15

AD-A232 245

MARYLAND

COLLEGE PARK CAMPUS

ON LOCKING AND ROBUSTNESS IN THE FINITE ELEMENT METHOD

Ivo Babuška
Institute for Physical Science and Technology
University of Maryland
College Park, Maryland 20742

and

Manil Suri
Department of Mathematics & Statistics
University of Maryland Baltimore County
Baltimore, Maryland 21228

DTIC
ELECTE
MAR 0 6 1991
S B D

BN-1112

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

May 1990

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
Technical Note BN-1112

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
On Locking and Robustness in the Finite Element Method

5. TYPE OF REPORT & PERIOD COVERED
Final life of contract

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Ivo Babuska[1] and Manil Suri[2]

8. CONTRACT OR GRANT NUMBER(s)
[1]ONR/N00014-89-J-1030
[2]AFOSR 89-0252

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Department of the Navy
Office of Naval Research
Arlington, VA 22217

12. REPORT DATE
May 1990

13. NUMBER OF PAGES
58

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)

15. SECURITY CLASS. (of this report)

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS

20. ABSTRACT

Abstract

A numerical scheme for the approximation of a parameter-dependent problem is said to exhibit *locking* if the accuracy of the approximations deteriorates as the parameter tends to a limiting value. A *robust* numerical scheme for the problem is one that is essentially uniformly convergent for all values of the parameter. We develop precise mathematical definitions for these terms, give their quantitative characterization and prove some general theorems involving locking and robustness. A model problem involving heat transfer is analyzed in detail using this mathematical framework and various related computational results are described. Applications of our theory to some different problems involving locking are presented.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ON LOCKING AND ROBUSTNESS IN THE FINITE ELEMENT METHOD

by

I. Babuška[1]
Institute of Physical Sciences and Technology
University of Maryland
College Park, Maryland 20742

and

Manil Suri[2]
Department of Mathematics & Statistics
University of Maryland Baltimore County
Baltimore, Maryland 21228

MD-90-14-IB

TR90-14

May 1990

[1]Research partially supported by the U.S. Office of Naval Research under Grant N00014-89-J-1030.

[2]Research partially supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant #AFOSR 89-0252.

Abstract

A numerical scheme for the approximation of a parameter-dependent problem is said to exhibit *locking* if the accuracy of the approximations deteriorates as the parameter tends to a limiting value. A *robust* numerical scheme for the problem is one that is essentially uniformly convergent for all values of the parameter. We develop precise mathematical definitions for these terms, give their quantitative characterization and prove some general theorems involving locking and robustness. A model problem involving heat transfer is analyzed in detail using this mathematical framework and various related computational results are described. Applications of our theory to some different problems involving locking are presented.

Accession For
NTIS GRA&I ☑
DTIC TAB ☐
Unannounced ☐
Justification
By
Distribution/
Availability Codes
Dist | Avail and/or Special
A-1

QUALITY INSPECTED 3

1. Introduction.

The mathematical formulation of various problems involves dependency on a crucial parameter t arising out of physical considerations. For example, plate and shell models involve the thickness d of the plate or shell, the analysis of elastic materials in general involves the Poisson ratio ν, heat transfer involves the ratio of conductivities μ in different directions, etc.

The numerical approximation of such parameter-dependent problems may suffer when t lies close to a limiting value t_0 (thickness $d \to 0$, Poisson ratio $\nu \to 0.5$, ratio μ of conductivities is very small $(\mu \to 0)$, etc.). Most a priori error estimates yield optimal asymptotic convergence rates for $t > t_0$ fixed. Since these estimates are not uniform in t, a degeneration often occurs for values of t close to t_0. This is manifested in actual computations, where the error may not decrease at the predicted rate for t close to t_0 for most practical choices of the discretization parameter. We refer to this phenomenon as *locking*. Various examples of locking have been reported in the engineering and mathematical literature, see for example [1], [6] - [10], [14] and others.

Locking involves the "shifting" of the asymptotic range of the calculations and will eventually disappear when the level of discretization is increased enough, depending upon the strength of the degeneracy. Unfortunately, using brute force may lead to an infeasible level of discretization required before convergence can be observed. Moreover, the prediction of the discretization level for a required error tolerance is now quite complicated.

The most ideal remedy to locking is to employ a method that is <u>robust</u>, i.e., one which is more or less uniformly convergent for all t. Various robust methods (particularly mixed methods) have been proposed and analyzed in

the context of locking, see for example [7] for the plate problem, [1], [9] for a beam problem, [6], [14] for Poisson ratio locking, among others.

Generally, in the above mentioned papers, locking was addressed in an ad hoc manner. In this paper our primary goal is to develop a systematic mathematical approach which will allow precise characterization of locking and robustness of a method as well as a quantitative measure of the locking and robustness strength. In the next section we present the definitions of these concepts.

Our definitions are quite general in nature, so that one can treat various different types of locking phenomena from the same point of view. The examples analyzed in Sections 2 - 5 (both theoretical and computational) illustrate the need for this flexibility.

In the next section, we also derive various general theorems for a special class of locking problems which are important in the context of applications and have been studied, for example, in [6]. These problems have a well-defined associated limit problem for $t \to t_0$, satisfying certain properties. An example is the case of elasticity, where we get the Stokes' problem in the limit as the Poisson ratio $\nu \to 0.5$. In [6], it was shown that a necessary condition for the absence of locking is the satisfaction of a suitable approximability condition for this limit problem. We show in Theorems 2.2, 2.3 that this condition is sufficient as well, under certain conditions.

Section 3 contains detailed results on locking and robustness for the model problem introduced in Section 2. Section 4 shows how the non-quasi-uniformity of meshes used can be interpreted in terms of locking. In Section 5, we present a simplified analysis for the beam problem which was analyzed in [1], [9]. In Section 6, we discuss a similar simplified analysis that is possible for the Stokes' problem, detailed results for which (using the theorems

from Section 2) are presented in [5].

2. A model problem and the definitions.

To show the main features of locking, we first discuss an illustrative model elliptic problem which arises in the study of (for example) heat transfer through highly orthotropic materials with the conductivity being very different in two perpendicular directions. Let the conductivities in the x_1 and x_2 directions be $k_1 = 1$ and $k_2 = 1/t$, respectively. We will be particularly interested in the case that t is close to 0.

We assume homogeneous Dirichlet boundary data on Γ_D and Neumann data on Γ_N (with $\Gamma = \Gamma_D \cup \Gamma_N$). The corresponding partial differential equation we study is

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{1}{t}\frac{\partial^2 u}{\partial x_2^2} = f(x,y) \quad \text{in} \quad \Omega \tag{2.1}$$

$$u = 0 \quad \text{on} \quad \Gamma_D \tag{2.2a}$$

$$\frac{\partial u}{\partial n_c} = g \quad \text{on} \quad \Gamma_N, \tag{2.2b}$$

where n_c is the usual conormal.

Let $H_D^1(\Omega) = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma_D\}$. Then this problem may be put in the variational form: Find $u \in H_D^1(\Omega)$ such that for all $v \in H_D^1(\Omega)$,

$$B_t^1(u,v) = F(v), \tag{2.3}$$

where

$$B_t^1(u,v) = \int_\Omega \left[\left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial v}{\partial x_1}\right) + \frac{1}{t}\left(\frac{\partial u}{\partial x_2}\right)\left(\frac{\partial v}{\partial x_2}\right)\right] dx_1 dx_2 \tag{2.4}$$

and

$$F(v) = \int_{\Omega} fv dx_1 dx_2 + \int_{\Gamma_N} gv ds.$$

Let $\Omega = (-1,1)^2$ for simplicity. We will be particularly interested in the case that $\Gamma_D = \emptyset$, $f = 0$, and g is chosen so that the true solution (unique up to a constant) is given by

$$u_t^1 = \sin x_1 e^{-\sqrt{t}x_2}. \tag{2.5}$$

As $t \to 0$, it is seen that u_t^1 has a well defined limit $\sin x_1$ which is constant in the x_2 direction.

We also consider (2.3) with B_t^1 replaced by its rotated form

$$B_t^2(u,v) = \int_{\Omega} \left[\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}\right)\left(\frac{\partial v}{\partial x_1} + \frac{\partial v}{\partial x_2}\right) + \frac{1}{t}\left(\frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_1}\right)\left(\frac{\partial v}{\partial x_2} - \frac{\partial v}{\partial x_1}\right)\right] dx_1 dx_2, \tag{2.6}$$

which corresponds to the case where the directions of the orthotropy do not coincide with the x_1, x_2 axes. By choosing g appropriately, we obtain as the solution of our new variational problem the rotated version of (2.5)

$$u_t^2 = \sin(x_1+x_2)e^{-\sqrt{t}(x_2-x_1)}, \tag{2.7}$$

which is once again unique up to a constant. As $t \to 0$, u_t^2 becomes constant in the $\eta = x_2 - x_1$ direction.

We now approximate the above problems by the finite element method. First we consider the h-version on a uniform square mesh on Ω with 1, 4 and 16 square elements. In Figures 2.1 - 2.3 we have plotted the percentage relative error versus number of degrees of freedom in a log-log scale, for $p = 1$, 2 and 4. In each case, results are shown for both u_t^1 and u_t^2, for $t = 10^{-1}$ and 10^{-6}. The dotted lines show the error in the H^1 norm, while the solid lines represent the error in the energy norm – their behavior is

very similar (see Theorem 2.4).

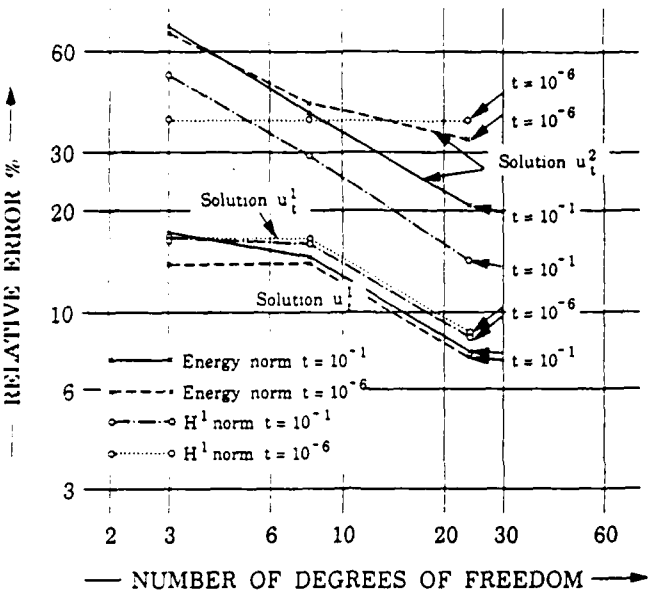

Figure 2.1. Relative error in u_t^1 and u_t^2 for the h-version, $p = 1$.

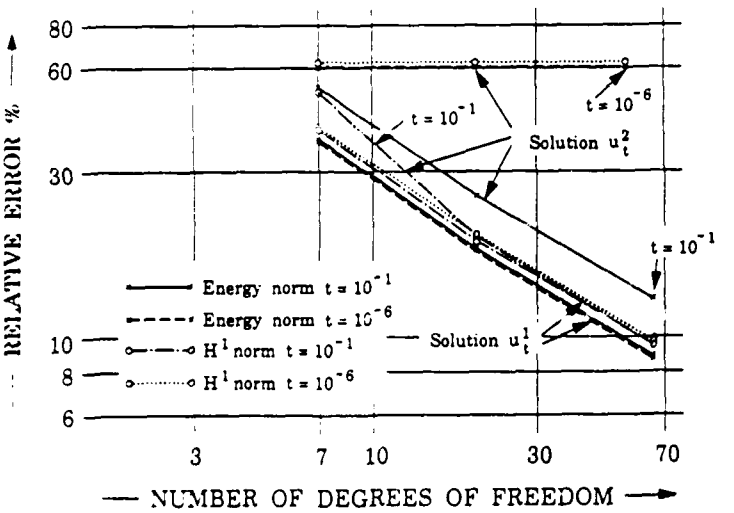

Figure 2.2. Relative error in u_t^1 and u_t^2 for the h-version, $p = 2$.

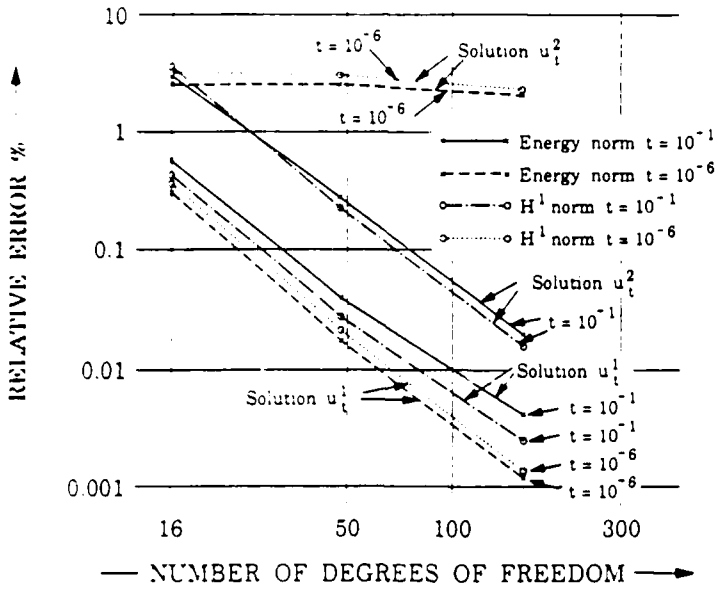

Figure 2.3. Relative error in u_t^1 and u_t^2 for the h-version, $p = 4$.

It is observed that for $t = 10^{-1}$, both cases lead to the error decreasing at the predicted asymptotic optimal rate. However, when $t = 10^{-6}$, only the unrotated case shows the same behavior. The error for the rotated case hardly decreases – this is described as a *locking effect*.

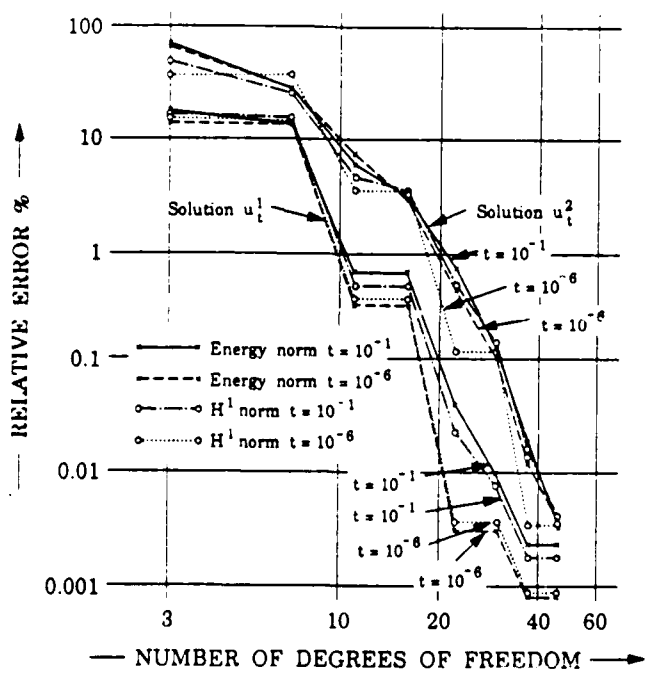

Figure 2.4.

Relative error in u_t^1 and u_t^2 for the p-version using one square element

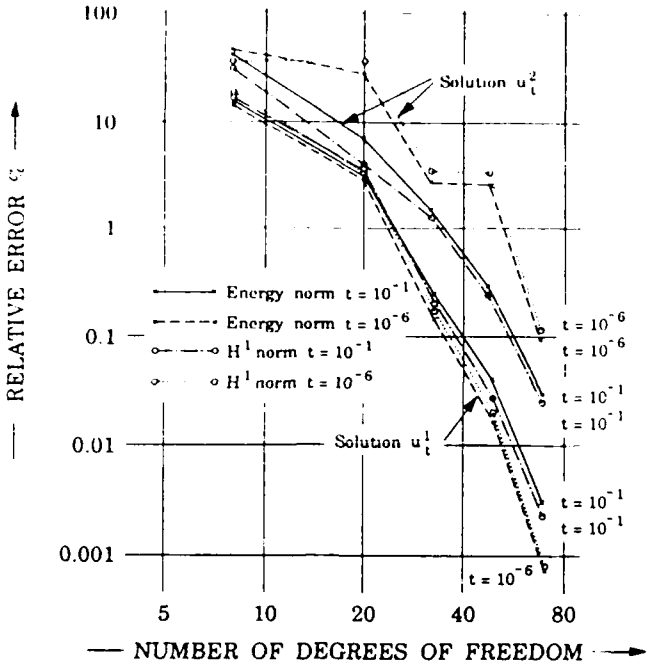

Figure 2.5.

Relative error in u_t^1 and u_t^2 for the p-version using four elements.

In Figures 2.4 - 2.5 the p-version has been used, with $h = 2,1$ (i.e., 1 and 4 square elements). Figure 2.4 shows that the p-version with one square is free of locking for both u_t^1 and u_t^2 since the value of t has no appreciable effect on the observed rate of convergence. In Figure 2.5, it is seen that the error curve for u_t^2 with $t = 10^{-6}$ is shifted upwards compared to the other curves, although the slope of this curve remains the same. This suggests that the *order* of convergence remains the same, but the constant in the error estimate is larger for t small. (This "shift" will, of course, occur for the h-version as well, but will only be visible when h is of the same order as t. Hence, in practice it is not observed.)

Figure 2.6 compares the h-version $(p = 1,2)$ and p-version (one square) for u_t^2 in terms of the $H^1(\Omega)$ norm error. This clearly illustrates the difference in terms of locking. It shows that the p-version is more *robust*.

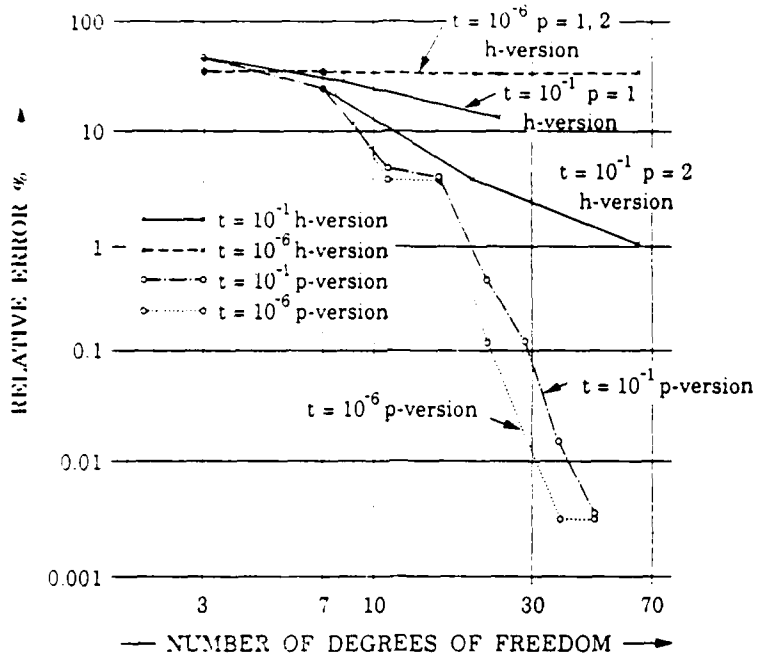

Figure 2.6. Comparison of the h- and p-version accuracies for u_t^2.

In Section 3, we shall prove various results to explain these phenomena. It should be noted that the above numerical results are only valid for the practical range of h and p we have worked with. If refinement is continued, the asymptotic convergence rate would eventually be observed for any fixed t.

We now develop mathematically precise definitions of locking and robustness.

Assume first that the parameter t is an element of a set S which is taken to be a semi-open interval of the form $(a,b]$ (with the limiting value of t being a). For each t, let $B_t(\cdot,\cdot)$ be a bounded symmetric bilinear form defined over a Hilbert space V such that the energy norm

$$\|u\|_{E,t} = (B_t(u,u))^{1/2} \tag{2.8}$$

satisfies

$$C_1(t)\|u\|_V \le \|u\|_{E,t} \le C_2(t)\|u\|_V \tag{2.9}$$

where $0 < C_1, C_2 < \infty$, though $C_1(t)/C_2(t)$ could $\to\infty$ or 0 as $t\to a$.

(Here, $\|\cdot\|_V$ represents the usual norm of the quotient space if, for example, we have a pure Neumann problem.)

Let $\mathcal{H} = \{H_t\}$ be a family of linear spaces, $H_t \subset H \subset V$, where H is a compactly imbedded subspace of V. For each H_t, let $\|\cdot\|_t$ denote an associated norm and let $H_t^B = \{u \in H_t, \|u\|_t \le 1\}$, where $\|u\|_H \le C\|u\|_t$. In our examples, H_t may often be identical to H.

We are interested in the approximation of "exact solutions" that lie in H_t. To this end, for each t, let $\mathcal{F}_t = \{V_t^N\}$ be a family of finite-dimensional subspaces of V with the dimension N being independent of t. (We assume that a V_t^N is given for each $N \in \mathcal{N}$, with $\mathcal{N}$ being unbounded.) For $u_t \in H_t$, we consider the sequence of approximations $u_t^N \in V_t^N$ given by

$$B_t(u_t^N, v) = B_t(u_t, v) \quad \forall\, v \in V_t^N, \tag{2.10}$$

i.e., a sequence of approximations to the *problem* $\mathcal{P}_t$ associated with bilinear form B_t and solution set H_t. The family $\{\mathcal{F}_t\}$ can be identified with an *extension procedure* $\mathcal{F}$, i.e., a rule which (for each t) gives a method regarding how the dimension N of the approximate subspace is increased. It is seen that (2.10) defines a projection of the space H_t into V_t^N. However, the exact form of (2.10) may not allow us to consider certain problems, for example, those involving inhomogeneous Dirichlet boundary conditions. A more general form of (2.10) is therefore

$$u_t^N = B_t^N u_t, \tag{2.11}$$

where $B_t^N : H_t \to V_t^N$ is a projection operator. The definitions that follow are equally valid for (2.11) as well.

In case we are using (2.11), we will assume the existence of an energy norm corresponding to (2.8) such that

$$\|u_t - u_t^N\|_{E,t} \le C \inf_{v \in V_t^N} \|u_t - v\|_{E,t} \tag{2.12}$$

for all N, t and C independent of t. Note that otherwise (2.12) follows immediately from (2.10).

Remark. We introduced the set H_t^B by help of the norm $\|\cdot\|_t$. This characterization is a very special one. We can, instead, define H_t^B to be any *reasonable* set. We will do this in Theorem 3.5.

We denote $R_0 = S \times \mathcal{N}$ and let $R \subset R_0$ be a subset of the pairs (t,N) under consideration.

Our final component is a family $\mathcal{E} = \{E_t\}$ of measures for the error $E_t : V \longrightarrow \mathbb{R}$. We are interested in the errors $E_t(u_t - u_t^N)$ for problems $\mathcal{P}_t$. Let us mention that some of B_t, H_t, $\mathcal{F}_t$, E_t may be independent of t (and usually are).

For any R, we now define the *locking ratio* $L(t,N) = L(t,N,R,\mathcal{H},\mathcal{F},\mathcal{E},\eta)$ $(0 < \eta \le b-a)$ for the problem $\{\mathcal{P}_t\}$, $a < t \le b$, by

$$L(t,N) = \frac{\sup_{u_t \in H_t^B} E_t(u_t - u_t^N)}{\inf_{\substack{t \\ (t,N) \in R_\eta}} \sup_{u_t \in H_t^B} E_t(u_t - u_t^N)}, \tag{2.13}$$

where $R_\eta = \{(t,N),\ t \ge a + \eta\} \cap R$.

Remark. The locking ratio compares the performance of the method at parameter value t to the best possible performance for *reasonable* values of t, which are characterized by $t \ge a + \eta$. In most applications, the infimum in the denominator over R_η is the same as that over R, so that η can be taken arbitrarily close to zero (and appears in the definition essentially as a technicality). This is because typically, the error in u_t^N is smallest when the problem is least singular.

Let us illustrate this ratio by considering the bilinear forms (2.4), (2.6) with $R = \{10^{-1}, 10^{-2}, 10^{-4}, 10^{-6}\} \times \mathcal{N}$. Let H_t^i $(i = 1,2)$ be spanned by the single element u_t^i, $i = 1,2$ (see (2.5) and (2.7)), and let $E_t(u) = \|u\|_{E,t}$.

We take $\mathcal{F}$ to be the h extension procedure (and let $\eta = 10^{-6}$). Then $L^1(t,N)$ and $L^2(t,N)$ are shown for various p in Tables 2.1, 2.2 using 1, 4, 9 and 16 square elements.

It is observed that for $p = 1$, $L^1(t,N)$ remains close to unity. Similar results are observed for higher p for L^1 (not reproduced here). On the other hand, it is seen that for all p, $L^2(t,N)$ seems to be unbounded as $t \longrightarrow 0$. (For t fixed, the locking ratio is of course bounded, but can be very large.)

Table 2.1. Locking ratio $L^1(t,N)$ and $L^2(t,N)$ for the h-version, $p = 1$ $(\eta = 10^{-6})$ and the energy norm

	t\N	3	8	15	24
$L^1(t,N)$	10^{-1}	1.21	1.06	1.05	1.05
	10^{-2}	1.02	1.01	1.01	1.01
	10^{-4}	1.00	1.00	1.00	1.00
	10^{-6}	1.00	1.00	1.00	1.00
$L^2(t,N)$	10^{-1}	1.02	1.00	1.00	1.00
	10^{-2}	1.00	1.07	1.23	1.41
	10^{-4}	1.00	1.08	1.29	1.53
	10^{-6}	1.00	1.08	1.29	1.53

Table 2.2. Locking ratio $L^2(t,N)$ for the h-version, $p = 2,3,4$ $(\eta = 10^{-6})$ and the energy norm

	t\N	7	20	39	64
p = 2	10^{-1}	1.00	1.00	1.00	1.00
	10^{-2}	1.64	2.67	2.92	2.97
	10^{-4}	1.77	4.81	10.06	15.90
	10^{-6}	1.77	4.87	10.67	18.76
p = 3	10^{-1}	1.00	1.00	1.00	1.00
	10^{-2}	1.04	1.46	2.17	2.40
	10^{-4}	1.05	1.79	4.98	10.43
	10^{-6}	1.05	1.79	5.11	11.74
p = 4	10^{-1}	1.15	1.00	1.00	1.00
	10^{-2}	1.01	1.73	1.83	1.50
	10^{-4}	1.00	7.83	15.17	15.50
	10^{-6}	1.00	9.01	43.33	101.56

In Table 2.3 we show $L^2(t,N)$ for the p-version for one and 16 square elements. We see that for one element, L^2 stays bounded while for 16 elements it becomes unbounded.

Table 2.3. Locking ratio $L^2(t,N)$ for the p-version ($\eta = 10^{-6}$) and the energy norm

# of elem.	t\N	3	7	11	16	22
1	10^{-1}	1.02	1.00	1.00	1.15	1.35
	10^{-2}	1.00	1.64	1.04	1.01	1.00
	10^{-4}	1.00	1.77	1.05	1.00	1.00
	10^{-6}	1.00	1.77	1.05	1.00	1.00

# of elem.	t\N	24	64	104	160
16	10^{-1}	1.00	1.00	1.00	1.00
	10^{-2}	1.41	2.97	2.39	1.52
	10^{-4}	1.53	15.91	10.43	15.56
	10^{-6}	1.53	18.76	11.74	101.53

Let us now present the definition of locking.

<u>Definition 2.1</u>. The extension procedure $\mathcal{F}$ is *free from locking* over the region R for the family of problems $\{\mathcal{P}_t,\ (t,N) \in R\}$ with respect to solution sets $\mathcal{H}$ and measures $\mathcal{E}$ iff there exists $0 < \eta \le b-a$ such that

$$\limsup_{N\to\infty}\left[\sup_{\substack{t\\(t,N)\in R}} L(t,N,R,\mathcal{H},\mathcal{F},\mathcal{E},\eta)\right] = M < \infty. \tag{2.14}$$

$\mathcal{F}$ shows *locking of order* $f(N)$ (where $\lim_{N\to\infty} f(N) = \infty$) iff for some η,

$$0 < \limsup_{N\to\infty}\left[\sup_{\substack{t\\(t,N)\in R}} L(t,N,R,\mathcal{H},\mathcal{F},\mathcal{E},\eta)\frac{1}{f(N)}\right] = C < \infty. \tag{2.15}$$

For the case that C is bounded (respectively, infinite), we say that the *order of locking* is *at most* (respectively, *at least*) $f(N)$. □

Related to the above definition, we may also define *robustness* as follows.

<u>Definition 2.2</u>. The extension procedure $\mathcal{F}$ is *robust* over the region $R \subset R_0$ for the family of problems $\{\mathcal{P}_t,\ (t,N) \in R\}$ with respect to solution sets $\mathcal{H}$ and measures $\mathcal{E}$ iff

$$\lim_{N\to\infty} \sup_{\substack{t \\ (t,N)\in R}} \sup_{u_t \in H_t^B} E_t(u_t - u_t^N) = 0.$$

It is *robust with uniform order* $g(N)$ iff

$$\lim_{N\to\infty} \sup \left(\left[\sup_{\substack{t \\ (t,N)\in R}} \sup_{u_t \in H_t^B} E_t(u_t - u_t^N) \right] \frac{1}{g(N)} \right) = C < \infty, \tag{2.16}$$

where $g(N) \longrightarrow 0$ as $N \longrightarrow \infty$. □

Definitions 2.1 and 2.2 allow great flexibility in the way various components may depend on t. This is because as illustrated by the examples in this and the succeeding sections, different formulations require different t dependencies.

The measure E_t in our definition can take various forms, for example, the maximum stresses, stress at a point, etc. The two main choices discussed here will be the V norm and the energy norm.

As $t \longrightarrow a$, the approximation may deteriorate due to reasons other than locking. For example, the exact solution of the Reissner-Mindlin plate model with uniform load has a boundary layer of increasing strength as $t \longrightarrow 0$. Hence the deterioration of the approximation is due to the loss in regularity of the solution as well as locking. The numerical resolution of these two effects is based on different strategies. Our definition permits us to effectively isolate the locking effects by sufficiently restricting the set $\mathcal{H}$ to exclude unsmooth solutions.

By locking, we typically understand the behavior of the extension procedure when t is much smaller than N^{-1} (for $a = 0$). Hence, instead of considering the region $R_0 = (0,b]\times\mathcal{N}$ (Figure 2.7a), we may be interested in a region of the type $R = \{(t,N),\ 0 < t \le b_N\}$ shown in Figure 2.7b. This is why we have R explicitly in our definition. For the examples discussed in this paper, however, we are able to prove all our results for the entire region $R = R_0$.

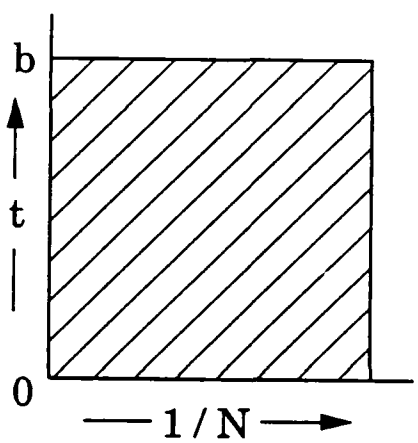

Figure 2.7a. The region R_0.

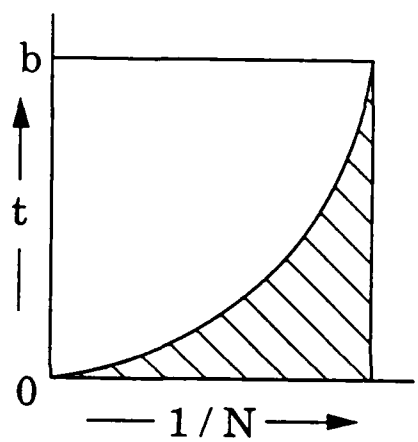

Figure 2.7b. The region R.

We will often need to bound below the denominator in (2.13). In this connection, the following definition of the n-width will be useful. (Here H^B is the unit ball in H.)

The n-width of the set H in V with respect to the measure E is defined as

$$d_N(V,H,E) = \inf_M \left\{ \sup_{u\in H^B} \inf_{v\in M} E(u-v) \mid M \subset V,\ \dim M = N \right\} \tag{2.17}$$

with M being a linear manifold. When $E(u)$ is given by $\|u\|_V$, we denote this by $d_N(V,H)$.

The choice of the measures $\|\cdot\|_t$ and E_t are not independent if we want to study problems which make sense mathematically. We will restrict our

attention to choices of $\mathcal{H}, \mathcal{E}, R$ such that for any $(t,N) \in R$,

$$d_N(V, H_t, E_t) \le CF_A(N), \tag{2.18}$$

where $F_A(N) \to 0$ as $N \to \infty$, F_A independent of t, C independent of N and t.

Condition (2.18) ensures that there are extension procedures for which it is at least *possible* that the robustness can be achieved. Related to (2.18) is the following condition, which ensures that our choice of extension procedure has a chance to avoid locking: For any $(t,N) \in R$,

$$C_1 F_B(N) \le \sup_{\substack{u_t \in H_t^B \\ (t,N) \in R}} \inf_{v \in V_t^N} E_t(u_t - v) \le C_2 F_B(N), \tag{2.7}$$

where $F_B(N) \to 0$ as $N \to \infty$, F_B independent of t, C independent of N and t. We say that $\{\mathcal{H}, \mathcal{F}, \mathcal{E}, R\}$ is F_B-admissible if (2.19) is satisfied. Note that (2.19) can be an unduly broad condition to assume – for example, if $E_t = \|\cdot\|_{E,t}$, then (2.19) already implies robustness with uniform order F_B, as well as absence of locking.

We now formulate

<u>Assumption A</u>. There exists $0 < \eta \le b - a$ such that for $(t,N) \in R_\eta$

$$C_1 F_0(N) \le \sup_{\substack{u_t \in H_t^B \\ (t,N) \in R_\eta}} E_t(u_t - u_t^N) \le C_2 F_0(N), \tag{2.20}$$

where $F_0(N) \to 0$ as $N \to \infty$, F_0 independent of t, $C_1, C_2 > 0$ independent of t, N, but in general depending on η (u_t^N is the finite element solution). □

Assumption A says that there exists an R_η on which the extension procedure is robust with uniform order $F_0(N)$. Under this assumption, the locking

ratio L takes a simpler form and Definition 2.1 is equivalent to saying that $\mathcal{F}$ is free from locking iff

$$(2.21) \qquad \limsup_{N\to\infty} \left[\sup_{\substack{t \\ (t,N)\in R}} \sup_{u_t \in H_t^B} E_t(u_t - u_t^N)(F_0(N))^{-1} \right] = M < \infty$$

(a similar reformulation holds for (2.15)).

The following theorem relates the concepts of locking and robustness and follows immediately from their definitions.

<u>Theorem 2.1</u>. Let S, $R_0 = S\times\mathcal{N}$, $R \subset R_0$, $\mathcal{H}, \mathcal{E}, \mathcal{F}$ be given and let Assumption A hold. Then the extension procedure $\mathcal{F}$ is free from locking over the region R iff it is robust over the region R with uniform order $F_0(N)$. Moreover, if $f(N)$ is such that

$$f(N)F_0(N) = g(N) \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty,$$

then $\mathcal{F}$ shows locking of order $f(N)$ iff it is robust with maximum uniform order $g(N)$. □

Note that $\mathcal{F}$ is non-robust iff it shows locking of order $(F_0(N))^{-1}$. In this case we say that $\mathcal{F}$ shows *complete locking*.

Let us now discuss a special case of the bilinear form $B_t(\cdot,\cdot)$ in terms of which many locking problems can be formulated. Let V be imbedded in the Hilbert space W, with norm $\|\cdot\|_W$ and inner product $(\cdot,\cdot)$. We assume, for $t \in S = (0,b]$

$$(2.22) \qquad B_t(u,v) = a(u,v) + \frac{1}{t}(Cu,Cv)$$

is a symmetric bilinear form defined on $V\times V$, where $a(\cdot,\cdot)$ is a symmetric bilinear form satisfying

$$(2.23) \qquad C_1\|u\|_V \le (a(u,u))^{1/2} \le C_2\|u\|_V$$

and $C : V \to W$ is a linear form satisfying

$$\|Cv\|_W \le C_3\|v\|_V. \tag{2.24}$$

Here C_1, C_2, C_3 are constants independent of t ($a(\cdot,\cdot)$ and C could be made t-dependent as long as (2.23) - (2.24) were satisfied). We have chosen $(a,b] = (0,b]$ for convenience.

(2.22), (2.23), and (2.24) give immediately that for any $t \in (0,b]$, the following version of (2.9) holds

$$C_1\|u\|_V \le \|u\|_{E,t} \le C_2 t^{-1/2}\|u\|_V \tag{2.25}$$

We shall give examples of problems that satisfy the form (2.22) in sections 3, 5, 6. For a number of such problems and the set $\mathcal{H} = \{H_t\}$, $H_t \subset H \subset V$, it is useful to define $\lim_{t\to 0} H_t^B = H_0^B \subset H \subset V$. We define it as follows: $u_0 \in H_0^B$ iff there exists a sequence $\{u_t\}$, $u_t \in H_t^B$, $t \in S$, and a constant C independent of t such that

$$\|u_t\|_{E,t} \le C \tag{2.26}$$

$$\|u_t - u_0\|_V \to 0 \quad \text{as} \quad t \to 0. \tag{2.27}$$

Assume now $u_0 \in H_0^B$, then by definition there are $u_t \in H_t^B$ such that (2.26) and (2.27) hold. Hence

$$\frac{1}{t}\|Cu_t\|_W^2 \le C$$

so that

$$\|Cu_t\|_W = O(t^{1/2}) \tag{2.28}$$

from which it follows that for any $u_0 \in H_0^B$ we have

$$Cu_0 = 0. \tag{2.29}$$

In many examples it turns out that $Cu_t \to 0$ at a rate $O(t)$ instead of

(2.28) (see the beam example in Section 5 and Poisson locking in Section 6), i.e.,

$$\|Cu_t\|_W = O(t) \quad (\text{or} \quad o(t)). \tag{2.30}$$

Obviously (2.30) is satisfied if Condition (α) below holds.

<u>Condition</u> (α). Let $\|\cdot\|_t$ be such that $\|u\|_{E,t} \le C\|u\|_t$. Then for any $u_t \in H_t^B$, there is a $u_0 \in H_0^B \cap H_t$ (u_0 depending on u_t) such that

$$\|u_t - u_0\|_t \le Ct^{1/2} \tag{2.31}$$

with C independent of t and u_t. □

Condition (α) is important in the context of various theorems and characterizes an aspect of the regularity of the problem.

We now prove some general theorems for the form (2.22) which hold when the spaces V_t^N are assumed independent of t.

<u>Theorem 2.2</u>. (A) Let $\mathcal{H} = \{H_t\}$, $H_t \subset H \subset V$, $\mathcal{F} = \{\mathcal{F}_t\}$, $\mathcal{F}_t = \{V_t^N\}$, $V_t^N = V^N$, $\mathcal{E} = \{E_t\}$, $E_t(u) = \|u\|_V$, $S = (0,b]$, $R = \{(t,N),\ 0 < t \le b_N\} \subset R_0$ and let $\lim_{t\to\infty} H_t^B = H_0^B$. Assume further that the quadruple $\{\mathcal{H},\mathcal{F},\mathcal{E},R\}$ is F_0-admissible. Then the extension process $\mathcal{F}$ is free from locking over R only if

$$g(N) = \sup_{u\in H_0^B} \inf_{\substack{w\in V^N \\ Cw=0}} \|u-w\|_V \le CF_0(N). \tag{2.32}$$

More generally, it shows locking of order $f(N)$ only if

$$g(N) \le CF_0(N)f(N). \tag{2.33}$$

Moreover, if

$$g(N) \ge CF_0(N)f(N) \tag{2.34}$$

then the extension process shows locking of at least order $f(N)$.

(B) Suppose, in addition, that Condition (α) holds. Then $\mathcal{F}$ is free from locking over R *iff* (2.32) holds. It shows locking of order $f(N)$ *iff*

$$C_1F_0(N)f(N) \le g(N) \le C_2F_0(N)f(N). \tag{2.35}$$

<u>Proof</u>. First, it is easy to see that Assumption A is satisfied with the same F_0 because for $t \ge \eta$ the energy norm $\|\cdot\|_{E,t}$ is equivalent to the norm $\|\cdot\|_V$. Suppose there is no locking (i.e., $\mathcal{F}$ is free of locking) over R. Let N be fixed. Then by (2.21), for any (t,N) we have for $u_t \in H_t^B$,

$$\|u_t - u_t^N\|_V \le CF_0(N).$$

Let now $u_0 \in H_0^B$ and $u_t \in H_t^B$ be such that (2.26) and (2.27) hold. Because V^N is finite-dimensional, there exists a subsequence of $\{u_t^N\}$ such that $u_{t_i}^N \to u_0^N \in V^N$ in V and

$$\|u_0 - u_0^N\|_V \le CF_0(N),$$

with $Cu_0 = Cu_0^N = 0$. Hence (2.32) holds. (2.33) follows similarly by replacing $F_0(N)$ by $F_0(N)f(N)$.

Next suppose that the locking is of order $o(f(N))$. Then by (2.33)

$$g(N) \le o(F_0(N)f(N)),$$

which contradicts (2.34). Hence the locking is of at least order $f(N)$.

Suppose now that Condition (α) holds. We have to show that if (2.32) holds there is no locking. So let for any $u_t \in H_t^B$, $u_0 \in H_0^B \cap H_t$ be such that (2.31) holds. Then we may write

$$u_t = u_0 + (u_t - u_0) = u_0 + \chi_t, \tag{2.36}$$

where $\chi_t \in H_t$ satisfies

$$\|\chi_t\|_t \le Ct^{1/2}. \tag{2.37}$$

Hence we have

$$\begin{aligned}\|u_t - u_t^N\|_{E,t} &\le C \inf_{v\in V^N} \|u_t - v\|_{E,t} \\ &\le C \inf_{v\in V^N} \|u_0 + \chi_t - v\|_{E,t} \\ &\le C\Big\{ \inf_{\substack{v_1\in V^N \\ Cv_1=0}} \|u_0 - v_1\|_V + \inf_{v_2\in V^N} \|\chi_t - v_2\|_{E,t}\Big\},\end{aligned}$$

using the fact that $u_0 \in H_0^B$ and hence $Cu_0 = 0$. This gives, by (2.32) and (2.25)

$$\le C\{F_0(N) + t^{-1/2} \inf_{v_2\in V^N} \|\chi_t - v_2\|_V\}.$$

Since χ_t satisfies (2.37), we use the F_0-admissibility (2.19) to get

$$\begin{aligned}&\le C\{F_0(N) + t^{-1/2}t^{1/2}F_0(N)\} \\ &\le CF_0(N).\end{aligned}$$

Then we see that

$$\|u_t - u_t^N\|_V \le C\|u_t - u_t^N\|_{E,t} \le CF_0(N)$$

so that on R, $L(t,N)$ remains bounded, i.e., there is no locking in the V norm.

Suppose now that (2.35) holds. Then we may replace $F_0(N)$ in the above by $F_0(N)f(N)$ so that

$$\|u_t - u_t^N\|_V \le C\|u_t - u_t^N\|_{E,t} \le CF_0(N)f(N)$$

which shows locking of order *at most* $f(N)$. Also, since (2.34) is satisfied,

the locking is of order *at least* $f(N)$, i.e., of order $f(N)$.

Conversely, if the locking is of order $f(N)$, then (2.33) holds. If $g(N) = o(F_0(N)f(N))$, then, just like before, we establish that

$$\|u_t - u_t^N\|_V \le o(F_0(N)f(N))$$

so that the locking is of order at most $o(f(N))$, a contradiction. □

We have actually proved a stronger result, namely,

<u>Theorem 2.3</u>. (A) Let the conditions of Theorem 2.2 (A) hold. Then $\mathcal{F}$ is free from locking with respect to the energy norm over R *only if* (2.32) holds. It shows locking of order $f(N)$ *only if* (2.33) holds. It shows locking of order *at least* $f(N)$ *if* (2.34) holds.

(B) Let Condition (α) be true. Then $\mathcal{F}$ is free from locking over R *iff* (2.32) holds. It shows locking of order $f(N)$ *iff* (2.35) holds.

<u>Proof</u>. As before, Assumption A is satisfied so that in both cases we can use the same $F_0(N)$. The rest of the proof is the same. □

Theorems 2.2 and 2.3 show that under certain circumstances, we need only check condition (2.32) (or (2.35)) to determine whether or not locking exists.

The energy norm and the V norm are two important error measures. The relation between them in terms of locking is expressed in the following theorem.

<u>Theorem 2.4</u>. (A) Let the assumptions of Theorem 2.2 (A) hold. Then $\mathcal{F}$ is free from locking over the region $R \subset R_0$ with respect to the V norm *if* it is free from locking with respect to the energy norm. It shows locking of order $f(N)$ in the V norm *only if* it shows locking of at least order $f(N)$ in the energy norm.

(B) Moreover, if, in addition, Condition (α) holds, then $\mathcal{F}$ is free from

locking with respect to the V norm *iff* it is free with respect to the energy norm. It shows locking of order $f(N)$ in the V norm *iff* it shows locking of order $f(N)$ in the energy norm.

Proof. Part (A) follows immediately using the fact that (2.20) holds with the same $F_0(N)$ for both $\|\cdot\|_V$ and $\|\cdot\|_{E,t}$. Part (B) follows by using Theorems 2.2 (B) and 2.3 (B). □

3. Locking and robustness results for the model problem.

In this section we state and prove some theoretical results for the model problem, which explain the computational observations presented in the previous section.

We first note that we may write for $t \in (0,\beta]$ $(\beta < 1)$

$$(3.1) \qquad B_t^1(u,v) = \int_\Omega \left[\left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) + \left(\frac{1}{t} - 1\right)\left(\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right)\right] dx_1 dx_2,$$

which is of the form (2.22) with

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \; dx_1 dx_2$$

$$Cu = \frac{\partial u}{\partial x_2}.$$

Replacing $\frac{t}{1-t}$ by t, we have

$$t \in (0,\beta/1-\beta] = (0,b].$$

Clearly (2.23) - (2.24) are satisfied $(W = L_2(\Omega))$.

Instead of analyzing the forms B_t^1 and B_t^2 on the same meshes as in Section 2, we will now deal with the analogous problem of considering only B_t^1

on unrotated and rotated meshes (Figure 3.1). This will simplify some notation. Although the problems analyzed now are not totally identical with the ones addressed in Section 2, they show the same features and are essentially equivalent.

Let $S \subseteq \mathbb{R}^2$ be a triangle or parallelogram and let $\mathcal{P}_p(S)$ denote the set of all polynomials of total degree (degree in each variable) $\leq p$ if S is a triangle (parallelogram). We will talk about spaces V^N of continuous piecewise polynomials of degree $\leq p$ on the meshes in Figure 3.1 satisfying $u|_S \in \mathcal{P}_p(S)$ for any S in the mesh, for any $u \in V^N$.

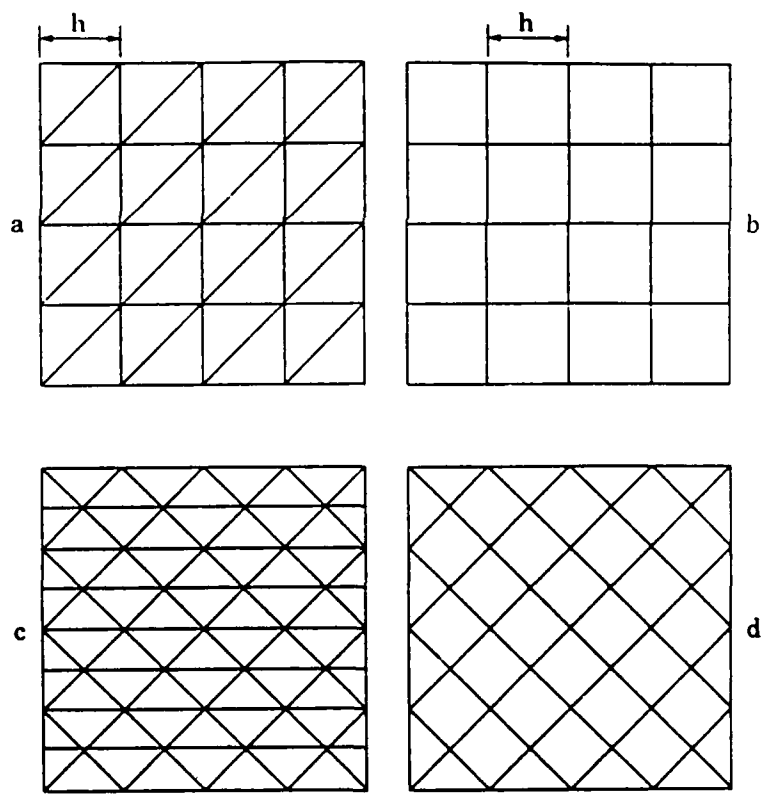

Figure 3.1. Unrotated and rotated meshes

(a) Unrotated triangular mesh (b) Unrotated rectangular mesh

(c) Rotated triangular mesh (d) Rotated rectangular mesh

We will also find it convenient to work with a suitably scaled version of (3.1), given by the change of variables

$$\hat{x}_1 = x_1, \quad \hat{x}_2 = x_2\sqrt{t}, \quad u(\hat{x}_1,\hat{x}_2) = u(x_1,x_2). \tag{3.2}$$

Under the transformation (3.2) we see that (3.1) may be written as

$$B_t(u,v) = \frac{1}{\sqrt{t}} \int_{\Omega_t} \nabla\hat{u}\cdot\nabla\hat{v}\, d\hat{x}_1 d\hat{x}_2$$

with $\Omega_t = \{(\hat{x}_1,\hat{x}_2) \mid |\hat{x}_1| \le 1, \ |\hat{x}_2| \le \sqrt{t}\}$ so that we just get Laplace's equation over a very thin domain. The corresponding meshes from Figure 3.1 now look as shown in Figure 3.2.

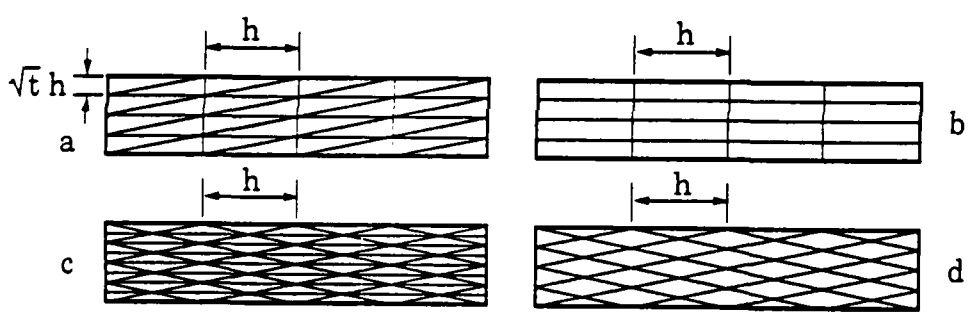

Figure 3.2. Scaled unrotated and rotated meshes.

(a) Scaled unrotated triangular mesh (b) Scaled unrotated rectangular mesh

(c) Scaled rotated triangular mesh (d) Scaled rotated rectangular mesh

The above families of uniform meshes are all examples of more general *quasiuniform* meshes. We will refer to triangular meshes satisfying the *minimum* (respectively, *maximum*) angle condition if the minimal angle of any triangle in the mesh is larger than ε (respectively, maximal angle is smaller than $\pi - \varepsilon$), where ε is fixed for the family of meshes (see [2]). Analogously, we will say that a rectangular mesh satisfies the *ratio condition*

if the ratio of the largest to the smallest sides in each rectangle is bounded above independently of the meshes.

Let us now look at the scaled triangular meshes. We observe that as $t \to 0$, the minimum angle condition is violated for both (a) and (c). However, we observe a clear difference in terms of the *maximum* angle condition. Mesh (a) satisfies the maximum angle condition *uniformly* with respect to t as $h \to 0$, while mesh (c) violates this condition as $t \to 0$. Similarly, the ratio condition for the rectangular mesh (b) holds uniformly as $t \to 0$ but not for the rotated mesh (d). The reason that meshes (c) and (d) lead to locking is essentially due to the violation of the maximum angle condition as $t \to 0$.

Let us define the following weighted norms over Ω in terms of the corresponding norms over Ω_t

$$\|u\|_{H_t^k(\Omega)} = \frac{1}{t^{1/4}}\|\hat{u}\|_{H^k(\Omega_t)}. \tag{3.3}$$

Note that $\|u\|_{H_t^1(\Omega)} = \|u\|_{E,t}$, the energy norm.

We will need the following lemma about n-widths.

<u>Lemma 3.1</u>. Let $\Omega \subset \mathbb{R}^i$, $i = 1,2$, be a polygonal domain (for $i = 1$, Ω is an interval). Then

$$d_N(H^1(\Omega), H^{k+1}(\Omega)) \approx N^{-k/i}, \quad k \geq 1, \tag{3.4}$$

where the equivalency constants depend on k and Ω but are independent of N.

For the proof, see [12].

<u>Lemma 3.2</u>. Let $\{V^N\} \subset H^1(\Omega)$ be subspaces of piecewise polynomials of fixed degree p on quasi-uniform triangular or rectangular meshes which satisfy the uniform maximum angle condition (for triangles) or the ratio condition (for

rectangles). Then

$$\sup_{\substack{u\in H^{k+1}(\Omega)\\ \|u\|_{H^{k+1}(\Omega)}\le 1}} \inf_{v\in V^N} \|u-v\|_{H^1(\Omega)} \ge CN^{-\min(p,k)/2}, \tag{3.5}$$

where C is independent of N, but depends on p,k,Ω and the characterization of the mesh (quasi-uniformity constant and maximum angle condition).

<u>Proof</u>. For $k \le p$ (3.5) follows from Lemma 3.1. Let $k > p$. In (3.5) we assumed that $V^N \subset H^1(\Omega)$, i.e., V^N is the space of conforming piecewise polynomials of degree p. Let us denote $\tilde{V}^{\tilde{N}} \subset L_2(\Omega)$ to be the space of piecewise polynomials of degree p without continuity conditions between elements. Then $C\tilde{N} \le N \le \tilde{N}$ and

$$\Phi(\tilde{N}) = \sup_{\substack{u\in H^{k+1}(\Omega)\\ \|u\|_{H^{k+1}(\Omega)}\le 1}} \inf_{v\in \tilde{V}^{\tilde{N}}} \left[\sum_{\mathcal{T}} \|u-v\|^2_{H^1(\mathcal{T})}\right]^{1/2} \le \sup_{\substack{u\in H^{k+1}(\Omega)\\ \|u\|_{H^{k+1}(\Omega)}\le 1}} \inf_{v\in V^N} \|u-v\|_{H^1(\Omega)},$$

where the sum above is over all single elements $\mathcal{T}$ of the mesh.

Hence it is sufficient to prove

$$\Phi(\tilde{N}) \ge CN^{-p/2}.$$

Obviously also $N = N(h) \approx h^{-2} \approx \tilde{N}$. Consider now a polynomial u of degree $p+1$ (total (triangles) or separate degrees (rectangles)). Then $u \in H^{k+1}(\Omega)$. Hence if (3.5) is not true then

$$\Phi(\tilde{N}) \le f(h)h^p \tag{3.6}$$

where $f(h) \to 0$ as $h \to 0$. Since the mesh is quasi-uniform with uniform angle condition the following inverse inequality holds on every element $\mathcal{T}$ for any polynomial of degree at most $p+1$

$$\|v\|_{H^s(\mathcal{T})} \le Ch^{t-s}\|v\|_{H^t(\mathcal{T})}, \quad 0 \le t \le s,$$

where C is independent of $\mathcal{T}, t, s, h$. Hence if (3.6) is satisfied then we have for $v_h \in \tilde{V}^{\tilde{N}}$

$$\left(\sum_{\mathcal{T}} \|u - v_h\|^2_{H^{p+1}(\mathcal{T})}\right)^{1/2} \le Cf(h),$$

assuming that $\|u\|_{H^{p+1}(\Omega)} = 1$. But since v_h is a polynomial of degree p, $\frac{\partial^{p+1} v_h}{\partial x_i^{p+1}} = 0$, $i = 1,2$ and hence $\|\frac{\partial^{p+1} u}{\partial x_i^{p+1}}\|_{L_2(\Omega)} \longrightarrow 0$ as $h \longrightarrow 0$, which is a contradiction. □

We now consider the case of the h-version on unrotated meshes followed by the analysis of the rotated meshes. For simplicity we will assume that either $\Gamma_N = \emptyset$ or $\Gamma_D = \emptyset$.

<u>Theorem 3.1</u>. Consider the problem (2.10) with B_t given by (3.1). Let V_t^N be the set of piecewise polynomials of degree $\le p$ $(p \ge 1)$ on an unrotated mesh of the form shown in Figure 3.1 a,b, with mesh spacing $h = h(N)$. Let H_t be the space of functions given by

$$H_t = \{u \mid \|u\|_{H_t^{k+1}(\Omega)} < \infty\}$$

with $k \ge p$. Let the error measure $E_t(u) = \|u\|_{E,t} = \|u\|_{H_t^1(\Omega)}$. Then the extension procedure $\mathcal{F}$ is free from locking over the region $R_0 = (0,b]\times\mathcal{N}$ and is robust with uniform order $g(N) = O(N^{-p/2})$ over this region.

<u>Proof</u>. Let $u \in H_t$. Then $\hat{u} \in H^{k+1}(\Omega_t)$, with $k \ge p$. Consider first the case that piecewise linear functions are used on a triangular mesh. Then the proof of the sufficiency of the maximum angle condition [2] easily gives

$$\inf_{\hat{v}\in\hat{V}_t^N} \|\hat{u} - \hat{v}\|_{H^1(\Omega_t)} \le Ch\|\hat{u}\|_{H^2(\Omega_t)}, \tag{3.7}$$

where $\hat{V}_t^N$ is the image of V_t^N and (due to the mesh satisfying the maximum

angle condition) C is independent of h and t.

Translating back to Ω, we obtain

$$\inf_{v \in V_t^N} \|u - v\|_{E,t} \le Ch\|u\|_{H_t^2(\Omega)}.$$

Now by (2.12) we have (since $h^{-2} \approx N$)

$$\|u - u^N\|_{E,t} \le Ch\|u\|_{H_t^2(\Omega)} \le CN^{-1/2}\|u\|_{H_t^k(\Omega)} \tag{3.8}$$

for all t. We now take $\eta = b$ in (2.13) which gives, using Lemma 3.2,

$$L(t,N) = \sup_{\substack{u \in H_t^2 \\ \|u\|_{H_t^2} \le 1}} \|u - u^N\|_{E,t} N^{1/2} \le C.$$

Hence, Definition 2.1 shows that $\mathcal{F}$ is free from locking. The robustness follows easily from (3.8).

For the case of piecewise polynomials of degree $p > 1$, the sufficiency of the maximum angle condition may once again be established using the arguments in [2]. This leads to an analog of (3.8) of the form

$$\|u - u^N\|_{E,t} \le Ch^p\|u\|_{H_t^{p+1}(\Omega)} \le CN^{-p/2}\|u\|_{H_t^{k+1}(\Omega)}.$$

Using Lemma 3.2 again, we establish the required results. The rectangular case follows similarly. □

<u>Remark</u>. The above theorem will hold for more general quasi-uniform meshes as well. It explains the results in Figures 2.1 - 2.3 for the unrotated case.

Note that Theorem 2.3 shows immediately that with the same choice of H_t, there is no locking (and uniform robustness) in the V (i.e., H^1) norm as well.

Another interesting question is to examine locking in the V norm when, instead of $H_t^{k+1}(\Omega)$, H_t is taken to be an unweighted space. For this case, we expect once again that there will be no locking. We prove this assertion for the case $p = 1$, using the equivalence between finite difference and finite element schemes.

<u>Theorem 3.2</u>. Consider the problem (2.10) with B_t given by (3.1) to be the approximate problem corresponding to (2.1) - (2.2) with $\Gamma_N = 0$. Assume that $\mathcal{F} = \{\mathcal{F}_t\}$ where $\mathcal{F}_t = \{V^N\}$ is the space of continuous, piecewise linear functions on the triangular mesh shown in Figure 3.1 a. Let $H_t = C^4(\bar{\Omega})$ (independently of t). Let the error measure be $E_t(u) = \|u\|_{H^1(\Omega)}$. Then $\mathcal{F}$ is free of locking over $R_0 = (0,b]\times\mathcal{N}$ and is robust with uniform order $O(N^{-1/2})$ there.

<u>Proof</u>. Consider the mesh shown in Figure 3.1 a, with Ω being given by $(-1,1)^2$ and the nodes being labeled A_{ij}, $i = 0,1,\ldots,M$, $j = 0,1,\ldots,M$, where $M = 2/h$ (A_{00} corresponding to the node with coordinates $(-1,-1)$).

Let u_t, the exact solution and u_t^N, the finite element solution at A_{ij} be denoted by u_{ij}, u_{ij}^N, respectively. Then by the usual relation between finite element and finite difference solutions for a regular triangular mesh, we have for i,j such that $A_{ij} \in \Omega$ with $u_{ij}^N = 0$ when $A_{ij} \in \Gamma$,

$$(3.9) \qquad -\Delta_t^{ij} u_t^N = t(-u_{i-1,j}^N + 2u_{i,j}^N - u_{i+1,j}) + (-u_{i,j-1}^N + 2u_{i,j}^N - u_{i,j+1}^N) = th^2 \bar{f}_{ij},$$

where

$$\bar{f}_{ij} = \frac{1}{h^2} \int_\Omega f\phi_{ij}\, dx_1 dx_2.$$

Here, ϕ_{ij} is the standard "hat function" at A_{ij}. We are given $u \in C^{(4)}(\bar{\Omega})$, i.e.,

$$\|u\|_{C^{(4)}(\bar{\Omega})} \le K.$$

Using this, we may show

$$\bar{f}_{ij} = f_{ij} + B_{ij}h^2,$$

where $f_{ij} = f(A_{ij})$ and B_{ij} satisfies $|B_{ij}| \le CK$ with C independent of u, h and t. Moreover, for the exact solution we have

$$-\Delta_t^{ij} u_t = h^2(tf_{ij} + D_{ij}h^2) \tag{3.10}$$

where again $|D_{ij}| \le CK$. Using (3.9) - (3.10), we see that with $\chi = u_t^N - u_t$, we have $\chi_{ij} = 0$ on Γ and

$$-\Delta_t^{i,j}\chi = h^4 H_{ij} \tag{3.11}$$

with $|H_{ij}| \le CK$. Let

$$\eta(x,y) = 1 - y^2$$

so that

$$\eta_{ij} = (jh)(2-jh) \ge 0.$$

Then $\eta_{ij} = 0$ for $j = 0, M$, and for A_{ij} in the interior of Ω,

$$-\Delta_t^{ij}\eta = 2h^2.$$

Let $\mathcal{K} = Ch^2K\eta - \chi$. Then

$$-\Delta_t^{ij}\mathcal{K} = 2Ch^4K - h^4H_{ij} \ge Ch^4K > 0. \tag{3.12}$$

Also, $\mathcal{K}_{ij} \ge 0$ on the boundary Γ of Ω. Suppose $\mathcal{K}_{ij} < 0$ in Ω for some $A_{ij} \notin \Gamma$. Then there exists a minimum value of $\mathcal{K}_{ij} < 0$ in Ω. At such a value, for any t,

$$-\Delta_t^{ij}\mathcal{K} \le 0,$$

contradicting (3.12). Hence, $\chi_{ij} \geq 0$ on $\bar{\Omega}$, i.e.,

$$\chi_{ij} \leq CKh^2\eta_{ij}.$$

Similarly, taking $\xi = Ch^2K\eta + \chi$, we see that

$$\chi_{ij} \geq -CKh^2\eta_{ij}.$$

Noting that $|\eta_{ij}| \leq 1$, we have

$$|\chi_{ij}| \leq CKh^2.$$

This shows that the difference between the interpolant of u_t and the finite element solution at the nodal points is $O(h^2)$, from which

$$\|u_t - u_t^N\|_{H^1(\Omega)} \leq Ch = O(N^{-1/2}). \tag{3.13}$$

Using Lemma 3.2. we see that we have the same rate as (3.5) with $p = 1$. Since (3.13) holds for all t, the theorem follows. □

<u>Remark</u>. In Theorem 3.2 we have used some assumptions only for the sake of simplicity. For example, we assumed $\Gamma = \Gamma_D$, which can be relaxed. Further, we assumed $H_t = C^{(4)}(\bar{\Omega})$ and triangular meshes, which can also be relaxed.

One could of course consider the case when $H_t = H^k(\Omega)$ and $E_t(u) = \|u\|_{E,t}$. To see why this combination is not appropriate for analysis, consider functions on Ω that are functions of the variable x_2 alone, i.e., $\Phi \in H^k(I)$. For such a function, by Lemma 3.1,

$$d_N(H^1(I), H^k(I)) \approx N^{-(k-1)}.$$

Now considering these functions as functions of both x_1 and x_2 we have $\Phi \in H^k(\Omega)$ so that we get from the above

$$d_N(H^1(\Omega), H^k(\Omega), E_t) \geq \frac{1}{\sqrt{t}} CN^{-(k-1)}.$$

Hence, since $d_N(H^1(\Omega), H^k(\Omega), E_t)$ does not converge to zero uniformly in t as $N \to \infty$, condition (2.18) is violated. Therefore, locking will exist (albeit in a trivial way) for any choice of approximation.

Let us now consider the effect of using rotated meshes. Figures 2.1 - 2.3 indicate that there is locking both in the V norm and the energy norm. To explain this, let us first prove

<u>Lemma 3.3</u>. Let the extension process $\mathcal{F}$ be based on the h-version using a continuous piecewise polynomial of fixed degrees $\leq p$ on a rotated mesh (Figure 3.1 c,d). Let u be a function defined on Ω such that $Cu = \partial u/\partial x_2 = 0$ and u is not a polynomial on Ω. Then there exists a constant $C > 0$, independent of N, such that

$$\inf_{\substack{w \in V^N \\ Cw=0}} \|u - w\|_V = \inf_{\substack{w \in \mathcal{P}_p(\Omega) \\ Cw=0}} \|u - w\|_V \geq C \quad \text{for all } N. \tag{3.8}$$

<u>Proof</u>. Consider a triangular mesh of the form shown in Figure 3.1 c. Let V^N be the space of piecewise polynomials of total degree $\leq p$ on this mesh and let $w \in V^N$ satisfy $Cw = 0$. Then over any element $\mathcal{T}_j$, w must be of the form

$$w = \sum_{i=0}^{p} a_{ij} x^i.$$

For any two triangles $\mathcal{T}_j, \mathcal{T}_k$ with a common side Γ_{jk}, we see that the variable x_1 is never fixed along Γ_{jk}. Hence the continuity of w implies that

$$a_{ij} = a_{ik}.$$

Repeating this argument, we see that $w \in \mathcal{P}_p(\Omega)$. Since $u \notin \mathcal{P}_p(\Omega)$, we obtain

(3.8). The same argument works for rectangular meshes. □

We now obtain

<u>Theorem 3.3</u>. Consider the problems (2.10), (3.1). Let V_t^N be the set of piecewise polynomials of fixed degree $\le p$ $(p \ge 1)$ on a rotated mesh of the form shown in Figure 3.1 c or d, with mesh spacing $h = h(N)$. Let the error measure be $\mathcal{E}_1 = \{E_t^1\}$, $E_t^1(u) = \|u\|_{H^1(\Omega)}$ or $\mathcal{E}_2 = \{E_t^2\}$, $E_t^2(u) = \|u\|_{E,t}$, and let $R = \{(t,N),\ 0 < t \le b_N\} \subset R_0$. Further, assume that the quadruple $(\mathcal{H},\mathcal{F},\mathcal{E},R)$ is F_0-admissible. Then the extension procedure $\mathcal{F}$ shows complete locking and is not robust with any uniform order over R.

<u>Proof</u>. We can directly apply Theorem 2.2(A) or Theorem 2.3(A). Lemma 3.3 shows that the condition (2.32) is not satisfied. Hence $\mathcal{F}$ cannot be free from locking over R. The lack of robustness follows as well from Lemma 3.3. □

Suppose we now choose H_t to be a space of the form $H^k(\Omega)$, as in Theorem 3.2. Then, since the space of admissible functions in Theorem 3.3 is more restrictive, we again see that there will be locking in the $H^1(\Omega)$ norm. Let us mention here that an alternate proof of Theorem 3.3 may be derived by considering the necessity of the maximum angle condition (see [2]).

Let us now examine what happens when we use the p-version instead of the h-version. We have

<u>Theorem 3.4</u>. Consider the problems (2.10), (3.1), with $\Gamma_D = 0$. Let V_t^N be the set of piecewise polynomials of degree $\le p = p(N)$ on *any* of the meshes shown in Figure 3.1 (i.e., rotated or unrotated), with fixed mesh size h independent of N. Let H_t^B be the space of functions given by

$$H_t^B = \{u \mid \|u\|_{H_t^{k+1}(\Omega)} \le 1\}$$

with $k > 0$. Let the error measure be $E_t(u) = \|u\|_{E,t}$. Then the p-version extension procedure $\mathcal{F}$ is free from locking over the region $R_0 = (0,b]\times\mathcal{N}$ and is robust with uniform order $f(N) = O(N^{-k/2})$ over this region.

<u>Proof</u>. Let $u \in H_t^{k+1}(\Omega)$. Denote $\tilde{\Omega} = \{|\hat{x}_1| < \frac{5}{4},\ |\hat{x}_2| < 1\}$, $\hat{\Omega} = \{|\hat{x}_1| < \frac{5}{4},\ |\hat{x}_2| < \frac{5}{4}\}$. By the extension approach of Nikolsky and Babich (see [11] Theorem 3.9) we extend u to $\tilde{u}$ on $\tilde{\Omega}$: for $-\frac{5}{4} < \hat{x}_1 < -1$,

$$\tilde{u}(\hat{x}_1,\hat{x}_2) = \sum_{\ell=1}^{k+1} \lambda_i u\left[-1-(\hat{x}_1+1)\frac{\ell}{k+1},\hat{x}_2\right]$$

with the exact form of the coefficients λ_i being described in [11]. $\tilde{u}$ is analogously extended on $1 < \hat{x} < \frac{5}{4}$ and then on $\hat{\Omega}$ to get $\hat{u}(\hat{x}_1,\hat{x}_2)$, which preserves the norm, i.e.,

$$\|\hat{u}\|_{H_t^{k+1}(\hat{\Omega})} \le C\|u\|_{H_t^{k+1}(\Omega)}.$$

Now we use the same approach as in [3] Lemma 3.1. We transform $\hat{\Omega}$ onto

$$Q = \{|\xi_1| < \tfrac{\pi}{2},\quad |\xi_2| < \tfrac{\pi}{2}\}$$

with the transformation

$$\hat{x}_1 = \frac{5}{4}\sin\xi_1$$

$$\hat{x}_2 = \frac{5}{4}\sin\xi_2$$

and let $v(\xi_1,\xi_2) = \hat{u}(\hat{x}_1,\hat{x}_2)$. Then obviously

$$\|v\|_{H_t^{k+1}(Q)} \le C\|\hat{u}\|_{H_t^{k+1}(\hat{\Omega})} \le C\|u\|_{H_t^{k+1}(\Omega)}.$$

Now function v can be extended periodically on $\tilde{R} = \{|\xi_1| < \pi,\ |\xi_2| < \pi\}$ and it is symmetric with respect to the lines $\xi_1 = \frac{\pi}{2}$, $\xi_2 = \frac{\pi}{2}$. Writing

$$v = \mathrm{Re} \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_{j\ell} e^{i(j\xi_1+\ell\xi_2)}$$

we denote

$$v_p = \mathrm{Re} \sum_{j=-p}^{p} \sum_{\ell=-p}^{p} a_{j\ell} e^{i(j\xi_1+\ell\xi_2)}$$

and get

$$\|v - v_p\|_{H_t^1(Q)} \le Cp^{-k} \|u\|_{H_t^{k+1}(\Omega)}.$$

Transforming v back into variables $\hat{x}_1$ and $\hat{x}_2$ and denoting it by V we see that V is a polynomial of degree p in x_1 and x_2. We get on Ω

$$\|u - V\|_{H_t^1(\Omega)} \le Cp^{-k} \|u\|_{H_t^{k+1}(\Omega)},$$

and because $\|u\|_{H_t^1(\Omega)} = \|u\|_{E,t}$ we have

$$\inf \|u - w\|_{E,t} \le CN^{-k/2} \|u\|_{H_t^{k+1}(\Omega)} \tag{3.9}$$

where the infimum is taken over all polynomials of degree $2p$ and where C depends on k but is independent of N, t and u.

For $t = \eta > 0$, $\|\cdot\|_{E,t}$ and $\|\cdot\|_{H_t^{k+1}(\Omega)}$ are equivalent to $\|\cdot\|_{H^1(\Omega)}$ and $\|u\|_{H^{k+1}(\Omega)}$, respectively, with the equivalency constants dependent on η. Using Lemma 3.1 we get

$$d_N(H^1(\Omega), H^{k+1}(\Omega)) \ge CN^{-k/2} \tag{3.10}$$

and the statement follows from (3.10) and (3.9). □

<u>Remark</u>. The statement and proof of Theorem 3.4 above carry over identically if instead of (3.3), we use the following weaker definition of the norm

$\|\cdot\|_{H_t^k(\Omega)}$,

$$(3.11) \qquad \|u\|^2_{H_t^k(\Omega)} = \sum_{0\le\alpha_1\le k} \|\frac{\partial^{\alpha_1}u}{\partial x^{\alpha_1}}\|_0^2 + t^{-1} \sum_{\substack{0\le\alpha_1+\alpha_2\le k \\ \alpha_2\ge 1}} \|\frac{\partial^{\alpha_1+\alpha_2}u}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\|_0^2.$$

In fact, Theorem 3.1 also holds with the above weaker definition – instead of using the maximum angle condition, one can establish it directly by considering interpolation on the region Ω. Hence the results in this section are valid if (3.3) is replaced by (3.11).

For the case of just one square, the above theorem guarantees that the rate of convergence is the same, independent of t, for both the rotated and unrotated case. This is clearly observed in Figure 2.4. Note that our function there was $u_t^1 = \sin x_1\, e^{-\sqrt{t}x_2}$ which satisfies

$$u_t^1 \in H_t^k(\Omega) \ \forall\ k.$$

Since this function is very well behaved, we can actually assert a stronger robustness result for it (see Theorem 3.5 below).

For the case of more than one square, Theorem 3.4 asserts that the *rate* of convergence is independent of t both with rotated and unrotated meshes. However, in actual practice, while the unrotated case will be insensitive to t, the rotated case will not. This is because it is known from [4] that the error for the finite element method using piecewise polynomials of degree $\le p$ on a quasi-uniform mesh with mesh size h is of the form

$$(3.12) \qquad \|u - u^N\|_{H^1(\Omega)} \le \frac{Ch^{\min(k,p)}}{p^k} \|u\|_{H^{k+1}(\Omega)}.$$

When the p-version is used, the factor $Ch^{\min(k,p)} = Ch^k$ (for p large) appears as a constant. In terms of locking, when t is bounded away from 0,

one observes the full effect of this constant. However, for t close to 0, Lemma 3.3 shows that this constant is much larger. This explains the data in Table 2.3. For a rigorous demonstration of this effect, we begin by proving some lemmas on the approximability of smooth solutions like (2.5).

<u>Lemma 3.4</u>. Let $f(x) \in C^{\infty}(I)$, $I = (-1,1)$, be such that for any $k \geq 0$,

$$\text{Max}|f^{(k)}(x)| \leq Ad^k, \tag{3.13}$$

with $0 < d \leq 1$ and A independent of k and d. Then

$$\inf_{v \in \mathcal{P}_p(I)} \|f - v\|_0 \leq CAd^{p+1}\rho(p), \tag{3.14}$$

where

$$\rho(p) = \frac{1}{2^{(p+1)}(p+1)!}.$$

Moreover, suppose that there is a $B > 0$ such that $f^{(p+1)}(x)$ does not change sign on I and

$$\min|f^{(p+1)}(x)| \geq Bd^{p+1}. \tag{3.15}$$

Then

$$\inf_{v \in \mathcal{P}_p(I)} \|f - v\|_0 \geq CBd^{p+1}\rho(p), \tag{3.16}$$

where C is independent of p. (Here $\|\cdot\|_0 = \| \ \|_{L_2(I)}$.)

<u>Proof</u>. We write

$$f(x) = \sum_{k=0}^{\infty} a_k L_k(x) \tag{3.17}$$

where $L_k(x)$ is the k-th Legendre polynomial. Then we have

$$L_k(x) = \frac{(-1)^k}{2^k k!} \frac{d^k}{dx^k} \{(1-x^2)^k\}$$

and coefficients a_k in (3.17) satisfy

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) L_k(x) dx.$$

Hence we have

$$(3.18) \quad |a_k| = \frac{2k+1}{2^{k+1} k!} \left| \int_{-1}^{1} (1-x^2)^k f^{(k)}(x) dx \right| \le A d^k \frac{2k+1}{2^{k+1} k!} \int_{-1}^{1} (1-x^2)^k \, dx.$$

Denoting

$$\mathcal{H}(k) = \int_{-1}^{1} (1-x^2)^k \, dx = \frac{(k!)^2}{(2k+1)!} 2^{k+1}$$

we get

$$C_1 k^{-1/2} \le \mathcal{H}(k) \le C_2 k^{-1/2}$$

so that

$$|a_k| \le \frac{C k^{1/2} A d^k}{2^k k!}.$$

Now

$$E_p^2 = \inf_{v \in \mathcal{P}_p(I)} \|f - v\|_0^2 = \sum_{k=p+1}^{\infty} a_k^2 \frac{2}{2k+1} \le CA^2 \sum_{p+1}^{\infty} d^{2k} \frac{1}{(2^k k!)^2}$$

$$\le \frac{CA^2}{((p+1)!)^2} \sum_{p+1}^{\infty} \left(\frac{d}{2}\right)^{2k} \le \frac{CA^2}{((p+1)!)^2} \left(\frac{d}{2}\right)^{2(p+1)},$$

since $\frac{d}{2} \le \frac{1}{2}$ and the sum is a geometric series. This gives (3.14). On the other hand,

$$E_p^2 \ge \max_{k \ge p+1} a_k^2 \frac{2}{2k+1}.$$

Hence using (3.15) and (3.18), we see that (3.16) holds. □

For any $0 < d \le 1$, let us denote by $\mathcal{L}_d$ the set of all functions

$f(x) \in C^\infty(I)$ such that for each $p \geq 1$ there exist at most M intervals $I_i^{(p)}$, $i = 1,\ldots,M$ (M independent of p), with $J^p = I - \bigcup_{j=1}^{M} I_j^{(p)}$ having measure $\geq \tau > 0$ (τ independent of p) and

$$\min_{x\in J^p} |f^{(p)}(x)| \geq Bd^p$$

and

$$\max_{x\in I} |f^{(p)}(x)| \leq Ad^p$$

with A,B independent of p. Then we have

<u>Lemma 3.5</u>. Let $\{\mathcal{T}^h\}$ be a sequence of quasi-uniform meshes on $I = (-1,1)$ and let $I = \bigcup_{j=1}^{N(h)} I_j^h$, $I_j^h = (x_j^h, x_{j+1}^h)$, $h_j = x_{j+1}^h - x_j^h$, $\tilde{C}_1 h \leq h_j \leq \tilde{C}_2 h$. Let $\{V_h^p\}$ denote a sequence of subspaces of $H^1(I)$ of piecewise polynomials of degree $\leq p$ on the mesh $\mathcal{T}^h$. Let $f \in \mathcal{L}_d$. Then

$$C_1 d^p h^p \rho(p-1) \leq \inf_{v\in V_h^p} |f - v|_{H^1(I)} \leq C_2 d^p h^p \rho(p-1), \tag{3.19}$$

where C_1, C_2 depend on τ, A, B, M, but are independent of d, h, p and where the first inequality holds provided h is sufficiently small.

<u>Proof</u>. Let $g(x) = f'(x)$ and let

$$e_{p-1,j}^h = \inf_{w\in\mathcal{P}_{p-1}(I_j^h)} \|g - w\|_{L_2(I_j^h)}. \tag{3.20}$$

Transforming I_j^h to the standard interval I we get by applying Lemma 3.4

$$e_{p-1,j}^h \leq Cd^{p+1} h^{p+(1/2)} \rho(p-1). \tag{3.21}$$

If $I_j^h \in J^p$, then also

(3.22) $$e^h_{p-1,j} \geq Cd^{p+1}h^{p+(1/2)}\rho(p-1).$$

Let now $\bar{w}(x)$ be the polynomial of degree $p-1$ which attains the infimum (3.20). Then on I^h_j let

$$v(x) = f(x_j) + \int_{x_j}^{x} \bar{w}(x)dx.$$

Because of the construction of $v(x)$, by expanding $f'(x)$ into Legendre polynomials we have

$$v(x_{j+1}) = f(x_{j+1}).$$

Hence in this manner we construct $v(x)$ on the entire I with $v(x) \in H^1(I)$. From (3.21) we have by squaring and summing over j

$$\|f'(x) - v'(x)\|_{L_2(I)} \leq Cd^{p+1}h^p\rho(p-1).$$

For h sufficiently small, we have $\frac{C}{h}$ intervals contained in J^p with C independent of p. Hence we also get using (3.22)

$$\|v'(x) - f'(x)\|_0 \geq Cd^{p+1}h^p\rho(p-1). \qquad \square$$

Let us note that the function $\sin x_1 \in \mathcal{L}_1$ and $e^{-\sqrt{t}x_2} \in \mathcal{L}_{\sqrt{t}}$. Hence the function u^1_t given by (2.5) satisfies

(3.23) $$u^1_t \in H_t = \mathcal{L}_1 \otimes \mathcal{L}_{\sqrt{t}}.$$

Note that for $\sin x_1$, $M = 1$ and (3.19) will hold for $h \leq 1$. Also, if $h > 1$, then (3.19) holds for p even.

We now prove the following result for the case when the solution set $\mathcal{H} = \{H_t\}$ where H_t is given by (3.23). Here H_t is in fact a countably normed

space and H_t^B is characterized by t, A, B, d, τ, M.

<u>Theorem 3.5</u>. Consider the problem in Theorem 3.4 with $H_t^B = \mathcal{L}_1 \otimes \mathcal{L}_{\sqrt{t}}$. Then the h-p extension procedure $\mathcal{F}$ using piecewise polynomials of degree $\le p_N$ on unrotated meshes of mesh size h_N, is free from locking over the region $R_0 = (0,b]\times\mathcal{N}$ and is robust with uniform order $f(N) = O(h_N^{p_N}\rho(p_N-1))$ there.

<u>Proof</u>. We consider for simplicity only the case $\Gamma_D = \emptyset$. Let $u_t \in H_t^B$ so that

$$u_t(x_1,x_2) = \xi(x_1)\chi(x_2)$$

with $\xi \in \mathcal{L}_1$, $\chi\in\mathcal{L}_{\sqrt{t}}$. Then for any $y = v(x_1)w(x_2) \in V^N$,

$$\begin{aligned}
(3.24)\quad \|u_t - y\|_{E,t}^2 &= \|\xi'\chi - v'w\|_0^2 + \tfrac{1}{t}\|\xi\chi' - vw'\|_0^2 \\
&\le \|\xi'\chi - v'\chi\|_0^2 + \|v'\chi - v'w\|_0^2 + \tfrac{1}{t}\|\xi\chi' - \xi w'\|_0^2 + \tfrac{1}{t}\|\xi w' - vw'\|_0^2 \\
&= \|\xi' - v'\|_0^2\|\chi\|_0^2 + \|\chi - w\|_0^2\|v'\|_0^2 + \tfrac{1}{t}\{\|\xi\|_0^2\|\chi' - w'\|_0^2 \\
&\quad + \|\xi - v\|_0^2\|w'\|_0^2\}.
\end{aligned}$$

Suppose $v, w \in V_h^p$ are the best approximations to ξ, χ, respectively, in the sense of Lemma 3.5, i.e.,

$$\|\xi' - v'\|_0 \le Ch^p\rho(p-1)$$

$$\|\chi' - w'\|_0 \le Ct^{p/2}h^p\rho(p-1).$$

Then (3.24) gives for $p \ge 1$,

$$(3.25)\qquad \|u_t - y\|_{E,t} \le Ch^p\rho(p-1)$$

uniformly in t. This proves the robustness result. To show that there is no locking, we must verify that (3.25) is the best rate possible. Now it may be easily shown that

$$\inf_{y\in V^N} \|u_t - y\|^2_{E,t} = \inf_{v,w\in V_h^p} \|\xi\chi - vw\|^2_{E,t}$$

$$\geq \inf_{v\in V_h^p} \|\xi'\chi - v'\chi\|_0^2 = \inf_{v\in V_h^p} \|\xi' - v'\|_0^2 \|\chi\|_0^2.$$

Using Lemma 3.5 then shows that (3.25) is the best rate possible. Then by Definition 2.1, there is no locking over R_0. □

<u>Remark</u>. Note that the above theorem also shows that there is no locking for the p-version.

We now look at the case when rotated meshes are used. We will distinguish between two cases. For the case that no refinement is present, i.e., only one element is used, we essentially get the unrotated case with one element. [We could understand this case also by considering the rotated solution u_t^2 as in Section 2 (as mentioned in the beginning of this section, this is equivalent).] We will observe no locking in this case. When refinement is present, Lemma 3.3 tells us that for the limit problem, the mesh behaves essentially like one with a single element. This will show locking for both the p-version and h-p version.

<u>Theorem 3.6</u>. Consider the problems (2.10), (3.1) ($\Gamma_D = \emptyset$). Let V_t^N be the set of piecewise polynomials of degree $p = p_N$ on a family of rotated meshes of the form shown in Figure 3.1 d, with mesh spacing $h = h_N$. Let $H_t^B = \mathcal{L}_1 \otimes \mathcal{L}_{\sqrt{t}}$ and $E_t(u) = \|u\|_{E,t}$ or $\|u\|_{H^1(\Omega)}$. Let $\mathcal{F}$ consist of the p-version (h_N fixed) or the h-p version. For the p-version on a single element, $\mathcal{F}$ is free from locking and uniformly robust with order $O(\rho(p_N-1))$. For a refined mesh with mesh size $h_N \leq 1$, the extension process $\mathcal{F}$ shows locking of at least order $(2/h_N)^{p_N}$ but is uniformly robust with order $\rho(p_N-1)$.

Proof. For a single element, we have $h = 2$. We may either consider the pair u_t^1, $B_t^1(\cdot,\cdot)$ or u_t^2, $B_t^2(\cdot,\cdot)$. In either case, Lemma 3.5 may be used, like in the proof of Theorem 3.5, to show that there is no locking and one gets uniform robustness of order $f(N) = O(\rho(p_N-1))$.

When refinement is present with $h \le 1$, we see that for the limit problem, by Lemma 3.3, the mesh behaves like a single element, i.e., one with mesh size 2. However, for t bounded away from 0, one gets the same convergence rate as in Theorem 3.5. This implies that

$$F_0(N) = \inf_{w\in V^N} \|u - w\|_V = O(h_N^{p_N}\rho(p_N-1)),$$

but for $Cu = 0$,

$$\inf_{\substack{w\in V^N \\ Cw=0}} \|u - w\|_V (F_0(N))^{-1} \approx O((\frac{2}{h_N})^{p_N}).$$

Hence by (2.34) we get locking of at least order $(2/h_N)^{p_N}$. The method $\mathcal{F}$ is still uniformly robust with the rate using one square, i.e., $O(\rho(p_N-1))$.□

In the above proof, we used the necessity of the condition (2.32) to prove that there was locking. We mention here that for this problem, Condition (α) does not hold in general so that (2.32) would not be sufficient except in special cases. For example, taking $H_t = \text{span}\{u_t^1\}$ gives

$$u_t^1 \longrightarrow u_0 = \sin x_1$$

and

$$\|u_t^1 - u_0\|_{E,t} \approx O(1)$$

so that if $\|\cdot\|_{E,t} \le C\|\cdot\|_t$, then

$$\|u_t^1 - u_0\|_t \approx O(1),$$

violating (α). One could, of course, define $\mathcal{H}$ such that we only look at a class of problems for which (α) holds.

Let us summarize the various locking phenomena we have dealt with in this section in a table. (U $\equiv$ Unrotated, R $\equiv$ Rotated.)

Extension Process	H_t	E_t	Locking	Robustness (uniform order)
h-version, degree p, U	$H_t^{k+1}(\Omega)$, $k \geq p$	H_t^1, H^1	No	$O(N^{-p/2})$
(degree 1)	$C^{(4)}(\bar{\Omega})$	H^1	No	$O(N^{-p/2})$
h-version, degree p, R	$H_t^{k+1}(\Omega)$, $k > 0$	H_t^1, H^1	Yes	None
	$H^{k+1}(\Omega)$, $k > 0$	H^1	Yes	None
p-version, U	$H_t^{k+1}(\Omega)$, $k > 0$	H_t^1, H^1	No	$O(N^{-k/2})$
p-version, R	$H_t^{k+1}(\Omega)$, $k > 0$	H_t^1, H^1	No	$O(N^{-k/2})$
p or h-p version, U	$\mathcal{L}_1 \times \mathcal{L}_{\sqrt{t}}$	H_t^1	No	$O(h_N^{p_N} \rho(p_N-1)$
p-version, R, 1 square	$\mathcal{L}_1 \times \mathcal{L}_{\sqrt{t}}$	H_t^1	No	$O(\rho(p_N-1))$
p or h-p version, R refined	$\mathcal{L}_1 \times \mathcal{L}_{\sqrt{t}}$	H_t^1	Yes	$O(\rho(p_N-1))$

We see from the above table the effect of changing various choices of the extension procedure, H_t and E_t.

Let us remark that our results will be valid for *any* rotated mesh. We have considered the case that the rotation is 45° for convenience, since locking will be observed most readily in this case. However, Lemma 3.3 is true for other angles of rotation as well.

4. Locking and non quasi-uniform meshes.

Our definition for locking in Section 2 allows considerable flexibility in terms of the various components $\mathcal{H}, \mathcal{E}, \mathcal{F}$, etc., which may depend on the parameter t. We now exploit this flexibility to analyze an example where locking is caused by the failure of an inverse estimate to hold when the meshes are non quasi-uniform.

Let $I = [0,1]$ and $t \in (0,\frac{1}{3}]$. For each N, let $h = N^{-1}$, $x_j = jh$, $j = 0,\ldots,N$. Define $\delta = h(1-2t)$, $\varepsilon = ht$ and let $\{\mathcal{T}_t^h\}$ be the sequence of meshes shown in Figure 4.1 consisting of nodal points x_i^ℓ, x_i, x_i^r. Let $V_t^N = V_t^h \subset H_0^1(I)$ be the space of piecewise linear functions on $\mathcal{T}_t^h$. This defines the family $\{\mathcal{F}_t\} = \mathcal{F}$. Given $u \in H_0^1(I)$, for $t > 0$, we define $u_t^h = u_t^N \in V_t^N = V_t^h$ by (2.10) with

$$B_t(u,v) = B(u,v) = \int_I uv\,dx,$$

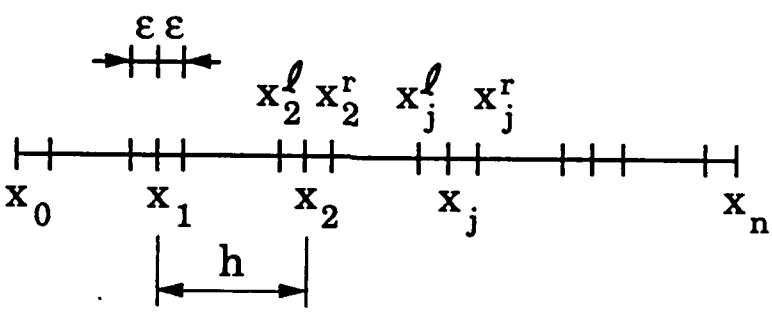

Figure 4.1. The mesh $\mathcal{T}_t^h$.

i.e., u_t^h is the L_2 projection of u on V_t^h. This defines our extension procedure $\mathcal{F}$. We choose $H_t = H^2(I) \cap H_0^1(I)$ and $E_t(u) = \|u\|_1$ and ask whether locking occurs as $t \to 0$. Note that the only component depending on t is $\mathcal{F} = \{\mathcal{F}_t\}$ (due to the mesh). We let $R_0 = \{(t,N),\ 0 < t \le \frac{1}{3}\}$.

<u>Theorem 4.1</u>. Let $\{P_t\}$, $\mathcal{H}$ and $\mathcal{E}$ be as above. Then $\mathcal{F}$ shows locking of

$O(N)$ for any region $R = \{(t,N),\ 0 < t \le b_N\} \subset R_0$ and is not robust with any uniform order.

<u>Proof</u>. First it is easy to see that the quadruple $(\mathcal{H},\mathcal{F},\mathcal{E},R)$ is F_0-admissible with $F_0(N) \approx CN^{-1}$ due to standard results about the $H^1(I)$ projection on V_t^N. For any $\eta \ge \eta_0 > 0$, the mesh is quasi-uniform on R_η so that the space V_t^N satisfies the inverse assumption. This leads to Assumption A being satisfied so that Theorem 2.1 holds with $F_0(N)$ as above.

Let us keep h fixed and let $t \to 0$. Then it may be easily verified that in the L_2 norm, $u_t^h \to u_0^h \in V_0^h$ where V_0^h is a set of discontinuous linear functions on $[(j-1)h, jh]$. Also u_0^h satisfies the limit problem

$$(4.1) \qquad \int_{(j-1)h}^{jh} u_0^h v dx = \int_{(j-1)h}^{jh} uv dx, \quad u_0^h(0) = u_0^h(1) = 0,$$

with v being a linear function on $[(j-1)h, jh]$, $j = 1,\ldots,N$.

Suppose $\mathcal{F}$ is robust with some uniform order. Then for $t > 0$, h fixed,

$$\|u_t^h\|_1 \le \|u\|_1 + \|u - u_t^h\|_1 \le C,$$

with C independent of t. Hence there exists $\{u_{t_k}^h\}$, a convergent subsequence such that

$$u_{t_k}^h \to \tilde{u}_0^h \quad \text{in} \quad H^1(I).$$

Since $u_t^h \to u_0^h$ in L_2, we must have

$$\tilde{u}_0^h = u_0^h.$$

But this is a contradiction, since $u_0^h \in V_0^h$, satisfying (4.1) will not be continuous in general, i.e., $u_0^h \notin H_0^1(I)$. Hence $\mathcal{F}$ is not robust with any

uniform order. Also, because $F_0(N) \approx N^{-1}$, $\mathcal{F}$ shows locking of order N. □

<u>Remark 4.1</u>. Note that if we choose $E(u) = \|u\|_0$, then $\mathcal{F}$ will be free from locking.

5. <u>The clamped loaded beam</u>.

In this section we look at the robustness of some discretization schemes for a one-dimensional problem involving the Timoshenko beam. This problem (along with various extension processes for it) has been analyzed in the context of locking in [1], [9]. Here, we present a simplified alternate analysis, using our theory, to prove some of the results in these references. We do this by showing that Condition (α) is satisfied, so that Theorems 2.2 (A) and (B) hold. We may therefore restrict attention to the limiting problem which is essentially a one-dimensional biharmonic problem.

The problem we look at is

$$-\phi_t'' + \frac{1}{t}(\phi_t - w_t') = f \quad \text{on} \quad I = (0,1) \tag{5.1}$$

$$\frac{1}{t}(\phi_t - w_t')' = g \quad \text{on} \quad I \tag{5.2}$$

$$\phi_t(0) = \phi_t(1) = w_t(0) = w_t(1) = 0. \tag{5.3}$$

Here, w_t and ϕ_t represent the vertical displacement and rotation, respectively, of the vertical fibers of the beam, which is subject to a vertical body force $-tg(x)$ (f can be related in applications to dislocations or moments). The thickness $\sqrt{t}$ is assumed to lie in the interval $(0,1]$.

We may cast (5.1) - (5.3) into the following variational form. Let $V = H_0^1(I) \times H_0^1(I)$. Then we wish to find $u_t = (\phi_t, w_t) \in V$ satisfying

$$B_t(u_t,v) = b(u_t,v) + \frac{1}{t}(Cu_t,Cv) = F(v) \tag{5.4}$$

for all $v = (\psi,z) \in V$, with $b(u_t,v) = (\phi_t',\psi')$, $Cu_t = \phi_t - w_t'$. Here $(\cdot,\cdot)$ represents the $L_2(I)$ inner product and F is given by

$$F(v) = \langle f,\psi\rangle + \langle g,z\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the duality between $H^1(I)$ and $H^{-1}(I)$.

It is easy to see [1] that

$$\frac{1}{3}\|u_t\|_V^2 \le \|u_t\|_{E,t}^2 \le \left(1+\frac{2}{t}\right)\|u_t\|_V^2 \tag{5.5}$$

with the energy norm being defined as usual. The bilinear form can be put in the form (2.22) if, as in (3.1), we take $\frac{t}{1-t}$ instead of t, giving for $u = (\phi,w)$

$$a(u,u) = b(u,u) + (Cu,Cu) \ge \frac{1}{3}\|u\|_V^2$$

so that (2.23) - (2.24) are satisfied.

In [1] it has been shown that for $f,g \in H^{-1}(I)$ and $0 < t \le 1$, the following a priori estimate holds for $k = 0,1,\ldots,$ for the problem (5.4) (or equivalently, (5.1) - (5.3))

$$\|u_t\|_{k+1} + \frac{1}{t}\|Cu_t\|_k \le C(\|f\|_{k-1}+\|g\|_{k-1}), \tag{5.6}$$

where C is a constant depending only on k and

$$\|u_t\|_{k+1} = \|\phi_t\|_{k+1} + \|w_t\|_{k+1}.$$

Hence a natural choice for the solution set $H_t = H_{t,k+1}$ is to define $H_{t,k+1} \subset H = (H^{k+1}(I) \cap H_0^1(I))^2$ by

$$H_{t,k+1} = \{u \mid \|u\|_{t,k+1} < \infty\} \tag{5.7}$$

with the norm $\|\cdot\|_t = \|\cdot\|_{t,k+1}$ given by

$$\|u\|_{t,k+1} = \|u\|_{k+1} + \frac{1}{t}\|Cu\|_k.$$

Using (5.1) - (5.2), we also have for $k = 0,1,\ldots,$

$$\|f\|_{k-1} + \|g\|_{k-1} \le C\|u_t\|_{t,k+1}. \tag{5.8}$$

From (5.4), we see that as $t \to 0$, the Kirchoff hypothesis

$$Cu = 0 \tag{5.9}$$

gets imposed so that any u_0 in the limiting set H_0^B must satisfy (5.9). Moreover, using (5.6) with $k = 0$, we see that if f,g are held fixed in (5.4) and $t \to 0$, then

$$\|Cu_t\|_0 = O(t).$$

Suppose now we are given a $u_t \in H_{t,k+1}^B$. We show that Condition (α) holds. First it is clear that $\|u\|_{E,t} \le \|u\|_{t,k+1}$ for $k \ge 0$. Define f,g by equations (5.1) - (5.3) so that by (5.8), $\|f\|_{k-1}$, $\|g\|_{k-1} \le C$ and denote by $(u_0,\zeta_0) = (\phi_0,w_0,\zeta_0)$ the solution of

$$\begin{aligned} -\phi_0'' + \zeta_0 &= f \quad \text{on} \quad \tau \\ \zeta_0' &= g \quad \text{on} \quad I \\ \phi_0 - w_0' &= 0 \quad \text{on} \quad I \\ \phi_0(0) = \phi_0(1) &= w_0(0) = w_0(1). \end{aligned} \tag{5.10}$$

(5.10) has the equivalent variational form; find $(u_0,\zeta_0) \in V\times S$ satisfying

$$\begin{aligned} b(u_0,v) + (\zeta_0,Cv) &= f(v) \quad \text{for all} \quad v \in V, \\ (Cu_0,\eta) &= 0 \quad \text{for all} \quad \eta \in S, \end{aligned} \tag{5.11}$$

with $S = L_2(I)$. Then (5.11) has a unique solution which satisfies

(5.12) $$\|u_0\|_{k+1} + \|\zeta_0\|_k \le C(\|f\|_{k-1} + \|g\|_{k-1}).$$

(5.12) is proven in [1] for $k = 0$ and can be obtained for higher k from (5.10) by differentiation. (5.12) shows that $u_0 \in H_0 \cap H_{t,k+1}$. Using (5.8),

(5.13) $$\|u_0\|_{k+1} + \|\zeta_0\|_k \le C\|u_t\|_{t,k+1}.$$

Now by (5.1) and (5.10), we have

(5.14) $$-(\phi_t-\phi_0)'' + \frac{1}{t}C(u_t-u_0) = \zeta_0,$$

which corresponds to (using (5.4) and (5.11))

(5.15) $$b((u_t-u_0),v) + \frac{1}{t}(C(u_t-u_0),Cv) = (\zeta_0,Cv)$$

from which, putting $v = u_t - u_0$, we get

(5.16) $$\|u_t - u_0\|^2_{E,t} \le \frac{t}{2}\|\zeta_0\|^2_0 + \frac{1}{2t}\|C(u_t-u_0)\|^2_0 \le \frac{t}{2}\|\zeta_0\|^2_0 + \frac{1}{2}\|u_t - u_0\|^2_{E,t}.$$

Hence

$$\|u_t - u_0\|^2_{E,t} \le t\|\zeta_0\|^2_0,$$

i.e.,

(5.17) $$\|u_t - u_0\|_{E,t} \le t^{1/2}\|\zeta_0\|_0.$$

Next, differentiating (5.14) and using (5.2), (5,10) we see

$$(\phi_t-\phi_0)''' = 0$$

so that

$$(\phi_t-\phi_0)'' = a_t, \quad (\phi_t-\phi_0)' = a_t x + b_t$$

where a_t, b_t are constants depending on t. Using (5.17),

$$a_t, b_t \le t^{1/2}\|\zeta_0\|_0$$

from which (using (5.1) - (5.3), (5.10) and (5.14)), the following estimate may be established

$$\|u_t - u_0\|_{t,k+1} \le t^{1/2}\|\zeta_0\|_k.$$

Using (5.13), this gives

$$\|u_t - u_0\|_{t,k+1} \le t^{1/2}\|u_t\|_{t,k+1}$$

so that (α) is satisfied. Hence we may take $H^B_{0,k+1}$ to be the set of all u_0's constructed from $u_t \in H^B_{t,k}$ through the equations (5.10).

We will be interested in locking in the $\|\cdot\|_V$ norm. Then by Theorem 2.2, a necessary and sufficient condition for the absence of locking is that for all $u \in H^B_{0,k+1}$,

$$\inf_{\substack{w\in V^N \\ Cw=0}} \|u - w\|_V \le C\, F_0(N) = \inf_{w\in V^N} \|u - w\|_V. \tag{5.18}$$

We will denote by $\{S^p_h\}$ a sequence of C^0 subspaces consisting of piecewise polynomials of degree $\le p$ on a quasi-uniform family of meshes on I.

<u>Theorem 5.1</u>. Consider the problem (5.4). Let $\{V^N_t = (S^p_h)^2\} = \mathcal{F}_t$, where $p \ge 1$ is kept fixed and the h extension is used ($h = h_N$). Let $\mathcal{H} = \{H_t\}$, $H_t = H_{t,k+1}$ be given by (5.7), with $k \ge p$. Let the error measure $E_t(u) = \|u\|_V$. Then the h-version extension procedure $\mathcal{F}$ shows locking of order h_N^{-1}. It is uniformly robust with order h_N^{p-1} for $p > 1$.

<u>Proof</u>. We must estimate the two infimums in (5.18). First we have for $u = (\phi, w) \in H^B_{0,k+1}$,

$$F_0(N) = \inf_{\psi, z\in S^p_h} \{\|\phi - \psi\|_1 + \|w - z\|_1\} \approx O(h_N^p) \tag{5.19}$$

by standard results, since $\phi, w \in H^{k+1}(I)$, $k \ge p$. Next, we estimate for $u = (\phi, w) \in H^B_{t,k+1}$ and $\phi - w' = 0$,

$$A_N = \inf_{\substack{\psi, z \in S_h^p \\ \psi - z' = 0}} \{\|\phi - \psi\|_1 + \|w - z\|_1\}.$$

Obviously,

$$A_N = \inf_{z \in S_h^p} \{\|w' - z'\|_1 + \|w - z\|_1\} \approx \inf_{z \in S_h^p} \|w - z\|_2. \tag{5.20}$$

For $p = 1$, $A_N = O(1)$. For $p \geq 2$, standard spline theory [13] shows that for $k \geq p$,

$$A_N = O(h_N^{p-1}). \tag{5.21}$$

(5.19), (5.21) together with Theorem 2.2 prove the result. □

Let us now consider the p-version.

<u>Theorem 5.2</u>. Consider the problem (5.4). Let $\{V_t^N = (S_h^p)^2\} = \mathcal{F}_t$, where h is fixed and the p-extension $(p = p_N)$ is used. Let $H_t = H_{t,k+1}$, $k > 0$, and $E_t(u) = \|u\|_V$. Then the p-version extension procedure $\mathcal{F}$ is free from locking and uniformly robust with order p_N^{-k}.

<u>Proof</u>. By standard p-version approximation theory we have, for $u = (\phi, w) \in H_{0,k+1}^B$,

$$F_0(N) = \inf_{\psi, z \in S_h^p} \{\|\phi - \psi\|_1 + \|w - z\|_1\} \approx O(p_N^{-k})$$

since $\phi, w \in H^{k+1}(I)$. Next, for $u \in H_{0,k+1}^B$, $Cu = 0$ so that $w' = \phi \in H^{k+1}(I)$, i.e., $w \in H^{k+2}(I)$. Hence

$$A_N = \inf_{z \in S_h^p} \|w - z\|_2 = O(p_N^{-(k+2-2)}) = O(p_N^{-k}).$$

This proves the theorem. □

The results in [1], [9] also indicate what happens when the measure E_t

is changed. We have summarized these results in a table. The set $H_t = H_{t,k+1}$ in each case (with $k \geq p$ for the h-version).

Table 3.1. Locking and robustness for different choices of $\mathcal{F}$ and E_t.

Extension Process $\mathcal{F}$	E_t	Locking	Robustness (Uniform order)
h-version	$\|u\|_V$	Yes	None
$p = 1$	$\|\phi\|_1$	Yes	None
[1]	$\|w\|_1$	Yes	None
	$\|\phi\|_0$	Yes	None
	$\|w\|_0$	Yes	None
h-version	$\|u\|_V$	Yes	$O(h_N^{p-1})$
$p \geq 2$	$\|\phi\|_1$	Yes	$O(h_N^{p-1})$
[1]	$\|w\|_1$	No	$O(h_N^{p})$
	$\|\phi\|_0$	Yes	$O(h_N^{p})$
	$\|w\|_0$	Yes, experimentally	$O(h_N^{p})$
p-version	$\|u\|_V$	No	$O(p_N^{-k})$
[9]	$\|\phi\|_1$	No	$O(p_N^{-k})$
	$\|w\|_1$	No	$O(p_N^{-k})$
	$\|\phi\|_0$	No	$O(p_N^{-(k+1)})$
	$\|w\|_0$	No	$O(p_N^{-(k+1)})$

<u>Remark 5.1</u>. Additional results in [1] deal with the h-version of a mixed formulation for which locking is eliminated. The reference [9] also analyzes an h-p extension procedure which is free from locking.

6. Nearly incompressible materials.

The procedure outlined in the previous section can be used to analyze "Poisson Locking" as well, which occurs in the case of the elasticity equations when the Poisson ratio ν is close to 0.5. More specifically, consider the following problem

$$-\Delta\vec{u}_\nu - \frac{1}{1-2\nu}\,\text{grad div}\,\vec{u}_\nu = 0 \quad \text{in} \quad \Omega \tag{6.1}$$

$$\sum_{j=1}^{2}(\varepsilon_{ij}(\vec{u}_\nu) + \delta_{ij}\frac{\nu}{1-2\nu}\,\text{div}\,\vec{u}_\nu)n_j = g_i, \quad 1 \le i \le 2 \quad \text{on} \quad \Gamma, \tag{6.2}$$

where $\{n_j\}_{j=1}^{2}$ is the unit outward normal to Γ and the strain tensor $\{\varepsilon_{ij}\}$ is given by

$$\varepsilon_{ij}(\vec{u}) = \frac{1}{2}\left(\frac{\partial}{\partial x_i}u_j + \frac{\partial}{\partial x_j}u_i\right).$$

It is assumed that

$$\int_\Gamma \vec{g}\cdot\vec{R}ds = 0$$

for any rigid motion R. Let denote $t = \frac{1-2\nu}{\nu}$ and $\vec{u}_t = \vec{u}_\nu$. As $t \to 0$, the constraint

$$C\vec{u} = \text{div}\,\vec{u} = 0$$

gets enforced. $\vec{u}_t$ converges to a limit $\vec{u}_0$ which satisfies

$$\begin{aligned} -\Delta\vec{u}_0 - \text{grad}\,\zeta_0 &= 0 \quad \text{in} \quad \Omega \\ \text{div}\,\vec{u}_0 &= 0 \quad \text{in} \quad \Omega \\ \sum_{j=1}^{2}\varepsilon_{ij}(\vec{u}_0)n_j + \nu\zeta_0 n_i &= g_i, \quad 1 \le i \le 2 \quad \text{on} \quad \Gamma \end{aligned} \tag{6.3}$$

which is the Stokes' equation, with $-\nu\zeta_0$ being the pressure.

The corresponding variational forms for (6.1) and (6.3) are respectively given by (5.4) and (5.11) with $V = \vec{H}_0^1(\Omega)$, $S = \{\eta \in L^2(\Omega), \int_\Omega \eta dx = 0\}$,

$$b(\vec{u},\vec{v}) = \iint_\Omega \sum_{i,j=1}^{2} \varepsilon_{ij}(\vec{u})\varepsilon_{ij}(\vec{v})dx,$$

$$F(\vec{v}) = \int_\Gamma \vec{g}\cdot\vec{v}ds.$$

If the domain is smooth, we obtain all the usual shift theorems (as in the case of the example in Section 5). Also the argument of equations (5.15) - (5.16) carries over identically, so that (5.17) will again hold. In fact, as we have shown in [5], Condition (α) will hold for this problem, even for certain cases when the domain is not smooth.

As a result, Theorem 2.2 applies again and we have the following result when an appropriate solution set $H_t = H$ is taken.

<u>Theorem 6.1</u>. The extension procedure $\mathcal{F}$ is *robust* with respect to error measure $E_t(\vec{u}) = \|\vec{u}\|_1$, and solution set H for the variational form of (6.1) - (6.2) with uniform order $g(N)$ given by

$$g(N) = \sup_{\substack{\vec{u}\in H^B \\ \operatorname{div}\vec{u}=0}} \inf_{\substack{\vec{w}\in V^N \\ \operatorname{div}\vec{w}=0}} \|\vec{u}-\vec{w}\|_1. \tag{6.4}$$

It is free from locking iff

$$g(N) \le CF_0(N) = C \sup_{\vec{u}\in H^B} \inf_{\vec{w}\in V^N} \|\vec{u}-\vec{w}\|_1 \tag{6.5}$$

and shows locking of order $f(N)$ iff

$$g(N) \approx CF_0(N)f(N).$$

Note that in [6], (6.5) was stated only as a necessary condition, whereas we

show that it is sufficient as well, which simplifies the subsequent analysis. In particular, the above result shows that all the numerical schemes analyzed in [6] which were shown to be robust for the limit problem will also be robust for the sequence of problems given by (6.1) - (6.2).

In [5], we use Theorem 6.1 and some results developed in Section 2 to investigate the robustness of various schemes for nearly incompressible materials. In particular, we analyze the locking that can result through the use of curved elements.

References

[1] Arnold, D.N., Discretization by finite elements of a model parameter dependent problem, Numer. Math. 37 (1981), 405-421.

[2] Babuška, I. and Aziz, A.K., On the angle condition in the finite element method, SIAM J. Numer. Anal. 13 (1976), 214-226.

[3] Babuška, I. and Suri, M., The optimal convergence rate of the p-version of the finite element method, SIAM J. Numer. Anal. 24 (1987), 750-776.

[4] Babuška, I. and Suri, M., The h-p version of the finite element method with quasi-uniform meshes, RAIRO Math. Mod. Num. Anal. 21 (1987), 199-238.

[5] Babuška, I. and Suri, M., Locking effects in the finite element approximation of elasticity problems (to appear)

[6] Brezzi, F. and Bathe, K.J., Studies of finite element procedures – the inf-sup condition, equivalent forms and applications, in Reliability of Methods for Engineering Analysis, K.J. Bathe and D.R.J. Owen (eds.), Pineridge Press, Swansea, U.K., 1986, 197-219.

[7] Brezzi, F., Bathe, K., Fortin, M., Mixed-interpolated elements for Reissner Mindlin plates, Int. J. Num. Met. Eng. 28 (1989), 1787-1801.

[8] Hughes, T.J.R., The Finite Element Method. Linear, Static and Dynamic Finite Element Analysis, Prentice Hall, New Jersey (1987).

[9] Li, L., Discretization of the Timoshenko beam problem by the p and h-p versions of the finite element method, Numer. Math. 57 (1990), 1-8.

[10] MacNeal, R.H. and Harder, R.L., A proposed standard set of problems to test finite element accuracy, J. Finite Elements Anal. Des 1 (1985), 3-20.

[11] Necas J., Les méthodes directes en théorie des equations elliptiques, Academia. Prague (1967).

[12] Pinkus, A., n-widths in Approximation Theory, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo (1985).

[13] Schumaker, K.L., Spline Functions: Basic Theory, John Wiley, New York (1981).

[14] Vogelius, M., An analysis of the p-version of the finite element method for nearly incompressible materials. Uniformly valid, optimal error estimates, Numer. Math. 41 (1983), 39-53.

The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

Further information may be obtained from **Professor I. Babuska**, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.