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Chapel Hill, North Carolina

INTERACTING HILBERT SPACE VALUED STOCHASTIC DIFFERENTIAL EQUATIONS

AND PROPAGATION OF CHAOS

by

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INTERACTING HILBERT SPACE VALUED STOCHASTIC DIFFERENTIAL EQUATIONS AND PROPAGATION OF CHAOS

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Abstract. Interacting Hilbert space valued stochastic differential equations are studied as an extension of Funaki's model for random strings to a system of interacting strings. The martingale problem for the corresponding McKean-Vlasov equation is solved. Special results when $H = L^2(G)$, G, a bounded domain in $\mathbb{R}^d$ are obtained.

Key words: Hilbert space valued stochastic differential equations, McKean-Vlasov equation, propagation of chaos.

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1. Introduction and notation. In a very interesting paper Funaki has studied a class of Hilbert space valued stochastic differential equations (SDE's) in connection with his investigations of random strings (Funaki [4]). The work of the present paper is a continuation or extension of [4] in the sense that the SDE's considered here may be regarded as an abstract model for interacting random strings. We do not know of specific examples of interacting strings in mechanics or physics to which our results apply. However, a possible application which provided us with our original motivation, is to the asymptotic behavior of voltage potentials of certain models of interacting spatially extended neurons. This application is not considered here since it is briefly discussed in the recent paper of Chiang et al. [1].

The propagation of chaos for interacting particle systems has been investigated in recent years by several authors (see Funaki [5] and the references given there). It is natural, in the context of the present paper, to consider the extension of such results for interacting Hilbert space valued SDE's.

The general notation and plan of the paper are given below.

For a complete separable metric space E, $\mathcal{P}(E)$ stands for the family of probability measures on the Borel sets of E equipped with the topology of weak convergence. For any separable Banach space B, C([0,T],B) denotes the space of B-valued continuous functions on [0,T] with norm $\|x\| := \sup_{0\le t\le T} \|x_t\|_B$. If H is a separable Hilbert space we write $\mathcal{C} := C([0,T],H)$ whenever it is convenient to do so.

We begin by considering an $\mathcal{H}$-valued SDE whose solution will be denoted by $X_\cdot^N = (X_\cdot^{N,1},\dots,X_\cdot^{N,N})$ where $\mathcal{H}$ is a Hilbert space which it is convenient to take as the N-fold direct sum of a basic Hilbert space H. This SDE is our interacting system with mean field interaction. The precise form of the SDE and the conditions on the coefficients will be given in Section 2. The existence of a unique solution is a simple consequence of a result of Dawson [2].

Our main concern is the study of the asymptotic behavior of the sequence of empirical measures $\Gamma^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^N}$, δ_x being the Dirac measure at $x \in \mathscr{C}$. The tightness of $P\circ(X_\cdot^{N,1})^{-1}$ in $\mathscr{P}(\mathscr{C})$ is proved in Theorem 2.1 and the tightness of $P\circ(\Gamma^N)^{-1}$ in $\mathscr{P}(\mathscr{P}(\mathscr{C}))$ is derived as a consequence in Theorem 2.2.

As is to be expected, the propagation of chaos of Γ^N in which we are interested leads to an SDE of the McKean-Vlasov type. The martingale problem for this SDE is introduced in Section 3. Uniqueness of the solution is proved in Theorem 3.2. The final propagation of chaos result is established in Section 4 (Theorem 4.8).

Section 5 is devoted to what we believe to be a new type of result for McKean-Vlasov SDE's. We assume that $H = L^2(G)$ where G is a bounded domain in $\mathbb{R}^d$. In this case we are able to show that there is a propagation of chaos in $C(0,T],C(G))$. (Theorems 5.3 and 5.4). There has been some recent interest in obtaining continuous versions of processes which are solutions of H-valued Ornstein-Uhlenbeck SDE's (e.g. see Iscoe et al. [6]). Theorem 5.4 referred to above, is a similar strengthening of the corresponding propagation of chaos result.

To obtain this stronger result in Section 5 we have had to assume that the eigenvalue λ_n of the operator A (which is throughout assumed to have a discrete spectrum) satisfy the condition $\lambda_n \sim cn^{1+\delta}$ $(c>0,\ \delta>0)$. This limitation is needed also for the continuous versions obtained by Funaki for his equation

[4]. With this restriction, Theorem 5.4 is applicable to the case when the generator of the semigroup T_t (see Section 5) is given by a strongly elliptic operator of order 2m, provided $2m > d$. Unfortunately, the last condition excludes the interesting case of the Laplacian in three dimensions. We believe, however, that a different approach to the problem based on the use of the Galerkin approximation might yield more general results in this direction. On the other hand, it is likely that such an approach would require the enlargement of the Hilbert space to some space of distributions and bring the results more in line with those of [1]. Our aim in this paper has been to obtain our results in $C([0,T],H)$ itself and to see under what conditions propagation of chaos takes place in the space of continuous functions $C([0,T]\times G)$ when H is taken to be $L^2(G)$.

2. Interacting systems of H-valued SDE's and tightness of Γ^n.

Let H be a separable Hilbert space and let A be a self-adjoint, non-negative operator on H with dense domain. Suppose that A satisfies condition (A.1) in the appendix, hence in particular for some $\theta<1$,

$$A^{-\theta} \text{ is nuclear.} \tag{2.1}$$

Let $\{\lambda_k\}$ be the eigenvalues of A and $\{\phi_k\}$ be the corresponding eigenvectors. Then $\{\phi_k\}$ forms a CONS. Let $T_t := e^{-tA}$ be the semigroup acting on H.

Let $W_t^1,\ldots,W_t^N$ be N-independent cylindrical Brownian motions on H (defined on some complete probability space $(\Omega,\mathcal{F},P)$).

Consider the following equation for an interacting system X_t^N with N-components $X_t^N = (X_t^{N,1},X_t^{N,2},\ldots,X_t^{N,N})$

$$dX_t^{N,i} = -AX_t^{N,i}dt + b(t,X_t^{N,i})dW_t^i + a(t,X_t^{N,i})dt + \frac{1}{N}\sum_{j=1}^{N} I(X_t^{N,i},X_t^{N,j})dt \tag{2.2}$$

where $b:[0,T]\times H \longrightarrow L(H)$, $a:[0,T]\times H \longrightarrow H$ and $I:H\times H \longrightarrow H$ are continuous mappings which satisfy

$$\|b(t,h)\phi_k\| \le C_{2.1} \tag{2.3}$$

$$\|a(t,h)\| \le C_{2.1}(1+\|h\|) \tag{2.4}$$

$$\|I(h_1,h_2)\| \le C_{2.1}(\|h_1\| + \|h_2\|) \tag{2.5}$$

$$\|b(t,h_1)\phi_k-b(t,h_2)\phi_k\| \le C_{2.2}(\|h_1-h_2\|) \tag{2.6}$$

$$\|a(t,h_1)-a(t,h_2)\| \le C_{2.2}(\|h_1-h_2\|) \tag{2.7}$$

$$\|I(h_1,h_1') - I(h_2,h_2')\| \le C_{2.2}\{\|h_1-h_2\| + \|h_1'-h_2'\|\}. \tag{2.8}$$

for some constants $C_{2.1}$, $C_{2.2}$ and $h,h_1,h_2,h_1',h_2' \in H$, $t \in [0,T]$.

Letting $\mathcal{H}$ denote N-fold direct sum of H, define

$$L((h_1,\dots,h_N)) = (Ah_1,\dots,Ah_N).$$

It is easily checked that L satisfies the condition imposed in the Appendix. $W_t := (W_t',\dots,W_t^N)$ becomes an $\mathcal{H}$-valued cylindrical Brownian motion and (2.2) can be written as

$$dX_t^N = -LX_t^N dt + \beta(t,X_t^N)dW_t + \alpha(t,X_t^N)dt$$

for appropriate β,α. It can be checked that β,α satisfy conditions (A.5)-(A.8). Thus we have that (2.2) admits a unique mild solution, with paths belonging to $C([0,T],\mathcal{H})$ (see Appendix). Let us recall that by a mild solution to (2.2), we mean that

(2.9) $X_t^{N,i}$

$$= T_t X_0^{N,i} + \int_0^t T_{t-s} b(s, X_s^{N,i}) dW_s^i + \int_0^t T_{t-s} a(s, X_s^{N,i}) ds + \frac{1}{N} \sum_{j=1}^{N} \int_0^t T_{t-s} I(X_s^{N,i}, X_s^{N,j}) ds.$$

Here, $X^{N,i} \in C([0,T],H)$. The following estimate on moments can be proved exactly as in Theorem A.3. Let

$$C_{2.3} := \sum_{k=1}^{\infty} \frac{1}{\lambda_k}$$

($C_{2.3} < \infty$ in view of (2.1)). Proceeding as in the proof of Theorem A.3, we can show that for constants C_p', C_p'' depending only on p, we get

$$E\|X_t^{N,i}\|^p \le C_p' E\|X_0^{N,i}\|^p + C_p'' C_{2.3} E\int_0^t (1+\|X_s^{N,i}\|^p + \frac{1}{N} \sum_{j=1}^{N} \|(X^{N,j}\|^p) ds$$

Summing over i, we get

$$\sum_{i=1}^{N} E\|X_t^{N,i}\|^p \le \sum_{i=1}^{N} C_p' E\|X_0^{N,i}\|^p + 2C_p'' C_{2.1}^p C_{2.3} \sum_{i=1}^{N} E\int_0^t (1+\|X_s^{N,i}\|^p) ds.$$

We can justify the use of Gronwall's lemma as in the proof of Theorem A.3, and thus get

$$\sum_{i=1}^{N} (1+E\|X_t^{N,i}\|^p) \le C_p' \exp\{2C_p'' C_{2.1}^p C_{2.3} T\} \sum_{i=1}^{N} (1+E\|X_0^{N,i}\|^p).$$

So we get for a constant $C_{2.4}$ (not depending on N)

$$\text{(2.10)} \qquad \sup_{t\le T} E\|X_t^{N,i}\|^p \le C_{2.4}\{1+E\|X_0^{N,i}\|^p + \frac{1}{N} \sum_{i=1}^{N} E\|X_0^{N,i}\|^p\}.$$

In particular, for p=2, we have

$$\text{(2.10)}' \qquad \sup_{t\le T} E\|X_t^{N,1}\|^2 \le C_{2.4}'\{1+E\|X_0^{N,1}\|^2 + \frac{1}{N} \sum_{i=1}^{N} E\|X_0^{N,i}\|^2\}.$$

We will fix a sequence of initial r.v.'s $X_0^N = (X_0^{N,1},\dots,X_0^{N,N})$ for the interacting system with N-components satisfying the following conditions:

(2.11) The law of $(X_0^{N,1},\dots,X_0^{N,N})$ is a symmetric measure on $H\times\dots\times H$.

$$(2.12) \qquad \nu_0^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_0^{N,i}} \longrightarrow \mu_0 \text{ in } \mathcal{P}(H), \text{ in probability.}$$

(2.13) There exists a constant $C_{2.5}$ such that $E\|X_0^{N,i}\|^2 \le C_{2.5}$ for all N.

Our problem is to investigate the asymptotics of

$$(2.14) \qquad \Gamma^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_\cdot^{N,i}} \in \mathcal{P}(\mathcal{C}).$$

We will prove that Γ^N converges in distribution to $\Gamma \in \mathcal{P}(\mathcal{C})$ where Γ is non-random.

We first prove

<u>Theorem 2.1</u>: Assume (2.1), (2.3)-(2.8) and (2.11)-(2.13). Then $P\circ(X_1^{N,1})^{-1}$ is tight (as elements of $\mathcal{P}(\mathcal{C})$).

<u>Proof</u>: Let us write

$$X_t^{N,1} = T_t X_0^{N,1} + V_t^N$$

where

$$V_t^N = \int_0^t T_{t-s} b(s,X_s^{N,1})dW_s + \int_0^t T_{t-s} a(s,X_s^{N,1})ds + \frac{1}{N}\sum_{j=1}^{N}\int_0^t T_{t-s} I(X_s^{N,1},X_j^{N,j})ds$$

$$= V_t^{N,1} + V_t^{N,2} + V_t^{N,3}, \text{ say.}$$

First note that (2.11), (2.12) imply that $P\circ(X_0^{N,1})^{-1}$ converges in $\mathcal{P}(H)$ to μ_0 and hence that $P\circ(X_0^{N,1})^{-1}$ is tight. From this it follows that $\mathcal{L}(TX_0^{N,1})$ is tight in $P(\mathcal{C})$ since $h_n \longrightarrow h$ in H implies $T_t h_n \longrightarrow T_t h$ uniformly in t. By construction, $V_1^N \in \mathscr{C}$. Thus to verify tightness of $\mathscr{L}(V_\cdot^N)$ in $\mathscr{P}(\mathscr{C})$, it suffices to prove (see Ethier-Kurtz, [3])

(2.15) For all $\epsilon > 0$, $0 \le t \le T$, $\exists\, K_{\epsilon,t} \subseteq H$, compact, such that

$$P(V_t^N \in K_{\epsilon,t}) \ge 1-\epsilon$$

(2.16) For $0<\delta<1$, $\exists$ r.v. $Z_N(\delta)$ such that $0\le t\le\delta$, $0\le s\le T$, $s+t\le T$

$$E[\|V_{t+s}^N - V_s^N\|^2 | \mathscr{F}_s^N] \le E[Z_N(\delta) | \mathscr{F}_s^N]$$

and

$$\lim_{\delta\downarrow 0} [\sup_N E[Z_N(\delta)] = 0.$$

Since A^{-r} is a compact operator for all $r>0$, (2.15) will follow if we prove that for some $r>0$

(2.17) $$E\|A^r V_t^N\|^2 \le C_{2.6}.$$

Then we can take $K_{\epsilon,t} = \{h \in H\colon \|A^r h\|^2 \le \frac{1}{\epsilon} \cdot C_{2.6}\}$. $K_{\epsilon,t}$ is compact as A^{-r} is compact and (2.17) implies

$$P(V_t^N \notin K_{\epsilon,t}) \le \frac{\epsilon}{C_{2.6}} E\|A^r V_t^N\|^2 \le \epsilon.$$

Now

(2.18) $$\begin{aligned} E\|A^r V_t^{N,1}\|^2 &= E\int_0^t \|A^r T_{t-s} b(s, X_s^{N,1})\|_{H\cdot S}^2\, ds \\ &\le C_{2.1} \int_0^t \|A^r T_{t-s}\|_{H.s}^2\, ds \\ &= C_{2.1} \sum_{k=1}^{\infty} \int_0^t \lambda_k^{2r} e^{-2\lambda_k(t-s)} ds \\ &\le C_{2.1} \sum_{k=1}^{\infty} \lambda_k^{2r} \cdot (2\lambda_k)^{-1} \\ &:= C_{2.7} \end{aligned}$$

and $C_{2.u} < \infty$ if $1-2r > \theta$. So we fix $r := \frac{1-\theta}{3}$.

Note that (2.11)-(2.13) imply $E\|X_0^{N,j}\|^2 \le C_{2.5}$ and hence in view of (2.10).

$$(2.19) \qquad \sup_{t\le T} E[\|X_t^{N,i}\|^2] \le 2C_{2.4}C_{2.5} := C_{2.8}.$$

Proceeding as in (2.18), one can show that for i=2,3.

$$(2.20) \qquad E\|A^r V_t^{N,i}\|^2 \le eC_{2.7}C_{2.8}C_{2.1}^2.$$

(2.18) and (2.20) imply (2.17).

For (2.16), fix $0<\delta<1$, $0\le t\le\delta$, $0\le s\le s+t\le T$. Then

$$(2.21) \quad E[\|V_{t+s}^{N,1}-V_s^{N,1}\|^2|\mathcal{F}_s^N] = E[\int_0^{t+s}\|T_{t+s-r}b(r,X_r^{N,1})-T_{s-r}b(r,X_r^{N,1})1_{(r\le s)}\|_{H\cdot S}^2 dr|\mathcal{F}_s^N]$$

$$\le C_{2.1}^2\{\int_s^{t+s}\|T_{t+s-r}\|_{H\cdot S}^2 dr + \int_0^s\|T_{t+s-r}-T_{s-r}\|_{H\cdot S}^2 dr$$

$$\le C_{2.1}^2\sigma(\delta)$$

where

$$\sigma(\delta) = \int_0^\delta\|Tr\|_{H\cdot S}^2 dr + \sup_{t\le\delta}\int_0^T \|T_{t+s} - T_r\|_{H\cdot S}^2 dr.$$

On the other hand

$$(2.22) \qquad \|V_{t+s}^{N,2}-V_s^{N,2}\|^2 \le \{\int_0^{t+s}\|T_{t+s-r}-T_{s-r}\cdot 1_{(s\le r)}\|_{H\cdot S}\|a(r,X_r^{N,s})\|dr\}^2$$

$$\le C_{2.1}^2\sigma(\delta) \cdot \int_0^T(1+\|X_r^{N,1}\|^2)dr.$$

Similarly

$$(2.23) \qquad \|V_{t+s}^{N,3}-V_s^{N,3}\|^2 \le 2C_{2.1}^2\sigma(\delta)\int_0^T(1+\|X_r^{N,1}+\frac{1}{N}\sum_{j=1}^N\|X_r^{N,j}\|^2)dr.$$

Putting together (2.21)-(2.23), we get that

$$E[\|V_{t+s}^N-V_s^N\|^2|\mathcal{F}_s^N] \le E[U^{N,1}(\delta)|\mathcal{F}_s^N]$$

where

$$U^{N,1}(\delta) = 9C_{2.1}^2\sigma(\delta)\int_0^T(1+\|X_r^{N,1}\|^2 + \frac{1}{N}\sum_{j=1}^{N}\|X_r^{N,j}\|^2)dr. \tag{2.24}$$

Now

$$EU^{N,1}(\delta) = 9C_{2.1}^2 T(1+2C_{2.8})\cdot\sigma(\delta) := C_{2.9}\sigma(\delta)$$

and thus to prove (2.16), it remains to show that $\sigma(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. Now $\int_0^\delta \|Tr\|_{H\cdot S}^2 ds \longrightarrow 0$ as $\delta \longrightarrow 0$ follows from the fact that

$$\int_0^\delta \|Tr\|_{H\cdot S}^2 dr \le \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty.$$

For the second term,

$$\begin{aligned}\sup_{t\le\delta} \int_0^T \|T_{t+r}-T_r\|_{H\cdot S}^2 dr &= \sup_{t\le\delta} \int_0^T \sum_k \{e^{-\lambda_k(t+r)} - e^{-\lambda_k r}\}^2 dr \\ &\le \sum_k (e^{-\lambda_k\delta}-1)^2 \int_0^T e^{-\lambda_k r} dr \\ &\le \sum_k (e^{-\lambda_k\delta}-1)^2 \cdot \frac{1}{\lambda_k} \\ &\longrightarrow 0\end{aligned}$$

as $\delta \longrightarrow 0$ by the Dominated Convergence Theorem, as $\Sigma\lambda_k^{-1} < \infty$. This completes the proof of (2.16) and hence it follows that $\mathcal{L}(X_\cdot^{N,1})$ is tight in $\mathcal{P}(\mathcal{C})$. □

As a consequence we have

<u>Theorem 2.2</u>: Assume (2.1), (2.3)-(2.8) and (2.11)-(2.13). Then

$$P\cdot(\Gamma^N)^{-1} \text{ is tight in } \mathcal{P}(\mathcal{P}(\mathcal{C})).$$

<u>Proof</u>: For each $\epsilon > 0$, let k_ϵ be a compact set in $\mathcal{C}$ such that

$$P(X_{\bullet}^{N,1} \in K_{\epsilon}^{c}) \le \epsilon^2.$$

Such a K_ϵ exists as $P\circ(X_{\bullet}^{N,1})^{-1}$ is tight in $\mathcal{P}(\mathcal{C})$. Then

$$P(\Gamma^N(K_\epsilon^c) > \epsilon) \le \frac{1}{\epsilon} E\Gamma^N(K_\epsilon^c)$$

$$= \frac{1}{\epsilon} \cdot \frac{1}{N} \sum_{i=1}^{N} P(X_i^{N,i} \in K_\epsilon^c)$$

$$\le \frac{1}{\epsilon} \cdot \epsilon^2 = \epsilon.$$

Let $\mathcal{K}_\epsilon = \{\Lambda \in \mathcal{P}(\mathcal{C}): \Lambda(K_{\epsilon 2^{-m}}) \ge 1-\epsilon 2^{-m} \ \forall\ m\ge 1\}$. Then $\mathcal{K}_\epsilon$ is compact in $\mathcal{P}(\mathcal{C})$ and

$$P(\Gamma^N \notin \mathcal{K}_\epsilon) \le \sum_{m=1}^{\infty} P(\Gamma^N(K^c_{\epsilon 2^{-m}}) > \epsilon 2^{-m})$$

$$\le \sum_{m=1}^{\infty} \epsilon 2^{-m} = \epsilon. \qquad \square$$

3. Martingale problem for the McKean-Vlasov equation.

For $\mu_1, \mu_2 \in \mathcal{P}(H)$, let $M(\mu_1,\mu_2)$ denote the class of probability measures λ on $H\times H$ with $\lambda(E\times H) = \mu_1(E)$ and $\lambda(H\times E) = \mu_2(E)$ for all Borel sets E in H. For $p>0$, let $\mathcal{P}_p(H) := \{\mu \in \mathcal{P}(H): \int \|h\|^p \mu(dh) < \infty\}$. Let $\rho_p : \mathcal{P}_p(H)\times\mathcal{P}_p(H) \longrightarrow [0,\infty)$ be defined by

$$\rho_p^p(\mu_1,\mu_2) = \inf\{ \int_{H\times H} \|h_1-h_2\|^p \lambda(dh_1 dh_2) : \lambda \in M(\mu_1,\mu_2)\}. \tag{3.1}$$

$\mathcal{P}_p(H)$ is a metric space with the metric ρ_p.

Let $\hat{I}: H\times\mathcal{P}_1(H) \longrightarrow H$ be defined by

$$\hat{I}(h,\mu) = \int I(h,h')\mu(dh'). \tag{3.2}$$

Then using (2.8), we can deduce

$$\|\hat{I}(h_1,\mu_1)-\hat{I}(h_2,\mu_2)\| \le C_{2.2}\{\|h_1-h_2\| + \rho_1(\mu_1,\mu_2)\}. \tag{3.3}$$

For a continuous function $t \longrightarrow \mu_t$ from $[0,t]$ into $\mathcal{P}_1(H)$, consider the equation

$$(3.4) \qquad dZ_t = -AZ_t dt + b(t,Z_t)dW_t + a(t,Z_t)dt + \hat{I}(Z_t,\mu_t)dt$$

with $E\|Z_0\|^2 < \infty$ where b,a,I are assumed to satisfy (2.3)-(2.8). Then with $\mathcal{H}=H$, $L=A$, $\beta=b$, $\alpha(t,h) = a(t,h) + \hat{I}(h,\mu_t)$, the conditions in the appendix. Thus we have that the equation (3.4) has a unique solution (Z_t) with $Z.\in C([0,T],H)$, $\sup_{t\in T} E\|Z_t\|^2 < \infty$. Further, the law of $(Z_\cdot)$ is uniquely determined by $b,a,I,(\mu_t),\mathcal{L}(Z_0)$.

Let us note that if Z is a process with paths in $C([0,T],H)$ with $\sup_{t\in T} E\|Z_t\|^p < \infty$, for $p>1$, then $t \longrightarrow \nu_t := \mathcal{L}(Z_t)$ is continuous from $[0,T] \longrightarrow \mathcal{P}_1(H)$. This follows from

$$\rho_1(\nu_s,\nu_t) \leq E\|Z_t-Z_s\|$$

and $Z_s \longrightarrow Z_t$ as $s \longrightarrow t$ pointwise and $\|Z_s-Z_t\|$ is uniformly integrable since $\sup_{t\leq T} E\|Z_t\|^p < \infty$.

Let us now consider the McKean-Vlasov equation

$$(3.5) \qquad dZ_t = -AZ_t dt + b(t,Z_t)dW_t + a(t,Z_t)dt + \hat{I}(Z_t,\mathcal{L}(Z_t))dt.$$

A process (Z_t) is said to be a solution to (3.5) if $Z_\cdot \in C([0,T],H)$,

$$(3.6) \qquad t \longrightarrow \mathcal{L}(Z_t) \text{ is a continuous function from } [0,T] \longrightarrow \mathcal{P}_1(H)$$

and (Z_t) is a solution to (3.4) with $\mu_t = \mathcal{L}(Z_t)$.

In the next section, we will prove that for any $\mu_0 \in \mathcal{P}_2(H)$, there exists a solution (Z_t) to (3.5) with $\mathcal{L}(Z_0) = \mu_0$. We will now prove uniqueness.

<u>Theorem 3.1</u>: Let $\mu_0 \in \mathcal{P}_2(H)$. Let (Z_t^1), (Z_t^2) be solutions to the McKean-Vlasov

equation (3.5) with $\mathcal{L}(Z_0^1)=\mathcal{L}(Z_0^2) = \mu_0$. Then $\mathcal{L}(Z_\cdot^1) = \mathcal{L}(Z_\cdot^2)$ (these are measures on $\mathcal{C} = C([0,T],H)$).

<u>Proof:</u> Let $\mu_t^1 = \mathcal{L}(Z_t^1)$, $\mu_t^2 = \mathcal{L}(Z_t^2)$. Then we have

$$dZ_t^i = -AZ_t^i dt + b(t,Z_t^i)dW_t^i + a(t,Z_t^i)dt + \hat{I}(Z_t^i,\mu_t^i)dt \tag{3.7}$$

where W^1, W^2 are cylindrical Brownian motions. Take another probability space with a cylindrical Brownian motion (W_t) and a.r.v. $\tilde{Z}_0$ with $\mathcal{L}(\tilde{Z}_0) = \mu_0$, $\tilde{Z}_0$ independent of (W_t). Let $(\tilde{Z}_t^i)$ be the solution to

$$d\tilde{Z}_t^i = -A\tilde{Z}_t^i dt + b(t,\tilde{Z}_t^i)dW_t + a(t,\tilde{Z}_t^i)dt + \hat{I}(\tilde{Z}_t^i,\mu_t^i)dt \tag{3.8}$$

with $\tilde{Z}_0^i = \tilde{Z}_0$. Since equation (3.7) admits a unique solution in law, it follows that $\mathcal{L}(\tilde{Z}_\cdot^i) = \mathcal{L}(Z_\cdot^i)$. So to complete the proof, it suffices to prove $\tilde{Z}_\cdot^1 = \tilde{Z}_\cdot^2$ a.s., and since these are continuous processes, that $E\|\tilde{Z}_t^1 = \tilde{Z}_t^2\|^2 = 0$ for all t. Now we have (see Appendix)

$$\tilde{Z}_t^i = S_t\tilde{Z}_0 + \int_0^t S_{t-s} b(t,\tilde{Z}_t^i)dW_t + \int_0^t S_{t-s} a(t,\tilde{Z}_t^i)dt + \int_0^t S_{t-s}\hat{I}(\tilde{Z}_t^i,\mu_t^i)dt$$

and hence, using (2.6), (2.7) and (3.3),

$$E\|Z_t^1 - Z_t^2\|^2 \le 3C_{2,2}\{\int_0^t \|S_{t-s}\|_{H.S}^2\{4E\|\tilde{Z}_s^1 = \tilde{Z}_s^2\|^2 + 2\rho_1^2(\mu_s^1,\mu_s^2)\}ds\}. \tag{3.9}$$

Note that

$$\rho_1^2(\mu_s^1,\mu_s^2) \le \rho_2(\mu_s^1,\mu_s^2) \le E\|\tilde{Z}_s^1-\tilde{Z}_s^2\|^2.$$

since $\mathcal{L}(\tilde{Z}_s^i) = \mathcal{L}(\tilde{Z}_s^i) = \mu_s^i$. Thus

$$E\|\tilde{Z}_t^1-\tilde{Z}_t^2\|^2 \le 18C_{2,2}\int_0^t \|S_{t-s}\|^2 E\|\tilde{Z}_s^1-\tilde{Z}_s^2\|^2 ds. \tag{3.10}$$

Since $\sup_{t\le T} E\|Z_t^i\| < \infty$ and $\int_0^T \|S_{t-s}\|^2 ds < \infty$, Remark A.2 yields $E\|\tilde{Z}_t^1-\tilde{Z}_t^2\|^2 \equiv 0$.

This completes the proof. □

Martingale problem.

For $f \in C_0^2(\mathbb{R}^n)$, let $U_n f\colon H \longrightarrow \mathbb{R}$ be defined by

$$(U_n f)(h) = f((h,\phi_1),\dots,(h,\phi_n)) \tag{3.11}$$

and let $\mathcal{D} = \{U_n f\colon f \in C_0^2(\mathbb{R}^n),\ n \geq 1\}$. Let $\mathcal{A}_t$ be defined by

$$\mathcal{A}_t(U_n f)(h) := \frac{1}{2}\sum_{ij=1}^{n}(b^*(t,h)\phi_i, b^*(t,h)\phi_j)(U_n f_{ij})(h) \tag{3.12}$$

$$+ \sum_{i=1}^{n}(\alpha(t,h) - \lambda_i h,\psi_i)(U_n f_i)(h)$$

where $f_i = \frac{\partial}{\partial x_i} f$, $f_{ij} = \frac{\partial}{\partial x_j} f_i$. Let

$$\mathcal{G}(U_n f)(h_1,h_2) = \sum_{i=1}^{n}(I(h_1,h_2),\phi_i)f_i(h_1) \tag{3.13}$$

and

$$\hat{\mathcal{G}}(U_n f)(h_1,\mu) = \int_H \mathcal{G}(U_n f)(h_1,h_2)\mu(dh_2) \tag{3.14}$$

for $h_1,h_2 \in H$, $\mu \in \mathcal{P}_1(H)$. As noted in the Appendix, if (Z_t) is a solution to the McKean-Vlasov equation (3.5), then for all $g \in \mathcal{D}$

$$g(Z_t) - \int_0^t \mathcal{A}_s g(Z_s)ds - \int_0^t \hat{\mathcal{G}} g(Z_s,\mu_s)ds$$

is a martingale with $\mu_s = \mathcal{L}(Z_s)$. This leads us to the following definition. Let $(\hat{Z}_t)$ be the canonical process on $\mathcal{C} = C([0,T],H)$.

A probability measure Λ on $\mathcal{C}$ is said to be a solution to the McKean-Vlasov martingale problem if

$$t \longrightarrow \Lambda\circ\hat{Z}_t^{-1} \text{ is continuous from } [0,T] \longrightarrow \mathcal{P}_1(H). \tag{3.15}$$

and

$$g(\hat{Z}_t) - \int_0^t \mathscr{A}_s(\hat{Z}_s)ds - \int_0^t \hat{\mathscr{F}}g(Z_s, \Lambda \circ Z_s^{-1})ds \tag{3.16}$$

is a Λ-martingale for all $g \in \mathscr{D}$.

It follows from Theorem A.2 that Λ is a solution to the McKean-Vlasov martingale problem if and only if Λ is the distribution of a solution to the McKean-Vlasov equation (3.5). These remarks and Theorem 3.1 lead us to

Theorem 3.2. Let $\mu_0 \in \mathscr{P}_2(H)$. Let Λ^1, Λ^2 be solutions to the McKean-Vlasov martingale problem with

$$\Lambda^1 \circ (\hat{Z}_0)^{-1} = \Lambda^2 \circ (\hat{Z}_0)^{-1} = \mu_0.$$

Then $\Lambda^1 = \Lambda^2$.

Remark 3.1: We can choose a countable subset $\mathscr{D}_0 \subseteq \mathscr{D}$ s.t. (3.16) is a martingale for all $g \in \mathscr{D}_0$ implies the martingale property for all $g \in \mathscr{D}$. Indeed, for each n, let E_n be a countable dense subset in $C_0^2(\mathbb{R}^n)$ and let $\mathscr{D}_0 = \{U_n f: f \in E_n, n \geq 1\}$.

Let $\hat{\mathscr{P}}(\mathscr{C})$ be the class of $\Lambda \in \mathscr{P}(\mathscr{C})$ such that (3.15) holds.

It is easy to see that for $g \in \mathscr{D}_0$, (3.16) is a martingale under $\Lambda \in \hat{\mathscr{P}}(\mathscr{C})$ if and only if

(3.17)

$$F(\Lambda) := \int_{\mathscr{C}} [g(\hat{Z}_t) - g(\hat{Z}_s) - \int_s^t \mathscr{A}_u g(\hat{Z}_u)du - \int_s^t \hat{\mathscr{F}}g(\hat{Z}_u, \Lambda \circ \hat{Z}_u^{-1})du] g_1(\hat{Z}_{r_1}) \dots g_m(\hat{Z}_{r_m}) d\Lambda$$

is zero for all $r_1 \leq r_2 \leq \dots \leq r_m \leq s \leq t$, $g_1, \dots, g_m \in \mathscr{D}_0$, $m \geq 1$. Also we can restrict $r_1, \dots, r_m, s, t$ to rationals.

Let $\mathscr{E}$ be the class of functionals $F: \hat{\mathscr{P}}(\mathscr{C}) \longrightarrow \mathscr{R}$ defined by (3.17), for $g, g_1, \dots, g_m \in \mathscr{D}_0$, $r_1 \leq \dots \leq r_m \leq s \leq t$ rationals. Then $\mathscr{E}$ is a countable class, and we have

Remark 3.2: $\Lambda \in \hat{P}(\mathscr{C})$ is a solution to the McKean-Vlasov martingale problem iff $F(\Lambda) = 0$ for all $F \in \mathscr{E}$.

4. Propagation of chaos in $C([0,T],H)$

We return to the setup of Section 2. We assume conditions (2.3)-(2.8) and (2.11)-(2.13) throughout this section. Thus by Theorem 2.2, $P\circ(\Gamma^N)^{-1}$ is tight (in $\mathscr{P}(\mathscr{P}(\mathscr{C}))$). We need to identify the limit points of this sequence. Let us fix a subsequence N' such that $P\circ(\Gamma^{N'})^{-1}$ converges, i.e. $\Gamma^{N'}$ converges in distribution to say Γ. Γ is a $\mathscr{P}(\mathscr{C})$ valued random variable.

We will show that $\Gamma = \Lambda_0$ a.s. where Λ_0 is the unique solution to the McKean-Vlasov martingale problem with $\Lambda_0\circ\hat{Z}_0^{-1} = \mu_0$. This will prove that $P\circ(\Gamma^N)^{-1}$ converges in $\mathscr{P}(\mathscr{P}(\mathscr{C}))$ to δ_{Λ_0}.

For $\Lambda \in \mathscr{P}(\mathscr{C})$, let

(4.1) $$d(\Lambda) := \int_0^T \int_{\mathscr{C}} \|\hat{Z}_t\|^2 d\Lambda\, dt.$$

Now

$$\begin{aligned} Ed(\Gamma^N) &= E\int_0^T \frac{1}{N} \sum_{i=1}^N \|X_t^{N,i}\|^2 dt \\ &\leq TC_{2.8} \end{aligned}$$

(see (2.19)) and hence by Fatou's lemma, $Ed(\Gamma) \leq TC_{2.8}$. In particular, $d(\Gamma) < \infty$ a.s.

Fix $F \in \mathscr{E}$ defined by (3.17). Then

(4.2) $$F(\Lambda) = \int_{\mathscr{C}} \int_{\mathscr{C}} G(\hat{Z}_\cdot, \hat{Z}'_\cdot) d\Lambda(\hat{Z}_\cdot) d\Lambda(\hat{Z}'_\cdot)$$

where

(4.3) $$G(\hat{Z}_\cdot, \hat{Z}'_\cdot) =: [g(\hat{Z}_t) - g(\hat{Z}_s) - \int_s^t \mathscr{A}_u g(\hat{Z}_u) du - \int_s^t \hat{\mathscr{G}} g(\hat{Z}_u, \hat{Z}'_u) du] \cdot \\ g_1(\hat{Z}_{r_1}) \hat{g}_2(\hat{Z}_{r_2}) \ldots \hat{g}_m(\hat{Z}_{r_m}).$$

Thus G is a continuous function on $\mathscr{C}\times\mathscr{C}$ and further

$$(4.4) \qquad |G(\hat{Z}_{\cdot},\hat{Z}'_{\cdot})| \leq C_F\{1 + \int_0^T \|\hat{Z}_u\|du + \int_0^T\|\hat{Z}'_u\|du\}$$

where C_F is a constant depending on F and $C_{2.1}$. Thus $F(\Lambda)$ can be defined by (4.2) for all Λ with $d(\Lambda) < \infty$. In particular, $F(\Gamma)$ is well defined.

Lemma 4.1: $E[|F(\Gamma^{N'})|] \longrightarrow E[|F(\Gamma)|]$

Proof: For $k\geq 1$, let

$$(4.5) \qquad F_k(\Lambda) := \int_{\mathscr{C}}\int_{\mathscr{C}} (Gv(-k))\wedge k d\Lambda d\Lambda.$$

Then F_k is a bounded continuous function on $\mathscr{P}(\mathscr{C})$ and hence $F_k(\Gamma^{N'}) \longrightarrow F_k(\Gamma)$ in distribution and $|F_k(\Lambda)| \leq k$. Thus

$$(4.6) \qquad E[|F_k(\Gamma^{N'})|] \longrightarrow E[|F_k(\Gamma)|].$$

Moreover,

$$(4.7) \qquad \begin{aligned} E|F_k(\Gamma^{N'})-F(\Gamma^{N'})| &\leq E\int|G|.1_{\{|G|>k\}}d\Gamma^{N'}d\Gamma^{N'} \\ &\leq \frac{1}{k} E\int |G|^2 d\Gamma^{N'}d\Gamma^{N'} \\ &\leq \frac{1}{k} E\{3C_F^2(1+2d(\Gamma^{N'}))\} \\ &\leq \leq \frac{1}{k} \cdot 3C_F^2(1+2TC_{2.8}). \end{aligned}$$

Similarly

$$(4.8) \qquad \begin{aligned} E|F_k(\Gamma) - F(\Gamma)| &\leq \frac{1}{k} E\{3C_F^2(1+2d(\Gamma))\} \\ &\leq \frac{1}{k} \cdot 3C_F^2(1+TC_{2.8}). \end{aligned}$$

The familiar $\epsilon/3$ argument and (4.6), (4.7) and (4.8) yield the result. □

Lemma 4.2: $E[F^2(\Gamma^N)] \longrightarrow 0.$

Proof: Note that

$$(4.9) \qquad F(\Gamma^N) = \frac{1}{N}\sum_{i=1}^{N}[g(X_t^{N,i})-g(X_s^{N,i})-\int_s^t \mathcal{A}_u g(X_s^{N,i})ds$$

$$-\int_s^t \frac{1}{N}\sum_{j=1}^{N}\mathcal{B}g(X_s^{N,i},X_s^{N,j})ds]g_1(X_{r_1}^{N,i})\ldots g_m(X_{r_m}^{N,i}).$$

By considering the martingale problem corresponding to the system of equations (2.9) (as in the Appendix), we can deduce that

$$M_t^i := g(X_t^{N,i}) - g(X_s^{N,i}) - \int_s^t \mathcal{A}_u g(X_s^{N,i})ds - \int_s^t \frac{1}{N}\sum_{j=1}^{N}\mathcal{B}g(X_s^{N,i},X_s^{N,j})ds$$

is a martingale, for $i \neq j$, M_t^i, M_t^j are orthogonal martingales (i.e. $M_t^i M_t^j$ is a martingale) and

$$\langle M_t^i, M_t^j\rangle_t = \int_0^t \sum_{k,j=1}^{n}(U_n f_k)(X_s^{N,i})(U_n f_j)(X_s^{N,i})(b^*(s,X_s^{N,i})\phi_k, b^*(s,X_s^{N,i})\phi_j)ds$$

where $g = U_n f$, $f_k = \frac{\partial}{\partial x_k} f$. Thus

$$(4.10) \qquad E[(M_t^i - M_s^i)(M_t^\ell - M_s^\ell)|\mathcal{F}_s^N] = 0 \qquad i \neq \ell$$

and

$$(4.11) \qquad E(M_t^i - M_s^i)^2 \leq C_{4.2}T$$

where $C_{4.2}$ is a constant depending on F and $C_{2.1}$. Thus

$$EF^2(\Gamma^N) \leq \frac{1}{N^2} N\, C_{4.2}T \longrightarrow 0. \qquad \square$$

Together, these Lemmas yield

Theorem 4.3: $F(\Gamma) = 0 \;\; \forall\, F \in \mathcal{L}$, a.s.

Proof: Note that

$$\{E|F(\Gamma)|\}^2 = \lim_{N'} \{E|F(\Gamma^{N'})|\}^2 \le \lim_{N'} E[F^2(\Gamma^{N'})] = 0.$$

Hence $F(\Gamma) = 0$ a.s. Since $\mathcal{E}$ is a countable class, the result follows. □

We now need to show that $\Gamma \in \hat{\mathcal{P}}(\mathcal{C})$ a.s., i.e. $\nu_t := \Gamma \circ (\hat{Z}_t)^{-1}$ belongs to $C([0,T],\mathcal{P}_1(H))$ a.s. Then we can invoke Remark 3.2 to conclude $\Gamma = \Lambda_0$. Let $\nu_t^N := \Gamma^N \circ (\hat{Z}_t)^{-1} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{N,i}}$. We first show that $P\circ(\nu_\bullet^N)^{-1}$ is tight in $\mathcal{P}(C([0,T],\mathcal{P}_1(H)))$. The following lemma will be used in the proof.

<u>Lemma 4.4</u>:

(4.12) $$\|(T_s-I)h\| \le \|(T_t-I)h\| \quad \forall\ s \le t,\ h \in H.$$

(4.13) $$\lim_{t\to 0} \sup_N E[\|(T_t-I)X_0^{N,1}\|] = 0.$$

<u>Proof</u>:

$$\|(T_s-I)h\|^2 = \sum_{k=1}^{\infty} (e^{-\lambda_k s}-1)^2(\phi_k,h)^2$$

$$= \sum_{k=1}^{\infty} (e^{-\lambda_k t}-1)^2(\phi_k,h)^2$$

$$= \|(T_t-I)h\|^2.$$

Since $P\circ(X_0^{N,1})^{-1} \longrightarrow \mu_0$ in $\mathcal{P}(H)$, we can assume without loss of generality that $X_0^{N,1} \longrightarrow X_0^1$ a.s., with $P\circ(X_0^1)^{-1} = \mu_0$. Moreover, $E\|X_0^{N,1}\|^2 \le C_{2.5}$ implies

(4.14) $$E\|X_0^{N,1} - X_0^1\| \longrightarrow 0.$$

Since $\|(T_t-I)\| \le 2$, we get from (4.14)

(4.15) $$\sup_t E\|(T_t-I)(X_0^{N,1}-X_0^1)\| \le E\|X_0^{N,1}-X_0^1\| \longrightarrow 0.$$

Given $\epsilon>0$, let N_0 be s.t. for $N\geq N_0$, L.H.S. in (4.15) is $\leq \epsilon/2$. Since for each N, $E\|(T_t-I)X_0^{N,1}\| \longrightarrow 0$ as $t \downarrow 0$, choose $t_N > 0$ s.t. $t\leq t_N$ implies $E\|(T_t-I)X_0^{N,1}\| \leq \epsilon$. Also let $t_0>0$ be s.t. $t\leq t_0$ implies $E\|(T_k-I)X_0^1\| \leq \epsilon/2$. Now for $t \leq t^* := \min(t_0, t_1, \ldots, t_{N_0})$, we have

$$\sup_N E\|(T_t-I)X_0^{N,1}\| \leq \epsilon. \qquad \square$$

To show the required tightness in the next result we will use the fact that if K is a compact set in $\mathcal{P}(H)$, then for all $C < \infty$,

$$K_1 = K \cap \{\mu \in \mathcal{P}_1(H) : \int \|h\|^2 \mu(dh) \leq C\}$$

is a compact subset in $\mathcal{P}_1(H)$.

<u>Theorem 4.5</u>: $P \circ (\nu_{\cdot}^N)^{-1}$ is tight in $\mathcal{P}(C([0,T],\mathcal{P}_1(H)))$.

<u>Proof</u>: We already know that $P\circ(\nu_t^N)^{-1}$ is tight in $\mathcal{P}(H)$. Moreover

$$E\{\int \|h\|^2 \nu_t^N(dh)\} = E\, \frac{1}{N} \sum_{i=1}^{N} \|X_t^{N,i}\|^2 \leq C_{2.8}.$$

Thus, using the comment made just before the statement of the theorem, for $\epsilon>0$, we can find a compact set $K_{\epsilon,t}$ in $\mathcal{P}_1(H)$ s.t.

$$P(\nu_t^N \in K_{\epsilon,t}) \geq 1-\epsilon.$$

In what follows we use the notation of Section 2. For $0\leq s\leq T$, $0\leq t\leq \delta$, $s+t\leq T$, from (2.21)-(2.23) we have

$$\text{(4.16)}\quad E[\|X_{t+s}^{N,1} - X_s^{N,1}\| \,|\mathcal{F}_s^N] \leq E[\|T_{t+s}X_0^{N,1} - T_sX_0^{N,1}\| \;|\mathcal{F}_s^N] + E[\|V_{t+s}^N - V_s^N\| \;|\mathcal{F}_s^N]$$

$$\leq E[\|(T_\delta - I)X_0^{N,1}\| \;|\mathcal{F}_s^N] + \{E[\|V_{t+s}^N - V_s^N\|^2 \;|\mathcal{F}_s^N]\}^{1/2}$$

$$\leq E[\tilde{U}^{N,1}(\delta) \;|\; \mathcal{F}_s^N]$$

where

$$(4.17)\quad \tilde{U}^{N,j}(\delta) = \|(T_\delta - I)X_0^{N,j}\|^2 + C_{2.1}[\sigma(\delta)]^{1/2}\int_0^T (1+\|X_r^{N,j}\|^2 + \frac{1}{N}\sum_{j=1}^{N}\|X_r^{N,j}\|^2)dr.$$

Now using Lemma 4.4, (2.19) and $\sigma(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$, we get

$$(4.18)\qquad \sup_N E\tilde{U}^{N,1}(\delta) \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0.$$

Moreover, $\rho_1(\upsilon^N_{t+s},\upsilon^N_s) \le \frac{1}{N}\sum_{j=1}^{N}\|X^{N,j}_{s+t} - X^{N,j}_s\|$ so that

$$E[\rho_1(\upsilon^N_{t+s},\upsilon^N_s)\,|\mathcal{F}^N_s] \le \frac{1}{N}\sum_{j=1}^{N} E[\|X^{N,j}_{s+t} - X^{N,j}_s\| \;\, |\mathcal{F}^N_s]$$

$$\le E[\frac{1}{N}\sum_{j=1}^{N}\tilde{U}^{N,j}(\delta)\;\, |\mathcal{F}^N_s]$$

$$\le E\tilde{U}^N(\delta)\;\, |\mathcal{F}^N_s]$$

where $\hat{U}^N(\delta) = \frac{1}{N}\sum_{j=1}^{N}\tilde{U}^{N,j}(\delta)$. Since $E\tilde{U}^{N,j}(\delta) = E\tilde{U}^{N,i}(\delta)$ by symmetry, (4.18) implies

$$\sup_N E(\hat{U}^N(\delta)) \longrightarrow \quad \text{as} \quad \delta \longrightarrow 0.$$

This completes the proof of the theorem, (see Ethier-Kurtz, [3]). □

We are now in a position to prove

<u>Theorem 4.6</u>: $\Gamma \in \hat{\mathcal{P}}(\mathcal{C})$ a.s.

<u>Proof</u>: Recall that $\Gamma^{N'}$ converges in distribution to Γ. Now by Theorem 4.5 $P\circ(\upsilon^{N'}_\bullet)^{-1}$ is tight in $\mathcal{P}(C[\cdot,T],\mathcal{P}_1(H)))$ and hence for s subsequence $\{N''\}$ of $\{N'\}$,

$$P\circ(\upsilon^{N''}_\bullet)^{-1} \quad \text{converges in} \quad \mathcal{P}(C([0,T],\mathcal{P}_1(H))).$$

Thus

$$P\circ(\Gamma^{N''}, \nu_{\bullet}^{N''})^{-1} \text{ converges in } \mathcal{P}(\mathcal{P}(\mathcal{C})\times C([0,T],\mathcal{P}_1(H)))$$

to say, $P\circ(\tilde{\Gamma},\tilde{\nu})^{-1}$. Since $\Gamma^{N''}\circ(\hat{Z}_t)^{-1} = \nu_t^{N''}$, it follows that $\tilde{\Gamma}\circ(\hat{Z}_t)^{-1} = \tilde{\nu}_t$.
Thus

$$t \longrightarrow \tilde{\Gamma}\circ(\hat{Z}_t) \text{ is continuous from } [0,T] \text{ to } \mathcal{P}_1(H).$$

i.e. $\tilde{\Gamma} \in \hat{\mathcal{P}}(\mathcal{C})$. Since $\mathcal{L}(\Gamma) = \mathcal{L}(\tilde{\Gamma})$, we conclude that

$$\Gamma \in \hat{\mathcal{P}}(\mathcal{C}). \qquad \square$$

<u>Theorem 4.8</u>: (a). Let $\mu_0 \in \mathcal{P}_2(H)$. Then there exists a unique solution Λ_0 to the McKean-Vlasov martingale problem with $\Lambda_0 \circ (\hat{Z}_0)^{-1} = \mu_0$.

(b) $\Gamma^N \longrightarrow \Lambda_0$ in probability (as a $\mathcal{P}(\mathcal{C})$-valued r.v.).

<u>Proof</u>: Theorems 4.3 and 4.7 imply that if Γ is any subsequential limit of $\{\Gamma^N\}$, then $\Gamma \in \hat{\mathcal{P}}(\mathcal{C})$ a.s. and

$$F(\Gamma) = 0 \quad \forall\, F \in \mathcal{E} \text{ a.s.}$$

Thus Γ is a solution to the McKean-Vlasov martingale problem with $\Gamma\circ(\hat{Z}_0)^{-1}=\mu_0$. This shows existence of Λ_0 as in (a). Uniqueness of Λ_0 follows from Theorem 3.2. So now we get $\Gamma = \Lambda_0$ a.s. Thus, all subsequential limits of $\{\Gamma^N\}$ are equal to Λ_0. Since $\{\Gamma^N\}$ is tight, this gives $P\circ(\Gamma^N)^{-1} \longrightarrow \delta_{\Lambda_0}$ in $\mathcal{P}(\mathcal{P}(\mathcal{C}))$, and hence $\Gamma^N \longrightarrow \Lambda_0$ in probability. $\square$

<u>5. Propagation of chaos in</u> $C([0,T];C(G))$.

Let us now assume that $H = L^2(G)$, where G is a bounded region in $\mathbb{R}^d$. Suppose further that the semigroup (T_t) generated by $-A$ satisfies the following additional conditions:

T_t is an integral operator given by a symmetric kernel $p(t,x,y)$:

(5.1) $$T_t g(x) = \int_G p(t,x,y)g(y)dy, \quad t>0.$$

We put $T_0 g(x) \equiv g(x)$, for $g \in L^2(G)$.

(5.2) $T_t g(x)$ is jointly continuous in (t,x) for $g \in C(G)$

(5.3) $$|p(t,x,y)| \leq C_{5.1}\, t^{-\delta} \quad \text{for} \quad 0<\delta<1$$

(5.4) $$\left| \frac{\partial}{\partial y^i} p(t,x,y) \right| \leq C_{5.2} t^{-\alpha} \exp(-C_{5.3}|y-x|^{\beta} t^{-r}), \quad t>0$$

with $\alpha - r/\beta < 1$ for each i

(5.5) $$\int |p(t,x,y)|dy \leq C_{5.3} \quad \text{for } t>0.$$

Under these conditions on (A, T_t), we will show that the $X_t^{N,i}(x)$ admit versions in $C([0,T],C(G))$ and that Γ^N, defined in Section 2 converges in probability as random elements in $\mathcal{P}(C([0,T],C(G)))$ and as a consequence,

$$\Lambda_0(C([0,T],C(G))) = 1$$

provided $\mu_0(C(G)) = 1$. We begin with a lemma:

<u>Lemma 5.1</u>: There exist constants $C_{5.4}$, $\delta_1>0$, $\delta_2>0$ (depending only on A) such that for $0<t_1 \leq t_2$, $x_1, x_2 \in G$

(5.6) $$\int_0^{t_2} \int_G \{p(t_1-s,x_1,y)1_{s\leq t_1} - p(t_2-s,x_2,y)\}^2 dyds \leq C_{5.4}\{|t_2-t_1|^{\delta_1} + |x_2-x_1|^{\delta_2}\}$$

<u>Proof</u>: From $p(s,x,y) = p(s,y,x)$ and the semigroup property one gets

(5.7) $$\int_G p(u,x,y)p(v\ x,y)dy = p(u+v,x,x).$$

Using this, the L.H.S. in (5.6) equals

$$(5.8)\quad \int_0^{t_2}\{p(2t_1-2s,x_1,x_1)1_{(s\le t_1)}+p(2t_2-2s,x_2,x_2)-2p(t_2+t_1-2s,x_1,x_r)1_{(s\le t_1)}\}ds.$$

By changing variables using substitutions $t_1-s = r$, $t_2-s = r$ and $\frac{t_2+t_1}{2} - s = r$, respectively in the three terms, (5.8) equals

$$\int_0^{t_1}p(2r,x_1,x_1)dr + \int_0^{t_2}p(2r,x_2,x_2)dr - 2\int_{(t_2-t_1)/2}^{(t_2+t_1)/2} p(2r,x_1,x_2)dr$$

$$= I_1 + I_2$$

where

$$I_j = \int_0^{t_j} p(2r,x_j,x_j)dr - 2\int_{(t_2-t_1)/2}^{(t_2+t_1)/2} p(2r,x_1,x_2)dr.$$

Let us write $I_1 \le I_{11} + I_{12} + I_{13}$, where

$$I_{11} = \int_0^{(t_2-t_1)/2} p(2r,x_1,x_2)dr$$

$$I_{12} = \int_{t_1}^{(t_2+t_1)/2} p(2r,x_1,x_2)dr$$

$$I_{13} = \int_0^{t_1}|p(2r,x_1,x_1) - p(2r,x_1,x_2)|dr.$$

From the inequality (5.3), it follows that

$$(5.9)\qquad I_{11} + I_{12} \le C_{5.5}|t_2-t_1|^{1-\delta} .$$

Now using (5.4) and $e^{-x} \le x^{-q}$ for $q>0$, we have

$$(5.10)\quad |p(2r,x_1,x_2)-p(2r,x_1,x_1)| = |\int_0^1 (x_2-x_1)\cdot\nabla_y p(2r,x_1,x_1+u(x_2-x_1))du|$$

$$\le C_{5.2}|x_2-x_1|t^{-\alpha}\exp(-C_{5.3}u^{\beta}|x_2-x_1|^{\beta}t^{-\upsilon}\}$$

$$\leq C_{5.2}|x_2-x_1|t^{-\alpha}|C_{5.3}|x_2-x_1|^{\beta}t^{-\nu}|^{-q}$$

$$= C_{5.2}C_{5.3}^{q}|x_2-x_1|^{1-\beta q}t^{-\alpha+qv} .$$

In (5.10) ∇_y is the gradient and the dot stands for the scalar product in $\mathbb{R}^d$. Since $\alpha = \frac{r}{\beta} < 1$, we can choose q such that $\delta_1 = 1-\beta q > 0$ and $\alpha-qr < 1$. Then we have

(5.11) $$I_{13} \leq C_{5.6}|x_2-x_1|^{\delta_1} .$$

Putting together (5.9), 5.11) we get (5.6). □

Lemma 5.2: Let $f: [0,T]\times\Omega \longrightarrow L(H,H)$ be a measurable $(\mathcal{F}_t^W)$-adapted process such that

(5.12) $$\|f\|_{L(H,H)} \leq C_{5.7} < \infty. \quad (H = L^2(G)).$$

Let $W_t^n = W_t(\varphi_n)$ and let

(5.13) $$\xi_t(x) := \sum_{n=1}^{\infty} \int_0^t[\int_G[f_s^*p(t,x,\cdot)](y)\phi_n(y)dy]dW_s^n .$$

Then $\xi_t(x)$ is a version of the $L^2(G)$-valued process $\xi_t=\int_0^t T_{t-s}f_s dW_s$.

Proof: Since $\{W_s^n: n\geq 1\}$ is a family of independent Wiener processes, the convergence of series appearing in (5.12) follows from

$$E \sum_{n=1}^{\infty} \int_0^t(f_s^*p(t-s,x,\cdot),\varphi_n)^2 ds = E\int_0^t\|f_s^*p(t-s,x,\cdot)\|^2_{L^2(G)} ds$$

$$\leq C_{5.7}\int_0^t\|p(t-s,x,\cdot)\|^2_{L^2(G)} ds$$

$$= C_{5.7}\int_0^t p(2t-2s,x,x)ds < \infty.$$

Moreover for $h=h(x) \in L^2(G)$, we have

$$(5.14) \qquad \int \xi_t(x)h(x)dx = \sum_{n=1}^{\infty} \int_0^t [\int (f_s^* p(t-s,x,\cdot),\varphi_n)h(x)dx]dW_s^n$$

$$= \sum_{n=1}^{\infty} \int_0^t (f_s^* T_{t-s}h,\varphi_n)dW^n$$

$$= \int_0^t (f_s^* T_{t-s}h, dW_s)$$

$$= (h, \int_0^t T_{t-s}dW_s).$$

Since (5.14) holds for all $h \in L^2(G)$, it follows that $\xi_t(x)$ is a version of ξ_t

□

We are now in a position to prove

<u>Theorem 5.3</u>: Suppose that (2.1), (2.3)-(2.8), (2.11)-(2.13), (5.1)-(5.5) hold. Further, suppose that

$$(5.15) \qquad X_0^{N,j}(\cdot) \in C(G), \quad 1 \le j \le N$$

and for some $p \ge 1$, such that $p\delta_1 > 2(1+\epsilon)$, $p\delta_2 > 2(1+\epsilon)$ with $\epsilon > 0$ (where δ_1, δ_2 are as in (5.6)),

$$E\|X_0^{N,1}\|^p \le C_{5.8}, \quad N \ge 1.$$

Then, the processes $X_t^{N,i}$ admit versions $\tilde{X}_t^{N,i}$ such that $\tilde{X}^{N,i} \in C([0,T],C(G)) \equiv C([0,T] \times G)$.

<u>Proof</u>: Note that in view of Theorem A, there exists a constant $C_{5.9}$, depending on $C_{2.1}$ and $C_{5.8}$ such that

$$(5.16) \qquad E\|X_t^{N,i}\|^p \le C_{5.9}$$

for all $t \in [0,T]$, $i \geq 1$. Now write

$$X_t^{N,1}(x) = Y_t^{N,1}(x) + Y_t^{N,2}(x) + Y_t^{N,3}(x) + Y_t^{N,4}(x)$$

where

$$Y_t^{N,1}(x) = \int p(t,x,y) X_0^{N,1}(y) dy$$

$$Y_t^{N,2}(x) = \left(\int_0^t T_{t-s} b(s,X_s^{N,1}) dW_s\right)(x)$$

(the version given by Lemma 5.2).

$$Y_t^{N,3}(x) = \left(\int_0^t T_{t-s} a(s,X_s^{N,1}) ds\right)(x)$$

$$= \int_0^t \int p(t-s,x,y) a(s,X_s^{N,1})(y) dy ds$$

and

$$Y_t^{N,4}(x) = \frac{1}{N} \sum_{j=1}^{N} \int_0^t \int p(t-s,x,y) I(X_s^{N,1},X_s^{N,j})(y) dy ds.$$

For $t_1 \leq t_2$, $x_1, x_2 \in G$, we have

(5.17) $$E|Y_{t_2}^{N,2}(x_2) - Y_{t_1}^{N,2}(x_1)|^p$$

$$\leq C_p E\left|\int_0^{t_2} \| b^*(s,X_s^{N,1})(p(t_2-s,x_2,\cdot) - p(t_1-s_1,x_1,\cdot)1_{(s \leq t_1)})\|^2 ds\right|^{p/2}$$

$$\leq C_p C_{2.1}^p \left|\int_0^{t_2} \| p(t_2-s,x_2,\cdot) - p(t_1-s,x_1,\cdot)1_{(s \leq t_1)}\|^2 ds\right|^{p/2}$$

$$\leq C_p C_{2.1}^p C_{5.4}^{p/2}(|t_2-t_1|^{\delta_1} + |x_2-x_1|^{\delta_2})^{p/2}$$

$$\leq C_{5.10}(|t_2-t_1|^{1+\epsilon} + |x_2-x_1|^{1+\epsilon})$$

On the other hand

(5.18) $$E|Y_{t_2}^{N,3}(x_2) - Y_{t_1}^{N,3}|^p$$

$$\leq E|\int_0^t\int_G\{p(t_2-s,x_2,y)b(s,X_s^{N,1})(y)-p(t_1-s,x_1,y)b(s,X_s^{N,1})(y)1_{(s\leq t_1)}\}dyds|^p$$

$$\leq E[\int_0^{t_2}\int_G\{p(t_2-s,x_2,y)-p(t_1-s,x_1,y)1_{(s\leq t_1)}\}^2 dsdy\ \int_0^{t_2}\|b(s,X_s^{N,1})\|^2 ds]^{p/2}$$

$$\leq E[C_{5.4}\{|t_2-t_1|^{\delta_1}+|x_2-x_1|^{\delta_2}\}\cdot\int_0^{t_2}(1+\|X_s^{N,1}\|^2)ds]^{p/2}$$

$$\leq C_p C_{5.4}^{p/2}\{|t_2-t_1|^{1+\epsilon}+|x_2-x_1|^{1+\epsilon}\cdot\int_0^{t_2}(1+E\|X_s^{N,1}\|^p)ds$$

$$\leq C_{5.11}\{|t_2-t_1|^{1+\epsilon}+|x_2-x_1|^{1+\epsilon}\}.$$

Similarly

(5.19) $$E|Y_{t_2}^{N,4}(x_2)-Y_{t_1}^{N,4}(x_1)|^p \leq C_{5.12}\{|t_2-t_1|^{1+\epsilon}+|x_2-x_1|^{1+\epsilon}\}.$$

Here $C_{5.10}, C_{5.11}, C_{5.12}$ do not depend on N. In view of (5.17), (5.18) and (5.19), it follows from a well-known result (see [4]) that $Y_t^{N,j}(x)$ admit versions $\tilde{Y}_N^{N,j}(x)$, j=2,3,4, such that

$$\tilde{Y}_{\cdot}^{N,j}(\cdot) \in C([0,T]\times G) \equiv C([0,T],C(G)).$$

In view of (5.2), $Y_{\cdot}^{N,1}(\cdot) \in C([0,T]\times G)$. Hence it follows that $X_{\cdot}^{N,1}(\cdot)$ admits a version $\tilde{X}_{\cdot}^{N,1}(\cdot) \in C([0,T],C(G))$. Similarly, we can construct versions $\tilde{X}^{N,j}$ of $X^{N,i}$, i=1,2,...,N.

Let $\tilde{\Gamma}^N = \frac{1}{N}\ \Sigma_1^N \delta_{\tilde{X}^{N,i}}$. We regard $\tilde{\Gamma}^N$ as a random element of $\mathcal{P}(\mathcal{C}([0,T],C(G)))$. Since $\tilde{X}_{\cdot}^{N,i}$ is a version of $X^{N,i}$, it follows that $\tilde{\Gamma}_{\cdot}^N$ is a version of Γ^N. We will now show that the sequence $\tilde{\Gamma}^N$ converges in probability in C([0,T],C(G)).

<u>Theorem 5.4</u>: Suppose that the conditions of Theorem 5.3 are satisfied. Further assume that

(5.20) $$P\circ(X_0^{N,1})^{-1} \longrightarrow \tilde{\mu}_0 \quad \text{in } \mathcal{P}(C(G)).$$

Then we have the following conclusions:

(i) $P\circ(\tilde{X}^{N,1})^{-1}$ is tight in $\mathcal{P}(C([0,T],C(G)))$;

(ii) $P\circ(\tilde{\Gamma}^{N})^{-1}$ is tight in $\mathcal{P}(\mathcal{P}(C([0,T,C(G))))$;

(iii) If Λ_0 is the unique solution to the McKean-Vlasov martingale problem with $\Lambda_0\circ(\tilde{Z}_0)^{-1} = \mu_0$, and $\tilde{\Lambda}_0$ is its restriction to $C([0,T],C(G))$, then $\tilde{\Gamma}^N \longrightarrow \tilde{\Lambda}_0$ in probability in $\mathcal{P}(C([0,T],C(G)))$.

Proof: The estimates (5.17), (5.18) and (5.19) imply that $P\circ(\tilde{Y}^{N,j})^{-1}$ are tight in $\mathcal{P}(C([0,T],C(G))$ for $j=2,3,4$.

From (5.5), it follows that if $g_n \longrightarrow g_0$ in $C(G)$ (uniformly) then $(T_t g_n)(x) \longrightarrow (T_t g_0)(x)$ uniformly in $(t,x) \in [0,T]\times G$. This observation and (5.20) imply that

$$P\circ(\tilde{Y}^{N,1}_{\cdot})^{-1} \quad \text{converges in } \mathcal{P}(C([0,T],C(G)))$$

and hence $P\circ(\tilde{X}^{N,1})^{-1}$ is tight in $\mathcal{P}(C[0,T],C(G))$ where

$$\tilde{X}^{N,1} = \tilde{Y}^{N,1} + \tilde{Y}^{N,2} + \tilde{Y}^{N,3} + \tilde{Y}^{N,4}.$$

This proves (i). (ii) follows from (i). The proof is the same as that of Theorem 2.2. We only have to replace $\mathcal{C}$ by $C([0,T],C(G))$. Now, $P(\Gamma^N)^{-1} = P\circ(\tilde{\Gamma}^N)^{-1}$ converges in $\mathcal{P}(\mathcal{C})$ (Section 4). Moreover, $C([0,T],C(G)) \subseteq C([0,T],L^2(G)) = \mathcal{C}$ and $P\circ(\tilde{\Gamma}^N)^{-1}$ is tight in $\mathcal{P}(C([0,T],C(G)))$. Hence it follows that $P\circ(\tilde{\Gamma}^N)^{-1}$ converges in $\mathcal{P}(C[0,T],C(G))$ to say $\tilde{\Lambda}_0$. Then $\tilde{\Lambda}_0$ is the restriction of Λ_0 to $C([0,T],C(G))$. This proves (5.21). □

Appendix

Let $\mathcal{H}$ be a separable, real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let L be a self-adjoint, non-negative operator on $\mathcal{H}$ with dense domain and such that L has a discrete spectrum $\{\rho_n\}$ with

(A.1) $\rho_n \sim cn^{1+\delta}$ $(c>0,\ \delta>0)$.

We denote the eigenfunctions of L by $\{\psi_n\}$ and the semigroup of which -L is the generator by S_t. Note that for $\theta > 1/1+\delta$ we have

(A.2) $$\sum_{n=1}^{\infty} \rho_n^{-\theta} < \infty.$$

a fact that will be used often in what follows. Note that we also have

(A.3) $$S_t \psi_k = e^{-\rho_k t} \psi_k.$$

Let (W_t) be a cylindrical Brownian motion on H, (defined on some complete probability space $(\Omega, \mathcal{F}, P)$.

Consider the SDE

(A.4) $$dY_t = -LY_t dt + \beta(t, Y_t) dW_t + \alpha(t, Y_t) dt,$$

where β: $[0,T] \times \mathcal{H} \longrightarrow L(\mathcal{H}, \mathcal{H})$, $\alpha: [0,T] \times \mathcal{H} \longrightarrow \mathcal{H}$ are continuous functions satisfying conditions as in Dawson [2].

(A.5) $$\|\beta^*(t,h)\psi_k\| \le C_{A.1}$$

(A.6) $$\|\alpha(t,h)\| \le C_{A.1}(1+\|h\|)$$

(A.7) $$\|(\beta^*(t,h_1)-\beta^*(t,h_2))\psi_k\| \le C_{A.2}\|h_1-h_2\|$$

(A.8) $$\|\alpha(t,h_1) - \alpha(t,h_2)\| \le C_{A.2}\|h_1-h_2\|$$

for constants $C_{A.1}, C_{A.2}$ and for all $k \ge 1$, $h, h_1, h_2 \in \mathcal{H}$, $0 \le t \le T$. β^* denotes the adjoint of β and $L(\mathcal{H},\mathcal{H})$ is the space of continuous linear operators from $\mathcal{H}$ to $\mathcal{H}$.

Under these conditions, (A.4) can not be interpreted as

$$Y_t = Y_0 + \int_0^t -LY_s ds + \int_0^t \beta(s, Y_s) dW_s + \int_0^t \alpha(s, Y_s) ds$$

as the stochastic integral $\int_0^t \beta(s, Y_s) dW_s$ may not be defined and a priori, Y may not belong to the domain of L. Instead, (A.4) is to be interpreted as

$$Y_t = S_t Y_0 + \int_0^t S_{t-s} \beta(s, Y_s) dW_s + \int_0^t S_{t-s} \alpha(s, Y_s) ds. \tag{A.9}$$

Since

$$\begin{aligned} E\int_0^t \|S_{t-s}\beta(s, Y_s)\|^2_{H \cdot S} ds &= E\int_0^t \|\beta^*(s, Y_s) S_{t-s}\|^2_{H \cdot S} ds \\ &= E\int_0^t \sum_{k=1}^{\infty} \|\beta^*(s, Y_s) S_{t-s} \psi_k\|^2 ds \\ &= E\int_0^t \sum_{k=1}^{\infty} e^{-2(t-s)\rho_k} \|\beta^*(s, Y_s) \psi_k\|^2 ds \\ &\leq C_{A \cdot 1} \sum_{k=1}^{\infty} \frac{(1 - e^{-2t\rho_k})}{2\rho_k} \\ &< \infty \end{aligned} \tag{A.10}$$

in view of (A.2), the stochastic integral appearing in (A.9) is well defined. $\| \cdot \|_{H.S}$ appearing above is Hilbert-Schmidt norm.

<u>Definition</u>: A measurable process (Y_t) is said to be a <u>mild</u> solution to (A.4) if

$$\int_0^T \|Y(t, \omega)\|^2 dt < \infty \quad \text{a.s.}$$

and if (A.9) is satisfied for all t.

The following result is due to Dawson [2].

<u>Theorem A.1</u>: Let $E\|Y_0\|^2 < \infty$ and Y_0 be independent of (W_t). Then (A.4) admits a unique (up to P-null sets) mild solution (Y_t) in the class of measurable processes satisfying

$$\text{(A.11)} \qquad \sup_{t \leq T} E\|Y_t\|^2 < \infty.$$

Further (Y_t) can be chosen to have paths in $C([0,T],\mathcal{H})$. See Remark A.3 below for an outline of the proof of Theorem A.1.

It is easy to see that (Y_t) satisfies (A.9) iff

$$Y_t^k := (Y_t,\psi_k) \text{ satisfy for } k \geq 1$$

$$\text{(A.12)} \quad Y_t^k = e^{-\rho_k t} Y_0^k + \int_0^t (e^{-\rho_k(t-s)} \beta^*(s,Y_s)\psi_k, dW_s) + \int_0^t e^{-\rho_k(t-s)} (\alpha(s,Y_s),\psi_k) ds.$$

An elementary computation shows that (A.12) is equivalent to

$$\text{(A.13)} \qquad dY_t^k = -\rho_k Y_t^k dt + (\beta^*(t,Y_t)\psi_k, dW_t) + (\alpha(t,Y_t),\psi_k) dt.$$

Thus, it follows that (Y_t) is a mild solution to (A.4) iff $Y_t^k := (Y_t,\psi_k)$ satisfy (A.13) for $k \geq 1$.

Martingale problem corresponding to (A.4).

For $f \in C_0^2(\mathbb{R}^n)$, let $U_n f : H \longrightarrow \mathbb{R}$ be defined by

$$\text{(A.14)} \qquad (U_n f)(h) = f((h,\psi_1),\ldots,(h,\psi_n)).$$

For $f \in C_0^2(\mathbb{R}^n)$, we write $f_i := \frac{\partial}{\partial x_i} f$, $f_{ij} = \frac{\partial}{\partial x_j} f_i$.

Let $\mathcal{D} = \{U_n f : f \in C_0^2(\mathbb{R}^n), n \geq 1\}$ and let

$$\text{(A.15)} \qquad \mathcal{L}_t(U_n f)(h)$$

$$:= \frac{1}{2} \sum_{i,j=1}^n (\beta^*(t,h)\psi_i, \beta^*(t,h)\psi_j)(U_n f_{ij})(h) + \sum_{i=1}^n (\alpha(t,h) - \rho_i h, \psi_i)(U_n f_i)(h).$$

If (Y_t) is a mild solution to (A.4) then (A.13) is satisfied, and hence for all $g \in \mathcal{D}$,

$$g(Y_t) - \int_0^t (\mathcal{L}_s g)(Y_s)ds \tag{A.16}$$

is a martingale, and hence $P_1 := P\circ Y^{-1}$ is a solution to the $(\mathcal{L}_t)$-martingale problem on $\Omega_1 := C([0,T],\mathcal{H})$. This means the following. Let (ξ_t) denote the canonical process on Ω_1. Then

$$g(\xi_t) - \int_0^t (\mathcal{L}_s g)(\xi_s)ds \tag{A.17}$$

is a P_1-martingale. We have the following converse to this observation.

<u>Theorem A.2</u>: Let P_1 be a solution to the $(\mathcal{L}_t)$-martingale problem $(P_1 \in \mathcal{P}(C([0,T],\mathcal{H})))$. Then P_1 is a 'weak solution' to (A.4), i.e. P_1 is the distribution of a mild solution to (A.4) on some probability space.

<u>Proof</u>: Using (A.16) for $g = U_n f$ for $f \in C_0^2(\mathbb{R}^n)$, it follows using standard arguments that

$$M_t^k := (\xi_t,\psi_k)-(\xi_0,\psi_k)-\int_0^t(\alpha(s,\xi_s)-\lambda_k\xi_s,\psi_k)dt \tag{A.18}$$

is a local martingale, with continuous paths and

$$\langle M^k,M^j\rangle_t = \int_0^t(\beta^*(s,\xi_s)\psi_k,\beta^*(s,\xi_s)\psi_j)ds. \tag{A.19}$$

As a consequence,

$$E \sup_{t\leq T} |M_t^k|^2 \leq (C_{A.1})^2 T \tag{A.20}$$

and (M_t^k) is a square-integrable martingale. Let

$$N_t^k = \rho_k^{-1/2}\, M_t^k.$$

Then using (A.20) and (A.2), one has

$$E \sup_{t\leq T} \| \sum_{k=m}^{n} N_t^k \psi_k \|^2 \longrightarrow 0 \quad \text{as } n,m \longrightarrow \infty.$$

Hence $N_t := \Sigma_k N_t^k \psi_k$ is an $\mathscr{H}$-valued martingale with continuous paths. Since

$$\langle N_t^k, N_t^j \rangle = \int_0^t \rho_k^{-1/2} \rho_j^{-1/2} (\beta^*(s,\xi_s)\psi_k, \beta^*(s,\xi_s)\psi_j) ds$$

$$= \int_0^t (\beta^*(s,\xi_s) A^{-1/2} \psi_k, \beta^*(s,\xi_s) A^{-1/2} \psi_{ij}) ds$$

it follows that

$$\langle\langle N,N \rangle\rangle_t = \int_0^t G_s^* G_s ds$$

where $G_s(\omega) = A^{-1/2} \beta(s,\xi_s(\omega))$.

Using arguments as in the proof of Theorem IV.3.5 in Yor [7], one can conclude that there exists a $\mathscr{H}$-valued cylindrical Brownian motion (B_t) (perhaps on an enlarged probability space) independent of ξ_0 such that

$$N_t = \int_0^t G_s dB_s$$

so that $N_t^k = \int_0^t (G_s^* \psi_k, dB_s) = \rho_k^{-1/2} (\beta^*(s,\xi_s)\psi_k, dB_s)$. Hence, $M_t^k = \int_0^t (\beta^*(s,\xi_s)\psi_k, dB_s)$. This and (A.18) imply that (ξ_t) is a solution to (A.13) and hence a mild solution to (A.4). □

We will discuss later the question of uniqueness of solution to the martingale problem. We first obtain an estimate on moments. Note that in (A.10) we have essentially proved

(A.21)
$$\int_0^T \|S_t\|_{H\cdot S}^2 dt = \sum_{k=1}^{\infty} \frac{1}{\rho_k} \{1-e^{-2T\rho_k}\}.$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{\rho_k} = C_{A.4}.$$

<u>Theorem A.3</u>: Let (Y_t) be a mild solution to (A.4) with $Y_\cdot \in C([0,T],\mathscr{H})$. Then for $p\geq 2$

$$\sup_{t\le T} E\|Y_t\|^p \le C_{A.5}\{1+E\|Y_0\|^p\} \tag{A.22}$$

where $C_{A.5}$ depends only on $C_{A.1}, C_{A.4}$ and p.

<u>Proof</u>: First note that

$$\|S_tY_0\|^p \le \|Y_0\|^p. \tag{A.23}$$

Next,

$$E\|\int_0^t S_{t-s}\beta(s,Y_s)dW_s\|^p \le C_{A.6}E[\int_0^t \|S_{t-s}\beta(s,Y_s)\|^2_{H.S}ds]^{p/2}$$

and hence

$$E\|\int_0^t S_{t-s}\beta(s,Y_s)dW_s\|^p \le C_{A.6}\cdot C^p_{A.1}\cdot C_{A.4} \tag{A.24}$$

as seen in (A.10). For the dt integral term one has

$$\begin{aligned} \|\int_0^t S_{t-s}\alpha(s,Y_s)ds\|^p &\le \|\int_0^t \|S_{t-s}\|_{H\cdot S}\|\alpha(s,Y_s)\|ds|^p \\ &\le |\int_0^t \|S_{t-s}\|^2_{H\cdot S}ds)(\int_0^t \|\alpha(s,Y_s)\|^2ds|^{p/2} \\ &\le (C_{A.4})^{p/2}C^p_{A.1}[\int_0^t(1+\|Y_s\|^2)ds]^{p/2} \\ &\le C_{A.7}\int_0^t(1+\|Y_s\|^p)ds. \end{aligned} \tag{A.25}$$

If we knew a priori that the L.H.S. in (A.22) is finite, we could combine (A.23), (A.24), (A.25), take expectations and complete the proof using Gronwall's inequality. To overcome this difficulty we proceed as follows. Let

$$g_k(\omega) = C(1 + \frac{1}{k}\sup_{t\le T}\|Y_t(\omega)\|^p)^{-1}$$

where C is chosen so that $\int g_k dP=1$ and let $dP_k = g_k dP$. Let E_k denote $\int\cdot dP_k$. Since $g_k\le 1$, it follows that $E_k(f) \le E(f)$ for $f \ge 0$ and hence (A.24) holds with E replaced by E_k. Taking expectation in (A.23), (A.25) with respect to P_k, and using Gronwall's lemma we get

(A.20) $$\sup_{t\le T} E_k\|Y_t\|^p \le C_{A.5}\{1+E_k\|Y_0\|^p\}.$$

Since $g_k \to 1$, (A.22) follows from (A.26) by dominated convergence. □

<u>Remark A.1</u>. Note that the result proved above gives a proof of the fact that any solution $Y_\cdot \in C([0,T],\mathscr{X})$ of (A.4) must satisfy (A.11).

<u>Lemma A.4</u>: Let $f \in L([0,T])$ and $f \ge 0$.

(i) Suppose g, δ are nonnegative, bounded, measurable functions on $[0,T]$ such that

(A.27) $$g(t) \le \int_0^t f(t-s)\{g(s) + \delta(s)\}ds, \quad t \in [0,T].$$

Then, there exists a finite measure μ on $[0,T]$ depending only on f, such that

$$\sup_{t\le T} g(t) \le \int_0^t \delta(u)\mu(du).$$

(ii) Suppose g_n, δ_n are sequences of bounded measurable functions such that

$$g_n(t) \le \int_0^T f(t-s)\{g_n(s) + \delta_n(s)\}ds, \quad n\ge 1, \ t\in[0,T]$$

Further, suppose $\delta_n(s) \le C$ and $\delta_n(s) \to 0$ as $n \to \infty$ $\forall s$. Then

$$\sup_{t\le T} g_n(t) \to 0 \quad \text{as} \quad n \to \infty$$

<u>Proof</u> (i) Without loss of generality, we can assume that g is increasing. Now, repeatedly using the inequality (A.27) we get

$$g(t) \le \int_0^t f(t-t_1)\{g(t_1)+\delta(t_1)\}dt,$$

$$\le \int_0^t f(t-t_1)\delta(t_1)dt_1 + \int_0^t\int_0^{t_1} f(t-t_1)f(t_1-t_2)\{g(t_2)+\delta(t_2)\}dt_2dt_1$$

$$\le$$

It follows that for any integer $k \geq 1$,

(A.28) $$g(T) \leq a_1 + \dots + a_k + b_k g(T),$$

where

(A.29) $$a_k = \int_0^T \dots \int_0^T f(s_1) \dots f(s_k) \delta(T - \Sigma_1^k s_i) 1_{(\Sigma_1^k s_i \leq T)} ds_1 \dots ds_k$$

and b_k is defined similarly with $\delta \equiv 1$. Now

$$a_k = \int \delta(u) \mu_k(du)$$

where $\mu_k \in \mathcal{P}([0,T])$ is defined for $B \in \mathcal{B}([0,T])$ by

$$u_k(B) = \int_0^T \dots \int_0^T 1_B(T - \Sigma_1^k s_i) f(s_1) \dots f(s_k) ds_1 \dots ds_k.$$

Note, $b_k = \mu_k([0,T])$. Now

$$\mu_k([0,T]) = \int_0^T \dots \int_0^T 1_{(\Sigma_1^k s_i \leq T)} f(s_1) \dots f(s_k) ds_1 \dots ds_k$$

$$\leq \exp(\alpha T) \prod_{i=1}^{k} \int_0^T \exp(-\alpha s_i) f(s_i) ds_i.$$

Choosing $\alpha > 0$ such that $\int_0^T \exp(-\alpha s) f(s) ds \leq ½$, we get

$$b_k = \mu_k([0,T]) \leq \exp(\alpha T)(½)^k.$$

Thus $b_k \longrightarrow 0$. Moreover, since $\Sigma_{k=1}^{\infty} \mu_k([0,T]) < \infty$ and thus

$$\mu(B) := \Sigma_{k=1}^{\infty} \mu_k(B)$$

defines a finite measure on $[0,T]$. Now

$$g(T) \leq \sum_{i=1}^{\infty} a_i = \sum_{k=1}^{\infty} \int \delta(u) \mu_k(du) = \int \delta(u) \mu(du).$$

This proves (i). (ii) is an easy consequence of (i). □

Remark A.2: It follows from the preceding lemma that if g is bounded nonnegative and measurable, f is nonnegative and integrable on $[0,T]$ and

$$g(t) \leq \int_0^t f(t-s)g(s)ds, \quad 0 \leq t \leq T$$

then $g(t) \equiv 0$.

Theorem A.5: Let $\mu_0 \in \mathcal{P}_2(\mathcal{H})$. Then there is a unique solution to the $((\mathcal{L}_t),\mu_0)$ martingale problem on $C([0,T],\mathcal{H})$.

Proof: In view of Theorem A.2, A.3 it suffices to prove that if (Y_t) is a mild solution to (A.4) with $P \circ Y_0^{-1} = \mu_0$, then the law of (Y_t) is uniquely determined. Let (Y_t) be a mild solution.

Let $\Delta = \{0=t_0<t_1<\ldots<t_m=T\}$ be a partition of $[0,T]$ with $|\Delta| = \max_i |t_{i+1}-t_i|$. For $t \in [0,T]$, $\Delta(t) := t_i$ if $t_i \leq t < t_{i+1}$. Let $\tilde{Y}_t \equiv \tilde{Y}_t^\Delta$ be defined as follows: for $t_0 \leq t \leq t_1$,

$$\tilde{Y}_t = S_t Y_0 + \int_0^t S_{t-s}\beta(s,Y_0)dW_s + \int_0^t S_{t-s}\alpha(s,Y_0)ds.$$

It follows that $\{\tilde{Y}_t: 0 \leq t \leq t_1\}$ is a functional of $\{Y_0,(W_\cdot)\}$. Then for $t_1 < t \leq t_2$

$$\tilde{Y}_t = \tilde{Y}_{t_1} + S_t Y_0 - S_{t_1} Y_0 + \int_{t_1}^t S_{t-s}\beta(s,\tilde{Y}_{t_1})dW_s + \int_{t_1}^t S_{t-s}\alpha(s,\tilde{Y}_{t_1})ds.$$

It again follows that $\{\tilde{Y}_t: t_1 < t \leq t_2\}$ is a functional of $\{Y_0,\tilde{Y}_{t_1},(W_\cdot)\}$ and hence that of $\{Y_0,(W_\cdot)\}$.

Having defined $\{\tilde{Y}_t: 0<t\leq t_i\}$, define $\{\tilde{Y}_t: t_i < t \leq t_{i+1}\}$ by

$$\tilde{Y}_t = \tilde{Y}_{t_i} + S_t Y_0 - S_{t_i} Y_0 + \int_{t_i}^t S_{t-s}\beta(s,\tilde{Y}_{t_i})dW_s + \int_0^t S_{t-s}\alpha(s,\tilde{Y}_{t_i})ds.$$

It then follows that $\{\tilde{Y}_t: 0 \leq t \leq T\} = F(Y_0,(W_\cdot))$ for a suitable functional F, and as a result, the law of $(\tilde{Y}_\cdot)$ is uniquely determined. Now, $\tilde{Y}$ satisfies

(A.30) $$\tilde{Y}_t = S_t Y_0 + \int_0^t S_{t-s}\beta(s,\tilde{Y}_{\Delta(s)})ds + \int_0^t S_{t-s}\alpha(s,\tilde{Y}_{\Delta(s)})ds.$$

Proceeding as in Theorem A.3, it can be shown that

(A.31) $$\sup_{t\leq T} E\|\tilde{Y}_t\|^2 \leq C_{A.8}E(1+\|Y_0\|^2) := C_{A.9}.$$

Using Lipschitz conditions (A.7), (A.8) on α,β we can conclude that

$$E\|Y_t-\tilde{Y}_t\|^2 \leq 2\int_0^t \|S_{t-s}\|^2_{H\cdot S}C_{A.2}E\|Y_s-\tilde{Y}_{\Delta(s)}\|^2 ds$$

$$\leq 2C_{A.2}\int_0^t \|S_{t-s}\|^2_{H\cdot S}\{E\|Y_s-\tilde{Y}_s\|^2 + E\|\tilde{Y}_s-\tilde{Y}_{\Delta(s)}\|^2\}ds.$$

Let $\delta_\Delta(s) = E\|\tilde{Y}_s-\tilde{Y}_{\Delta(s)}\|^2$. We will show that if we take Δ_n such that $|\Delta_n| \longrightarrow 0$, then $\delta_{\Delta_n}(s) \longrightarrow 0$. This along with (A.31) and Lemma A.4 implies, (writing $\tilde{Y}_n \equiv \tilde{Y}^{\Delta_n}$).

(A.32) $$\sup_{t\leq T} E\|Y_t-\tilde{Y}^n_t\|^2 \longrightarrow 0.$$

It will then follow that the finite dimensional distributions of $(\tilde{Y}^n_\cdot)$ converge to (and thus determine) the corresponding finite dimensional distributions of $(Y_\cdot)$.

Fix t,Δ and let $r=\Delta(t)$. Again we write $\tilde{Y} \equiv \tilde{Y}^\Delta$. Then

$$E\|\tilde{Y}_t-\tilde{Y}_r\|^2 \leq 3\{E\|(S_{t-r}-I)S_rY_0\|^2 + \int_r^t \|S_{t-s}\|^2_{H.S}C^2_{A.1}(a+C_{A.4})ds$$

and hence

(A.33) $$\delta_\Delta(t) \leq 3\{\int_0^{|\Delta|} \|S_u\|^2_{H.S}du + E\|(S_{t-\Delta(t)}-I)Y_0\|^2\}.$$

Now if $|\Delta_n| \longrightarrow 0$, $\int_0^{|\Delta_n|} \|S_u\|^2_{H.S}du \longrightarrow 0$ and since

$$\|(S_{t-\Delta_n(t)}-I)Y_0\|^2 \leq 2\|Y_0\|^2$$

and

$$\lim_n \|(S_{t-\Delta_n(t)}-I)Y_0\|^2 \longrightarrow 0 \quad \text{a.s.}$$

the dominated convergence theorem implies that the second term on the R.H.S. of (A.33) goes to zero. Hence

$$\delta_{\Delta_n}(t) \longrightarrow 0 \text{ for each } t.$$

This completes the proof.

Remark A.3: Uniqueness of mild solution to (A.4), in the class of measurable processes can be proved by getting the moment estimate (A.22) (with p=2) for any two solutions Y^1, Y^2 and then using the Lipschitz conditions and Remark A.2. The existence of a $C([0,T,\mathcal{H})$ solution can be proved as follows. Let π_n be the orthogonal projection onto the linear span of $\{\psi_1, \ldots, \psi_n\}$ (See (A.3) and let Y^n be the solution to

$$\text{(A.34)} \qquad dY_t^n = -A\pi_n Y_t^n + \pi_n \beta(t, Y_t^n) dW_t + \pi_n \alpha(t, Y_t^n) dt$$

$$Y_0^n = \pi_n Y_0.$$

The existence (and uniqueness) of solution to (A.34) can be proved via routine methods as it is essentially an equation for a finite dimensional process. Again our Lipschitz conditions imply that $E\|Y_t^n - Y_t^m\|^2 \longrightarrow 0$. Further arguments as in Theorem 2.1 will yield that $\{Y_\cdot^n\}$ is "tight" in $C([0,T],\mathcal{H})$. If $\{Y_\cdot\}$ is a weak-limit, it can be proved that Y is a mild solution to (A.4).

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