2.4c}$$

The purpose of this constraint is to ensure that we do not surpass the codeword length n by permitting higher values of the ℓ_{nn}s.

It remains to find expressions for all $P(E_{nn})$, for $nn = 1, 2, \cdots, 2^m$. This is equivalent to finding the probability of having ρ demodulator outputs correct and $m - \rho$ ones in error during a particular transmitted symbol (of identical order for all receivers), for $\rho = 0, 1, \cdots, m$. These probabilities should be a function of ρ, m, k, and q. We denote them by $P_s(\rho, m, q, k)$. First we find $P(E_0) = P_{cc\ldots c} = P_s(m, 0, q, k)$, that is, the probability of decoding correctly all the simultaneous symbols in all receivers ; recall E_0 corresponds to the vector event $(0, 0, \cdots, 0)$. Because of the symmetry we get

$$\begin{aligned} P_{\underbrace{c\ldots c}_{m}} &= P(c|\underbrace{c\ldots c}_{m-1}) \cdot P_{\underbrace{c\ldots c}_{m-1}} \\ &= P(c|\underbrace{c\ldots c}_{m-1}) \cdot P(c|\underbrace{c\ldots c}_{m-2}) \ldots P(c|cc) \cdot P(c|c) \cdot P(c) \\ &= \prod_{j=1}^{m} [1 - P_h(q-j+1, k-j+1)]. \end{aligned} \tag{2.5}$$

In the above and in subsequent expressions the following notation is used: $P(c|\underbrace{c\ldots c}_{i})$ or $P(e|\underbrace{c\ldots c}_{i})$, $i = 1, 2, .., m-1$, denote the conditional probabilities of a particular symbol of a single receiver being correct or incorrect given that symbols of the same order of i other receivers are correct; similarly $P(\underbrace{e\ldots e}_{j}|\underbrace{c\ldots c}_{i})$ denotes the conditional probability of a particular symbol of j receivers being incorrect given that symbols of the same order of i other receivers are correct; and $P(\underbrace{c\ldots c}_{i})$ or $P(\underbrace{e\ldots e}_{i})$ denote the the unconditional

(absolute) probabilities of a particular symbol of i receivers being correct or incorrect, respectively. We now find $P(E_{2^m-1}) = P_{ee\cdots e} = P_s(0,m,q,k)$. We have

$$
\begin{aligned}
P_{\underbrace{e\ldots e}_{m}} &= P(e|\underbrace{e\ldots e}_{m-1})P_{\underbrace{e\ldots e}_{m-1}} \\
&= \left[1 - P(c|\underbrace{e\ldots e}_{m-1})\right] \cdot P_{\underbrace{e\ldots e}_{m-1}} \\
&= \left[1 - \frac{P(e\ldots e|c)\cdot P(c)}{P_{\underbrace{e\ldots e}_{m-1}}}\right] \cdot P_{\underbrace{e\ldots e}_{m-1}} \\
&= P_{\underbrace{e\ldots e}_{m-1}} - P(c)\cdot P(\underbrace{e\ldots e}_{m-1}|c)
\end{aligned}
$$

$$
\Rightarrow P_s(0,m,q,k) = P_s(0,m-1,q,k) - [1 - P_h(q,k)]\cdot P_s(0,m-1,q-1,k-1) \qquad (2.6)
$$

In these expressions Equation (2.6) is a recursive formula for finding $P_s(0,m,q,k)$. The solution of this equation, as shown in Appendix A, is

$$
P_s(0,m,q,k) = 1 + \sum_{i=1}^{m}\left[(-1)^i\binom{m}{i}\prod_{j=1}^{i}[1 - P_h(q-j+1,k-j+1)]\right] \qquad (2.7)
$$

Proceeding one step further we obtain the more general expression

$$
\begin{aligned}
P(E_{nn}) &= P_s(\rho, m-\rho, q, k) \\
&= P_{\underbrace{e\ldots e}_{m-\rho}\ \underbrace{c\ldots c}_{\rho}} \\
&= P(\underbrace{e\ldots e}_{m-\rho}|\underbrace{c\ldots c}_{\rho})P_{\underbrace{c\ldots c}_{\rho}} \\
&= P_s(0, m-\rho, q-\rho, k-\rho)\cdot P_s(\rho, 0, q, k) \\
&= \prod_{j=1}^{\rho}[1 - P_h(q-j+1,k-j+1)] \\
&\quad \cdot \left[1 + \sum_{i=1}^{m-\rho}(-1)^i\binom{m-\rho}{i}\prod_{j=1}^{i}[1 - P_h(q-\rho-j+1,k-\rho-j+1)]\right].
\end{aligned}
$$

(2.8)

The equality $P(\underbrace{e\ldots e}_{m-\rho}|\underbrace{c\ldots c}_{\rho}) = P_s(0, m-\rho, q-\rho, k-\rho)$ is obtained by observing that conditioning on a particular symbol of ρ receivers being correct is equivalent to reducing the number of available frequencies for hopping q and the number of users k by ρ. The last equality in (2.8) is then obtained through substitution from (2.5) and (2.7). Equations (2.3) and (2.8), together with the constraints posed by (2.4a), (2.4b), (2.4c), give the solution to our problem. As it becomes clear from (2.3), the exact evaluation of $P(l, m-l|k)$ requires the computation of $2^m - 1$ dependent sums, in which the limits should be found through a Diofantine analysis of (2.4a),(2.4b), and (2.4c). In addition, the summands are powers of $P(E_{nn})$, which can be computed through (2.8). Due to those computational requirements, exact expressions are nearly impossible to evaluate, for $m \geq 4$. However, as the next sections indicate, useful approximations have been developed with satisfactory accuracy in different ranges of the various system parameters of interest.

2.2. Inclusion of AWGN and Errors-Only or Erasures-Only RS Decoding

The expression in (2.3) remains valid in its general form; however, the expressions for $P_{\underbrace{e\ldots e}_{\mu}\ \underbrace{c\ldots c}_{\nu}}$ change in order to take the effects of AWGN into account. Let us find first $P_{\underbrace{c\ldots c}_{\nu}}$. For that we get

$$P_{\underbrace{c\ldots c}_{\nu}} = (1 - P_N)^{\nu} \cdot P^{(h)}_{\underbrace{c\ldots c}_{\nu}} \tag{2.9}$$

where $P^{(h)}_{\underbrace{c\ldots c}_{\nu}}$ denotes the previously evaluated $P_{\underbrace{c\ldots c}_{\nu}}$ in (2.5) by counting errors (or erasures) from hits only, and P_N is the symbol error probability of the MFSK due to AWGN. To find $P_{\underbrace{e\ldots e}_{\mu}}$ we write

$$\begin{aligned} P_{\underbrace{e\ldots e}_{\mu}} &= P(e|\underbrace{e\ldots e}_{\mu-1}) \cdot P_{\underbrace{e\ldots e}_{\mu-1}} \\ &= P_{\underbrace{e\ldots e}_{\mu-1}} - P(c) \cdot P(\underbrace{e\ldots e}_{\mu-1}|c) \end{aligned}$$

$$\Rightarrow P_e(\mu,q,k) \;=\; P_e(\mu-1,q,k)-(1-P_N)[1-P_h(q,k)]\cdot P_e(\mu-1,q-1,k-1). \quad (2.10)$$

The recursive equation (2.10) is similar to (2.6) and thus has as solution the expression

$$P_e(\mu,q,k) \;=\; 1 \;+\; \sum_{i=1}^{\mu}\left[(-1)^i\binom{\mu}{i}\prod_{j=1}^{i}\{(1-P_N)[1-P_h(q-j+1,k-j+1)]\}\right] \quad (2.11)$$

From that we obtain easily the more general result

$$\begin{aligned}
P(E_{nn}) \;&=\; P_{\underbrace{e\ldots e}_{\mu}\,\underbrace{c\ldots c}_{\nu}} \;=\; P(\mu,\nu) \;=\; P(\underbrace{e\ldots e}_{\mu}|\underbrace{c\ldots c}_{\nu})\cdot P_{\underbrace{c\ldots c}_{\nu}} \\
&=\; P_e(\mu,q-\nu,k-\nu)\cdot P_{\underbrace{c\ldots c}_{\nu}} \\
&=\; (1-P_N)^{\nu}\cdot\prod_{j=1}^{\nu}[1-P_h(q-j+1,k-j+1)] \\
&\quad\cdot\left[1\;+\;\sum_{i=1}^{\mu}(-1)^i\binom{\mu}{i}\prod_{j=1}^{i}\{(1-P_N)[1-P_h(q-\nu-j+1,k-\nu-j+1)]\}\right]
\end{aligned} \quad (2.12)$$

which gives the desired expression. For our problem, $\mu+\nu=m$.

The probability P_N of MFSK symbol errors due to AWGN given by

$$P_N \;=\; \sum_{m=1}^{M-1}\binom{M-1}{m}\frac{(-1)^{m+1}}{m+1}e^{-\frac{m}{m+1}\frac{r\cdot\log_2 M\,E_b}{N_0}} \quad (2.13)$$

where E_b/N_0 is the information symbol signal-to-noise ratio and r is the code rate of the system ($r=k_c/n$).

2.3. Errors and Erasures RS Decoding

In this section, we analyze the case in which all receivers employ combined errors and erasures decoding. Thus during the minimum distance RS decoding correction of both errors and erasures is attempted. We assume that erasures happen whenever a hit from other users occurs and they are detected; errors are caused by AWGN only.

It is well-known that any RS code can decode correctly any received word having s erasures, t errors if these numbers satisfy

$$2t + s \leq d_{\min} - 1 = n - k_c \tag{2.14}$$

We start the analysis for this case by noting that the basic "simultaneous" events in the multi-receiver are now defined by $x = (x_1, \ldots, x_m)$, where x_i, $i = 1, \ldots, m$, can take values c, e, s corresponding to correct, error, or erasure symbol, respectively. Again, because of symmetry, the probability of finding in a particular symbol (or dwell-time) some receivers in error, some in correct reception, and some in erasure is independent of the particular order of the receivers.

For a given symbol we define

$$Pr(\ n_e \textit{ receivers in error, } n_s \textit{ receivers in erasure, } n_c \textit{ receivers correct }\ )$$

$$= P_{\underbrace{e \ldots e}_{n_e} \underbrace{s \ldots s}_{n_s} \underbrace{c \ldots c}_{n_c}} = P(n_e, n_s, n_c) \tag{2.15}$$

where, of course,

$$n_e + n_s + n_c = m. \tag{2.16}$$

To find $P(l, m-l|k)$ we notice that the basic difference from the treatment in Section 2.1, is that now we have three different states in every symbol. So, instead of having $2^m - 1$ sums, we have $3^m - 1$ sums. If, we define as E_{nn} the event according to which the m demodulator outputs during the same symbol correspond to the ternary representation of nn, where 0 denotes correct reception, 1 denotes an erasure, and 2 denotes an error, and set ℓ_{nn} to denote the number of times the event E_{nn} occurs in one codeword, then we can write

$$P(l, m-l|k) = \sum_{\ell_1} \sum_{\ell_2} \cdots \sum_{\ell_{3^m-2}} \sum_{\ell_{3^m-1}} \binom{n}{\ell_1} \binom{n-\ell_1}{\ell_2} \cdots \binom{n - \sum_{nn=1}^{3^m-2} \ell_{nn}}{\ell_{3^m-1}}$$

$$\cdot P_1^{\ell_1} \cdot P_2^{\ell_2} \ldots P_{3^m-1}^{\ell_{3^m-1}} \cdot (P_0)^{n - \sum_{nn=1}^{3^m-1} \ell_{nn}}$$

(2.17)

The range of values of $\ell_1, \ell_2, \cdots, \ell_{3^m-1}$ can be found from the m inequalities:

$$0 \leq \sum_{nn=1}^{3^m-1} a_{nn}^{(i)} \cdot \ell_{nn} \leq n - k_c \text{ for } i = 1, 2, \cdots, l \tag{2.18a}$$

and

$$n - k_c + 1 \leq \sum_{nn=1}^{3^m-1} a_{nn}^{(i)} \cdot \ell_{nn} \text{ for } i = l+1, l+2, \cdots, m \tag{2.18b}$$

where $a_{nn}^{(i)}$ takes values $\in \{0, 1, 2\}$, depending on what type of event (correct symbol, erasure, or error) is implied for receiver $i \in \{1, \ldots, m\}$ from the i-th component of the ternary representation of nn (identifying the event E_{nn}); that is, $a_{nn}^{(j)} = 0$ for correct reception, $a_{nn}^{(j)} = 1$ for erasures, and $a_{nn}^{(j)} = 2$ for errors. To the above inequalities we must add the condition

$$0 \leq \sum_{nn=1}^{3^m-1} \ell_{nn} \leq n \tag{2.18c}$$

which ensures that the total number of symbols remains smaller than n, the codeword (or packet) length.

In order to evaluate $P(l, m-l|k)$, we need to calculate expressions of the form $P_{\underbrace{e \ldots e}_{n_e} \underbrace{s \ldots s}_{n_s} \underbrace{c \ldots c}_{n_c}}$, where $n_e + n_s + n_c = m$. We can write

$$P_{\underbrace{e \ldots e}_{n_e} \underbrace{s \ldots s}_{n_s} \underbrace{c \ldots c}_{n_c}} = P(\underbrace{s \ldots s}_{n_s}|\underbrace{e \ldots e}_{n_e} \underbrace{c \ldots c}_{n_c}) \cdot P(\underbrace{e \ldots e}_{n_e}|\underbrace{c \ldots c}_{n_c}) \cdot P(\underbrace{c \ldots c}_{n_c}) \tag{2.19}$$

where

$$P(\underbrace{c \ldots c}_{n_c}) = \prod_{j=1}^{n_c} \{(1 - P_N)[1 - P_h(q - j + 1, k - j + 1)]\} \tag{2.20a}$$

$$P(\underbrace{e \ldots e}_{n_e}|\underbrace{c \ldots c}_{n_c}) = \prod_{j=1}^{n_e} \{P_N[1 - P_h(q - n_c - j + 1, k - n_c - j + 1)]\} \tag{2.20b}$$

$$P(s\underbrace{\cdots}_{n_s}s|e\underbrace{\cdots}_{n_e}e\ c\underbrace{\cdots}_{n_c}c)$$

$$= 1 + \sum_{i=1}^{n_s}(-1)^i\binom{n_s}{i}\prod_{j=1}^{i}[1 - P_h(q - n_e - n_c - j + 1, k - n_e - n_c - j + 1)] \tag{2.20c}$$

so that

$$P(E_{nn}) = P_{e\underbrace{\cdots}_{n_e}e\ s\underbrace{\cdots}_{n_s}s\ c\underbrace{\cdots}_{n_c}c} = P(n_e, n_s, n_c)$$

$$= \prod_{j=1}^{n_c}\{(1 - P_N)[1 - P_h(q - j + 1, k - j + 1)]\}$$

$$\cdot \prod_{j=1}^{n_e}\{P_N[1 - P_h(q - n_c - j + 1, k - n_c - j + 1)]\}$$

$$\cdot \left[1 + \sum_{i=1}^{n_s}(-1)^i\binom{n_s}{i}\prod_{j=1}^{i}[1 - P_h(q - n_e - n_c - j + 1, k - n_e - n_c - j + 1)]\right] \tag{2.21}$$

2.4. Asynchronous FH/SSMA Case

It is straightforward to extend the results of the hop-synchronous case to the hop-asynchronous one, if we assume that all symbols within each codeword are interleaved. In the asynchronous case, in addition to the full hits that strike a particular user, we have partial ones, as well; this increases the probability of a hit to $P_h = 1 - (1 - 2/q)^{k-1}$, as stated at the beginning of this section. Substituting this for P_h in the above expressions we obtain the desired expressions for the multireception probabilities $P(l, m - l|k)$.

3. A NEW APPROXIMATION FOR $P(l, m - l|k)$

In the previous section, we derived exact expressions for the probabilities $P(l, m - l|k)$ $(l = 0, 1, \cdots, m,\ m \leq k)$. Here we develop an approximation method based on the Gaussian multivariate distribution. First we present a general approximation technique for $P(l, m - l|k)$ for systems with general interference covariance matrices. Then we exploit

the specific form of the covariance matrix of the interference for the symmetric FH/SSMA problem to obtain a closed form expression. Finally, we derive the necessary covariance matrices for the various cases of interest.

3.1 Gaussian Approximation

The computation of $P(l, m-l|k)$ is essentially a combinatoric problem, which involves keeping track of the bit erasures or errors and declares an error when the count for a particular user exceeds the correcting capability of the code. This is essentially the multi-dimensional extension of the binomial counting experiment, which, however, results in excessive computational complexity, as shown in the previous section. Here we we approximate the required "multi-nomial" probability distribution with a multidimensional Gaussian.

Let us define the random variables $x_{\ell i}$ $(1 \leq \ell \leq n,)$, where n is the number of bits per packet for the ith among m users of interest $(m \leq k)$ so that

$x_{\ell i}$ = 1, if the ℓth bit (or symbol) of user i is incorrect (due to an erasure or an error, depending on the decoding which occurs with known probability p)

$x_{\ell i}$ = 0, if the ℓth bit of user i is correct with probability $(1-p)$.

The actual calculation of p presents no difficulty and will be carried out in Section 3.3 for the various case of interest. Now define for user i $(1 \leq i \leq m)$ the RV x_i such that

$$x_i = \sum_{\ell=1}^{n} x_{\ell i}.$$

x_i is the number of bit errors or erasures that sender i suffers among his n bits within the packet (slot). Thus, $0 \leq x_i \leq n$. $x_{\ell i}$ is independent of $x_{\ell' i}$ when $\ell \neq \ell'$, because of the random FH patterns assumed. Consequently, x_i is the sum of n i.i.d. random variables and this tends towards a Gaussian distribution for large n [mean np, variance $np(1-p)$]. If we consider any linear combination of the x_is, say

$$\bar{z} = \sum_{i=1}^{m} a_i x_i$$

then

$$\overline{z} = \sum_{i=1}^{m} a_i \left(\sum_{\ell=1}^{n} x_{\ell i} \right)$$

$$= \sum_{\ell=1}^{n} \sum_{i=1}^{k} a_i \; x_{\ell i}$$

$$= \sum_{\ell=1}^{n} z_\ell .$$

It turns out that the z_ℓs are either independent of each other or one-dependent (i.e., $z_{\ell-1}$, z_ℓ, $z_{\ell+1}$ are dependent but z_ℓ is independent of $z_{\ell+2}, z_{\ell-2}, z_{\ell+3}, z_{\ell-3}, \cdots$). Independence arises if the system is hop synchronous and one-dependence when relative hop offsets are permitted, as in the hop-asynchronous systems (see Section 3.3.2). In either case, $\overline{z}$ is a sum of i.i.d. or one-dependent RVs and, as $n \to \infty$, tends to have a Gaussian distribution. Consequently, all x_is are jointly Gaussian if the codeword length n is sufficiently large. Define the m-dimensional column vectors

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix} \quad \text{and} \quad \underline{\mu} = \begin{bmatrix} np \\ np \\ \cdot \\ \cdot \\ \cdot \\ np \end{bmatrix} . \tag{3.0}$$

Then we have the multivariate Gaussian probability density function (pdf)

$$p_{\underline{x}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^m |\Sigma|}} \; e^{-(\underline{x}-\underline{\mu})^T \Sigma^{-1} (\underline{x}-\underline{\mu})} \tag{3.1}$$

where Σ is the $m \times m$ covariance matrix with diagonal elements

$$a = E\{(x_i - np)^2\} \tag{3.2}$$

and off diagonal elements (all of which are equal due to the symmetry of our problem)

$$b = E\{(x_i - np)(x_j - np)\} \tag{3.3}$$

$$= E\left\{ \left[\sum_{\ell=1}^{n} (x_{\ell i} - p) \right] \left[\sum_{\ell=1}^{n} (x_{\ell j} - p) \right] \right\} .$$

a and b are calculated in Section 3.3 on the basis of the signaling scheme (FH hop-synchronous or hop-asynchronous) or on the basis of the decoding scheme and the presence of AWGN. We know that, for user i to be successful,

$$0 \leq x_i \leq e$$

where e is the erasure or error correcting capability of the RS code ($e = n - k_c$ or $e = (n - k_c)/2$, respectively, for RS).

If user i is unsuccessful,

$$e < x_i \leq n.$$

Hence,

$$P_G(l, m - |k) = \int_0^e \cdots \int_0^e \int_{e+1}^n \cdots \int_{e+1}^n p_{\underline{x}}(\underline{x}) dx_1 \ldots dx_k. \tag{3.4}$$

In the above equation, there are l integrals of the form $\int_0^e$ and $m - l$ integrals of the form $\int_{e+1}^n$. We define

$$\underline{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e+1 \\ \vdots \\ e+1 \end{bmatrix} \begin{matrix} \}l \\ \\ \}m-l \end{matrix}$$

$$\underline{x}_2 = \begin{bmatrix} e \\ \vdots \\ e \\ n \\ \vdots \\ n \end{bmatrix} \begin{matrix} \}l \\ \\ \}m-l \end{matrix}$$

and have

$$P_G(l, m - l/k) = \int_{\underline{x}_1}^{\underline{x}_2} \frac{1}{\sqrt{(2\pi)^m |\Sigma|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1} (\underline{x}-\underline{\mu})} d\underline{x}. \tag{3.5}$$

This integral can be simplified, if the exponent is converted from a quadratic to a sum-of-squares form. This can be materialized through a linear transformation that diagonalizes

Σ^{-1}. Now Σ takes the form

$$\Sigma = \begin{bmatrix} a & b & b & \dots & \dots & b \\ b & a & & \vdots & \vdots & \vdots \\ b & \vdots & a & \vdots & \vdots & \vdots \\ \vdots & \vdots & & a & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ b & b & \dots & \dots & \dots & a \end{bmatrix}_{m \times m} .$$

Therefore,

$$\Sigma = \begin{bmatrix} a-b & & & \\ & a-b & 0 & \\ 0 & & \ddots & \\ & & & a-b \end{bmatrix}_{m \times m} + \begin{bmatrix} b & b & . & . & . & b \\ b & & & & & \\ \vdots & & & & & \\ b & . & . & . & . & b \end{bmatrix}_{m \times m}$$

and, consequently,

$$\Sigma = (a-b)I + b \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1} [1 \quad 1 \quad 1 \quad \dots]_{1 \times m}$$

or, equivalently,

$$\Sigma = A + \underline{u}\,\underline{u}^T$$

where $u^T = \sqrt{b}[1\ 1\ \dots\ 1]_{1 \times m}$ and $A = (a-b)I$. Moreover,

$$\Sigma^{-1} = A^{-1} - \frac{(A^{-1}\underline{u})(\underline{u}^T A^{-1})}{1 + \underline{v}^T A^{-1} \underline{u}}$$

$$= \frac{1}{(a-b)} I - \frac{\underline{u}\,\underline{u}^T}{(a-b)(a+(m-1)b)}$$

so that, if we define

$$\bar{a} = \frac{1}{a-b} \tag{3.6}$$

$$\bar{b} = \frac{b}{(a-b)[a+(m-1)b]} \tag{3.7}$$

then

$$\Sigma^{-1} = \bar{a}\, I - \begin{bmatrix} \bar{b} & \bar{b} & . & . & . & \bar{b} \\ \bar{b} & \bar{b} & & & & \\ \bar{b} & & & & & \\ \vdots & & & & & \\ \bar{b} & . & . & & & \bar{b} \end{bmatrix}_{m \times m} . \tag{3.8}$$

One way to diagonalize Σ^{-1} is to find its eigenvalues and the corresponding eigenvectors and then create a transformation matrix with the eigenvectors as its columns. The eigenvalues of Σ^{-1} are given by the equation

$$\det(\lambda\, I - \Sigma^{-1}) = 0.$$

But

$$\lambda\, I - \Sigma^{-1} = (\lambda - \bar{a})I + \begin{bmatrix} \bar{b} & \bar{b} & . & . & \bar{b} \\ \bar{b} & & & & \\ \vdots & & & & \\ \vdots & & & & \\ \bar{b} & & & & b \end{bmatrix}_{m \times m}$$

$$= (\lambda - \bar{a})I + B$$

where B is of rank 1, and

$$\begin{aligned} \det(\lambda\, I - \Sigma^{-1}) &= (\lambda - \bar{a})^m \det(I + \frac{1}{(\lambda - \bar{a})} B) \\ &= (\lambda - \bar{a})^m \left\{ 1 + \text{trace}\left(\frac{B}{(\lambda - \bar{a})} \right) \right\} \\ &= (\lambda - \bar{a})^m \left[1 + \frac{m\, \bar{b}}{(\lambda - \bar{a})} \right] \\ &= (\lambda - \bar{a})^{m-1} (\lambda - (\bar{a} - m\, \bar{b})). \end{aligned}$$

Hence

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = \bar{a} = \lambda_a \tag{3.9}$$

and

$$\lambda_m = \bar{a} - m\, \bar{b} = \lambda_b \tag{3.10}$$

where

$$\lambda_a = \frac{1}{a-b}$$

and

$$\lambda_b = \frac{1}{a+(m-1)b}.$$

Finally, notice that

$$det(\Sigma) = (a-b)^{m-1}[a+(m-1)b] \tag{3.11}$$

3.2 The Integral Transform Method

Consider the exponent of the integrand in (3.5). After a shift of variables to account for the mean, it can be simplified to

$$-\frac{1}{2}\underline{x}^T\Sigma^{-1}\underline{x} = -\frac{1}{2(a-b)}\left[\sum_{j=1}^{m} x_j^2 - \frac{b}{a+(m-1)b}\left(\sum_{j=1}^{m} x_j\right)^2\right].$$

Using $u_j = \frac{x_j}{\sqrt{a-b}}$ makes the exponent

$$-\frac{1}{2}\left[\sum_{j=1}^{m} u_j^2 - \frac{b}{a+(m-1)b}(\sum_{j=1}^{m} u_j)^2\right].$$

The cross product terms come from the factor

$$\left(\sum_{j=1}^{m} u_j\right)^2.$$

But the square in the above equation can be eliminated with the help of the following integral transform (see [5]):

$$e^{\frac{1}{2}\sigma^2\phi^2} = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} dy\ e^{\phi y}e^{-\frac{y^2}{2\sigma^2}} \tag{3.12}$$

with

$$\sigma^2 = \frac{b}{a+(m-1)b} \tag{3.13}$$

and

$$\phi = \sum_{j=1}^{m} u_j. \tag{3.14}$$

After some manipulations in which we use (3.11)-(3.14), we obtain the basic result

$$P_G(l, m-l|k) = \int_{-\infty}^{+\infty} dy \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} [\Phi(z_1) - \Phi(z_2)]^l \cdot [1 - \Phi(z_1) + \Phi(z_2)]^{m-l} \tag{3.15}$$

where

$$z_1 = \frac{e - np - y\sqrt{b}}{\sqrt{a-b}} \tag{3.16a}$$

$$z_2 = \frac{-np - y\sqrt{b}}{\sqrt{a-b}} \tag{3.16b}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-u^2/2} du$$

and $e - np$ is the erasure (or error) correcting capability of the block code employed, shifted by the mean np. Eq. (3.15) gives a method for calculating $P_G(l, m-l|k)$ with linear computational complexity in m. Note that, for the case $b < 0$ (b being the off-diagonal elements of the covariance matrix), (3.15) involves Φ evaluated at complex arguments. This is perfectly legitimate and results in an additional small computational effort. In this case, the real part of the entity $[\Phi(z_1) - \Phi(z_2)]^l \cdot [1 - \Phi(z_1) + \Phi(z_2)]^{m-l}$ is involved in the integral of (3.15).

3.3. Derivation of Received Code Vector Means and Covariances

The method developed in Sections 3.1 and 3.2 for approximating the probability $P(l, m-l|k)$ requires only knowledge of the three quantities np, a, and b. These quantities are the mean of the received code vector $\underline{x}$ and the diagonal and off-diagonal terms of its covariance matrix Σ [see (3.0)-(3.3)]. Here we evaluate these quantities for FH signaling and the various types of system conditions (hop-synchronous or hop-asynchronous, presence of AWGN) and decoding schemes (erasures-only, errors-only, and errors/erasures decoding) of RS codes.

Each transmitter sends data in packets with n RS symbols per packet. For the jth symbol of the ith user, the RV x_{ji} takes value 1, if that symbol is either erased or in error, and 0 otherwise. The probability of x_{ji} being 1 is p (to be evaluated separately later for the two cases). The expression $x_i = \sum_{j=1}^{n} x_{ji}$ counts the number of errors or erasures per packet and has mean np; we also need a, as defined in (3.2) and b as defined in (3.3).

Each transmitter hops randomly between q available frequencies with one M-ary symbol transmitted per frequency hop. A "hit" takes place when the frequency at which a particular user chooses to transmit is also used by one or more other users during a dwell time. The occurrence of this event can be detected by listening to the channel and conducting a threshold test; then a symbol erasure is declared. Between different users, the duration of a hop (or dwell time) may be assumed perfectly synchronized, or more realistically, as involving relative delays. Noise (AWGN) may also be present in general, but this is dealt with a little later.

We first evaluate the desired quantities for the noiseless hop-synchronous case in Section 3.3.1, then for the noiseless hop-synchronous case in Section 3.3.2; for these cases errors-only and erasures-only decoding are treated together; finally, we incorporate the effects of AWGN and treat errors/erasures decoding in Section 3.3.3.

3.3.1 Noiseless Hop-Synchronous FH/SSMA

Since we only have multiple access interference, the decoding employed is erasures only; however the results are also applicable to errors-only decoding after a trivial change in the parameter ϵ. The probability of a hit is

$$p = 1 - \left(1 - \frac{1}{q}\right)^{k-1} \tag{3.17}$$

and

$$a = E\{(x_i - np)^2\} = np(1-p) \tag{3.18}$$

$$\begin{aligned} b &= E\{(x_i - np)(x_j - np)\} \\ &= E\left\{\left[\sum_{\ell=1}^{n}(x_{\ell i} - p)\right]\left[\sum_{\ell=1}^{n}(x_{\ell j} - p)\right]\right\} \end{aligned}$$

$$= \sum_{\ell=1}^{n} \left(E\{x_{\ell i}\, x_{\ell j}\} \; - \; p^2 \right).$$

But

$$E\{x_{\ell i} x_{\ell j}\} = Pr\{x_{\ell i} \; = \; 1, \; x_{\ell j} \; = \; 1\}$$

$$= \; Pr\{A_{ij}^{\ell}, \; x_{\ell i} \; = \; 1, \; x_{\ell j} \; = \; 1\} \; + \; Pr\{\overline{A_{ij}^{\ell}}, \; x_{\ell i} \; = \; 1, \; x_{\ell j} \; = \; 1\}$$

where A_{ij}^{ℓ} is the event that users i and j hop to the same frequency during the ℓth hop. Therefore,

$$E\left\{x_{\ell i} x_{\ell j}\right\} = 1 \cdot Pr\{A_{ij}^{\ell}\} + \left[\sum_{m=1}^{k-3} Pr\{\; m \; users \; hit \; i, \;\; user \; j \; gets \; hit \mid \overline{A}_{ij}^{\ell}\} \right] \cdot Pr\{\overline{A}_{ij}^{\ell}\}$$

$$= \frac{1}{q} + \left(1 - \frac{1}{q}\right) \left[\sum_{m=1}^{k-3} \binom{k-2}{m} \left(\frac{1}{q}\right)^{m} \left(1 - \frac{1}{q}\right)^{k-2-m} \left(1 - \left(1 - \frac{1}{q-1}\right)^{k-2-m}\right) \right]$$

$$= \frac{1}{q} + \left(1 - \frac{1}{q}\right) \left\{ 1 - \left(1 - \frac{1}{q}\right)^{k-2} - \left[1 - \left(1 - \frac{1}{q}\right) \frac{1}{q-1}\right]^{k-2} + \left[\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{q-1}\right)\right]^{k-2} \right\} \tag{3.19}$$

and

$$b \; = \; n \; E\{x_{\ell i} \; x_{\ell j}\} \; - \; np^2. \tag{3.20}$$

3.3.2 Noiseless Hop-Asynchronous FH/SSMA Channel

In Section 3.3.1, it was assumed that hops between users were perfectly synchronized, i.e., that there was no overlap between the lth hop of user i and the $(l+1)$th hop of user j, for any i, j. This assumption is not realistic, since we assume that the system is distributed and not centrally controlled. It is time now to relax that assumption.

Assume that each hop is of total duration T. Let us set a reference point for the beginning of each hop. Then we assume that the the displacement in time of the beginning of the hop of any user is an RV uniformly distributed between $-T/2$ and $+T/2$.

First we note that p, the probability that any user is hit while in the lth hop, changes as follows:

$$p = 1 - \left(1 - \frac{2}{q}\right)^{k-1}. \tag{3.21}$$

Note that x_{li} is independent of x_{mi}, for $l \neq m$, due to the random assumption for the hopping pattern. However, because of chip overlap, x_{lj} is not independent of x_{mi}, when m and l differ only by 1. Therefore, we have

$$a = E\{(x_i - np)^2\} = np(1-p) \tag{3.22}$$

$$\begin{aligned} b &= E\{(x_i - np)(x_j - np)\} \\ &= E\left\{\left[\sum_{\ell=1}^{n}(x_{\ell i} - p)\right]\left[\sum_{\ell=1}^{n}(x_{\ell j} - p)\right]\right\} \end{aligned}$$

$$\begin{aligned} = \; & E\{(x_{1i} - p)(x_{1j} - p) + (x_{1i} - p)(x_{2j} - p) \\ & + (x_{2i} - p)(x_{1j} - p) + (x_{2i} - p)(x_{2j} - p) + (x_{2i} - p)(x_{3j} - p) \\ & \cdots \\ & + (x_{ni} - p)(x_{(n-1)j} - p) + (x_{ni} - p)(x_{nj} - p)\}. \end{aligned}$$

which, since $E\{(x_{li} - p)(x_{(l+m)j} - p)\} = 0$ for any i, j l, $m = 2, 3, \ldots$, reduces to

$$b = n(\rho_1 - p^2) + (2n-2)(\rho_2 - p^2) \tag{3.23}$$

where $\rho_1 = E\{x_{li}x_{lj}\}$ and $\rho_2 = E\{x_{li}x_{(l+1)j}\}$.

Next we derive expressions for ρ_1 and ρ_2. In the following, whenever we refer to a hit, we mean on the lth hops of users i and j.

$$\begin{aligned} \rho_1 &= Pr\{x_{\ell i} = 1, x_{\ell j} = 1\} \\ &= Pr\{x_{\ell i} = 1, x_{\ell j} = 1 | \text{i, j hit each other on } \ell\text{th hop}\} \\ &\quad \cdot Pr\{\text{i, j hit each other on } \ell\text{th hop}\} \\ &+ Pr\{\, x_{\ell i} = 1, x_{\ell j} = 1 | \text{i, j do not hit each other on } \ell'\text{th hop}\} \\ &\quad \cdot Pr\{\text{i, j do not hit each other on } \ell'\text{th hop}\} \end{aligned}$$

The first term equals $1/q$. The second term is equal to

$$\left(1 - \frac{1}{q}\right) Pr\{x_{\ell i} = 1, x_{\ell j} = 1 | \text{i, j do not hit each other}\}.$$

We write the la.ter probability as

$$\sum_G Pr\{\text{ i hit by } G, \text{ i not hit by } \overline{G}, \text{ j is hit} | \text{ i, j do not hit each other}\}$$

$$= \sum_G Pr\{\text{ i hit by } G | \text{ i, j do not hit each other}\}$$

$$\cdot Pr\{\text{ j gets hit } | \text{ i hit by } G, \text{ i, j do not hit each other}\}$$

where G refers to all possible groups of the remaining $k-2$ users and all nits pertain to the ℓth hop.

Consider a particular user u from group G and the probability of u hitting j, given that u hits i and i and j do not hit each other. We note that there will be two consecutive hops (dwell times) of u that overlap in time with the lth hop of i. Out of these two, one or both hit i. Therefore, there are two possible situations: (1) both hops of u overlap with the lth hop of j with probability 1/2 (given the uniform distribution of the chip delay of the users) and (2) only one of these two hops overlaps with j with probability 1/2. Note that one of the above possibilities has to take place, since the lth hops of i and j do overlap to some extent, under the asynchronous FH/SS system assumption.

Consider the probability that u hits the lth hop of j, given that (1) above holds, u hits i, and i and j do not hit each other. This probability is

$$\frac{1}{2} \cdot \frac{1}{q-1} + \frac{1}{2} \cdot \frac{1}{q-1} = \frac{1}{q-1}$$

Also, consider the probability that u hits the lth hop of j, given that (2) above holds, that u hits i, and that i and j do not hit each other. This will be

$$\frac{1}{q} + \frac{1}{2} \cdot \frac{1}{q-1} = \frac{3}{2} \cdot \frac{1}{q-1}.$$

From all the above we surmise that the probability of u hitting the lth hop of j, given that u hits the lth hop of i and that i and j do not hit each other on the lth hop, is

$$\frac{1}{2}\cdot\frac{1}{q-1} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{q-1} = \frac{5}{4}\cdot\frac{1}{q-1}.$$

Finally,

$$\begin{aligned}
\alpha_1 &= Pr\{x_{li} = 1, x_{lj} = 1 |\text{ i j do not hit each other}\} \\
&= \sum_{l=1}^{k-2}\binom{k-2}{l}\left(\frac{2}{q}\right)^l\left(1-\frac{2}{q}\right)^{k-2-l}\left[1-\left(1-\frac{5}{4(q-1)}\right)^l\left(1-\frac{2}{q-1}\right)^{k-2-l}\right] \\
&= 1-\left(1-\frac{2}{q}\right)^{k-2} \\
&\quad -\left\{\left[\frac{2}{q}\left(1-\frac{5}{4(q-1)}\right)+\left(1-\frac{2}{q}\right)\left(1-\frac{2}{q-1}\right)\right]^{k-2}-\left[\left(1-\frac{2}{q}\right)\left(1-\frac{2}{q-1}\right)\right]^{k-2}\right\}
\end{aligned} \tag{3.24}$$

and thus

$$\rho_1 = \frac{1}{q}+\left(1-\frac{1}{q}\right)\alpha_1. \tag{3.25}$$

We now derive $\rho_2 = E\{x_{li}x_{(l+1)j}\}$. Here, when talking of a hit, we refer to the lth hop of i and to the $(l+1)$th hop of j. We have

$$\begin{aligned}
\rho_2 &= Pr\{x_{li} = 1\ x_{(l+1)j} = 1\} \\
&= Pr\{x_{li} = 1, x_{(l+1)j} = 1 |\text{ i, j overlap in time}\}Pr\{\text{ i, j overlap in time}\} \\
&\quad + Pr\{x_{li} = 1, x_{(l+1)j} = 1 |\text{ i, j do not overlap in time}\}Pr\{\text{ i, j do not overlap in time}\}
\end{aligned}$$

In the above equation,

$$Pr\{\text{ i, j overlap in time}\} = Pr\{\text{ i, j do not overlap in time}\} = 1/2$$

and

$$Pr\{x_{li} = 1, x_{(l+1)j} = 1 |\text{ i, j do not overlap in time}\} = Pr\{x_{li} = 1\}Pr\{x_{(l+1)j} = 1\} = p^2$$

Moreover,

$$
\begin{aligned}
Pr\{x_{li} &= 1, x_{(l+1)j} = 1 | \text{ i, j overlap in time}\} \\
&= Pr\{x_{li} = 1, x_{(l+1)j} = 1 | \text{ i, j, overlap, hit each other on } \ell + 1\text{th hop}\} \\
&\quad \cdot Pr\{ \text{ i, j hit each other on } \ell + 1\text{th hop} | \text{i, j overlap}\} \\
&\quad + Pr\{x_{li} = 1, x_{(l+1)j} = 1 | \text{ i, j overlap, do not hit each other on } \ell + 1\text{th hop}\} \\
&\quad \cdot Pr\{ \text{ i, j do not hit each other on } \ell + 1\text{th hop} | \text{i, j overlap}\}
\end{aligned}
$$

From this point on, the evaluation of the quantities of interest proceeds in a similar way to that for ρ_1 and thus it will be omitted. It turns out that

$$\alpha_2 = Pr\{x_{li} = 1, x_{(l+1)j} = 1 | \text{ i, j overlap, do not hit each other}\} = \alpha_1 \tag{3.26}$$

where α_1 is given by (3.24), so that the final result is

$$\rho_2 = \frac{1}{2} \cdot \left[\frac{1}{q} + \left(1 - \frac{1}{q}\right) \alpha_2\right] + \frac{1}{2} p^2. \tag{3.27}$$

The desired quantity b is now obtained from (3.24)-(3.25) and (3.26)-(3.27) upon substitution in (3.23).

3.3.3 Inclusion of AWGN at Front End of Each Receiver

So far, we have assumed that our channel is noiseless and that we can employ erasures-only decoding, since symbol erasures can be detected. If we incorporate noise, the possibility of a symbol error exists. Symbol error can only take place, if there is no erasure due to multiple-access interference or to hits. Now we shall employ errors/erasures decoding. We maintain for user i the RV z_{li} which is set to 1, if the lth bit is an erasure, and the w_{li}, which is set to 1, if it is in error. Define $x_{li} = z_{li} + 2w_{li}$. Let

$$x_i = \Sigma_{l=1}^{n} x_{li}$$

We know that, for user i to be successful,

$$0 \leq x_i \leq n - k_c.$$

Using arguments similar to those of the noiseless case, we can prove once again that the x_is are jointly Gaussian, each being the sum of one-dependent variables. We need, therefore, to find $E\{x_i\}$; for this we need $E\{w_{li}\}$ and the probability of M-ary symbol error p_N, which is

$$p_N = p_N(M) = \sum_{m=1}^{M-1} \binom{M-1}{m} \frac{(-1)^{m+1}}{m+1} e^{-\frac{m}{m+1}\frac{r\log_2 M E_b}{N_0}} \tag{3.28}$$

where r is the code rate and E_b/N_0 the signal-to-noise ratio. Then

$$\begin{aligned} E\{x_i\} &= \Sigma_{l=1}^{n} E\{z_{li}\} + 2\, E\{w_{li}\} \\ &= n \cdot [p + 2 \cdot p_N \cdot (1-p)] \\ &= np' \end{aligned} \tag{3.29a}$$

where

$$p' = p + (1-p)p_N \tag{3.29b}$$

and p is given by (3.21) or (3.17) for hop-asynchronous or hop-synchronous FH/SSMA systems, respectively.

Note that the probability of an erasure $E\{z_{li}\}$ does not change with the presence of noise.

Further we need to calculate a' which is

$$\begin{aligned} a' &= E\{x_i^2\} - E\{x_i\}^2 \\ &= n \cdot \left[E\{(z_{li} + 2w_{li})^2\} - p'^2\right] \\ &= n \cdot \left[E\{z_{li}^2 + 4\, w_{li}^2\} + 4\, E\{z_{li}w_{li}\} - p'^2\right] \\ &= n \cdot \left[\, p + 4\,(1-p) \cdot p_N - p'^2\right]. \end{aligned} \tag{3.30}$$

We also need b', which is given by

$$b' = n(\rho_1' - p'^2) + (2n-2)(\rho_2' - p'^2) \tag{3.31}$$

and for whose computation ρ_1' and ρ_2' are necessary. The former is defined as

$$\rho_1' = E\{x_{li}x_{lj}\} = E\{z_{li}z_{lj} + 4z_{li}w_{lj} + 4w_{li}w_{lj}\}.$$

$E\{z_{li}z_{lj}\}$ is the same with that of the noiseless case, which was dealt with in the previous section and found to be equal to ρ_1, which is calculated there. Moreover,

$$\begin{aligned} E\{z_{li}w_{lj}\} &= Pr\{z_{li}=1, w_{lj}=1\} \\ &= Pr\{z_{li}=1, z_{lj}=0\}\cdot q \\ &= [Pr\{z_{li}=1\} - Pr\{z_{li}=1, z_{lj}=1\}]\cdot p_N \\ &= (p-\rho_1)p_N \end{aligned}$$

where ρ_1 is given by (3.25). Finally,

$$\begin{aligned} E\{w_{li}w_{lj}\} &= Pr\{w_{li}=1, w_{lj}=1\} \\ &= Pr\{z_{li}=0, z_{lj}=0\}\cdot p_N^2 \\ &= \left(1-\frac{1}{q}\right)\left[\left(1-\frac{2}{q}\right)\left(1-\frac{2}{q-1}\right)\right]^{k-2} p_N^2 \end{aligned}$$

Combining all the above we obtain

$$\rho_1' = \rho_1 + 4(p-\rho_1)p_N + 4\left(1-\frac{1}{q}\right)\left[\left(1-\frac{2}{q}\right)\left(1-\frac{2}{q-1}\right)\right]^{k-2} p_N^2. \tag{3.32}$$

The other quantity of interest, $\rho_2{}' = E\{x_{li}x_{(l+1)j}\}$, is calculated in an identical manner and is found to be

$$\rho_2{}' = E\{z_{li}z_{(l+1)j} + 4z_{li}w_{(l+1)j} + 4w_{li}w_{(l+1)j}\}$$

where

$$E\{z_{li}z_{(l+1)j} = \rho_2$$

$$\begin{aligned} E\{z_{li}w_{(l+1)j}\} &= Pr\{z_{li}=1, w_{(l+1)j}=1\} \\ &= Pr\{z_{li}=1, z_{(l+1)j}=0\}\cdot q \\ &= [Pr\{z_{li}=1\} - Pr\{z_{li}=1, z_{(l+1)j}=1\}\cdot p_N \\ &= (p-\rho_2)p_N \end{aligned}$$

and

$$
\begin{aligned}
E\{w_{li}w_{(l+1)j}\} &= Pr\{w_{li}=1, w_{(l+1)j}=1\} \\
&= Pr\{z_{li}=0, z_{(l+1)j}=0\}\cdot p_N^2 \\
&= Pr\{w_{li}=1, w_{lj}=1\} \\
&= \left(1-\frac{1}{q}\right)\left[\left(1-\frac{2}{q}\right)\left(1-\frac{2}{q-1}\right)\right]^{k-2} p_N^2
\end{aligned}
$$

Combining the above we obtain

$$
\rho_2' \;=\; \rho_2 + 4(p-\rho_2)p_N + 4\left(1-\frac{1}{q}\right)\left[\left(1-\frac{2}{q}\right)\left(1-\frac{2}{q-1}\right)\right]^{k-2} p_N^2. \tag{3.33}
$$

The desired quantity b' is now obtained from (3.32)-(3.33) and (3.29) upon substitution in (3.31). and

The above results are valid for **erasures/errors decoding**, when information about the state of the channel (side information) pertaining to hits from other users is available. If **error-only decoding** is employed (in the absence of side information), the results are simplified considerably. In that case, we use the previous definition of x_{li} to denote the presence of a error (not an erasure) at the lth hop. Then the necessary quantities p', a', and b' are obtained from the quantities p, ρ_1, and ρ_2 of the asynchronous case (no AWGN) of the previous section as

$$
p' = p + (1-p)p_N, \tag{3.34}
$$

where p is given by (3.21) or (3.17) and p_N is given by (3.28),

$$
a' = np'(1-p'), \tag{3.35}
$$

and

$$
b' = n[\rho_1 + (1-\rho_1)p_N^2 - p'^2] + (2n-2)[\rho_2 + (1-\rho_2)p_N^2 - p'^2]. \tag{3.36}
$$

4. APPROXIMATION BASED ON THE INDEPENDENCE ASSUMPTION

The assumption of independence between the packet errors of the users is commonly made for simplifying the evaluation of $P(l, m-l|k)$. The relevant expressions are given here

for reference, because they will be used for computing the numerical results. Let e and t be the erasure and error correcting capability, respectively, of the block code employed, ϵ the probability of symbol erasure, p' the probability of symbol error for errors-only decoding, p the probability of symbol error for errors and erasures decoding, $q(M)$ the probability of an M-ary FSK symbol error due to AWGN, and $p_e(k)$ the probability of a codeword (or receiver) error for the typical user. Also, let $P_{iid}(l, m-l|k)$ be the probability that exactly $m-l$ users suffer packet error, given that k users transmit, and that the probability is computed using the independence assumption. Then we have

$$P_{iid,T}(l, m-l|k) = \binom{m}{l} P_{iid}(l, m-l|k) \tag{4.1}$$

and

$$P_{iid}(l, m-l|k) = [p_e(k)]^{m-l} \, [1-p_e(k)]^l. \tag{4.2}$$

where

$$p_e(k) = \sum_{l_1=e+1}^{n} \binom{n}{l_1} \epsilon^{l_1} \, (1-\epsilon)^{n-l_1} \tag{4.3}$$

with $e = n - k_c$ for **erasures-only decoding**;

$$p_e(k) = \sum_{l=t+1}^{n} \binom{n}{l_1} p'^{l_1} \, (1-p')^{n-l_1} \tag{4.4}$$

where $t = (n-k_c)/2$, for **errors-only decoding**; and

$$p_e(k) = \sum_{l_1=0}^{n} \sum_{l_2=\max\{e+1-2l_1,0\}}^{n-l_1} \binom{n}{l_1} \binom{n-l_1}{l_2} p^{l_1} \, \epsilon^{l_2} \, (1-p-\epsilon)^{n-l_1-l_2} \tag{4.5}$$

where $e = n - k_c$, for **errors and erasures decoding** For example for asynchronous FH/SSMA with AWGN the parameters ϵ, p', and p take the values

$$\epsilon = 1 - \left(1 - \frac{2}{q}\right)^{k-1} \tag{4.6}$$

$$p' = \epsilon + (1-\epsilon)p_N \tag{4.7}$$

and

$$p = (1-\epsilon)P_N \tag{4.8}$$

where p_N is given by (2.13).

We can show that, under light traffic conditions (i.e., $k/q << 1$), the approximation based on the independence assumption (4.2) is valid for all l, m, and k. To this end it suffices to show that $P(E_{nn}) = P_s(\rho, m-\rho, k, q)$ of (2.8) can be approximated by the corresponding expression, when all receivers operate independently. Indeed, for $k << q$ we have

$$\begin{aligned} P_s(\rho, m-\rho, q, k) &\simeq [1 - P_h(q,k)]^\rho \cdot \left[1 + \sum_{i=1}^{m-\rho} (-1)^i \binom{m-\rho}{i} \prod_{j=1}^{i} (1 - P_h(q,k))\right] \\ &= [1 - P_h(q,k)]^\rho \cdot [1 - (1 - P_h(q,k))]^{m-\rho} \\ &= [1 - P_h(q,k)]^\rho P_h(q,k)^{m-\rho} \end{aligned} \tag{4.9}$$

5. NUMERICAL RESULTS

In this section we present our numerical results and comparisons. As explained before, the exact numerical evaluation of $P(l, m-l|k)$ becomes prohibitive for $m \geq 4$ and so only results for $m = 2$ and $m = 3$ can be presented. Indeed, as it is clear from equation (3), $2^m - 1$ nested sums are necessary for the computation of $P(l, m-l|k)$ in the case of errors-only or erasures-only decoding, whereas $3^m - 1$ such sums are necessary for the case of errors/erasures decoding [see (17)]. Since in each sum the number of terms varies between 0 and t or $t+1$ and n, as many as $(t+1)^{2^m-1}$ or even $(t+1)^{3^m-1}$ terms may be necessary, assuming $n = 2t$. This implies that, for a $(32, 16)$ RS extended code and $m = 2$ receivers, 3 and 8 nested sums are necessary, respectively, for a total of 17^3 or 17^8 terms; while for $m = 3$, 7 and 26 nested sums are necessary for a maximum of 17^7 or 17^{26} terms. The corresponding numbers for $m = 4$ are 15 and 80 nested sums (i.e., a maximum of 17^{15} and 17^{80} terms), which are prohibitive for most computers.

All results presented in this section pertain to asynchronous FH/SSMA systems employing RS $(32, 16)$ extended codes with 32-ary FSK data modulation and noncoherent

demodulation. Thus $M = n = 32$ and $k_c = 16$ for the modulation and code parameters of interest. The performance of different minimum-distance decoding algorithms is evaluated including errors-only, erasures-only, and errors/erasures correction decoding. The effect of AWGN is taken into account in all cases. The number of frequencies available for hopping is q and m is the number of FH/SS receivers of interest. In all cases, results based on the exact expressions $P(l, m-l|k)$ and results based on the independence approximation (IROA) $P_I(l, m-l|k)$, or the Gaussian approximation $P_G(l, m-l|k)$, are presented. Moreover, the results for the packet (codeword) probabilities $P(l, m-l|k)$ and $P_I(l, m-l|k)$ are presented in different subtables as the number of correct packets l changes ($l = m, m-1, \cdots, 1, 0$).

In Tables 1 and 2 we compare the performance of $m = 2$ and $m = 3$ receivers employing different forms of minimum-distance decoding, in particular errors-only, erasures-only, and errors/erasures decoding, for different values of the information bit signal-to-noise ratio E_b/N_0. The total number of contending users is $k = 10$ and the number of frequencies $q = 100$. The superiority of erasures decoding and errors/erasures decoding over errors-only decoding is established for the range of values of E_b/N_0 considered. The limiting values of the packet probabilities are essentially achieved already for $E_b/N_0 = 10$ dB, at which point all errors are caused by other-user interference. The approximation based on the IROA appears to be very close to the exact result for most cases. The accuracy of the IROA appears to be better for $P(2, 0|k)$ and $P(3, 0|k)$ than it is for $P(1, 1|k)$, $P(2, 1|k)$ or $P(1, 2|k)$; it is also better for errors-only or erasures-only decoding than for errors/erasures decoding. Finally, accuracy improves, as E_b/N_0 increases. The exact values of $P(l, m-l|k)$ are missing from Table 2 because the computation is prohibitive; for $m = 3$, 26 nested sums and a maximum of 17^{26} terms are necessary in (17).

Tables 3 and 4 illustrate the performance of $m = 2$ and $m = 3$ receivers, respectively, for FH/SSMA systems employing errors-only decoding, as the total number of users k and hopping frequencies q vary. The information bit signal-to-noise ratio $E_b/N_0 = 10$ dB. Comparing the exact results with those obtained under the IROA we observe that the accuracy of the latter is better for $P(2, 0|k)$ and $P(3, 0|k)$ than for the rest of the $P(l, m-l|k)$s. For fixed q, as the number of contending users k increases, the approximation becomes less accurate. However, for large values of k, both $P(l, m-l|k)$ and $P_I(l, m-l|k)$

become very small. On the other hand, for fixed k, the accuracy improves as q increases. Overall, the accuracy of the IROA improves as the ratio k/q decreases. This verifies the analysis of Section 3 under light traffic conditions.

Similarly, Tables 5 and 6 illustrate the performance of $m = 2$ and $m = 3$ receivers, respectively, for FH/SSMA systems employing erasures-only decoding, as the total number of users k and hopping frequencies q vary. The information bit signal-to-noise ratio $E_b/N_0 = 10$ dB. Similar observations to the ones made for Tables 3 and 4 are valid here. Moreover, the results of Tables 5 and 6 are uniformly better than those of Tables 3 and 4, since erasures-correction decoding is considerably more powerful than errors-correction decoding for for RS codes (RS codes can correct twice as many erasures as errors).

An interesting fact that holds true in the numerical analysis we have performed is that $P(m, 0|k)$ is larger than $P_I(m, 0|k)$ in all cases. In other words, IROA seems to give "optimistic" results in comparison to the exact analysis.

In Tables 7 to 10 the exact results for $P(l, m - l|k)$ are compared to those obtained via the Gaussian approximation method described in Section 3, the latter is denoted by $P_G(l, m - l|k)$. The same system assumptions presented at the beginning of this section hold. As before the results for the packet (codeword) probabilities $P(l, m - l|k)$ and $P_G(l, m - l|k)$ are presented in different subtables as the number of correct packets l changes. There is actually a one-to-one correspondence between Tables 1 and 7 and Tables 2 and 8. There are also similarities in the organization of the results between Tables 3 and 9 and Tables 4 and 10. However, in Tables 9 and 10 the number of simultaneous transmissions k changes whereas the number of frequencies is held constant at $q = 100$ and the results are presented for all three types of decoding considered in this report: errors-only, errors/erasures, and erasures-only decoding.

As it becomes clear from Tables 7 to 10 the Gaussian approximation is not as close to the exact results as the IROA is, at least for the cases of $m = 2$ and $m = 3$ receivers and $n = 32$. It appears that the accuracy of the approximation improves as the number of simultaneous users k increases and in general it is better for larger values of the exact probabilities.

6. CONCLUSIONS

For FH/SSMA communications, we presented exact expressions for the multireception probabilities $P(l, m-l|k)$ for hop-synchronous and hop-asynchronous systems and various types of RS decoding. The effects of AWGN were taken into consideration. However, these expressions are very difficult to evaluate for $m \geq 4$, as they require computation of $2^m - 1$ or $3^m - 1$ sums.

We also established the validity of the IROA for the case $q >> k$. Additionally, our numerical analysis indicated that IROA is a good approximation for the multireception probabilities, for $m = 2$ and $m = 3$. The accuracy of the approximation depends on the specific values of q and k; it improves with decreasing k/q; for $q >> k$, the corresponding results are almost identical to the exact ones. Therefore, it appears that IROA gives useful results, while requiring minimal numerical effort.

The kind of behavior observed so far is expected to be similar for higher values of m, as well. However, this is only a conjecture at this time and additional work on the derivation of computationally efficient techniques for the evaluation of $P(l, m-l|k)$ is necessary to prove this claim.

Note that, in many practical applications, the generated traffic is light. In such cases, the condition $q >> k$ is easily satisfied and the IROA can be used to obtain the multireception probabilities. However, there are also many practical situations, in which q and k are of comparable magnitude or even $k \geq q$ and in which the IROA can not be validated. In such situations, another method for evaluating the multireception probabilities is needed, since the values of m that are of interest can be considerably larger than 3 (the practical computational limit of the exact approach presented in this report).

In our report we also introduced an approximation based on central limit theorems for multivariate distributions. This Gaussian approximation has low computational complexity and promises to have good accuracy for large n (number of symbols per codeword or packet). Unfortunately, as our extensive comparisons with the exact results indicated in Section 5, for the nominal value $n = 32$ the Gaussian approximation does not yield uniformly satisfactory results. We conjecture that the accuracy of the approximation improves as n increases but we can not prove this since the computational effort for evaluating the

exact results is prohibitive for large n.

Finally, notice that the approach presented in this report for obtaining the exact results is only applicable to FH/SS systems. Both the IROA and Gaussian approximations are applicable to direct-sequence (DS) MA systems but an approach for the exact evaluation of $P(l, m-l|k)$ is needed in order to validate the accuracy of the two approximations in the DS case

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APPENDIX A

A1. Derivation of (2.3)

Here we derive the expression for $P(l, n_l - l|k)$ as a function of the probabilities $P_{e\cdots c}$. Let q, k denote the number of frequency slots and contending users in the slot, respectively.

For receiver i, $1 \leq i \leq m$, let $\underline{e}_i$ be a vector having 0 in the positions of correct symbol reception and 1 in the positions of erroneous reception, that is,

$$\underline{e}_i = (e_{i1}, e_{i2}, \cdots, e_{in}).$$

For hard decision decoding, the ith receiver decodes correctly the received packet iff

$$0 \leq \sum_{j=1}^{n} e_{ij} \leq t$$

while it decodes erroneously iff

$$t + 1 \leq \sum_{j=1}^{n} e_{ij} \leq n.$$

Let us now turn our attention to the interreceiver operation. For the jth transmitted symbols, for $1 \leq j \leq n$, we define the vector event E_j as

$$E_j = (e_{1j}, e_{2j}, \cdots, e_{mj}).$$

As each e_{ij} takes on two possible values, there is a total of 2^m possible E_j, for each j. As slotted operation is assumed throughout the paper, statistics are the same from symbol to symbol, so that the description of the system is independent of the particular symbol j. Thus we can drop the dependence from j in our notation and arbitrarily assign events E to vectors of symbol events $(e_1, e_2, \cdots, e_m)$. However, for simplicity we choose the correspondence

$$E_{nn} = (e_1, e_2, \cdots, e_m)$$

so that $(e_1, e_2, \cdots, e_m)$ is equal to the binary representation of nn. If we define by l_{nn} the number of times a particular event E_{nn} occurs as the index of symbols within a codeword

j varies ($1 \leq j \leq n$), we obtain, due to the memoryless operation assumption, the result given by (2.3).

A2. Solution to Recursion (2.6)

Here we prove that $P_s(0, m, q, k)$, given in (2.7), is the solution to the recursive equation described by (2.6). For compactness, we denote the binomial coefficients $\binom{m}{i}$ by $C_{m,i}$.

First we observe that, for the binomial coefficients $C_{m,i}$, the following recursion is true

$$C_{m+1,i} = C_{m,i-1} + C_{m,i}. \tag{A2.1}$$

Then, by direct substitution of (2.7) to the right hand side of (2.6), we get

$$\begin{aligned}
&1 + \sum_{i=1}^{m-1} (-1)^i C_{m-1,i} \prod_{j=1}^{i} [1 - P_h(q-j+1, k-j+1)] \\
&\quad - [1 - P_h(q,k)] \left\{ 1 + \sum_{i=1}^{m-1} (-1)^i C_{m-1,i} \prod_{j=1}^{i} [1 - P_h(q-j, k-j)] \right\} = \\
&= 1 - m[1 - P_h(q,k)] + \sum_{i=2}^{m-1} (-1)^i (C_{m-1,i} + C_{m-1,i-1}) \prod_{j=1}^{i} [1 - P_h(q-j+1, k-j+1)] \\
&\qquad + (-1)^m C_{m-1,m-1} \prod_{j=1}^{m} [1 - P_h(q-j+1, k-j+1)] \\
&= 1 + \sum_{i=1}^{m} (-1)^i C_{m,i} \prod_{j=1}^{m} [1 - P_h(q-j+1, k-j+1)]
\end{aligned} \tag{A2.2}$$

From (A2.2) we see that (2.6) has as solution the expression given in (2.7).

Table 1

Error probabilities computed using exact and IROA models for asynchronous FH/SSMA with RS (32,16) coding, $k=10$ and $q=100$.

(a) Probabilities $P(2,0|k)$ (exact) and $P_I(2,0|k)$ (under IROA)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	IROA	Exact	IROA	Exact	IROA
6	0.6320	0.6255	0.9900	0.9900	0.9998	0.9998
8	0.8490	0.8465	0.9999	0.9999	0.9999	0.9999
10	0.8633	0.8610	0.9999	0.9999	0.9999	0.9999
12	0.8634	0.8612	0.9999	0.9999	0.9999	0.9999
∞	0.8635	0.8612	0.9999	0.9999	0.9999	0.9999

(b) Probabilities $P(1,1|k)$ (exact) and $P_I(1,1|k)$ (under IROA)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	IROA	Exact	IROA	Exact	IROA
6	0.1588	0.1653	$4.176x\,10^{-5}$	$4.983x\,10^{-3}$	$5.777x\,10^{-5}$	$5.778x\,10^{-5}$
8	$7.101x\,10^{-2}$	$7.354x\,10^{-2}$	$3.249x\,10^{-6}$	$1.055x\,10^{-5}$	$3.341x\,10^{-6}$	$3.341x\,10^{-6}$
10	$6.462x\,10^{-2}$	$6.687x\,10^{-2}$	$2.510x\,10^{-6}$	$2.558x\,10^{-6}$	$2.511x\,10^{-6}$	$2.51x\,10^{-6}$
12	$6.454x\,10^{-2}$	$6.678x\,10^{-2}$	$2.501x\,10^{-6}$	$2.502x\,10^{-6}$	$2.501x\,10^{-6}$	$2.501x\,10^{-6}$
∞	$6.454x\,10^{-2}$	$6.678x\,10^{-2}$	$2.501x\,10^{-6}$	$2.502x\,10^{-6}$	$2.501x\,10^{-6}$	$2.501x\,10^{-6}$

(c) Probabilities $P(0,2|k)$ (exact) and $P_I(0,2|k)$ (under IROA)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	IROA	Exact	IROA	Exact	IROA
6	$5.018x\,10^{-1}$	$4.371x\,10^{-2}$	$9.901x\,10^{-3}$	$2.508x\,10^{-5}$	$1.284x\,10^{-8}$	$3.339x\,10^{-9}$
8	$8.915x\,10^{-3}$	$6.389x\,10^{-3}$	$1.460x\,10^{-5}$	$1.113x\,10^{-10}$	$1.198x\,10^{-10}$	$1.116x\,10^{-11}$
10	$7.438x\,10^{-3}$	$5.193x\,10^{-3}$	$9.637x\,10^{-8}$	$6.545x\,10^{-12}$	$7.660x\,10^{-11}$	$6.305x\,10^{-12}$
12	$7.421x\,10^{-3}$	$5.179x\,10^{-3}$	$1.458x\,10^{-10}$	$6.258x\,10^{-12}$	$7.616x\,10^{-11}$	$6.258x\,10^{-12}$
∞	$7.421x\,10^{-3}$	$5.179x\,10^{-3}$	$7.616x\,10^{-11}$	$6.252x\,10^{-12}$	$7.616x\,10^{-11}$	$6.258x\,10^{-12}$

Table 2

Error probabilities computed using exact and IROA models for asynchronous FH/SSMA with RS (32,16) coding, k=10 and q=100.

(a) Probabilities $P(3,0|k)$ (exact) and $P_I(3,0|k)$ (under IROA)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	IROA	IROA	Exact	IROA
6	0.5099	0.4947	0.9851	0.9998	0.9998
8	0.7857	0.7788	0.9999	0.9999	0.9999
10	0.8051	0.7990	0.9999	0.9999	0.9999
12	0.8054	0.7993	0.9999	0.9999	0.9999
∞	0.8054	0.7992	0.9999	0.9999	0.9999

(b) Probabilities $P(2,1|k)$ (exact) and $P_I(2,1|k)$ (under IROA)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	IROA	IROA	Exact	IROA
6	0.1121	0.1308	$4.958x\,10^{-3}$	$5.776x\,10^{-5}$	$5.778x\,10^{-5}$
8	$6.334x\,10^{-2}$	$6.766x\,10^{-2}$	$1.055x\,10^{-5}$	$3.340x\,10^{-6}$	$3.341x\,10^{-6}$
10	$5.816x\,10^{-2}$	$6.205x\,10^{-2}$	$2.558x\,10^{-6}$	$2.511x\,10^{-6}$	$2.511x\,10^{-6}$
12	$5.809x\,10^{-2}$	$6.198x\,10^{-2}$	$2.501x\,10^{-6}$	$2.501x\,10^{-6}$	$2.501x\,10^{-6}$
∞	$5.809x\,10^{-2}$	$6.198x\,10^{-2}$	$2.501x\,10^{-6}$	$2.501x\,10^{-6}$	$2.501x\,10^{-6}$

(c) Probabilities $P(1,2|k)$ (exact) and $P_I(1,2|k)$ (under IROA)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	IROA	IROA	Exact	IROA
6	$3.678x\,10^{-2}$	$3.457x\,10^{-2}$	$2.496x\,10^{-5}$	$1.284x\,10^{-8}$	$3.339x\,10^{-9}$
8	$7.67x\,10^{-3}$	$5.878x\,10^{-3}$	$1.113x\,10^{-10}$	$1.198x\,10^{-10}$	$1.116x\,10^{-11}$
10	$6.466x\,10^{-3}$	$4.819x\,10^{-3}$	$6.545x\,10^{-12}$	$7.660x\,10^{-11}$	$6.305x\,10^{-12}$
12	$6.452x\,10^{-4}$	$4.806x\,10^{-3}$	$6.258x\,10^{-12}$	$7.615x\,10^{-11}$	$6.258x\,10^{-12}$
∞	$6.452x\,10^{-3}$	$4.806x\,10^{-3}$	$6.258x\,10^{-12}$	$7.615x\,10^{-11}$	$6.258x\,10^{-12}$

(d) Probabilities $P(0,3|k)$ (exact) and $P_I(0,3|k)$ (under IROA)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	IROA	IROA	Exact	IROA
6	$1.340x\,10^{-2}$	$9.139x\,10^{-3}$	$1.256x\,10^{-7}$	$6.117x\,10^{-12}$	$1.929x\,10^{-13}$
8	$1.242x\,10^{-3}$	$5.107x\,10^{-4}$	$1.175x\,10^{-15}$	$7.264x\,10^{-13}$	$3.728x\,10^{-17}$
10	$9.718x\,10^{-4}$	$3.742x\,10^{-4}$	$1.674x\,10^{-17}$	$1.169x\,10^{-12}$	$1.583x\,10^{-17}$
12	$9.687x\,10^{-4}$	$3.727x\,10^{-4}$	$1.566x\,10^{-17}$	$8.808x\,10^{-13}$	$1.565x\,10^{-17}$
∞	$9.687x\,10^{-4}$	$3.727x\,10^{-4}$	$1.566x\,10^{-17}$	$9.580x\,10^{-13}$	$1.565x\,10^{-17}$

Table 3

Error probabilities computed using exact and IROA models for asynchronous FH/SSMA with RS (32,16) coding and error-correction decoding (E_b/N_0 = 10 dB)

(a) Probabilities $P(2,0|k)$ (exact) and $P_I(2,0|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	$4.4x\,10^{-8}$	$9.6x\,10^{-9}$	$1.6x\,10^{-29}$	$1.3x\,10^{-29}$	0.0	0.0	0.0	0.0
50	0.9200	0.9172	0.1065	0.0967	$9.07x\,10^{-7}$	$7.06x\,10^{-7}$	$2.4x\,10^{-29}$	$1.9x\,10^{-29}$
100	0.9989	0.9989	0.8635	0.8612	$7.36x\,10^{-2}$	$6.98x\,10^{-2}$	$2.6x\,10^{-10}$	$2.3x\,10^{-10}$

(b) Probabilities $P(1,1|k)$ (exact) and $P_I(1,1|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	$9.8x\,10^{-5}$	$9.8x\,10^{-5}$	$4.1x\,10^{-15}$	$3.7x\,10^{-15}$	0.0	0.0	0.0	0.0
50	$3.77x\,10^{-2}$	$4.05x\,10^{-2}$	0.2045	0.2142	$8.39x\,10^{-4}$	$8.39x\,10^{-4}$	$4.9x\,10^{-15}$	$4.3x\,10^{-15}$
100	$5.45x\,10^{-4}$	$5.51x\,10^{-4}$	$6.45x\,10^{-2}$	$6.67x\,10^{-2}$	0.1906	0.1944	$1.5x\,10^{-5}$	$1.5x\,10^{-5}$

(c) Probabilities $P(0,2|k)$ (exact) and $P_I(0,2|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	0.999	0.999	0.999	0.999	1.0	1.0	1.0	1.0
50	$4.64x\,10^{-3}$	$1.79x\,10^{-3}$	0.4846	0.4748	0.9983	0.9983	0.999	0.999
100	$6.09x\,10^{-6}$	$3.04x\,10^{-7}$	$7.42x\,10^{-3}$	$5.18x\,10^{-3}$	0.5451	0.5413	0.999	0.999

Table 4

Error probabilities computed using exact and IROA models for asynchronous FH/SSMA with RS (32,16) coding and error-correction decoding (E_b/N_0 = 10 dB)

(a) Probabilities $P(3,0|k)$ (exact) and $P_I(3,0|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	$9.3x\,10^{-11}$	$9.4x\,10^{-13}$	0.0	0.0	0.0	0.0	0.0	0.0
50	0.8863	0.8784	$3.97x\,10^{-2}$	$3.0x\,10^{-2}$	$1.2x\,10^{-9}$	$5.9x\,10^{-10}$	0.0	0.0
100	0.9983	0.9983	0.8054	0.7993	$2.1x\,10^{-2}$	$1.8x\,10^{-2}$	$4.9x\,10^{-15}$	$3.4x\,10^{-15}$

(b) Probabilities $P(2,1|k)$ (exact) and $P_I(2,1|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	$4.35x\,10^{-8}$	$9.62x\,10^{-9}$	$1.6x\,10^{-29}$	$1.3x\,10^{-29}$	0.0	0.0	0.0	0.0
50	$3.37x\,10^{-2}$	$3.88x\,10^{-2}$	$6.7x\,10^{-2}$	$6.7x\,10^{-2}$	$9.1x\,10^{-7}$	$7.1x\,10^{-7}$	$2.4x\,10^{-29}$	$1.8x\,10^{-29}$
100	$5.39x\,10^{-4}$	$5.51x\,10^{-4}$	$5.8x\,10^{-2}$	$6.2x\,10^{-2}$	$5.2x\,10^{-2}$	$5.14x\,10^{-2}$	$2.5x\,10^{-10}$	$2.2x\,10^{-10}$

(c) Probabilities $P(1,2|k)$ (exact) and $P_I(1,2|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	$9.8x\,10^{-5}$	$9.8x\,10^{-5}$	$4.1x\,10^{-15}$	$3.7x\,10^{-15}$	0.0	0.0	0.0	0.0
50	$3.9x\,10^{-3}$	$1.7x\,10^{-3}$	0.1378	0.1476	$8.4x\,10^{-4}$	$8.4x\,10^{-4}$	$4.9x\,10^{-15}$	$4.3x\,10^{-15}$
100	$5.98x\,10^{-6}$	$3.04x\,10^{-7}$	$6.5x\,10^{-3}$	$4.8x\,10^{-3}$	0.1386	0.1430	$1.5x\,10^{-5}$	$1.5x\,10^{-5}$

(d) Probabilities $P(0,3|k)$ (exact) and $P_I(0,3|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	0.999	0.999	0.999	0.999	1.0	1.0	1.0	1.0
50	$7.2x\,10^{-4}$	$7.57x\,10^{-5}$	0.3467	0.3271	0.9975	0.9975	0.999	0.999
100	$1.14x\,10^{-7}$	$1.68x\,10^{-10}$	$9.69x\,10^{-4}$	$3.72x\,10^{-4}$	0.4065	0.3982	0.999	0.999

Table 5

Error probabilities computing using exact and IROA models for asynchronous FH/SSMA with RS (32,16) coding and erasures-correction decoding (E_b/N_0 = 10 dB)

(a) Probabilities $P(2,0|k)$ (exact) and $P_I(2,0|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	$4.61x\,10^{-2}$	$3.77x\,10^{-2}$	$5.94x\,10^{-13}$	$6.01x\,10^{-13}$	0.0	0.0	0.0	0.0
50	0.9999	0.9999	0.9863	0.9863	0.1565	0.1532	$7.49x\,10^{-13}$	$7.49x\,10^{-13}$
100	0.9999	0.9999	0.9999	0.9999	0.9797	0.9797	$9.17x\,10^{-3}$	$8.99x\,10^{-3}$

(b) Probabilities $P(1,1|k)$ (exact) and $P_I(1,1|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	0.1481	0.1565	$7.75x\,10^{-7}$	$7.75x\,10^{-7}$	0.0	0.0	0.0	0.0
50	$6.06x\,10^{-7}$	$6.06x\,10^{-7}$	$6.81x\,10^{-3}$	$6.85x\,10^{-3}$	0.2349	0.2382	$8.65x\,10^{-7}$	$8.65x\,10^{-7}$
100	$2.44x\,10^{-11}$	$2.44x\,10^{-11}$	$2.50x\,10^{-6}$	$2.50x\,10^{-6}$	$1.01x\,10^{-2}$	$1.01x\,10^{-2}$	$8.56x\,10^{-2}$	$8.58x\,10^{-2}$

(c) Probabilities $P(0,2|k)$ (exact) and $P_I(0,2|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	0.6576	0.6493	0.9999	0.9999	1.0	1.0	1.0	1.0
50	$9.73x\,10^{-11}$	$3.67x\,10^{-13}$	$8.81x\,10^{-5}$	$4.75x\,10^{-5}$	0.3734	0.3703	0.9999	0.9999
100	$6.42x\,10^{-17}$	$5.97x\,10^{-22}$	$7.62x\,10^{-11}$	$6.26x\,10^{-12}$	$1.36x\,10^{-4}$	$1.05x\,10^{-4}$	0.8195	0.8194

Table 6

Error probabilities computed using exact and IROA models for asynchronous FH/SSMA with RS (32,16) coding and erasures-correction decoding (E_b/N_0 = 10 dB)

(a) Probabilities $P(3,0|k)$ (exact) and $P_I(3,0|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	$1.31x\,10^{-2}$	$7.32x\,10^{-3}$	$4.47x\,10^{-19}$	$4.66x\,10^{-19}$	0.9697	0.9696	0.0	0.0
50	0.9999	0.9999	0.9796	0.9795	$6.39x\,10^{-2}$	$5.99x\,10^{-2}$	$6.48x\,10^{-19}$	$6.48x\,10^{-19}$
100	0.9999	0.9999	0.9999	0.9999	0.9697	0.9697	$9.06x\,10^{-4}$	$8.52x\,10^{-4}$

(b) Probabilities $P(2,1|k)$ (exact) and $P_I(2,1|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	$3.29x\,10^{-2}$	$3.04x\,10^{-2}$	$5.94x\,10^{-13}$	$6.04x\,10^{-13}$	$9.95x\,10^{-3}$	$1.00x\,10^{-2}$	0.0	0.0
50	$6.06x\,10^{-7}$	$6.06x\,10^{-7}$	$6.72x\,10^{-3}$	$6.80x\,10^{-3}$	$9.26x\,10^{-2}$	$9.32x\,10^{-2}$	$7.49x\,10^{-13}$	$7.49x\,10^{-13}$
100	$2.44x\,10^{-11}$	$2.44x\,10^{-11}$	$2.50x\,10^{-6}$	$2.50x\,10^{-6}$	$9.95x\,10^{-3}$	$1.00x\,10^{-2}$	$8.27x\,10^{-3}$	$8.52x\,10^{-4}$

(c) Probabilities $P(1,2|k)$ (exact) and $P_I(1,2|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	0.1152	0.1261	$7.75x\,10^{-7}$	$7.75x\,10^{-7}$	0.0	0.0	0.0	0.0
50	$9.73x\,10^{-11}$	$3.67x\,10^{-13}$	$8.64x\,10^{-5}$	$4.72x\,10^{-5}$	0.1423	0.1449	$8.65x\,10^{-7}$	$8.65x\,10^{-7}$
100	$6.42x\,10^{-17}$	$5.97x\,10^{-22}$	$7.61x\,10^{-11}$	$6.26x\,10^{-12}$	$1.34x\,10^{-4}$	$1.03x\,10^{-4}$	$7.74x\,10^{-2}$	$7.77x\,10^{-2}$

(d) Probabilities $P(0,3|k)$ (exact) and $P_I(0,3|k)$ (under IROA)

q	$k = 5$		$k = 10$		$k = 20$		$k = 50$	
	Exact	IROA	Exact	IROA	Exact	IROA	Exact	IROA
10	0.5424	0.5232	0.9999	0.9999	1.0	1.0	1.0	1.0
50	$1.37x\,10^{-12}$	$2.23x\,10^{-19}$	$1.70x\,10^{-6}$	$3.28x\,10^{-7}$	0.2314	0.2254	0.9999	0.9999
100	$2.78x\,10^{-12}$	$1.46x\,10^{-32}$	$9.58x\,10^{-13}$	$1.56x\,10^{-17}$	$2.22x\,10^{-6}$	$1.07x\,10^{-6}$	0.7422	0.7417

Table 7

Error probabilities computed using exact and Gaussian models for asynchronous FH/SSMA with RS (32,16) coding, k=10 and q=100.

(a) Probabilities $P(2,0|k)$ (exact) and $P_G(2,0|k)$ (Gaussian)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	Gauss	Exact	Gauss	Exact	Gauss
6	0.6320	0.4916	0.9900	0.9865	0.9998	0.9963
8	0.7782	0.7782	0.9999	0.9885	0.9999	0.9894
10	0.8633	0.7979	0.9999	0.9885	0.9885	0.9885
12	0.7982	0.7982	0.9999	0.9885	0.9999	0.9885
∞	0.8635	0.7982	0.9999	0.9885	0.9999	0.9885

(b) Probabilities $P(1,1|k)$ (exact) and $P_G(1,1|k)$ (Gaussian)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	Gauss	Exact	Gauss	Exact	Gauss
6	0.1588	0.2237	$4.176x\,10^{-5}$	$6.728x\,10^{-3}$	$5.777x\,10^{-5}$	$2.848x\,10^{-3}$
8	$1.035x\,10^{-1}$	$1.034x\,10^{-1}$	$3.249x\,10^{-6}$	$5.683x\,10^{-3}$	$3.341x\,10^{-6}$	$5.242x\,10^{-3}$
10	$6.462x\,10^{-2}$	$9.454x\,10^{-2}$	$2.510x\,10^{-6}$	$5.718x\,10^{-3}$	$2.511x\,10^{-6}$	$5.712x\,10^{-3}$
12	$9.443x\,10^{-2}$	$9.443x\,10^{-2}$	$2.501x\,10^{-6}$	$5.718x\,10^{-3}$	$2.501x\,10^{-6}$	$5.718x\,10^{-3}$
∞	$6.454x\,10^{-2}$	$9.443x\,10^{-2}$	$2.501x\,10^{-6}$	$5.718x\,10^{-3}$	$2.501x\,10^{-6}$	$5.718x\,10^{-3}$

(c) Probabilities $P(0,2|k)$ (exact) and $P_G(0,2|k)$ (Gaussain)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	Gauss	Exact	Gauss	Exact	Gauss
6	$5.018x\,10^{-2}$	$5.997x\,10^{-2}$	$9.901x\,10^{-3}$	$4.671x\,10^{-5}$	$1.284x\,10^{-8}$	$1.175x\,10^{-6}$
8	$1.484x\,10^{-2}$	$1.484x\,10^{-2}$	$1.460x\,10^{-5}$	$4.707x\,10^{-5}$	$1.198x\,10^{-10}$	$3.445x\,10^{-5}$
10	$7.438x\,10^{-3}$	$1.296x\,10^{-2}$	$9.637x\,10^{-8}$	$4.932x\,10^{-5}$	$7.660x\,10^{-11}$	$4.912x\,10^{-5}$
12	$1.294x\,10^{-2}$	$1.294x\,10^{-2}$	$1.458x\,10^{-10}$	$4.935x\,10^{-5}$	$7.616x\,10^{-11}$	$4.934x\,10^{-5}$
∞	$7.421x\,10^{-3}$	$1.294x\,10^{-2}$	$7.616x\,10^{-11}$	$4.935x\,10^{-5}$	$7.616x\,10^{-11}$	$4.934x\,10^{-5}$

Table 8

Error probabilities computed using exact and Gaussian models for asynchronous FH/SSMA with RS (32,16) coding, k=10 and q=100.

(a) Probabilities $P(3,0|k)$ (exact) and $P_G(3,0|k)$ (Gaussian)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	Gauss	Gauss	Exact	Gauss
6	0.5099	0.3191	0.9798	0.9998	0.9944
8	0.7857	0.6876	0.9829	0.9999	0.9843
10	0.8051	0.7146	0.9828	0.9999	0.9828
12	0.8054	0.7149	0.9828	0.9999	0.9828
∞	0.8054	0.7149	0.9828	0.9999	0.9828

(b) Probabilities $P(2,1|k)$ (exact) and $P_G(2,1|k)$ (Gaussian)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	Gauss	Gauss	Exact	Gauss
6	0.1121	0.1714	$6.682x\,10^{-3}$	$5.776x\,10^{-5}$	$2.844x\,10^{-3}$
8	$6.334x\,10^{-2}$	$9.059x\,10^{-2}$	$5.636x\,10^{-3}$	$3.340x\,10^{-6}$	$3.208x\,10^{-3}$
10	$5.816x\,10^{-2}$	$8.333x\,10^{-2}$	$5.669x\,10^{-3}$	$2.511x\,10^{-6}$	$5.663x\,10^{-3}$
12	$5.809x\,10^{-2}$	$8.324x\,10^{-2}$	$5.669x\,10^{-3}$	$2.501x\,10^{-6}$	$5.669x\,10^{-3}$
∞	$5.809x\,10^{-2}$	$8.325x\,10^{-2}$	$5.669x\,10^{-3}$	$2.501x\,10^{-6}$	$5.669x\,10^{-3}$

(c) Probabilities $P(1,2|k)$ (exact) and $P_G(1,2|k)$ (Gaussian)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	Gauss	Gauss	Exact	Gauss
6	$3.678x\,10^{-2}$	$5.163x\,10^{-2}$	$4.637x\,10^{-5}$	$1.284x\,10^{-8}$	$1.171x\,10^{-6}$
8	$7.67x\,10^{-3}$	$1.287x\,10^{-2}$	$4.654x\,10^{-5}$	$1.198x\,10^{-10}$	$3.417x\,10^{-5}$
10	$6.466x\,10^{-3}$	$1.121x\,10^{-2}$	$4.873x\,10^{-5}$	$7.660x\,10^{-11}$	$4.853x\,10^{-5}$
12	$6.452x\,10^{-4}$	$1.119x\,10^{-2}$	$4.875x\,10^{-5}$	$7.615x\,10^{-11}$	$4.875x\,10^{-5}$
∞	$6.452x\,10^{-3}$	$1.119x\,10^{-2}$	$4.874x\,10^{-5}$	$7.615x\,10^{-11}$	$4.874x\,10^{-5}$

(d) Probabilities $P(0,3|k)$ (exact) and $P_G(0,3|k)$ (Gaussian)

E_b/N_0 (dB)	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	Gauss	Gauss	Exact	Gauss
6	$1.340x\,10^{-2}$	$8.183x\,10^{-3}$	$3.295x\,10^{-7}$	$6.117x\,10^{-12}$	$1.353x\,10^{-9}$
8	$1.242x\,10^{-3}$	$1.973x\,10^{-3}$	$5.319x\,10^{-7}$	$7.264x\,10^{-13}$	$2.741x\,10^{-7}$
10	$9.718x\,10^{-4}$	$1.751x\,10^{-3}$	$5.951x\,10^{-7}$	$1.169x\,10^{-12}$	$5.889x\,10^{-7}$
12	$9.687x\,10^{-4}$	$1.748x\,10^{-3}$	$5.959x\,10^{-7}$	$8.808x\,10^{-13}$	$5.957x\,10^{-7}$
∞	$9.687x\,10^{-4}$	$1.748x\,10^{-3}$	$5.957x\,10^{-7}$	$9.580x\,10^{-13}$	$5.957x\,10^{-7}$

Table 9

Error probabilities computed using exact and Gaussian models for asynchronous FH/SSMA with RS (32,16) coding, $E_b/N_0=10dB$ and $q=100$.

(a) Probabilities $P(2,0|k)$ (exact) and $P_G(2,0|k)$ (Gaussian)

k	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	Gauss	Exact	Gauss	Exact	Gauss
5	0.9989	0.9042	0.9990	0.9044	0.9999	0.9045
10	0.8635	0.7979	0.9152	0.9885	0.9999	0.9885
20	$7.36x\,10^{-2}$	$4.0151x\,10^{-2}$	0.8921	0.9722	0.9797	0.9722
50	$2.6x\,10^{-10}$	$1.077x\,10^{-11}$	$7.25x\,10^{-6}$	$3.891x\,10^{-3}$	$9.17x\,10^{-3}$	$3.885x\,10^{-3}$

(b) Probabilities $P(1,1|k)$ (exact) and $P_G(1,1|k)$ (Gaussian)

k	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	Gauss	Exact	Gauss	Exact	Gauss
5	$5.45x\,10^{-4}$	$4.5293x\,10^{-2}$	$7.21x\,10^{-6}$	$4.521x\,10^{-2}$	$2.44x\,10^{-11}$	$4.516x\,10^{-2}$
10	$6.45x\,10^{-2}$	$9.454x\,10^{-2}$	$8.27x\,10^{-4}$	$5.718x\,10^{-3}$	$2.50x\,10^{-6}$	$5.712x\,10^{-3}$
20	0.1906	0.1616	$1.33x\,10^{-1}$	$1.381x\,10^{-2}$	$1.01x\,10^{-2}$	$1.379x\,10^{-2}$
50	$1.50x\,10^{-5}$	$4.931x\,10^{-6}$	$6.31x\,10^{-3}$	$6.247x\,10^{-2}$	$8.56x\,10^{-2}$	$6.249x\,10^{-2}$

(c) Probabilities $P(0,2|k)$ (exact) and $P_G(0,2|k)$ (Gaussian)

k	errors decoding		errors/erasures decoding		erasures decoding	
	Exact	Gauss	Exact	Gauss	Exact	Gauss
5	$6.09x\,10^{-6}$	$5.170x\,10^{-3}$	$8.32x\,10^{-5}$	$5.176x\,10^{-3}$	$6.42x\,10^{-17}$	$5.168x\,10^{-3}$
10	$7.42x\,10^{-3}$	$1.296x\,10^{-2}$	$3.75x\,10^{-6}$	$4.932x\,10^{-5}$	$7.62x\,10^{-11}$	$4.912x\,10^{-5}$
20	0.5451	0.6366	$2.87x\,10^{-2}$	$1.879x\,10^{-4}$	$1.36x\,10^{-4}$	$1.867x\,10^{-4}$
50	0.9999	0.9999	0.9123	0.8711	0.8195	0.8711

Table 10

Error probabilities computed using exact and Gaussian models for asynchronous FH/SSMA with RS (32,16) coding, $E_b/N_0=10dB$ and $q=100$.

(a) Probabilities $P(3,0|k)$ (exact) and $P_G(3,0|k)$ (Gaussian)

k	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	Gauss	Gauss	Exact	Gauss
5	0.9983	0.8632	0.8635	0.9999	0.8635
10	0.8054	0.7146	0.9828	0.9999	0.9828
20	$2.10x\,10^{-2}$	$7.880x\,10^{-3}$	0.9585	0.9697	0.9686
50	$4.00x\,10^{-15}$	$9.0393x\,10^{-16}$	$1.99x\,10^{-4}$	$9.06x\,10^{-4}$	$1.983x\,10^{-4}$

(b) Probabilities $P(2,1|k)$ (exact) and $P_G(2,1|k)$ (Gaussian)

k	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	Gauss	Gauss	Exact	Gauss
5	$5.49x\,10^{-4}$	$4.095x\,10^{-2}$	$4.087x\,10^{-2}$	$2.44x\,10^{-11}$	$4.083x\,10^{-2}$
10	$5.80x\,10^{-2}$	$8.333x\,10^{-2}$	$5.669x\,10^{-3}$	$2.50x\,10^{-6}$	$5.664x\,10^{-3}$
20	$5.20x\,10^{-2}$	$3.227x\,10^{-2}$	$1.363x\,10^{-2}$	$9.95x\,10^{-3}$	$1.361x\,10^{-2}$
50	$2.50x\,10^{-10}$	$1.078x\,10^{-11}$	$3.691x\,10^{-3}$	$8.27x\,10^{-3}$	$3.687x\,10^{-3}$

(c) Probabilities $P(1,2|k)$ (exact) and $P_G(1,2|k)$ (Gaussian)

k	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	Gauss	Gauss	Exact	Gauss
5	$5.98x\,10^{-6}$	$4.336x\,10^{-3}$	$4.340x\,10^{-3}$	$6.42x\,10^{-17}$	$4.333x\,10^{-3}$
10	$6.50x\,10^{-3}$	$1.121x\,10^{-2}$	$4.873x\,10^{-5}$	$7.61x\,10^{-11}$	$4.853x\,10^{-5}$
20	0.1386	0.1234	$1.855x\,10^{-4}$	$1.34x\,10^{-4}$	$1.843x\,10^{-4}$
50	$1.50x\,10^{-5}$	$4.931x\,10^{-6}$	$5.878x\,10^{-2}$	$7.74x\,10^{-2}$	$5.88x\,10^{-2}$

(d) Probabilities $P(0,3|k)$ (exact) and $P_G(0,3|k)$ (Gaussian)

k	errors decoding		errors/erasures decoding	erasures decoding	
	Exact	Gauss	Gauss	Exact	Gauss
5	$1.14x\,10^{-7}$	$8.346x\,10^{-4}$	$8.361x\,10^{-4}$	$2.78x\,10^{-12}$	$8.345x\,10^{-4}$
10	$9.69x\,10^{-4}$	$1.751x\,10^{-3}$	$5.951x\,10^{-7}$	$9.58x\,10^{-13}$	$5.889x\,10^{-7}$
20	0.4065	0.5073	$2.416x\,10^{-6}$	$2.22x\,10^{-6}$	$2.384x\,10^{-6}$
50	0.9990	09999	0.8124	0.7422	0.8123