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ESTIMATING THE ASYMPTOTIC VARIANCE WITH BATCH MEANS

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Abstract

We show that the asymptotic variance constant in a stochastic simulation cannot be estimated consistently from batch means when the number of batches is held fixed as the run length increases.

Keywords: simulation, estimation, batch means, consistency.

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1. Introduction

In this paper ~~we show~~ that there does not exist a procedure to consistently estimate the asymptotic variance constant in a stochastic simulation using batch means when the number of batches is held fixed as the run length increases. Thus, if consistency is desired, then the number of batches must increase as the run length increases.

To be precise, we must first specify what we mean by an estimation procedure. To be interesting, an estimation procedure should apply to a large family of stochastic processes. Hence, let $X \equiv \{X(t) : t \geq 0\}$ be a measurable mapping from a measure space $(\Omega, \mathcal{F})$ into $D \equiv D[0, \infty)$, the space of right-continuous real-valued functions on the interval $[0, \infty)$ with left limits, endowed with the usual Skorohod topology and associated Borel σ-field; e.g., see Ethier and Kurtz [3]. Of course, we want the underlying space $(\Omega, \mathcal{F})$ to be sufficiently rich; it suffices to let $\Omega = D$ and $X(t)$ be the projection or coordinate map. We consider the set $\mathcal{P}$ of all probability measures P on $(\Omega, \mathcal{F})$ such that there exist finite deterministic constants $\mu \equiv \mu(P)$ and $\sigma \equiv \sigma(P)$ such that

$$n^{1/2}\left[\frac{1}{n}\int_0^{[nt]} X(s) - \mu\right] \Rightarrow \sigma B(t) \text{ as } n \to \infty , \tag{1}$$

where $\Rightarrow$ denotes weak convergence in D with respect to P and $B \equiv \{B(t) : t \geq 0\}$ is standard (zero-drift, unit diffusion coefficient) Brownian motion. Our goal is to estimate σ^2, but we want our procedure to apply to all $P \in \mathcal{P}$. In other words, the procedure should apply to all stochastic processes X in D satisfying the functional central limit theorem (FCLT) (1).

To apply the method of batch means, we specify the number m of batches and the total run length T. We then construct our estimates from the m non-overlapping intervals of length T/m; i.e., let the ith batch mean be

$$\bar{X}_i(T) = \frac{m}{T}\int_{(i-1)T/M}^{iT/m} X(s)\,ds \ , \ 1 \le i \le m \ . \tag{2}$$

We now want a procedure for combining the m observations $\bar{X}_1(T), \ldots, \bar{X}_m(T)$ in such a way that σ^2 is consistently estimated as $T \to \infty$. This "combining transformation" should not depend on the "fine structure" of the process X. In particular, it should not depend on μ and σ^2. Thus, in this context we say that an *estimation procedure* is a family of measurable mappings

$$g_T : R^m \to R \quad \text{for } T > 0 \ , \tag{3}$$

such that the estimate of σ^2 is $g_T(x_1, \ldots, x_m)$ when the total run length is T and $\bar{X}_i(t) = x_i$, $1 \le i \le m$. Note that g_T can depend on T, but is independent of P.

We say that an estimation procedure is $\mathcal{P}$*-consistent* if for each $P \in \mathcal{P}$

$$g_T(\bar{X}_1(T), \ldots, \bar{X}_m(T)) \Rightarrow \sigma^2(P) \text{ as } T \to \infty \ . \tag{4}$$

Here $\Rightarrow$ denotes weak convergence with respect to P in R, which is equivalent to convergence in probability since $\sigma^2(P)$ is deterministic. Since we have a negative result, we focus on this weak consistency. We would have strong consistency if the convergence was w.p.1 with respect to P.

Here is our main result. It applies to any m.

Theorem 1. There does not exist an estimation procedure that is $\mathcal{P}$-consistent.

In Section 2 we show what happens with the standard variance estimator. We see that we do not get consistency for σ^2 for any fixed m, but we can get as close as we wish by letting m be suitably large. In Section 3 we prove Theorem 1.

Theorem 1 has applications to sequential stopping. It shows that the sufficient conditions in Glynn and Whitt [5] for asymptotic validity are *not satisfied* when the number of batches is held fixed as the run length grows.

For a fixed run length, our analysis shows that it is desirable to pick the number of batches as large as possible without seriously violating the assumption that the batches are independent and identically distributed (i.i.d.) with a normal distribution. However, the i.i.d. normal assumption typically holds only as an approximation and then only when there are large batch sizes. Statistical tests can be used to validate the assumption, but repeated tests of significance on the same data are fraught with peril, both theoretically and empirically. Hence, Schmeiser [9] suggested using a relatively small fixed number of batches, e.g., about 20. This avoids the complications above and gives relatively robust confidence intervals. However, we show that this is achieved at the expense of consistency for the variance estimator. Asymptotically valid confidence intervals are obtained anyway of course by cancellation methods, i.e., using the t distribution. For further discussion, see Schmeiser [9], Goldsman and Meketon [6], Sargent, Kang and Goldsman [8], Glynn and Iglehart [4] and Damerdji [2].

2. The Standard Estimator

The standard estimation procedure is specified by

$$g_T^*(x_1, \ldots, x_m) = \frac{T}{m(m-1)} \sum_{i=1}^{m} \left[x_i - \frac{1}{m} \sum_{k=1}^{m} x_k \right]^2 \tag{5}$$

for all $T > 0$, $m \geq 2$ and $(x_1, \ldots, x_m) \in R^m$. Let $\stackrel{d}{=}$ denote equality in distribution.

Theorem 2. Under (1),

$$g_T^*(\bar{X}_1(T), \ldots, \bar{X}_m(T)) \Rightarrow \sigma^2 m^2 g_1^*([B(i/m) - B((i-1)/m)]\ ,\ 1 \leq i \leq m)$$
$$\stackrel{d}{=} \frac{\sigma^2 \chi_{m-1}^2}{m-1} \text{ in } R\ ,$$

where χ_{m-1}^2 is a chi-square random variable with $m-1$ degrees of freedom.

Proof. Note that

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$$g_T^*(\bar{X}_1(T), \ldots, \bar{X}_m(T)) = \frac{m}{m-1} \sum_{i=1}^{m} \left[\sqrt{T} \frac{\int_{(i-1)T/m}^{iT/m} X(s)\,ds}{T} - \sqrt{T} \frac{\int_0^T X(s)\,ds}{mT} \right]^2$$

$$\Rightarrow \sigma^2 \frac{m}{m-1} \sum_{i=1}^{m} \left[B(i/m) - B((i-1)/m) - \frac{B(1)}{m} \right]^2 \stackrel{d}{=} \frac{\sigma^2 \chi^2_{m-1}}{m-1} \text{ by (1)} . \quad \blacksquare$$

Note that $\sigma^2 \chi^2_{m-1}/m-1$ has mean σ^2 and variance $2\sigma^4/(m-1)$; see p. 168 of Johnson and Kotz [7]. Moreover, as m increases,

$$\frac{\sigma^2 \chi^2_m}{m} \Rightarrow \sigma^2 \tag{6}$$

and

$$\sqrt{m} \left[\frac{\sigma^2 \chi^2_m}{m} - \sigma^2 \right] \Rightarrow N(0, 2\sigma^4) \tag{7}$$

where $N(m, \sigma^2)$ denotes a normally distributed random variable with mean m and variance σ^2. Hence, we can get as close as we want if we choose m suitably large. Moreover, we can obtain consistency under extra regularity conditions if $m \to \infty$ and $T \to \infty$ so that $T/m \to \infty$; see Goldsman and Meketon [6] and Damerdji [2]. In fact, Damerdji even proves strong consistency for a class of stochastic processes.

3. Proof of Theorem 1

To establish the negative result, it suffices to restrict attention to probability measures P such that X coincides with σB where B is standard Brownian motion. Then, for any m and T,

$$(\bar{X}_1(T), \ldots, \bar{X}_m(T)) = \left[\frac{m}{T} \sigma B(T/m), \ldots, \frac{m}{T} \sigma [B(T) - B(T(m-1)/m)] \right] , \tag{7}$$

so that the batch means are distributed exactly as m i.i.d. normal random variables with mean 0 and variance $\sigma^2 m/T$. Without loss of generality, we can remove the m/T factor by

considering the transformed functions

$$\tilde{g}_T(x_1, \ldots, x_m) = g_T\left[\sqrt{\frac{T}{m}}x_1, \ldots, \sqrt{\frac{T}{m}}x_m\right] . \tag{8}$$

Note that

$$g_T(\bar{X}_1(T), \ldots, \bar{X}_m(T)) \stackrel{d}{=} \tilde{g}_T(\sigma \mathbf{N}) \quad \text{for all } T , \tag{9}$$

where $\sigma \mathbf{N} \equiv (\sigma N_1, \ldots, \sigma N_m)$ and $\mathbf{N}$ is a fixed vector of i.i.d. standard (mean 0, variance 1) normal random variables.

Now consistency requires that

$$\tilde{g}_T(\sigma \mathbf{N}) \Rightarrow \sigma \text{ as } T \to \infty \tag{10}$$

for all $\sigma \geq 0$, but this cannot happen for two or more different positive values of σ, say σ_1 and σ_2. To see this, first note that the convergence in probability for σ_1 in (10) implies that there is a sequence $\{T_n : n \geq 1\}$ of deterministic positive numbers with $T_n \to \infty$ such that

$$\tilde{g}_{T_n}(\sigma_1 \mathbf{N}) \to \sigma_1 \quad \text{w.p.1 as } n \to \infty ; \tag{11}$$

see Theorem 4.2.3 of Chung [1]. By (10), $\tilde{g}_{T_n}(\sigma_2 \mathbf{N}) \Rightarrow \sigma_2$ as $n \to \infty$. Hence, there is a deterministic subsequence $\{T'_n : n \geq 1\}$ of $\{T_n : n \geq 1\}$ such that

$$\tilde{g}_{T'_n}(\sigma_i \mathbf{N}) \to \sigma_i \quad \text{w.p.1 as } n \to \infty \tag{12}$$

for both $i = 1$ and 2. Hence, for $i = 1$ and 2, $\tilde{g}_{T'_n}(x) \to \sigma_i$ for almost all x with respect to the law of $\sigma_i \mathbf{N}$, which implies that $\tilde{g}_{T'_n}(x) \to \sigma_i$ for almost all x with respect to Lebesgue measure on R^m, since $\sigma_i N$ has a positive density with respect to Lebesgue measure. However, it is not possible to have $\tilde{g}_{T'_n}(x)$ simultaneously converge almost everywhere with respect to Lebesgue measure to two different limits. (The set of convergence to one limit must be contained in the null set of non-convergence for the other limit.)

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