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AN ALGORITHM TO FIND A K_5 MINOR

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An algorithm to find a K_5 minor

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1 Introduction

Kuratowski's Theorem states that a graph is planar if and only if it does not contain a homeomorph of $K_{3,3}$ or K_5. For this reason, homeomorphs of either $K_{3,3}$ or K_5 are called *Kuratowski homeomorphs*. Because of the connection with planarity, there has been much interest in algorithms that find Kuratowski homeomorphs in non-planar graphs. Determining whether a graph contains a Kuratowski homeomorph can be solved in linear time by well-known planarity testing algorithms [HT74]. Williamson [Wil84] provided a linear-time algorithm to extract a Kuratowski homeomorph from a non-planar graph. Asano [Asa85] (see also Kaschube [Kas84]) provided a linear-time algorithm to test whether a graph contains a homeomorph of $K_{3,3}$. Fellows and Kaschube [FP] have obtained a linear algorithm that constructs a $K_{3,3}$ homeomorph when one is present. Khuller et al. [KMV89] describe parallel algorithms to find Kuratowski homeomorphs and $K_{3,3}$ homeomorphs, when such homeomorphs are present.

Extending Kuratowski's result, Wagner showed that a graph is planar if and only if it does not contain a K_5 minor or $K_{3,3}$ minor [Wag37a]. Robertson and Seymour [RS86] (see also [RS85]) have shown that, for any fixed graph H, there exists an $O(n^3)$ algorithm that determines whether a graph contains a minor isomorphic to H. Moreover, they have shown that there are polynomial-time algorithms for the fixed subgraph-homeomorphism problem; that is, there are polynomial time algorithms testing whether a graph contains a subgraph homeomorphic to a fixed graph. However, their algorithms contain large constants and so are far from practical.

This paper describes an algorithm that determines whether a graph contains a minor isomorphic to K_5, and constructs such a minor, if it exists, in $O(n^2)$ time. This algorithm complements the result by Asano for obtaining a $K_{3,3}$ homeomorph. The algorithm is also the first practical polynomial-time algorithm for finding a (non-trivial) minor in a graph, and so provides an indication that the work by Robertson and Seymour may indeed lead to practical polynomial-time algorithms for graph minors.

In section three, we prove a structural characterization of graphs that do not have a K_5 minor. This characterization is similar to a characterization of Wagner [Wag37b]. We show that, to characterize graphs without a K_5 minor, it suffices to consider 3-connected, non-planar graphs with

at most $3n - 6$ edges; these graphs must contain a $K_{3,3}$ homeomorph. The main structural result is that in any 3-connected graph G containing a $K_{3,3}$ homeomorph at least one of the following must hold: 1) G contains a K_5 minor, 2) G is isomorphic to an 8-cycle with crossing chords, or 3) the red branch (blue branch) vertices of the $K_{3,3}$ homeomorph are a 3-cut that divides the blue branch (red branch) vertices into 3 separate components. In section four, we present the algorithm. The algorithm first recursively decomposes the graph into its (augmented) three-connected components. Then the algorithm applies Williamson's linear-time algorithm [Wil84] on these components to test for planarity and construct a Kuratowski homeomorph in any non-planar graph. If Williamson's algorithm returns a $K_{3,3}$ homeomorph, then either the graph has a K_5 minor, or the graph is isomorphic to W. or we are able to recursively apply the algorithm on the augmented components induced by the 3-cut in the $K_{3,3}$ homeomorph. We obtain the representation of the K_5 minor by attaching appropriate paths to the $K_{3,3}$ homeomorph; the methods in the proofs of the structural results are applied to obtain these paths. Before presenting these results, we first present preliminary notation and definitions.

2 Preliminaries

Let G be an undirected simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$ with cardinalities n and m, respectively. Let $e = xy$ denote the edge between the two vertices x and y. An *elementary subdivision* of G is a graph obtained from G by removing some edge $e = xy$ and adding a new vertex z together with two new edges xz and zy. A *homeomorph* of G is a graph obtained from G by a succession of elementary subdivisions. In the literature, homeomorphs are also called *subdivisions*. Additionally, a graph H is a *topological subgraph* of G if G contains a homeomorph of H. Two graphs are *homeomorphic* if they are both homeomorphs of the same graph. A *contraction* of an edge $e = uv$ in G (denoted G/uv) is made by identifying vertices u and v with a new vertex whose neighborhood is the union of the neighborhoods of u and v (resulting multiple edges and loops are deleted). A graph H is a *minor* of G if a graph isomorphic to H if H can be obtained from G by a series of vertex deletions, edge deletions, and edge contractions. Observe that the minor order is transitive; that is, if G_1 is a minor of G_2, and G_2 is a minor of G_3,

then G_1 is a minor of G_3. Note also that, if G contains a homeomorph of H, then H is a minor of G.

Suppose G contains a minor isomorphic to H. We associate with each vertex v of H a set, called the *branch set of* v, which consists of those vertices of G that have been merged by edge contractions to form the vertex v. Observe that the branch sets depend both upon the choice of edge deletions, vertex deletions, and edge contractions used to obtain the minor, and upon the isomorphism between H and the minor of G. To describe a minor explicitly, it suffices to present the branch sets and the isomorphism. Such a presentation is called a *model* of the minor H in G. In the case that H is a complete graph, any bijection between the vertex set of H and the collection of branch sets produces an isomorphism, so we shall ignore the isomorphism.

For any $S \subset V(G)$, $G[S]$ denotes the subgraph of G induced by S. A set $S \subset V(G)$ is a *k-cut* if $G - S$ is disconnected and $|S| = k$. A k-cut S of G is *strong* if $G - S$ has more than two components. A graph with at least k vertices is k-connected if it has no $(k-1)$-cut. A *k-connected component* of G is a maximal k-connected subgraph of G. In this paper, W denotes an 8-cycle with crossing chords (see Figure 1). The *neighborhood* of a vertex v in G, denoted $N_G(v)$, is the set of vertices in G adjacent to v; $d_G(v) = |N_G(v)|$ is the *degree* of v. Suppose H is a graph with minimum degree at least three, and G is a homeomorph of H. The vertices of G with degree at least three are called *branch vertices.* The *interior vertices* of a path P_{ax} between a and x are the vertices in $V(P_{a,x}) - \{a, x\}$. Given a set $S \subset V(G)$, a path in G is an *outside path* of S if every one of it is interior vertices is contained in $V(G - S)$. Suppose P is a path connecting to vertices u and v. *Contracting u to v along P* means contract the edges of P between u and v.

3 Structural results

In this section, we characterize graphs that do not contain a K_5 minor. Because planar graphs do not contain a K_5 minor, we need only consider non-planar graphs. The following theorem shows that we may further restrict our attention to graphs with at most $3n - 6$ edges. See Gyori [Gy82] for another proof of this and related results.

Theorem 1 *If G has at least $3n-5$ edges, then G contains a K_5 minor.*

Proof: The proof is an induction on $n = |V(G)|$. The only graph with $n = 5$ vertices and at least $3n-5$ edges is K_5; this provides the basis for the induction. Suppose $n > 5$ and $m = |E(G)| \geq 3n-5$.

Let v be any vertex of G. If $G[N(v)]$ has minimum degree three then, by a result of Dirac [Dir60], $G[N(v)]$ contains a K_4 minor which together with v produces a K_5 minor in G. Hence we may assume there is a vertex $u \in N(v)$ with at most two neighbors in $N(v)$. Contract the edge uv to form $G' = G/_{uv}$, with $m' = |E(G')|$ and $n' = |V(G')| = n-1$. Now $m' \geq m-3$, since u has at most two neighbors in $N(v)$. Therefore, $m' \geq 3n-8 = 3n'-5$ and we apply the inductive hypothesis to conclude that G' contains a K_5 minor. The transitivity of the minor order implies that G also contains a K_5 minor. □

The proof of Theorem 1 yields an $O(n^2)$ algorithm that constructs a K_5 minor in any graph G with at least $3n-5$ edges. Select an arbitrary vertex $v \in G$ and examine its neighborhood. If $G[N(v)]$ has minimum degree at least three, then an $O(n)$ algorithm by Asano [Asa85] (see also [LG80]) finds a K_4 homeomorph in $G[N(v)]$ which together with v produces the desired K_5 minor. Otherwise, select a vertex $u \in N(v)$ with at most two neighbors in $N(v)$. Contract uv and continue on the resulting smaller graph. There are at most n iterations each requiring $O(n)$ time, so the algorithm requires $O(n^2)$ time in the worst case.

Let G be a graph with a cutset $C \subset V(G)$. A branch set of some minor of G *crosses* C if it contains vertices in two different components of $G - C$. Note that since branch sets induce a connected graph in G, any branch set crossing C must contain a vertex in C. Because branch sets are disjoint, at most $|C|$ branch sets cross C. In particular this implies that a graph G has a K_5 minor if and only if a 2-connected component of G contains a K_5 minor. This observation extends to 2-cuts, as the next theorem proves.

Suppose $C \subset V(G)$ separates G into p ($p \geq 2$) components $G_1, \ldots, G_p$. For $1 \leq i \leq p$, let $G_i \cup K(C)$ be the graph obtained from $G[V(G_i) \cup C]$ by adding an edge between any pair of non-adjacent vertices in C. The graphs $G_1 \cup K(C), \ldots, G_p \cup K(C)$ are called the *augmented components* induced by C.

Theorem 2 *Suppose H is a 3-connected graph, and G is a 2-connected graph with a 2-cut C. Then G has an H minor if and only if some augmented component of G induced by C has an H minor.*

Proof: Suppose some augmented component $G_i \cup K(C)$ contains an H minor. To show that G also has an H minor, it suffices to show that the augmented component $G_i \cup K(C)$ is a minor of G. Consider a path P with internal vertices from $V(G) - V(G_i) - C$ connecting the two vertices in C; such a path must exist since G is 2-connected. The graph obtained from $G[V(G) \cup C \cup V(P)]$ by contracting the edges in the path P is the desired minor isomorphic to $G_i \cup K(C)$.

On the other hand, suppose that some 3-connected graph H is a minor of G. Let C be a 2-cut of G. We must show that some augmented component of G induced by C contains an H minor. Consider the branch sets of H in G. Observe that at most two of these branch sets cross C. If two branch sets are completely contained in different components of $G - C$, then H has a cut consisting of those vertices whose branch sets cross C, contradicting that H is 3-connected. Hence, we may assume that there is one component of $G - C$, say G_i, that completely contains all non-crossing branch sets. Finally, replace any crossing branch set B with $B \cap (V(G_i) \cup C)$. The resulting collection of branch sets form an H minor in $G_i \cup K(C)$. □

The ideas in the previous proof can be extended to 3-cuts provided that the 3-cuts are strong. The following theorem presents the extension needed.

Theorem 3 *Suppose G is a 3-connected graph with a strong 3-cut C. Then G has a K_5 minor if and only if some augmented component of G induced by C has a K_5 minor.*

Proof: Let $G_1, G_2, \ldots, G_p, p \geq 3$ be the components of $G - C$. Suppose some augmented component induced by C contains a K_5 minor. We may assume that $G_1 \cup K(C)$ contains a K_5 minor. To show that G also has a K_5 minor, it suffices to show that the augmented component $G_1 \cup K(C)$ is a minor of G. Because G is 3-connected, there is a set of three disjoint paths in $G[V(G_2 \cup C)]$ from $v \in G_2$ to C. The paths can be contracted in G to produce two edges among the vertices of C. If C induces a clique in this minor of G, then $G_1 \cup K(C)$ is a minor of G. Otherwise, let x and y be the only non-adjacent vertices of C in this minor. Consider a path P with internal vertices from $V(G_3)$ connecting x and y; such a path must exist since G is 3-connected. Contracting the edges

in the path P now yields a minor of G containing $G_1 \cup K(C)$.

In the other direction, suppose G contains a K_5 minor. Consider the branch sets determined by this minor. At most three of these branch sets cross C. If two branch sets are entirely contained in different components of $G - C$, then K_5 contains a cut consisting of those vertices whose branch vertices cross C contradicting the 4-connectivity of K_5. Therefore, some component of $G-C$, say G_i, contains every non-crossing branch set. Replace every crossing branch set B with $B \cap (V(G_1) \cup C)$. The resulting collection of branch sets form a K_5 minor in $G_i \cup K(C)$. □

The previous results suggest the following recursion for he algorithm. Given a non-planar graph, compute its connectivity. If its connectivity is at most two, recursively apply the algorithm on the augmented components induced by a 2-cut. If its connectivity is at least three, recursively apply the algorithm on the augmented components induced by a strong 3-cut. The difficulty is that strong 3-cuts may not exist. The remainder of this section addresses this difficulty.

Observe that, by Kuratowski's Theorem, a non-planar graph with no K_5 minor must contain a homeomorph of $K_{3,3}$. Suppose S is a homeomorph of $K_{3,3}$ in a 3-connected non-planar graph G. Let $\{a, b, c\}$ and $\{x, y, z\}$ be the bipartition of the branch vertices of S corresponding to a 2-coloring of $K_{3,3}$. Branch vertices a, b, and c are red vertices; branch vertices x, y, and z are blue vertices. For convenience, we define $R = \{a, b, c\}$ and $B = \{x, y, z\}$. Paths in S connecting branch vertices are *branch paths* and are denoted P_{uv} where $u \in R$, $v \in B$. For example, P_{ax} denotes the branch path of S connecting a and x. Each branch vertex v of S determines a set of three branch paths incident to v called the *branch-fan* of S at v. Let $F(v)$ represent the branch-fan of S at the branch vertex v. A branch fan $F(v)$ is an *R-branch-fan* or *B-branch-fan* according to whether $v \in R$ or $v \in B$. Branch paths with distinct endpoints are *parallel.* Note that two vertices in S are either in the same branch-fan, or are interior vertices of parallel branch-paths. An interior vertex of a branch path determines two branch vertices called *branch-ends.* For example, any interior vertex w of the branch path P_{ax} has branch ends a and x.

Provided certain extra paths exist, a K_5 minor can be obtained from the homeomorph S of $K_{3,3}$. The following lemmas determine which extra paths are sufficient. There are two important cases. For the first case, suppose there are two disjoint outside paths, one path between two vertices

in $\{a,b,c\}$ (say a and b) and the other between two vertices of $\{x,y,z\}$ (say x and y). The union of S with these two paths is a homeomorph of the L graph (see Figure 2.) The L graph has a K_5 minor obtained by contracting the edge between the branch vertices of degree three. For the other case, suppose there is an interior vertex $v \in P_{ax}$ that has outside paths to b and y. The unions of S and these two paths is a homeomorph of M (see Figure 3). The graph M has a K_5 minor obtained by contracting za and cx; the branch vertices are $\{v,(az),(cx),b,y\}$. By the transitivity of the minor order, a graph that has an L or M minor has a K_5 minor. The strategy of the lemmas is to produce either an L or M minor, by using a $K_{3,3}$ homeomorph and the additional structure implied by the hypothesis of each lemma.

Lemma 1 *Suppose G is a 3-connected graph containing a homeomorph S of $K_{3,3}$. If a vertex in $G - S$ has three outside paths to S with endpoints not all in the same branch-fan, then G contains a K_5 minor.*

Proof: Let w be a vertex in $G - S$ with three outside paths to S having endpoints t, u, and v such that t, u, v are not contained in one branch-fan. We may assume that the outside paths from w to S are disjoint because otherwise we can contract w toward their intersections. At least one of t,u, or v must be an interior vertex of a path in S, otherwise they would necessarily be in the same branch-fan. For example, if $\{t,u,v\} = \{a,x,y\}$ then they are all in $F(a)$. Without loss of generality, let t be an interior vertex of P_{ax}. Vertices u and v cannot both be in $F(a)$ since t is in $F(a)$; similarly u and v cannot both be in $F(x)$. Now there are three cases to consider:

Case 1: $\{u,v\} \cap (F(a) \cup F(x)) = \emptyset$. That is, neither u nor v is in the same branch fan as t. In this case, u and v must be interior vertices of branch paths of S, If u and v appear on different branch paths of S, then contracting u to its blue branch end, contracting v to its red branch end, and contracting w to t produces a graph containing an M minor. So we may assume that u and v appear on the same branch path, say P_{by}. We may further assume that the order of vertices on P_{by} is b,u,v,y. Contract u to b and v to y along P_{by}, and contract w to t along the outside path connecting them. These contractions produces a graph homeomorphic to M (see Figure 4). Hence, G contains a K_5 minor.

Case 2: $\{u, v\} \cap P_{ax} \neq \emptyset$. Without loss of generality, let $u \in P_{ax}$. Because both t and u are in $F(a) \cap F(x)$, we may assume that v an interior vertex of P_{by}. By switching the labels of v and t, this reduces to the previous case.

Case 3: $\{u, v\} \cap (F(a) \cup F(x) - P_{ax}) \neq \emptyset$. Without loss of generality, assume $u \in F(x) - P_{ax}$; by relabeling b and c if necessary, we may assume that $u \in P_{bx}$. In this case v cannot be in $F(x)$, and so must be in $F(y) - \{b, c\}$ or $F(z) - \{b, c\}$. Contract u to b along P_{bx} and, contract v to its blue branch end (either y or z) along the branch path of S containing v. Because these contractions occur along distinct branch paths of S, no conflicts occur. Now contract w to t along the outside path connecting them. All of these contractions form a homeomorph of M (see Figure 5). Hence, K_5 is a minor of G. □

Lemma 2 *Let S be a $K_{3,3}$ homeomorph in a 3-connected graph G. Suppose there exists an outside path with endpoints in distinct branch-paths of an R-branch-fan, and there is an outside path with endpoints in distinct branch-paths of a B-branch-fan. Then G contains a K_5 minor.*

Proof: Let P_1 be the outside path with endpoints in an R-branch-fan, and let P_2 be the outside path with endpoints in a B-branch-fan. Without loss of generality, we may assume that the endpoints of P_1 are in $F(a)$, and the endpoints of P_2 are in $F(x)$. Since the endpoints must be in different branch paths, the vertex a is not an endpoint of P_1 and x is not an endpoint of P_2. We now consider two cases depending on whether P_1 and P_2 intersect.

Case 1: P_1 and P_2 do not intersect. Contract each endpoint of P_1 to its corresponding blue branch end. Similarly, contract each endpoint of P_2 to its corresponding red branch end. If these contractions involve distinct vertices, then this produces a homeomorph of L, hence a K_5 minor, in G (see Figure 6). If the contractions are not disjoint, then P_1 and P_2 have endpoints in the same branch path. Because P_1 is in $F(a)$ and P_2 is in $F(x)$ this implies that P_1 and P_2 have an endpoint in P_{ax}, say $u \in P_1 \cap P_{ax}$ and $v \in P_2 \cap P_{ax}$. Because the endpoints of P_1 and P_2 are in distinct branch paths, both u and v are interior vertices of P_{ax}. In this case, the following contractions produce a homeomorph of M in G: contract u to v along P_{ax}, contract the other endpoint of P_1 to its blue branch end, and contract the other endpoint of P_2 to its red branch end.

Case 2: P_1 and P_2 intersect. If P_1 and P_2 intersect in a vertex $u \in G - S$, then Lemma 1 applies since u then has three outside paths to S with endpoints not all in the same branch fan. Hence, we may assume that P_1 and P_2 only intersect in a vertex of S, which must be an interior vertex of P_{ax}. The other endpoints of P_1 and P_2 must be different because P_1 and P_2 end in distinct branch paths of different branch fans. Contracting the other endpoint of P_1 to its blue branch end, and contracting the other endpoint of P_2 to its red branch end, produces a homeomorph of M. Hence G contains a K_5 minor. □

Lemma 3 *Let S be a $K_{3,3}$ homeomorph in a 3-connected graph G. Suppose there are two outside paths of S such that one path connects two interior vertices of parallel branch-paths of S and the other path connects two vertices in S in distinct branch-paths. If the two paths do not have the same two endpoints in S, then G contains a K_5 minor.*

Proof: Let P_1 be the path that connects two interior vertices of parallel branch-paths of S, and let P_2 be the path that connects two vertices in S in distinct branch-paths. Without loss of generality, assume P_1 ends in P_{ax} and P_{by}. As in the previous lemma, if P_1 and P_2 intersect in a vertex $u \in G - S$, then Lemma 1 applies since u has three outside paths to S consisting of those fragments of P_1 and P_2 that connect u to S and that have distinct endpoints in S. It remains to consider the following two cases:

Case 1: The endpoints of P_1 and P_2 are in distinct branch paths. Without loss of generality, assume that the endpoints of P_2 have distinct red branch ends. In this case, contract the endpoints of P_1 to their corresponding blue branch ends (x and y), and contract the endpoints of P_2 to their corresponding red branch ends. The resulting minor of G contains a homeomorph of L (see Figure 7).

Case 2: At least one endpoint of P_1 and P_2 are in the same branch path. Without loss of generality, assume one endpoint of P_2 is in P_{ax}. If this endpoint is a branch vertex of S, then the contractions in the previous paragraph produce a minor of G containing a homeomorph of L. Hence we may assume that this endpoint of P_2 is an interior vertex of P_{ax}. If the endpoints of P_1 and P_2 in P_{ax} are distinct, contract the endpoint of P_1 in P_{ax} to the endpoint of P_2 in P_{ax} along P_{ax}. If the other endpoints of P_1 and P_2 appear on different branch paths of S, then contract one of them

to it blue branch end and contract the other to its red branch end. This produces a minor of G containing a homeomorph of M. If the other endpoints both appear on P_{by}, then contract the endpoint nearest b to b along P_{by}, and contract the other endpoint to y along P_{by}. The result is a minor of G homeomorphic to M (see Figure 8). □

Lemma 4 *Let S be a homeomorph of $K_{3,3}$ in G. Let a, b, and c be the red branch vertices of S, and let x, y, and z be the blue branch vertices of S. If neither $\{x, y, z\}$ nor $\{a, b, c\}$ is a 3-cut of G, then G contains a K_5 minor.*

Proof: Because $\{a, b, c\}$ is not a cut of G, there is a path P_{xy} from x to y avoiding vertices in $\{a, b, c\}$. Let u_1 be the vertex in $P_{xy} \cap F(x)$ furthest away from x along P_{xy} (u_1 may equal x). Let v_1 be the next vertex of P_{xy} after u_1 that is in S. The subpath P_1 of P_{xy} connecting u_1 to v_1 is an outside path of S, and its endpoints are not contained in any one branch path because this would imply that v_1 is in $F(x)$, contradicting the choice of u_1. Indeed this argument shows that u_1 and v_1 are in different blue branch fans. Suppose, without loss of generality, that $v_1 \in F(y)$.

To each of u_1 and v_1 there corresponds a single red branch end; hence there is a red branch vertex, say c, that is not the red branch end of either u_1 or v_1. Because $\{x, y, z\}$ is not a cut of G, there is a path P_{cb} from c to b avoiding vertices in $\{x, y, z\}$. Let u_2 be the vertex in $P_{cb} \cap F(c)$ furthest away from c along P_{cb} (u_2 may equal c). Let v_2 be the next vertex of P_{cb} after u_2 that appears in S. The subpath P_2 of P_{cb} connecting u_2 to v_2 is an outside path of S, and its endpoints are not contained in the same red branch fan. Furthermore, $u_2 \notin \{u_1, v_1\}$, since c is not a branch end of either of u_1 or v_1.

If u_1 and v_1 are contained in parallel branch paths, then by Lemma 3, K_5 is a minor of G. This means that u_1 and v_1 are in the same red branch fan. A similar argument applies if u_2 and v_2 are contained in parallel branch paths. Hence, u_2 and v_2 are in the same blue branch fan. This reduces to to Lemma 2, and G contains a K_5 minor minor. □

The following known result is an easy corollary of the above lemma (see, for example, a proof by Young [You71]).

Corollary 1 *If G is 4-connected and non-planar, then G contains a K_5 minor.*

Before we present the main structural theorem, we require a lemma on graphs containing a W homeomorph.

Lemma 5 *Suppose G contains a W homeomorph T, and there exists a path in G outside T that has endpoints in distinct branch paths of T. Then G contains a K_5 minor.*

Proof: Let $W \cup e$ denote any graph formed by adding an edge to W. Observe that any $W \cup e$ graph contains a K_5 minor. Let path P be an outside path of T, with endpoints in distinct branch paths of T. In all cases, $T \cup P$ is contractible to $W \cup e$. □

We now come to our characterizing theorem. Wagner was the first to provide a characterization of graphs containing no K_5 minor, which he applied to prove the equivalence of the Four Color Theorem and the Hadwiger conjecture for $k = 5$ [Wag37b],[Wag60]. A *clique-sum* of graphs G and H is obtained by identifying the vertices of a clique in G and H together, and removing some (or none) of the edges in the clique. Note that this is the reverse of our notion of augmented components.

Theorem 4 (Wagner) *Every graph with no K_5 minor may be obtained by means of clique-sums, starting from planar graphs and copies of W.*

Seymour [Sey81] describes Wagner's result, and Young [You71] presents an alternative proof of the equivalence theorem. Halin [Hal67] elaborated on Wagner's methods, and in doing so made an observation closer to our own characterization: Any graph G must have either a K_5 minor, a subdivision of W, or a strong 3-cut. The following theorem refines Halin's observation, and is also another method of obtaining Wagner's characterization.

Theorem 5 *Suppose that G is a 3-connected graph containing a $K_{3,3}$ homeomorph S with red branch vertices a, b, c and blue branch vertices x, y, z. Then at least one of the following must hold:*

1. *G contains a K_5 minor.*
2. *$\{a, b, c\}$ separates G such that x, y, and z are in three separate components.*

3. $\{x, y, z\}$ separates G such that a, b, and c are in three separate components.

4. G is isomorphic to W, an 8-cycle with cross edges.

Proof: Let G be a 3-connected graph containing a homeomorph S of $K_{3,3}$, with red branch vertex set $R = \{a, b, c\}$ and blue branch vertex set $B = \{x, y, z\}$. We prove the theorem by supposing that conditions two and three do not hold, and showing that this forces a K_5 minor in all cases except when G is isomorphic to W.

Lemma 4 guarantees that if G does not contain a K_5 minor, either $G - B$ or $G - R$ is disconnected. By symmetry, we may assume $G - B$ is disconnected. Because a, b, and c are not all in different components of $G - B$, two of them, say b and c, are in the same component of $G - B$. Let P_{bc} be a path connecting b to c in $G - B$. Let u_1 be the vertex in $P_{bc} \cap F(b)$ furthest from b along P_{bc}, and let v_1 be the next vertex in S after u_1 along P_{bc}. The subpath P_1 of P_{bc} connecting u_1 to v_1 is an outside path. By the choice of u_1, v_1 is not in $F(b)$. Because a is disconnected from b and c in $G - B$, $v_1 \notin F(a)$. Hence, $v_1 \in F(c) - B$ and $u_1 \in F(b) - B$. Similarly, because x, y, and z are not all in different components of $G - R$, at least two of them, say y and z, are in the same component of $G - R$. Let P_{yz} be a path connecting y to z in $G - R$. Let u_2 be the vertex in $P_{yz} \cap F(y)$ furthest from y along P_{yz}, and let v_2 be the next vertex in S after u_2 along P_{yz}. The subpath P_2 of P_{yz} connecting u_2 and v_2 is an outside path of S, with $u_2 \in F(y) - R$ and $v_2 \in (F(x) \cup F(z)) - R$.

Thus we have two outside paths P_1 and P_2. Observe that the endpoints of P_1 appear in distinct branch paths of S; likewise, the endpoints of P_2 appear in distinct branch paths of S. If u_1 and v_1 are contained in a B-branch-fan, and if u_2 and v_2 are contained in an R-branch-fan, then Lemma 2 implies that K_5 is a minor of G. Otherwise, P_1 and P_2 must both connect parallel branch-paths of S. If $\{u_1, v_1\} \neq \{u_2, v_2\}$, and at least one of P_1 or P_2 connects two interior vertices of parallel branch-paths of S, then G contains a K_5 minor by Lemma 3. Therefore, if G does not contain a K_5 minor, then $\{u_1, v_1\} = \{u_2, v_2\}$ and, P_1 and P_2 connect the same interior vertices of one pair of parallel branch paths of S, say P_{by} and P_{cz}. Suppose that P_1 contains an interior vertex $v \notin \{u_1, v_1\}$. By the 3-connectivity of G, v has three disjoint paths to S. Moreover, the existence of P_1 guarantees that these three disjoint paths can be chosen so that all of them do not end in the

same branch path of S. In this case, Lemma 1 guarantees that G contains a K_5 minor. Therefore, we may assume that P_1 has no interior vertices; that is, P_1 is the edge u_1v_1. The same argument applies to P_2, so that $P_1 = P_2$ is a single edge.

Let $T = S \cup P_1$. Note that T is a homeomorph of W. We now show that G is isomorphic to W, first by showing that $G[T]$ is isomorphic to W, then by showing $G - T$ is empty. Suppose branch path P_{st}, joining branch vertices s and t in T, contains an interior vertex r. Since G is 3-connected, there is a path O in $G - \{s,t\}$ from r to some vertex in $T - P_{st}$. Let q_1 be the vertex in $P_{st} \cap O$ furthest from r along O, and let q_2 be the next vertex in T after q_1 along O. Let Q be the subpath of O from q_1 to q_2. The path Q and the W homeomorph T in G satisfy the conditions of Lemma 5. Hence K_5 is a minor of G, unless $G[T]$ is isomorphic to W and contains no subdivided paths.

It remains to show that there are no vertices in $G - T$. Suppose there is a vertex w in $G - T$. Because G is 3-connected, there are three vertex-disjoint paths from w to three vertices x, y, and z, in T. At least two endpoint vertices in T, say x and y, are non-adjacent, because W (and hence T) contains no K_3 subgraph. From the paths from w to x and y, we can construct a path P_{xy}, outside T, so that P_{xy} and T satisfy the hypothesis of Lemma 5. Thus there are no vertices in $G - T$ if K_5 is not a minor of G. Hence G is isomorphic to W. □

4 The algorithm

In this section we present an algorithm that determines whether a graph has a K_5 minor and returns a model of the minor if it exists. The algorithm *Find-K_5-minor* runs in $O(n^2)$ time. Applying Theorems 1 and 2, Find-K_5-minor quickly reduces to the consideration of 3-connected augmented components with at most $3n-6$ edges. The crux of the algorithm is the application of Williamson's algorithm [Wil84] that, given a graph G, determines whether G is planar and, if G is non-planar, returns a Kuratowski homeomorph — either K_5 or $K_{3,3}$. If Williamson's algorithm determines that G is planar or returns a K_5 homeomorph, then the algorithm is done. In the remaining case, Williamson's algorithm returns a $K_{3,3}$ homeomorph S. The algorithm checks whether the red or blue branch vertices of S are a strong 3-cut of G separating the other color class into three separate components. If so, the algorithm recursively determines whether the augmented components formed

by the strong 3-cut contain a K_5 minor. If not, then by Theorem 5, G is isomorphic to W, or G contains a K_5 minor. If G is not isomorphic to W, the algorithm calls the algorithm *Construct-K_5-model*, described below. Construct-K_5-model constructs and returns the branch sets of a model of a K_5 minor in G, constructively applying the proofs of the structural results.

Algorithm: Find-K_5-minor

INPUT: A graph G, with $n = |V|$ vertices, and $m = |E|$ edges.

OUTPUT: Either report that G does not contain a K_5 minor, or return a model of a K_5 minor in G.

1. Determine whether $m \geq 3n - 5$. If so, find a model of a K_5 minor using the algorithm described after the proof of Theorem 1.

2. If any vertex v is a 1-cut of G, then recursively apply Find-K_5-minor on each augmented component of $G - v$. Return the model of a K_5 minor found in any augmented component or, if all augmented components report no K_5 minor, report that there is no K_5 minor in G and halt.

3. If there is a 2-cut $\{u, v\}$ in G, then recursively apply the algorithm on each augmented component of $G - \{u, v\}$. Return the model of a K_5 minor found in any augmented component or, if all augmented components report no K_5 minor, report that there is no K_5 minor in G and halt.

4. Apply the Williamson algorithm to test planarity. If G is planar, then report that G does not contain a K_5 minor and halt. If Williamson's algorithm returns a K_5 homeomorph, then return this homeomorph and halt. Otherwise, let S be the $K_{3,3}$ homeomorph returned by the algorithm. Proceed to the next step.

5. Test whether G is isomorphic to W. If $G \cong W$, report that G has no K_5 minor and halt. Otherwise proceed to the next step.

6. Determine whether the red (blue) branch vertices R (B) of S separate the blue (red) branch vertices into 3 separate components. If so, then recursively apply Find-K_5-minor to each augmented component of $G - R$ ($G - B$). Otherwise proceed to the next step.

7. If this step is reached, then by Theorem 5 G must contain a K_5 minor. Apply the algorithm Construct-K_5-model, described below. The algorithm is given G and the $K_{3,3}$ homeomorph S, and returns the model of the K_5 minor in G. (For the decision algorithm, it is enough to report that a K_5 minor exists.)

Theorem 1 confirms the correctness of step 1. When the algorithm operates on several augmented components of G, then the algorithm returns a model of a K_5 minor in G if and only if there is a K_5 minor (returned) in some augmented component of G. Theorems 2 and 3 provide proof of the correctness of the recursions in steps 2, 3 and 6. The correctness of Williamson's algorithm verifies the correctness of step 4. By Theorem 5, either the conditions tested in steps 5 and 6 hold, or there is a K_5 minor in G, as assumed in step 7. Thus, at least for the decision algorithm, Find-K_5-minor determines there is a K_5 minor in G if and only if K_5 is a minor of G. To prove the correctness of the overall algorithm Find-K_5-minor, it remains to describe the algorithm Construct-K_5-minor.

Algorithm: Construct-K_5-model.

INPUT: A 3-connected graph G that is not isomorphic to W and contains a $K_{3,3}$ homeomorph S with branch vertices $\{a, b, c, x, y, z\}$, such that neither the red branch vertices (a, b, and c) nor the blue branch vertices (x, y, z) of S form a strong 3-cut separating the other color class into different components.

OUTPUT: A model of a K_5 minor in G.

1. Determine the components of $G - \{a, b, c\}$ and $G - \{x, y, z\}$, to establish which branch vertices of S are in the same induced components. Without loss of generality, suppose b and c are in the same component of $G - \{x, y, z\}$ and y and z are in the same component of $G - \{a, b, c\}$. Construct a bc-path P_{bc} in $G - \{x, y, z\}$, and construct a subpath P_1 from P_{bc}, starting at the vertex furthest from b along P_{bc} that intersects $F(b)$ and ending with the next vertex of P_{bc} in S, that is, in $F(c)$. Construct a yz-path P_{yz} in $G - \{a, b, c\}$. Construct path P_2 from P_{yz}, starting at the vertex furthest from y along P_{yz} that intersects $F(y)$ and ending at the next vertex of P_{yz} intersecting S, that is, in $F(z)$. Both P_1 and P_2 are outside paths of S, and neither path joins vertices on the same branch path of S.

2. Determine whether P_1 and P_2 have the same endpoints in S. If P_1 and P_2 do not share both endpoints, then proceed to the next step. Otherwise, test to determine whether P_1 and P_2 have any interior vertices. If either path has an interior vertex, proceed to step 4. If not, then P_1 and P_2 are both the same edge — proceed to step 5.

3. Determine whether at least one of P_1 and P_2 joins vertices in a parallel path. If so, constructively apply the proof of Lemma 3 to find and return the model of the K_5 minor. Otherwise, P_1 joins vertices in distinct branch paths of a red branch-fan, since it cannot be within a blue branch-fan; likewise, P_2 joins vertices in distinct branch paths of a blue branch-fan. Construct and return the model of the K_5 minor using the proof of Lemma 2.

4. In this case, paths P_1 and P_2 have the same endpoints, $\{u, v\}$, and at least one path, say P_1, has an interior vertex w. Construct a path, in $G - \{u, v\}$, from w to some vertex t in $S - \{u, v\}$. The paths from w to $\{t, u, v\}$ in S satisfy the conditions of Lemma 1. Apply the proof of that lemma to construct and return a model of the K_5 minor in G.

5. In this case, $P_1 = P_2$ is an edge between $u \in P_{by}$ and $v \in P_{cx}$. Let $T = S \cup P_1$; T is a homeomorph of W. Test to see if $G[T]$ is isomorphic to W. If so, then proceed to step 7. Otherwise, proceed to the next step.

6. In this case, some branch path of T is subdivided, or $G[T]$ is isomorphic to W plus an edge. In the latter case, it is easy to find and return the model of the K_5 minor. In the former case, suppose t is an interior vertex of a branch path, say P_{ax}. Form a path O in $G - \{a, x\}$, from t to any vertex in $T - P_{ax}$. Construct a subpath Q from O, starting from the vertex furthest from t along O in P_{ax}, and ending at the next vertex of O in T. Construct the model of the K_5 minor from T and path Q, using the methods of Lemma 5.

7. In this case, $G[T]$ is isomorphic to W but G is not, so there must be a vertex w in $G - T$. Construct paths (they need not be disjoint) from w to 3 vertices in T. Construct a path P outside T, joining non-adjacent vertices in T, as done in the proof of Theorem 5. Using the path P and the methods of Lemma 5, construct and return the model of the K_5 minor.

Construct-K_5-model rests on constructively applying the proofs of the lemmas and theorems of the previous section. However, each step of the constructive implementations of these proofs is either a search for a particular type of vertex, or the construction of a particular path. All paths are constructed using simple depth-first search methods, and the application of the proofs can all be implemented in linear-time. Thus, the algorithm Construct-K_5-model requires $O(n)$ time.

4.1 Complexity analysis of Find-K_5-minor

The algorithm Find-K_5-minor proceeds by a divide and conquer method. The major problem with the 'divide and conquer' is that the algorithm divides the problem into subproblems of undetermined size. For example, a strong 3-cut may divide the graph into one augmented component of order $n-2$ and two augmented components of order four.

Step one of the algorithm first tests to determine whether $m \geq 3n-5$; this takes linear time. If $m \geq 3n-5$, then the algorithm described after the proof of Theorem 1 constructs the model of the K_5 minor in $O(n^2)$ time. We may obtain all 1-cuts in linear time [Tar72], and the algorithms of Hopcroft and Tarjan [HT73] or Miller and Ramachandran [MR88] can be used to obtain 2-cuts in linear time. The application of Williamson's algorithm requires $O(n)$ time. If Williamson's algorithm returns a K_5 homeomorph or reports that the graph is planar, we are done. Otherwise, we have a $K_{3,3}$ homeomorph, and we test the condition of Theorem 5. Testing if G is isomorphic to W is a constant-time operation. In step 6, we determine whether x, y, and z (a, b, and c) are pair-wise in distinct components of $G-\{a,b,c\}$ ($G-\{x,y,z\}$), by applying depth-first search on the appropriate subgraphs. This requires $O(n)$ time. In step 7, we implement construct-K_5-model, in $O(n)$ time.

If the algorithm does not make any recursive calls in processing a graph, it requires $O(n)$ time. If it makes a recursive call in the second or third step, then the recursion for a graph of order n is:

$$T(n) = T(n_1) + T(n - n_1 + 1) + cn$$

The value cn indicates the linear time required to count edges and find cuts. the previous steps of the algorithm prior to the recursive call. For the recursion of step 6, we may restrict our

consideration in this case to a recursion on exactly three subproblems, since additional augmented components induced by the cut could be combined with the third component in a recursive call.

$$T(n) = T(n_1) + T(n_2) + T(n - n_1 - n_2 + 6) + c'n$$

The value $c'n$ indicates the linear time required to count edges, find cuts, test planarity, and test for a strong 3-cut in the $K_{3,3}$ homeomorph. Variables n_1, n_2 and $n - n_1 - n_2 + 6$ indicate the size of each augmented component, that is, the size of the subproblems upon which the recursive algorithm is applied. The values are bounded by

$$4 \leq n_1, n_2, n - n_1 - n_2 + 6 \leq n - 2$$

The recurrences imply a worst-case complexity of n^2.

Theorem 6 *The algorithm Find-K_5-minor has complexity $O(n^2)$.*

To see how the recurrence might require kn^2 steps, consider the case where both n_1 and n_2 are minimum, i.e., size 4. Then the recurrence becomes $T(n) = 2T(4) + T(n-2) + cn$, and the best that $T(n)$ can do in this case is $T(n) = O(n^2)$. One might hope that, in general, the case of obtaining the K_5 minor from the $K_{3,3}$ would not require $O(n)$ iterations of finding a $K_{3,3}$ homeomorph, that perhaps one bad iteration might balance with a good one, or that graphs couldn't have the form implied by the worst-case recurrence. However, consider the graph in Figure 9. Two vertices are attached to a set of three vertices in the grid. Those three vertices are a strong 3-cut. The Williamson algorithm might return a $K_{3,3}$ homeomorph that has the three vertices of the attachment as the branch vertices. Then the recursion would be on a subgraph with only two vertices deleted and two subgraphs of order four; this is the recursion stated earlier in the paragraph.

There are further reasons to suspect that the $O(n^2)$ complexity for finding a K_5 minor would be difficult to improve. While the $K_{3,3}$ homeomorph algorithms ([Asa85], [FP]) have linear-time implementations, they depend on a characterization of a graph into triconnected components. Here, the characterization is a decomposition into components induced by strong 3-cuts. But the general problem of finding all separating 3-cuts in a graph currently requires $O(n^2)$ [KR87] (while

triconnectivity algorithms can be accomplished in linear time). The best way to improve this algorithm would be to improve methods for finding all strong 3-cuts in a graph.

5 Conclusion

The corresponding problem of finding a K_5 homeomorph in a graph seems far less tractable. Indeed, since these sorts of algorithms hinge on characterizations of graphs that do not contain the given substructure, it is pertinent to ask whether any such characterization has been obtained for graphs that do not contain a K_5 homeomorph. The situation in that area is bleak, although work by the authors [KM] is a small advance in this area.

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References

[Asa85] T. Asano. An approach to the subgraph homeomorphism problem. *Theoretical Computer Science*, 38:249–267, 1985.

[Dir60] G. A. Dirac. In abstrakten Graphen vorhandene vollständige 4-Graphen und ihre Untereilungen. *Math. Nachr.*, 22:61–85, 1960.

[FP] M. R. Fellows and P. Kaschube. Searching for $K_{3,3}$ in linear time. *Journal of Linear and Multilinear Algebra*, to appear.

[Gy82] E. Györi. On the edge number of graphs with Hadwiger number 4 and 5. *Periodica Mathematica Hungarica*, 13:21–27, 1982.

[Hal67] R. Halin. Zur Klassification der endlichen Graphen nach H. Hadwiger und K. Wagner. *Math. Annalen*, 172:46–78, 1967.

[HT73] J. E. Hopcroft and R. E. Tarjan. Dividing a graph into triconnected components. *SIAM J. Computing*, 4:507–518, 1973.

[HT74] J. E. Hopcroft and R. E. Tarjan. Efficient planarity testing. *J. ACM*, 21:549–568, 1974.

[Kas84] P. A. Kaschube. Finding a subgraph homeomorphic to $K_{3,3}$ in polynomial time. Graduation Thesis, Dept. of Mathematics, University of California, San Diego, 1984.

[KM] A. Kezdy and P. McGuinness. Do $3n - 5$ edges force a subdivision of K_5? *J. Graph Theory*, to appear.

[KMV89] S. Khuller, S. G. Mitchell, and V. V. Vazirani. Processor efficient parallel algorithms for the Two Disjoint Paths problem, and for finding a Kuratowski homeomorph. *Proceedings of FOCS Conference*, pages 300–305, 1989.

[KR87] A. Kanevsky and V. Ramachandran. Improved algorithms for graph four-connectivity. *Proc. 28th IEEE Symp. on Foundation of Computer Science*, pages 252–259, 1987.

[LG80] P. C. Lui and R. C. Geldmacher. An $O(\max(m, n))$ algorithm for finding a subgraph homeomorphic to K_4. *Congress Numeratium*, 29:597–609, 1980.

[MR88] G. L. Miller and V. Ramachandran. A new graph triconnectivity algorithm and its parallelization. *Combinatorica*, 7:105–114, 1988.

[RS85] N. Robertson and P. D. Seymour. Disjoint Paths — a survey. *SIAM J. Algebraic and Discrete Methods*, 6:300–305, 1985.

[RS86] N. Robertson and P. D. Seymour. Graph minors XIII. The Disjoint Paths problem. Manuscript, 1986.

[Sey81] P. D. Seymour. On Tutte's extension of the Four-Colour problem. *Journal of Combinatorial Theory, Series B*, 31:82–94, 1981.

[Tar72] R. E. Tarjan. Depth-first search and linear graph algorithms. *SIAM Journal of Computing*, 1:146–160, 1972.

[Wag37a] K. Wagner. Über eine Erweiterung eines Satzes von Kuratowski *Deut. Math*, 2:280–285, 1937.

[Wag37b] K. Wagner. Über eine Eigenshaft der ebenen Komplexe. *Math. Annalen*, 114:570–590, 1937.

[Wag60] K. Wagner. Bemerkungen zu Hadwigers Vermutung. *Math. Annalen*, 141:433–451, 1960.

[Wag64] K. Wagner. Beweis einer Abschwächung der Hadwiger-Vermutung. *Math. Annalen*, 153:139–141, 1964.

[Wil84] S. G. Williamson. Depth-first search and Kuratowski subgraphs. *J. ACM*, 31:681–693, 1984.

[You71] H. P. Young. A quick proof of Wagner's equivalence theorem. *J. London Math. Soc. (2)*, 3:661–664, 1971.

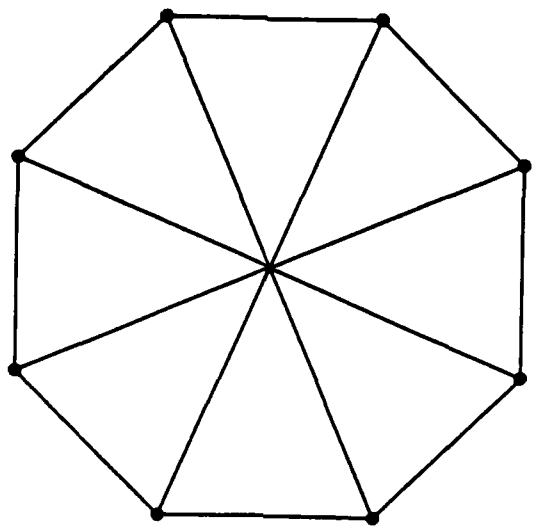

Figure 1: The graph W.

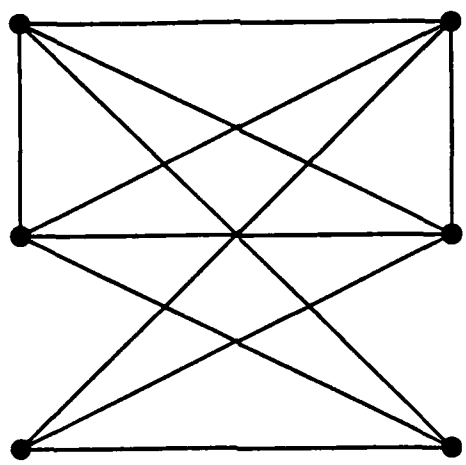

Figure 2: The graph L.

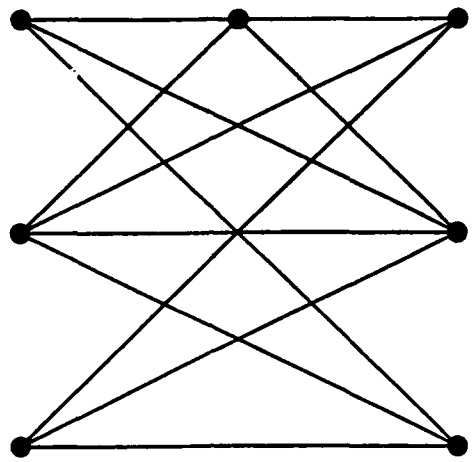

Figure 3: The graph M.

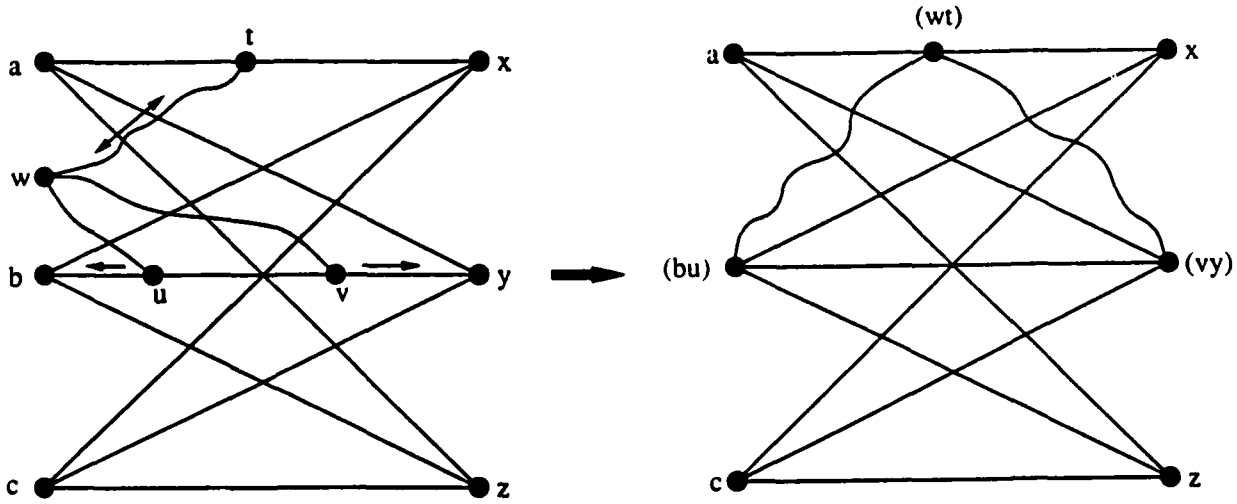

Figure 4: Lemma 1 case 1, with vertices u and v in P_{by}.

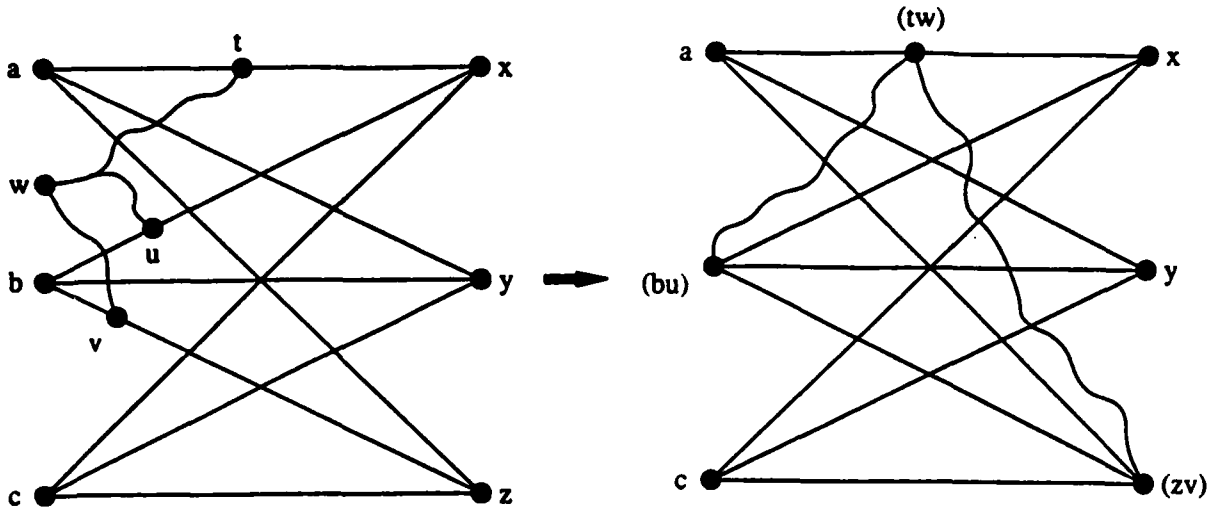

Figure 5: Lemma 1 case 3, with vertex u in P_{bx} and v in P_{bz}.

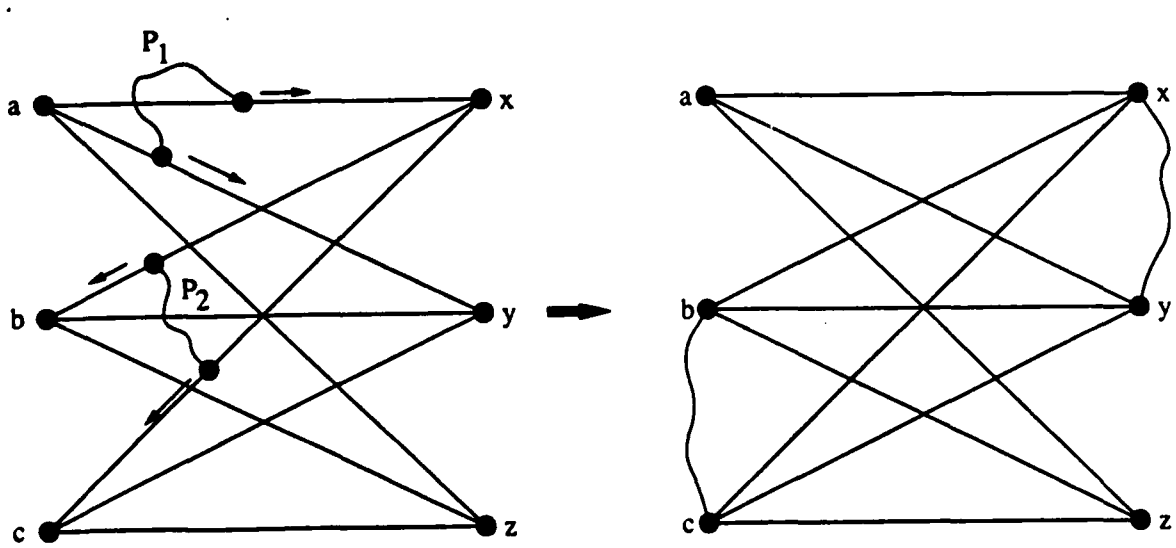

Figure 6: Lemma 2, case 1. Paths P_1 and P_2 contract to branch ends without conflict.

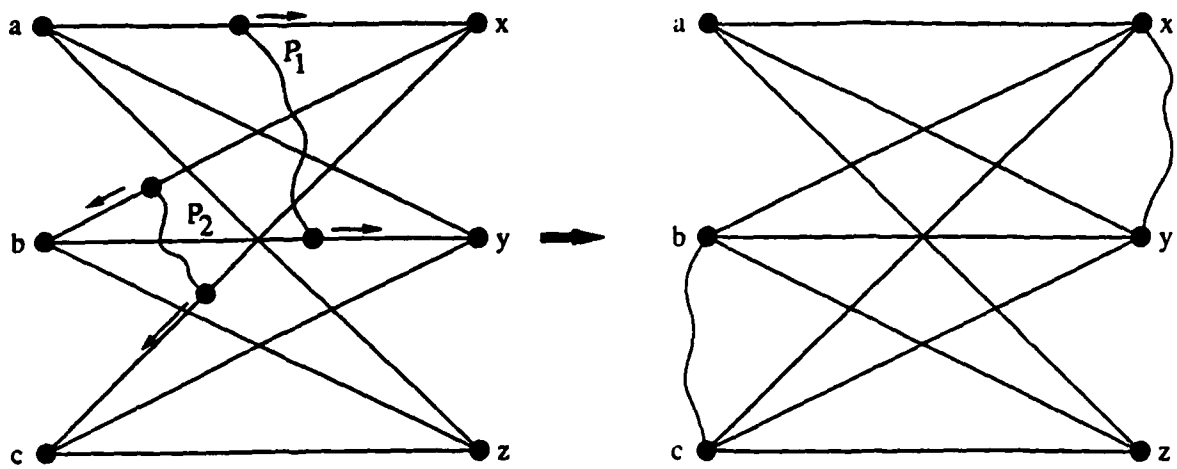

Figure 7: Lemma 3, case 1. Paths P_1 and P_2 have endpoints in distinct branch paths.

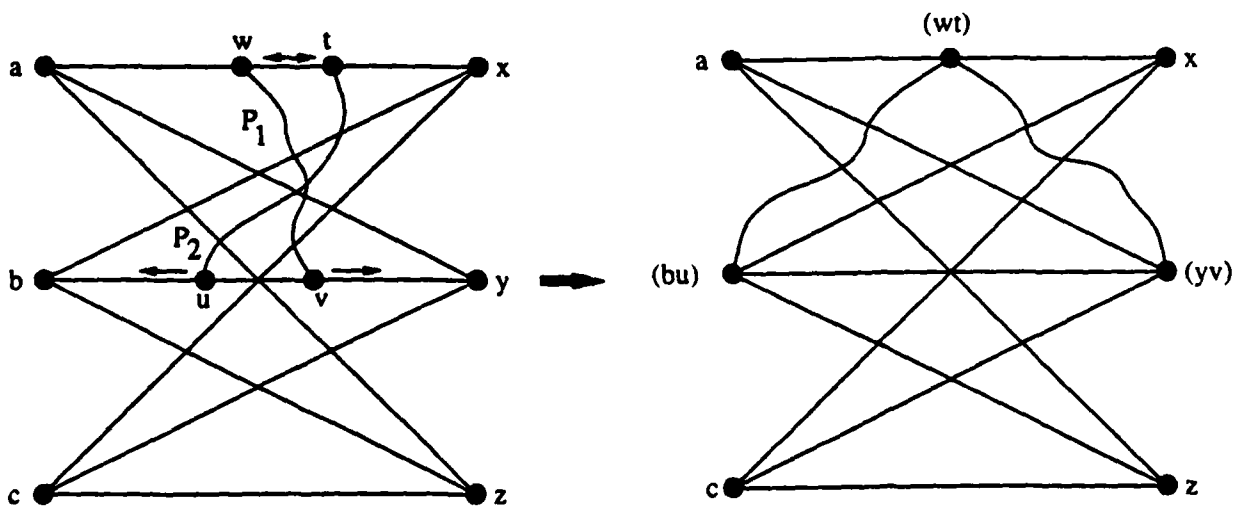

Figure 8: Lemma 3 case 2: Contract to form a homeomorph of M.

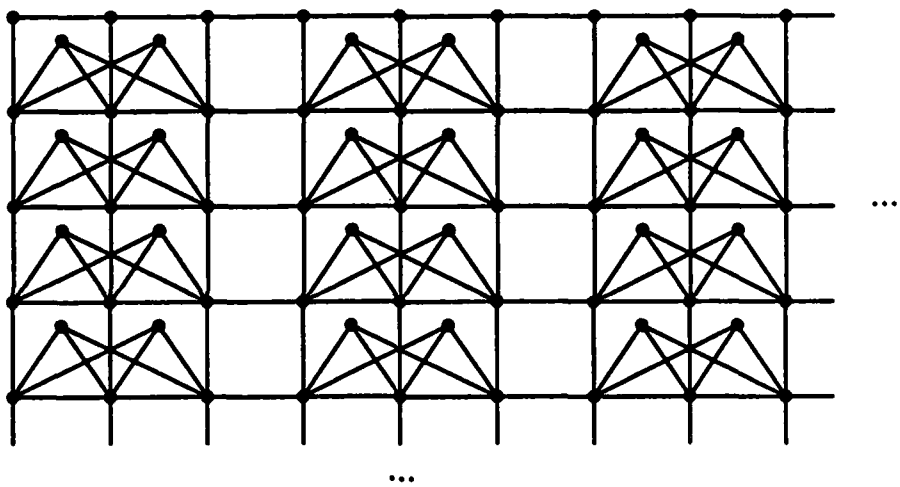

Figure 9: A graph that may require $O(n)$ decomposition iterations.