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Prepared for the
Fifth International Power Transmission and Gearing Conference
sponsored by the American Society of Mechanical Engineers
Chicago, Illinois, April 25-27, 1989
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COMPUTER-AIDED DESIGN OF BEVEL GEAR TOOTH SURFACES

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SUMMARY

This paper presents a computer-aided design procedure for generating bevel gears. The development is based on examining a perfectly plastic, cone-shaped gear blank rolling over a cutting tooth on a plane crown rack. The resulting impression on the plastic gear blank is the envelope of the cutting tooth. This impression and envelope thus form a conjugate tooth surface. Equations are presented for the locus of points on the tooth surface. The same procedures are then extended to simulate the generation of a spiral bevel gear. The corresponding governing equations are presented.

INTRODUCTION

Bevel gears are the principle means of motion and power transmission between intersecting shafts. Their use is widespread. The geometrical characteristics of bevel gears have long been documented by the American Gear Manufacturers' Association (AGMA, 1964) and others (Dudley, 1962; Dyson, 1969; Bonsignore, 1976; Litvin et al., 1975, 1982, 1983; Baxter, 1966; Huston and Coy, 1981,1982a,b). Recent advances in computer-aided design present opportunities for a new look at the geometry of these gears. These computer-based procedures also provide a means for optimizing the geometry.

In a previous paper (Chang, Huston, and Coy, 1984) we have demonstrated the feasibility of this procedure to determine a spur gear tooth profile. The basic idea was to use the envelope of a family of curves to develop an involute spur gear tooth profile. The study demonstrated that the envelope of an inclined straight-line segment on a rolling gear blank is an involute of a circle. The inclined line segment in turn represented the rack tooth of a hob cutter. Such a procedure provides a means for analytically and numerically determining the tooth profile, given a cutter profile.

*Work funded under NASA Grant NAG3-188.
In the current paper we extend these ideas to the generation of both straight and spiral bevel gears. The paper itself is divided into four sections with the section following the Symbols providing preliminary ideas useful in this sequel. The next two sections describe the formulation of string and spiral bevel gear tooth surfaces. This is followed by a discussion of applications.

SYMBOLS

The following symbols are used in the section for development of a straight bevel gear tooth.

\( R_m \)  
Mean radius of crown gear in pitch plane

\([R]\)  
Coordinate transformation matrix; from \( S \) to \( S \)

\([R_1]\)  
Coordinate transformation matrix; from \( S_1 \) to \( S \)

\([R_2]\)  
Coordinate transformation matrix; from \( S_2 \) to \( S_1 \)

\([R_3]\)  
Coordinate transformation matrix; from \( S \) to \( S_2 \)

\( \bar{r} \)  
Position vector in \( S \)

\( \bar{r}_g \)  
Position vector in \( \bar{S} \)

\( r_{ij} \)  
Components of \([R]\)

\( S(X,Y,Z) \)  
Cartesian coordinate system stationary with ground

\( \bar{S}(\bar{X},\bar{Y},\bar{Z}) \)  
Cartesian system stationary with generated gear

\( S_1(X_1,Y_1,Z_1) \)  
Intermediate Cartesian coordinate system

\( S_2(X_2,Y_2,Z_2) \)  
Intermediate Cartesian coordinate system

\( \bar{X}_p \)  
Projection of \( \bar{X} \) axis on crown gear pitch plane

\( \alpha \)  
Angular position of gear referenced to system \( S \)

\( \alpha_0 \)  
Initial angular position of gear

\( \gamma \)  
Pitch angle of pinion

\( \Theta \)  
Angle of rotation of generated gear

\( \phi \)  
Complement of pressure angle of cutter

\( \psi \)  
Cutter's orientation angle on pitch plane

The following symbols are used in the section for development of a spiral bevel gear tooth.

\( \bar{B} \)  
Derivative of \( \bar{\tau} \) with respect to \( \phi_2 \)
$b_i$ Components of $\bar{b}$

$d_{ij}$ $\partial e_{ij}/\partial \phi_2$

$[E]$ Coordinate transformation matrix; from $S_2$ to $S_c$

$e_{ij}$ Components of $[E]$

$H$ Horizontal machine setting of cutter

$h$ Addendum of cutter tooth (pitch to tip)

$N_g$ Number of teeth in crown gear

$N_p$ Number of teeth in generated gear

$[R_{c1}]$ Coordinate transformation matrix; from $S_1$ to $S_c$

$[R_{ig}]$ Coordinate transformation matrix; from $S_g$ to $S_i$

$[R_{gp}]$ Coordinate transformation matrix; from $S_p$ to $S_g$

$[R_{p2}]$ Coordinate transformation matrix; from $S_2$ to $S_p$

$r_c$ Mean radius of cutter in pitch plane

$r_c$ Position vector in $S_c$

$r_g$ Position vector in $S_g$

$r_p$ Position vector in $S_p$

$r_1$ Position vector in $S_1$

$r_2$ Position vector in $S_2$

$S_c(X_c,Y_c,Z_c)$ Cartesian coordinate system stationary with cutter with origin at $O_c$

$S_1(X_1,Y_1,Z_1)$ Cartesian coordinate system stationary with gear with origin at $O_1$

$S_2(X_2,Y_2,Z_2)$ Cartesian coordinate system stationary with pinion with origin at $O_2$

$S_g(X_g,Y_g,Z_g)$ Cartesian coordinate system stationary with ground with origin at $O_g$

$S_p(X_p,Y_p,Z_p)$ Cartesian coordinate system stationary with ground with origin at $O_p$

$\bar{T}$ Coordinate translation transformation matrix; from $S_2$ to $S_c$
\( \mathbf{T}_{1C} \) Coordinate translation transformation matrix; from \( S_C \) to \( S_1 \)
\( \mathbf{T}_{gp} \) Coordinate translation transformation matrix; from \( S_p \) to \( S_g \)
\( t_1 \) Components of \( \mathbf{T} \)
\( V \) Vertical machine setting of cutter
\( w \) Angular speed ratio
\( Z_0 \) \( Z_C \) Intercept of cutter surface
\( \alpha \) Surface coordinate of cutter
\( \gamma \) Root angle of pinion
\( \gamma_0 \) Pitch angle of pinion
\( \theta \) Angle of rotation of system \( S_C \) reference to \( S_1 \)
\( \phi_1 \) Angle of rotation of gear
\( \phi_2 \) Angle of rotation of gear blank
\( \phi_{10} \) Initial angular position of gear
\( \psi_0 \) Complement of pressure angle of cutter edge
\( \omega_g \) Rotation rate of gear
\( \omega_p \) Rotation rate of pinion

**ENVELOPE OF A FAMILY OF SURFACES**

Consider a family of surfaces represented by an equation of the form
\[ F(x_1, x_2, x_3, t) = 0 \] with a parameter \( t \). Let \( S \) be a surface of the family and let it be intersected by neighboring surfaces \( S' \). If \( S \) and \( S' \) correspond to the values \( t \) and \( t + \Delta t \), the curve is represented by the simultaneous equations

\[
\begin{align*}
F(x_1, x_2, x_3, t) &= 0 \\
F(x_1, x_2, x_3, t + \Delta t) &= 0
\end{align*}
\]

(1)

It may also be represented by the equations

\[
\begin{align*}
F(x_1, x_2, x_3, t) &= 0 \\
\frac{F(x_1, x_2, x_3, t + \Delta t) - F(x_1, x_2, x_3, t)}{\Delta t} &= 0
\end{align*}
\]

(2)
The surface \([F(x_1,x_2,x_3,t + \Delta t) - F(x_1,x_2,x_3,t)]/\Delta t = 0\) goes through the curve common to the two surfaces \(F(x_1,x_2,x_3,t) = 0\) and \(F(x_1,x_2,x_3,t + \Delta t) = 0\). When \(\Delta t\) approaches zero, the intersection curve will approach a limiting curve. This curve is given by

\[
\begin{align*}
F(x_1,x_2,x_3,t) &= 0 \\
\frac{\partial F}{\partial t}(x_1,x_2,x_3,t) &= 0
\end{align*}
\]

Equation (3), when \(t\) is fixed, represents a curve on the surface of the family. The same equation, with \(t\) variable, will represent a family of curves and will generate a surface. This surface is the envelope of the given family. The result of eliminating \(t\) is the equation of the envelope (Graustein, 1935).

In the two-dimensional case the envelope of a family of curves is a curve tangent to the given family. For example, the envelope given by an inclined straight-line segment on a rolling gear blank is found to be an involute of a circle (Chang, Huston, and Coy, 1984).

In the following sections, these procedures are generalized to simulate the straight bevel and spiral bevel gear tooth surface generation process. The family of surfaces created by the cutter generates an envelope in the gear blank. The envelope in the gear blank then forms a conjugate tooth surface.

**DEVELOPMENT OF A STRAIGHT BEVEL GEAR TOOTH SURFACE**

The cutter used for the surface generation of a straight bevel gear tooth is called the basic crown rack. Figure 1 depicts the machining model of the straight bevel gear tooth generation process. The basic crown rack \(R\) is considered to be fixed on the imaginary crown gear pitch plane \(C\) as a rack step, as depicted in Fig. 2. The generated gear blank \(G\) is a cone with the vertex at the machine center \(O\). The gear blank is allowed to roll on the crown gear pitch plane \(C\). The crown gear pitch plane is an imaginary fixed plane. When the gear blank rolls over the crown rack step, the envelope of the basic crown rack forms the tooth surface of the straight bevel gear.

The coordinate system used to describe the crown gear is \(S(X,Y,Z)\) with origin at \(O\). The coordinate system \(S(X,Y,\bar{Z})\) is fixed on the gear blank with origin at \(O\). The gear blank is allowed to rotate through an angle \(\theta\) about \(\bar{Z}\). The pitch angle of the gear blank is denoted as \(\gamma\). The initial position of the gear blank is defined by angle \(\alpha_0\), which is the angle between the axis \(X\) and the projection of the \(\bar{X}\) axis on the crown gear pitch plane, denoted by \(X_p\). During the gear blank rolling motion the angular position of the gear blank is defined by angle \(\alpha\), where \(\alpha\) is measured between \(\bar{X}_p\) and \(X\). From this geometry we obtain the relation between \(\theta\) and \(\alpha\) as

\[
\alpha = \alpha_0 - \theta \tan \gamma
\]

The coordinate system \(S\) may be obtained by coordinate transformation of the system \(S\) in the following steps:
The indicated rotation matrices are

\[
[R_1] = \begin{bmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{bmatrix} \tag{5}
\]

\[
[R_2] = \begin{bmatrix}
\cos \gamma & 0 & -\sin \gamma \\
0 & 1 & 0 \\
\sin \gamma & 0 & \cos \gamma
\end{bmatrix} \tag{6}
\]

\[
[R_3] = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix} \tag{7}
\]

The position vectors \( \vec{F} \) with the components \((x,y,z)\) and \( \vec{F}_g \) with the components \((\hat{x},\hat{y},\hat{z})\) locating a typical point in \( S \) and \( S' \) are related by the expression

\[
\vec{F} = [R_1][R_2][R_3]\vec{F}_g = [R]\vec{F}_g \tag{8}
\]

From Eqs. (5) to (7) the elements of \([R] \), \( r_{ij} (i,j=1,3) \), are

\[
\begin{align*}
 r_{11} &= \cos \alpha \cos \gamma \\
 r_{12} &= -\cos \alpha \sin \gamma \sin \theta - \sin \alpha \cos \theta \\
 r_{13} &= -\cos \alpha \sin \gamma \cos \theta + \sin \alpha \sin \theta \\
 r_{21} &= \sin \alpha \cos \gamma \\
 r_{22} &= -\sin \alpha \sin \gamma \sin \theta + \cos \alpha \cos \theta \\
 r_{23} &= -\sin \alpha \sin \gamma \cos \theta - \cos \alpha \sin \theta \\
 r_{31} &= \sin \gamma \\
 r_{32} &= \sin \theta \cos \gamma \\
 r_{33} &= \cos \theta \cos \gamma
\end{align*} \tag{9}
\]
EQUATION FOR CROWN RACK AND STRAIGHT BEVEL GEAR TOOTH SURFACE

The basic crown rack may be viewed as a straight cutting blade reciprocating in the radial direction. The tooth surface of the rack thus forms an inclined plane passing through the crown gear center. The pitch plane of the cutter surface is shown in Fig. 2. The normal plane view in Fig. 3 shows the rack profile. The equation of the rack surface may be expressed in terms of system $S$ as

$$x \tan \psi - y - z \cot \phi = 0 \quad (10)$$

where $\psi$ is the angle between the cutter face and the $XZ$ plane, and $\phi$ is the complement of the pressure angle of the cutter as shown in Fig. 2. The cutter's surface may be expressed in terms of the gear blank's system $S$ by substituting Eqs. (8) and (9) into Eq. (10). This leads to

$$\hat{x}[(\tan \psi \cos \alpha - \sin \alpha)\cos \gamma - \cot \phi \sin \gamma]$$
$$+ \hat{y}[(\sin \gamma \sin \theta(\sin \alpha - \tan \psi \cos \alpha) - \cos \theta(\cos \alpha + \tan \psi \sin \alpha)$$
$$- \cot \phi \cos \gamma \sin \theta]$$
$$+ \hat{z}[(\sin \gamma \cos \theta(\sin \alpha - \tan \psi \cos \alpha) - \sin \theta(\cos \alpha + \tan \psi \sin \alpha)$$
$$- \cot \phi \cos \gamma \cos \theta] = 0 = F(\hat{x}, \hat{y}, \hat{z}, \theta) \quad (11)$$

Following this procedure, to determine the envelope of the cutter surface on the generated gear blank, we evaluate the partial derivative of Eq. (11) with respect to parameter $\theta$, the rotation angle of the gear blank. This produces the relation

$$\hat{x}[(\sin \gamma(\tan \phi \sin \alpha + \cos \alpha))]$$
$$+ \hat{y}[(\sin \theta (1 - \sin \gamma \tan \gamma)(\cos \alpha + \tan \psi \sin \alpha)$$
$$+ \cos \theta(\sin \gamma - \tan \gamma)(\sin \alpha - \tan \psi \cos \alpha) - \cot \phi \cos \gamma \cos \theta]$$
$$+ \hat{z}[(\cos \theta(1 - \sin \gamma \tan \gamma)(\cos \alpha + \tan \psi \sin \alpha)$$
$$- \sin \theta(\sin \gamma - \tan \gamma)(\sin \alpha - \tan \psi \cos \alpha) + \cot \phi \cos \gamma \sin \theta] = 0 \quad (12)$$

Let $\hat{x}$ be an independent variable. Then, from Eqs. (11) and (12), $\hat{y}$ and $\hat{z}$ may be evaluated in terms of $\hat{x}$. Observe that since the coefficients are functions of $\theta$, the tooth surface has the parametric form

$$\begin{align*}
\hat{x} &= \hat{x} \\
\hat{y} &= \hat{y}(\hat{x}, \theta) \\
\hat{z} &= \hat{z}(\hat{x}, \theta)
\end{align*} \quad (13)$$
where $x$ and $\theta$ are the surface coordinates. Equation (13) represents the envelope of the rack relative to the gear blank. This represents the tooth surface of the straight bevel gear.

DEVELOPMENT OF A SPIRAL BEVEL GEAR TOOTH SURFACE

A spiral bevel gear tooth surface is developed in a similar manner. Figure 4 depicts a circular cutter generating a spiral bevel gear. The cutter is mounted on the cradel of a generating machine. When the cutter rotates about its own axis, it forms a surface that simulates a crown gear. As the cradle, and hence the cutter, rotates about $Z_0$ at the rate $\omega_g$ and the gear blank rotates about $Z_p$ at the rate $\omega_p$, the cutter will generate a spiral bevel gear. The cutting speed is independent of $\omega_g$ and $\omega_p$. It is not related to the kinematics of tooth generation. The relation between $\omega_g$ and $\omega_p$ is simply

$$\frac{\omega_p}{\omega_g} = \frac{N_g}{N_p}$$

(14)

where $N_g$ and $N_p$ are the numbers of teeth in the crown gear and generated gear, respectively.

The coordinate system used to describe the crown gear is $S_1(x_1,y_1,z_1)$ with origin at $O_1$. The crown gear $G$ with frame $S_1$ fixed in $G$ rotates through an angle $\phi_1$ about $Z_0$ with respect to a global coordinate system $S_0(x_0,y_0,z_0)$ with the origin at $O_0$ as in Fig. 4. The position vectors $\vec{r}_1$, with the components $(x_1,y_1,z_1)$, and $\vec{r}_g$, with the components $(x_g,y_g,z_g)$, locating a point in $S_1$ and $S_g$ are related by the expression (see Fig. 5).

$$\vec{r}_1 = [R_{1g}] \vec{r}_g$$

(15)

where $[R_{1g}]$ is an orthogonal transformation matrix given by

$$[R_{1g}] = \begin{bmatrix} \cos \phi_1 & \sin \phi_1 & 0 \\ -\sin \phi_1 & \cos \phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(16)

Let a coordinate system $S_c(x_c,y_c,z_c)$ be fixed on the cutter with origin at $O_c(H,V,O)$. Let $H$ and $V$ be the horizontal and vertical machine settings (see Fig. 5). The cutter rotates through an angle $\theta$ about axis $Z_c$. Position vectors $\vec{r}_c$, with the components $(x_c,y_c,z_c)$ locating a point relative to $S_c$, and $\vec{r}_1$ are related by the expression

$$\vec{r}_c = [R_{c1}] \vec{r}_1 + [R_{c1}] \vec{r}_{1c}$$

(17)
where \([R_{c1}]\) and \(\bar{T}_{1c}\) are

\[
[R_{c1}] = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(18)

and

\[
\bar{T}_{1c} = \begin{bmatrix}
H \\
V \\
0
\end{bmatrix}
\]  

(19)

Let the coordinate system \(S_2(X_2,Y_2,Z_2)\) be fixed in the gear blank. Let \(S_2\) rotate through an angle \(\phi_2\) about \(Z_p\) with respect to a second global coordinate system \(S_p(X_p,Y_p,Z_p)\) (see Figs. 4, 6, and 7). The position vectors \(\bar{T}_p\), with the components \((x_p,y_p,z_p)\), and \(\bar{T}_2\), with the components \((x_2,y_2,z_2)\), locating a point in \(S_p\) and \(S_2\) are related by the expression

\[
\bar{T}_p = [R_{p2}] \bar{T}_2
\]  

(20)

where the transformation matrix \([R_{p2}]\) from \(S_2\) to \(S_p\) is given by

\[
[R_{p2}] = \begin{bmatrix}
\cos \phi_2 & -\sin \phi_2 & 0 \\
\sin \phi_2 & \cos \phi_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(21)

The two global coordinate systems \(S_p\) and \(S_2\) are related by the root angle \(\gamma\) of the generated gear and the addendum of the cutter tooth \(h\) (see Figs. 4, 6, and 8). Hence, \(\bar{T}_g\) and \(\bar{T}_p\) are related by the expression

\[
\bar{T}_g = [R_{gp}] \bar{T}_p + \bar{T}_{gp}
\]  

(22)

where

\[
[R_{gp}] = \begin{bmatrix}
1 & 0 & 0 \\
0 & \sin \gamma & -\cos \gamma \\
0 & \cos \gamma & \sin \gamma
\end{bmatrix}
\]  

(23)

and
\[ \Phi_2 \sin \gamma_0 = \Phi_1 \cos(\gamma_0 - \gamma) \]  

where \( \gamma_0 \) is the pitch angle of the generated gear. Integrating Eq. (25) with respect to time then leads to the relation

\[ \phi_1 = \frac{\sin \gamma_0}{\cos(\gamma_0 - \gamma)} \Phi_2 + \Phi_{10} \]  

where \( \Phi_{10} \) is a constant determined by initial conditions.

Let the constant parameter \( w \) be defined as

\[ w = \frac{\sin \gamma_0}{\cos(\gamma_0 - \gamma)} \]  

Combining Eqs. (14), (25), and (27) leads to the relation

\[ \frac{N_0}{N} = \frac{w_p}{w} \quad \frac{\Phi_2}{\Phi_1} = \frac{1}{w} \]  

CIRCULAR CUTTER SURFACE

In Fig. 9 the rotation of the cutter with a straight blade describes a conical surface of revolution with vertex angle \( (\pi - 2\gamma_0) \). The mean radius of the head cutter measured in the plane \( Z_C = 0 \) is \( R_C \). The apex of the cone is at \( V \) with coordinates \( (0, 0, z_0) \) in \( S_C \). The coordinates \( (x_C, y_C, z_C) \) of an arbitrary point \( C \) on the surface of revolution can then be expressed by the surface coordinates \( z_C \) and \( \alpha \) as

\[
\begin{align*}
x_C &= (z_0 - z_C) \cot \psi_0 \cos \alpha \\
y_C &= (z_0 - z_C) \cot \psi_0 \sin \alpha
\end{align*}
\]
Equation (29) may also be expressed in the form

\[ f(x_c, y_c, z_c) = \tan^2 \psi_0 (x_c^2 + y_c^2) - (z_0 - z_c)^2 = 0 \]  \hspace{1cm} (30)

The cutter surface may be expressed in the gear blank system \( S_2 \) by substituting from Eqs. (15), (20), and (22) into Eq. (17). This leads to

\[ \mathbf{r}_c = [R_{c1}][R_{g1}][R_{g2}] \mathbf{r}_2 + [R_{c1}][R_{g1}] \mathbf{r}_{gp} + [R_{c1}] \mathbf{r}_{1c} = [E] \mathbf{r}_2 + \mathbf{t} \]  \hspace{1cm} (31)

where \([E]\) and \(\mathbf{t}\) are defined as

\[ [E] = [R_{c1}][R_{g1}][R_{g2}] \]  \hspace{1cm} (32)

and

\[ \mathbf{t} = [R_{c1}][R_{g1}] \mathbf{t}_{gp} + [R_{c1}] \mathbf{t}_{1c} \]  \hspace{1cm} (33)

By substituting from Eqs. (16), (18), (19), (23), and (24) into Eqs. (32) and (33), the elements of \([E]\) and \(\mathbf{t}\), \(e_{ij}\) and \(t_1(1, j, = 1, 3)\), are found to be

\[
\begin{align*}
    e_{11} &= \cos \phi_2 \cos(\Theta + \phi_1) + \sin \gamma \sin \phi_2 \sin(\Theta + \phi_1) \\
    e_{12} &= -\sin \phi_2 \cos(\Theta + \phi_1) + \sin \gamma \cos \phi_2 \sin(\Theta + \phi_1) \\
    e_{13} &= -\cos \gamma \sin(\Theta + \phi_1) \\
    e_{21} &= -\cos \phi_2 \sin(\Theta + \phi_1) + \sin \gamma \sin \phi_2 \cos(\Theta + \phi_1) \\
    e_{22} &= \sin \phi_2 \sin(\Theta + \phi_1) + \sin \gamma \cos \phi_2 \cos(\Theta + \phi_1) \\
    e_{23} &= -\cos \gamma \cos(\Theta + \phi_1) \\
    e_{31} &= \cos \gamma \sin \phi_2 \\
    e_{32} &= \cos \gamma \cos \phi_2 \\
    e_{33} &= \sin \gamma \\
\end{align*}
\]  \hspace{1cm} (34)

and

\[
\begin{align*}
    t_1 &= H \cos \Theta + V \sin \Theta \\
    t_2 &= -H \sin \Theta + V \cos \Theta \\
    t_3 &= -h \\
\end{align*}
\]  \hspace{1cm} (35)
Hence, the cutter surface expressed in the $S_2$ coordinate system may be obtained by substituting from Eqs. (31), (34), and (35) into Eq. (29). That is,

\[
\begin{align*}
(e_{11} \tan \psi_0 + e_{31} \cos \alpha)x_2 &= (e_{12} \tan \psi_0 + e_{32} \cos \alpha)y_2 + (e_{13} \tan \psi_0 + e_{33} \cos \alpha)z_2 \\
(z_0 - t_3 \cos \alpha - t_1 \tan \psi_0)
\end{align*}
\]

and

\[
\begin{align*}
(e_{21} \tan \psi_0 + e_{31} \sin \alpha)x_2 &= (e_{22} \tan \psi_0 + e_{32} \sin \alpha)y_2 + (e_{23} \tan \psi_0 + e_{33} \sin \alpha)z_2 \\
(z_0 - t_3 \sin \alpha - t_2 \tan \psi_0)
\end{align*}
\]

**SPIRAL BEVEL GEAR TOOTH SURFACE EQUATION**

Equations (36) represent the generating surface seen by the gear blank. In determining the envelope of the cutter surface on the gear blank, it is useful to evaluate the partial derivative of $\tilde{r}_c$ in Eq. (31) with respect to the parameter $\phi_2$. That is,

\[
\frac{\partial \tilde{r}_c}{\partial \phi_2} = \begin{bmatrix} \frac{\partial \tilde{r}_2}{\partial \phi_2} \\ \frac{\partial \tilde{r}_1}{\partial \phi_2} \\ \frac{\partial \tilde{r}_3}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{r}_1}{\partial \phi_2} \\ \frac{\partial \tilde{r}_2}{\partial \phi_2} \\ \frac{\partial \tilde{r}_3}{\partial \phi_2} \end{bmatrix}
\]

(37)

The second term is zero since $\tilde{r}_2$ is fixed in $S_2$. From Eq. (35) the last term is also zero because the cutting speed is independent of the gear blank rotation. Let the component of the first term be $d_{1j}$. Then $d_{1j}$ can be determined from Eq. (34), and Eq. (37) may be rewritten in the form

\[
\frac{\partial \tilde{r}_c}{\partial \phi_2} = (d_{1j}) \tilde{r}_2 \quad (1 \leq j \leq 3)
\]

(38)

where

\[
\begin{align*}
d_{11} &= -\sin \phi_2 \cos(\Theta + \Phi_1)(1 - w \sin \gamma) + (\sin \gamma - w) \cos \phi_2 \sin(\Theta + \Phi_1) \\
d_{12} &= -\cos \phi_2 \cos(\Theta + \Phi_1)(1 - w \sin \gamma) - (\sin \gamma - w) \sin \phi_2 \sin(\Theta + \Phi_1) \\
d_{13} &= -w \cos \gamma \cos(\Theta + \Phi_1) \\
d_{21} &= \sin \phi_2 \sin(\Theta + \Phi_1)(1 - w \sin \gamma) + (\sin \gamma - w) \cos \phi_2 \cos(\Theta + \Phi_1) \\
d_{22} &= \cos \phi_2 \sin(\Theta + \Phi_1)(1 - w \sin \gamma) - (\sin \gamma - w) \sin \phi_2 \cos(\Theta + \Phi_1) \\
d_{23} &= -w \cos \gamma \sin(\Theta + \Phi_1) \\
d_{31} &= \cos \gamma \cos \phi_2 \\
d_{32} &= -\cos \gamma \sin \phi_2 \\
d_{33} &= 0
\end{align*}
\]

(39)
Equation (38) represents the derivative of the equation giving transformation from the cutter system to the gear blank system. It is useful for deriving the constraint equation of the cutter's motion. The derivative of the cutter conical surface with respect to $\psi_2$, the rotation angle of the gear blank, is (from Eq. (30))

$$f(x_c, y_c, z_c) = 2\tan^2\psi_0 \left( x_c \frac{\partial x_c}{\partial \psi_2} + y_c \frac{\partial y_c}{\partial \psi_2} \right) + 2(z_0 - z_c) \frac{\partial z_c}{\partial \psi_2} = 0 \quad (40)$$

If $z_c$ is not equal to $z_0$, we may substitute from Eqs. (29) and (38) into Eq. (40). This leads to

$$[\tan \psi_0(d_{11} \cos \alpha + d_{21} \sin \alpha) + d_{31}]x_2 + [\tan \psi_0(d_{12} \cos \alpha + d_{22} \sin \alpha) + d_{32}]y_2 + [\tan \psi_0(d_{13} \cos \alpha + d_{23} \sin \alpha) + d_{33}]z_2 = 0 \quad (41)$$

Equation (41) is the constraint equation of the cutter motion on the gear blank. Combining Eqs. (36) and (41) forms a set of simultaneous equations representing the envelope of the cutter relative to the gear blank. The solution of the simultaneous equations describes the tooth surface impression created by the cutter. Observe that since the coefficients are functions of $((X, \phi_1))$, the solution has the parametric form

$$\begin{align*}
x_2 &= x_2(\alpha, \phi_1) \\
y_2 &= y_2(\alpha, \phi_1) \\
z_2 &= z_2(\alpha, \phi_1)
\end{align*} \quad (42)$$

where $\alpha$ and $\phi_1$ are the surface coordinates.

**DISCUSSION**

Equations (11) to (13) and Eqs. (36), (41), and (42) determine the cutter envelopes and hence the gear tooth surface for straight and spiral bevel gears, respectively. They form the basis for a numerical and computer graphic representation of the tooth surface. In this context these equations represent an extension of the procedure described earlier (Chang, Huston, and Coy, 1984) from spur gear to bevel gear. The difference here, however, is that the equations are more detailed and extensive. Hence, numerical and computer graphic analyses are needed.

In a general sense the method simulates the kinematic relation of a cone-shaped gear blank and the cutter. With the rotation angle of the gear blank being the parameter, the envelope of the cutter profile on the gear blank describes the gear tooth profile. The straight and spiral bevel gear tooth surfaces are the forms given in Eqs. (13) and (42). The machine settings, the cutter radius, and the mean cone distance are the parameters that can be varied in a design optimization analysis. An accurate finite element mesh can also be obtained by discretizing the tooth surface.
In summary, it is believed by the authors that the analysis presented herein can form a basis for numerical design studies as well as for stress and deformation studies. Finally, the method may be readily extended to the study of nonintersecting shaft (hypoid) gears.

REFERENCES


Fig. 1. Straight Bevel Gear Tooth Surface Generation

Fig. 2. The Pitch Plane

Fig. 3. The Normal Plane

Fig. 4. Machine Setting for Spiral Bevel Gear Manufacturing
Fig. 5. Relation of Crown Gear and Cutter

Fig. 6. Relation of Generating Gear and Generated Gear

Fig. 7. Relation of Coordinate System $S_1$ and $S_2$
Fig. 8. Relation of Coordinate System $S_o$ and $S_p$

Fig. 9. Surface of Revolution of the Cutter
This paper presents a computer-aided design procedure for generating bevel gears. The development is based on examining a perfectly plastic, cone-shaped gear blank rolling over a cutting tooth on a plane crown rack. The resulting impression on the plastic gear blank is the envelope of the cutting tooth. This impression and envelope thus form a conjugate tooth surface. Equations are presented for the locus of points on the tooth surface. The same procedures are then extended to simulate the generation of a spiral bevel gear. The corresponding governing equations are presented.