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PERTURBED EQUATION

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# SHOCK-LAYER BOUNDS FOR A SINGULARLY PERTURBED EQUATION

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## ABSTRACT

The size of the shock-layer governed by a conservation law is studied. The conservation law is a parabolic reaction-convection-diffusion equation with a small parameter multiplying the diffusion term and convex flux. Rigorous upper and lower bounding functions for the solution of the conservation law are established based on maximum-principle arguments. The bounding functions demonstrate that the size of the shock-layer is proportional to the parameter multiplying the diffusion term

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**1. Introduction.** Upper and lower bounding functions are presented. The bounds demonstrate that solutions to the singularly perturbed hyperbolic partial differential equation

$$(1) \quad P[u] := \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} - \epsilon \frac{\partial^2 u}{\partial x^2} - R(u) = 0$$

with a pre-existing shock have shock-layers of width  $O(\epsilon)$ . The analysis is performed in the style of Howes [4, 5, 6]. It begins with a multiple-scales asymptotic analysis. This provides the appropriate local scalings and indicates candidate forms for a bounding function. A bounding function is constructed from these candidates. Maximum principle arguments are then used to rigorously establish bounds for the solution. In this way, upper and lower bounds for solutions to equation (1) are established for any  $\epsilon$ .

This presentation will concentrate on a comparison between solutions to (1) and solutions to the corresponding reduced equation

$$(2) \quad P_0[U] := \frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} - R(U) = 0$$

obtained by setting  $\epsilon = 0$ . The result will be a bound on the difference between the solution to this reduced equation and the solution to equation (1).

There are implications of this analysis for the computational aspects of the problem as well as for the physics modeled by conservation laws. The bounding functions result in an upper bound on the size of the shock-layer. They isolate the internal-layer region in which viscosity is important from the convection-dominated outer region. This validates the assumption that the lack of resolution of the physics in the shock-layer effects the solution in smooth regions very little under certain circumstances; thus, the hyperbolic equation (2) may be substituted for equation (1).

This work sharpens the bounds of Howes [5, 7] which apply in a more general setting. The shock-birth region will not be studied here.

**2. Problem Specification.** This paper is concerned with presenting bounds for the solution  $u$  to the quasilinear parabolic equation (1) on the domain

$$(3) \quad D := \{(x, t) | 0 \leq x \leq b, 0 \leq t < T\},$$

subject to

$$(4) \quad u(x, 0) = g(x), \quad 0 < x < b;$$

$$(5) \quad u(0, t) = \alpha(t), \quad 0 < t < T; \text{ and}$$

$$(5) \quad u(b, t) = \beta(t), \quad 0 < t < T.$$

Let the portion of the boundary along which the data is specified be denoted by

$$\Pi := \{(x, t) | 0 \leq x \leq b, t = 0\} \cup \{(x, t) | 0 \leq t < T, x = 0, b\}.$$

The boundary and initial data are continuous and sufficiently smooth so that  $u$  is uniquely defined [1]. In particular, the boundary data  $\alpha$  and  $\beta$  have derivatives bounded independent of  $\epsilon$ . Corner-layers in  $u$  are prohibited by assuming the compatibility conditions

$$(7) \quad \alpha(0) = g(0), \quad g(b) = \beta(0)$$

$$(8) \quad \frac{d\alpha}{dt} + \frac{d}{dx}f(g) - R(g) = 0, \quad \text{for } (x, t) = (0, 0);$$

and

$$(9) \quad \frac{d\beta}{dt} + \frac{d}{dx}f(g) - R(g) = 0, \quad \text{for } (x, t) = (b, 0).$$

For simplicity, it is assumed that all boundaries are inflow boundaries; hence,  $\alpha(t)f'(\alpha(t)) > 0$  and  $\beta(t)f'(\beta(t)) < 0$ . Also assume that there is a single shock-layer in the initial data that is contained in an  $O(\epsilon \ln(\epsilon))$  neighborhood  $\pi_0$  of  $(\Gamma_0, 0)$ , where  $\Gamma_0$  is the location of the shock in  $U$  at  $t = 0$ . The domain of the initial viscous-layer is  $\pi_0 := \{(x, t) | t = 0 \text{ and } |x - \Gamma_0| < \epsilon \ln(\epsilon)\}$ .

The solutions to the parabolic problem will be compared to a weak solution of the hyperbolic equation (2). Let  $U$  be the weak solution of (2) with boundary data (5-6) that is the solution to (1) in the limit as  $\epsilon > 0$  tends to zero (denoted as  $\epsilon \downarrow 0$ ). The initial condition will reflect a shock emanating from  $(t, x) = (0, \Gamma_0)$ . Thus, the initial condition for  $U$  is

$$(10) \quad U(x, 0) = g_0(x), \quad 0 < x < b,$$

where the difference  $g - g_0$  is zero except in  $\pi_0$ . The relationship between  $g$  and  $g_0$  in  $\pi_0$  will be discussed in more detail in the proof of Theorem 4.4.

Let the path of the shock in  $U$  be given by the curve  $(x, t) = (\Gamma(t), t)$ . It is natural to describe the values of  $U$  at the shock as

$$U_R(t) = \lim_{x \downarrow \Gamma(t)} U(x, t)$$

and

$$U_L(t) = \lim_{x \uparrow \Gamma(t)} U(x, t).$$

For ease of presentation we will assume that  $U_L > U_R$ . The solution  $U$  to (2) will satisfy the entropy condition

$$(11) \quad U_L(t) > S(t) > U_R(t)$$

where the speed  $S(t)$  of the shock is given by the Rankine-Hugoniot jump condition [8]

$$(12) \quad S(t) = \frac{f(U_L(t)) - f(U_R(t))}{U_L(t) - U_R(t)}.$$

The entropy condition may be written as

$$(13) \quad \mu(t) = U_L(t) - U_R(t) \geq \mu_0,$$

for a constant  $\mu_0 > 0$  that is independent of  $\epsilon$ . It is assumed that the shock is part of the initial data and exists for the entire domain considered; thus,  $\mu(t)$  is defined for all  $t > 0$ . Notice that the initial condition for  $\Gamma$  may be obtained from  $g_0$  and that  $\Gamma$  satisfies the ordinary differential equation  $d\Gamma(t)/dt = S(t)$ . Thus,  $\Gamma$  is uniquely defined.

**3. Asymptotic Analysis.** Some of the relevant physics is presented here. The discussion includes introducing the appropriate scales and equations when equation (1) is used to model a shock-layer. These results are obtained using a heuristic analysis that treats  $\epsilon$  as a small parameter. The results are exploited in the construction of the bounding function and are made more rigorous in Section 4. First some of the properties of the inviscid solution will be discussed.

**3.1. Derivation of Bounding Function.** In this section we will derive a canonical form for the bounding function. The assumptions utilized in the derivation are used only to motivate the equation governing the canonical form. The assumptions are not necessarily valid. The validity of the bounding function will be established when it is used in Section 4.

The canonical form for the solution of the viscous problem is derived here. Large gradients in the neighborhood of a shock-layer are resolved by using the spatial scale  $1/\epsilon$ . This scale is combined with shock-following to obtain the internal-layer coordinates

$$\zeta = \frac{x - \Gamma}{\epsilon} \quad \text{and} \quad \tau = t.$$

The transformation defined by these coordinates is applied to equation (1) to obtain

$$\frac{\partial f(\hat{u})}{\partial \zeta} - S(t) \frac{\partial \hat{u}}{\partial \zeta} - \frac{\partial^2 \hat{u}}{\partial \zeta^2} = \epsilon \left( \frac{\partial \hat{u}}{\partial t} + R(\hat{u}) \right),$$

where  $\hat{u}(\zeta, \tau) = u(x, t)$ . This suggests the regular expansion

$$\hat{u} = \hat{u}_0 + \epsilon \hat{u}_1 + \dots$$

for  $\hat{u}$ . We assume that this expansion is *a priori* valid in the shock layer and use identification in  $\epsilon$  to obtain the equation

$$\frac{\partial f(\hat{u}_0)}{\partial \zeta} - S(t) \frac{\partial \hat{u}_0}{\partial \zeta} - \frac{\partial^2 \hat{u}_0}{\partial \zeta^2} = 0$$

for  $\hat{u}_0$ . This equation is integrated with respect to  $\zeta$  once to obtain

$$f(\hat{u}_0) - S(t)\hat{u}_0 - \frac{\partial \hat{u}_0}{\partial \zeta} = \text{const.}$$

To make this equation easier to solve, we approximate the term  $S - f/\hat{u}_0$  by  $\kappa\zeta^P$ , and we set the constant in the right-hand-side of the above equation to zero. The solution to this equation is the exponential

$$(14) \quad \exp(-\kappa\zeta^{P+1}).$$

Since the magnitude of the bounding function should be decaying away from the shock-layer, we assume  $\kappa > 0$ , and substitute  $|\zeta|$  for  $\zeta$ . This is the candidate form for the bounding function.

Guided by the form (14), let the general form for the bounding function  $\omega$  be

$$(15) \quad \omega = a\sqrt{t} + \theta e^{-K\hat{x}^p}.$$

Here  $\hat{x}(\Gamma(t), x, \epsilon)$  is a linear measure of the distance between  $(x, t)$  and  $(\Gamma(t), t)$ . The power of the distance function is some positive  $p$ , and  $\theta$  is a non-negative function of  $t$ .

**4. Comparison Theorem.** In this section we will show that the function  $\omega + U$  is an upper bound for  $u$ . The statement of this result is in the form of a comparison theorem. Several lemmas needed in the proof of Theorem 4.4 will be presented first. The first lemma is a maximum principle stated in the form most useful for the proof of the theorem. Lemma 4.2 demonstrates how to choose the parameters in  $\omega$  so that the maximum principle is satisfied within each of the regions  $\Omega_0(t) := \{x|0 \leq x \leq \Gamma(t)\}$  and  $\Omega_1(t) := \{x|\Gamma(t) < x\}$ . Finally, we see how the parameters in  $\omega$  can be chosen to satisfy the maximum principle on  $(x, t) = (\Gamma, t)$ .

**4.1. Maximum Principle.** This lemma is a modification of the the Nagumo-Westphal Lemma [11] to include functions that are  $C^\infty$  except on a set of measure zero where they may be only  $C^0$ . A condition on the spatial derivative replaces the condition involving a parabolic operator on this set. This lemma is a direct extension of the result by Nagumo and Westfal; thus, it is presented without proof.

**LEMMA 4.1.** *Let  $z(x, t)$  be a continuous function that is differentiable except on a finite number of curves. Suppose*

$$(16) \quad P[z] \geq 0$$

in the regions where  $z$  is differentiable, while  $z$  satisfies

$$(17) \quad (z)_x^-(X, t) = \lim_{x \rightarrow X} \frac{\partial z(x, t)}{\partial x} > \lim_{x \rightarrow X} \frac{\partial z(x, t)}{\partial x} = (z)_x^+(X, t)$$

for curves  $(x, t) = (X(t), t)$  on which  $z$  is continuous but not differentiable. When these conditions are satisfied along with

$$(18) \quad z \geq u \quad \text{for } (x, t) \in \Pi,$$

then  $z \geq u$  throughout  $D$ .

The implications of this lemma at discontinuities on the choice of the bounding function are demonstrated in Figure 1. The jump in the first derivative with respect to  $x$  is larger on

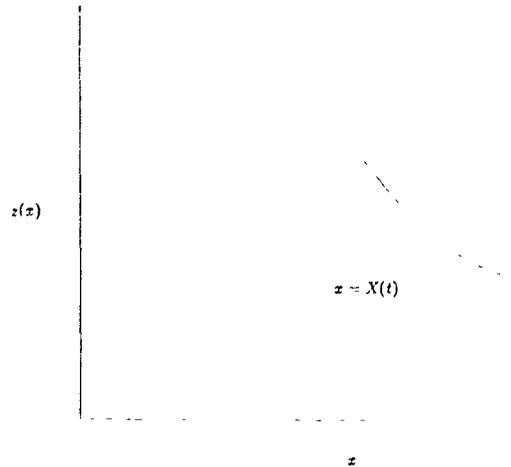


FIG. 1. Discontinuous Upper Bounding Function

the left of  $(x, t) = (X, t)$  than on the right.

**4.2. Analysis of  $\omega$ : Continuous Regions.** The results in this section require a convex flux function. Let  $Q(v) = \frac{\partial f(v)}{\partial v}$ . The assumption of convex flux means

$$Q'(v) > \text{const}$$

for all  $v$  and for a positive constant independent of  $\epsilon$  where  $Q'(v) = \frac{\partial Q(v)}{\partial v}$ . We also assume that  $Q'(v)$  is bounded above.

The shock speed  $S(t)$  and the values of  $Q(u)$  are related. The mean value theorem states that

$$(19) \quad S(t) = Q(\hat{U})$$

for some intermediate function  $\hat{U}(t)$  in the open interval  $]U_R, U_L[$ . We will assume the slightly more restrictive case of  $\hat{U}$  in the closed interval  $[U_R + \Delta, U_L - \Delta]$ , where  $\Delta$  is a positive constant. This relationship between  $S$  and  $Q$  will be exploited in the proof of the following lemma.

LEMMA 4.2. *There is an  $x_1$  close enough to  $\Gamma$  and an  $\epsilon_0$  small enough, such that for each region  $\Omega_i$  we may construct a specific form of  $\omega$  to satisfy inequality (16).*

Proof. First some algebraic details are discussed. Let  $\hat{x} = |x - \Gamma|/\nu$ . Differentiation of  $\omega$  results in

$$\omega_t = a\nu + \left( \theta' + \frac{\theta p K S}{\nu} \hat{x}^{p-1} \right) e^{-K\hat{x}^p},$$

$$\omega_x = -\frac{\theta p K}{\nu} \hat{x}^{p-1} e^{-K\hat{x}^p},$$

and

$$\omega_{xx} = \left( -\frac{\theta p(p-1)K}{\nu} \hat{x}^{p-2} + \frac{\theta p^2 K^2}{\nu^2} \hat{x}^{2(p-1)} \right) e^{-K\hat{x}^p}.$$

We use the mean value theorem to obtain  $f(U + \omega)_x = Q(U + \omega)\omega_x + (Q(U) + \omega Q')U_x$ , where  $Q'$  is evaluated at some function between  $U$  and  $U + \omega$ . The parabolic operator applied to the bounding function is

$$P[U + \omega] = \omega_t + Q\omega_x + Q'\omega U_x - \epsilon U_{xx} - \epsilon \omega_{xx} - R(U + \omega)$$

or

$$P[U + \omega] = \epsilon \tau_1 + \tau_2 + \frac{\epsilon}{\nu^2} \tau_3 + \frac{1}{\nu} \tau_4,$$

where

$$\tau_1 = a - U_{xx} + a\theta Q'U_x,$$

$$\tau_2 = [\theta' + \theta Q U_x] e^{-K\hat{x}^p} - R,$$

$$\tau_3 = -\theta K^2 p^2 \hat{x}^{2(p-1)} e^{-K\hat{x}^p}$$

and

$$\tau_4 = \theta K p [(S - Q)\hat{x}^{p-1} + (p-1)\hat{x}^{p-2}] e^{-K\hat{x}^p}.$$

We will consider two cases based on the two forms of  $\omega$  to verify that (16) is satisfied:

Case I. When  $x \in \Omega_0$  we let  $\omega = \omega_0$ , where

$$\omega_0 = a\epsilon t.$$

We choose  $a$  large enough and  $T$  small enough so that  $\tau_1 > 0$ . Inequality (16) holds in this case since  $\tau_2, \tau_3, \tau_4 = 0$ .

Case II. When  $x \in \Omega_1$  we let  $\omega = \omega_1$ , where

$$\omega_1 = a\epsilon t + \mu e^{-\kappa \hat{x}^2}.$$

Here,  $\delta$  and  $\kappa$  are positive constants independent of  $\epsilon$ , and  $\mu$  is a positive function of  $t$ . The term  $\tau_1$  is positive from the choices in Case I; thus, it is sufficient to consider only  $\tau_2, \tau_3$  and  $\tau_4$ . We choose  $\nu \gg \epsilon$ . For example, we could use  $\nu = \epsilon |\ln(\epsilon)|$ . Then, we assume that  $\epsilon_0$  is small enough that With this choice of  $\nu$ , it is sufficient to show that  $\tau_4$  is bounded below by a positive constant that is independent of  $\epsilon$ .

First we will establish a lower bound for  $S(t) - Q(U + \omega)$ . These properties are based on the fluid dynamics properties of the problem. Namely,  $Q(u)$  is the speed at which the characteristics of  $u$  travel. The characteristics will be traveling faster than the shock for  $x$  to the left of  $\Gamma$ , and slower than the shock for  $x$  to the right of  $\Gamma$ . This means that  $S(t) - Q(U + \omega)$  is positive for  $\hat{x} > b_0(\kappa)$ . The relation between  $\hat{x}$  and  $S - Q$  is depicted in Fig 2. (The precise shape of the curve cannot be known without more information about  $Q$ .) The location of  $b_0(\kappa)$  moves closer to the origin as  $\kappa$  increases. Also, multiplying  $S - Q$  by  $\hat{x}$  reduces the variation near the origin. It is clear that we can choose  $\kappa$  large enough such that

$$1 + (S - Q)\hat{x} > \frac{1}{2}.$$

□

We now have specific forms for the bounding function in each of the domains. Next we must show that we can satisfy inequality (17) and  $C^0$  continuity with  $U + \omega$  with the scaling and  $\kappa$  of the previous lemma.

**4.3. Analysis of  $\omega$ : Discontinuities.** First we will establish the  $C^0$  continuity by choosing the parameters for the bounding function. Condition (18) will also be satisfied by the choices in this lemma.

**LEMMA 4.3.** *We may choose the parameters in  $\omega_1$  to satisfy (18) and such that  $\omega$  is  $C^0$ .*

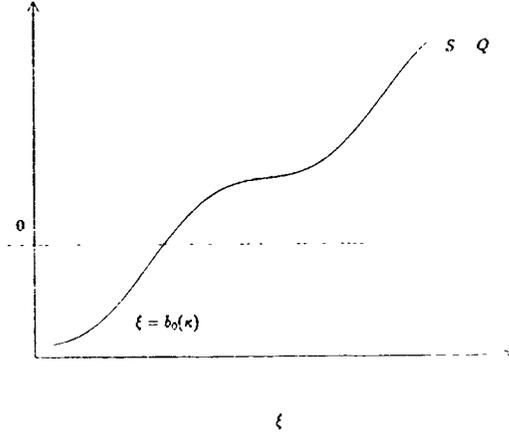


FIG. 2. Example form for S-Q

**Proof.** To obtain  $C^0$  continuity let  $\Theta = \mu$ .

The result will follow providing (18) is satisfied for  $(x, t) \in \Pi$ . This relation clearly holds except possibly for  $(x, t) \in \pi_0$ . In this region it is necessary to impose some restriction on  $g - g_0$ . Assume that the viscous profile of the shock-layer is of an exponential type that is bounded by the function  $\omega$  presented in this proof. This is not a severe restriction, since a layer with a profile like that of  $\tanh[-(x/\epsilon)]$  (a solution to Burgers' equation) is suitable.  $\square$

**THEOREM 4.4.** Assume that  $g$  and  $g_0$  with their first and second derivatives are bounded independent of  $\epsilon$  except that  $g_0$  has a jump at  $(x, t) = (\Gamma_0, 0)$  and  $g$  may have derivatives bounded depending on  $\epsilon$  for  $(x, t) \in \pi_0$ . Suppose  $\omega$  is constructed from  $\omega_0$  and  $\omega_1$  of Lemma 4.2. Also assume that  $U_x$  is continuous across  $(x, t) = (\Gamma, t)$ . Then there is a positive  $\epsilon_0$  such that

$$(20) \quad U + \omega \geq u$$

for  $(x, t) \in D$  when  $\epsilon_0 > \epsilon > 0$ .

**Proof.** We will have only one case to consider based on imposing (17) on the curve  $(x, t) = (\Gamma(t), t)$ . Consider a modified distance function  $\hat{x} = |x - \Gamma + \delta\nu|/\nu$ . Observe that for positive  $\delta$ , the term  $\omega_x$  from equation (4.2) is negative and has magnitude  $O(\delta)$ . If we take  $\delta$  asymptotically close to zero, then we still satisfy this condition. Thus, by taking the limit, we may return to the original basis function since  $\lim_{\delta \rightarrow 0} \hat{x} = \hat{x}$ .  $\square$

**Remark 2.** The constraint on  $U_x$  across  $(x, t) = (\Gamma, t)$  can be relaxed somewhat. This constraint was imposed so that  $\delta$  could be taken asymptotically close to zero in the proof.

With a more careful choice of  $\delta$  this constraint can be eased.

COROLLARY 4.5. *Under the conditions of the theorem,*

$$(21) \quad |u - U| = O\left(\mu \exp\left[-\frac{\kappa|x - \Gamma(t)|}{\nu}\right]\right) + O(\epsilon),$$

when  $\epsilon_0 > \epsilon > 0$  and  $\nu(\epsilon)$  is chosen as above.

This result follows directly using symmetric arguments to obtain a lower bound.

**5. Implications.** A direct result of this theorem is an upper bound on the size of the shock-layer. In this context, the shock-layer is defined as the region in which the solution to (1) differs from the solution to (2) by more than a specified amount. Namely, it is the region in which

$$(22) \quad |u - U| > \sigma$$

for some positive  $\sigma$ . As reflected in the following corollaries, there are different results depending on whether  $\sigma$  is a function of  $\epsilon$  or not. This result follows directly from the theorem. The following corollary extends this result to the case when  $\sigma$  is independent of  $\epsilon$  and is a direct result of Corollary 4.5.

COROLLARY 5.1. *Suppose the conditions of Theorem 4.5 obtain. Let  $(\hat{x}(t), t)$  be the independent variables for which Inequality (22) is satisfied. If  $\sigma$  is a constant independent of  $\epsilon$  then there is an  $\epsilon_0$  small enough so that  $|u - U| < \sigma$  when*

$$|\hat{x}(t) - \Gamma(t)| < O(\nu)$$

for  $\epsilon_0 > \epsilon > 0$ .

When we define the shock-layer as the region such that

$$|u(\hat{x}, t) - U(\hat{x}, t)| > \sigma,$$

then there is a positive  $\epsilon_0$  such that

$$|\hat{x}(t) - \Gamma(t)| < O(\epsilon \ln(\epsilon))$$

for  $\epsilon_0 > \epsilon > 0$ .

Physically motivated domain decomposition algorithms can be based on ideas presented in the analysis discussed herein. The computational domain can be partitioned into subdomains that have different physical behavior. Inside each of these regions different modeling equations (and hence different numerical methods) are used so computational effort is concentrated on the relevant physics [10], [2]. Global error bounds have been developed for these domain decomposition methods [9]. In addition, the general idea of using different modeling equations can be extended to systems of equations [3].

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