High-frequency reflection and scattering of sound by ellipsoidally embossed surfaces

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Earlier results for coherent reflection and incoherent scattering by embossed rigid or free surfaces [V. Twersky, J. Acoust. Soc. Am. 29, 209–225 (1957); 73, 85–94 (1983)] are applied to ellipsoidal bosses with axes large compared to wavelength. The asymptotic procedures used originally for circular cylinders and spheres, and, subsequently, for elliptic cylinders [J. E. Burke and V. Twersky, J. Acoust. Soc. Am. 40, 883–895 (1966)] are generalized to triaxial ellipsoids, and results for the corresponding bosses are obtained by the image method.

Illustrative curves for both isotropic and anisotropic surfaces exhibit the influences of boss shape and orientation on the coherent reflection coefficients and incoherent differential scattering cross sections per unit area with emphasis on forward-(specular) and backscattered effects.

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INTRODUCTION

In a previous article, results for scattering by relatively arbitrary bosses on rigid or free (pressure release) base planes were applied numerically to aligned embossed rigid or free surfaces [V. Twersky, J. Acoust. Soc. Am. 29, 209–225 (1957); 73, 85–94 (1983)] are applied to ellipsoidal bosses with axes large compared to wavelength. The asymptotic procedures used originally for circular cylinders and spheres, and, subsequently, for elliptic cylinders [J. E. Burke and V. Twersky, J. Acoust. Soc. Am. 40, 883–895 (1966)] are generalized to triaxial ellipsoids, and results for the corresponding bosses are obtained by the image method.

Illustrative curves for both isotropic and anisotropic surfaces exhibit the influences of boss shape and orientation on the coherent reflection coefficients and incoherent differential scattering cross sections per unit area with emphasis on forward-(specular) and backscattered effects.

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1. NOTATION

We write the position vector as
\[ \mathbf{r} = \mathbf{r}(\theta, \phi), \]
\[ \mathbf{r}(\theta, \phi) = \sin \theta (\cos \phi \mathbf{x} + \sin \phi \mathbf{y}) + \cos \theta \mathbf{z}, \]
and take the incident wave for a rigid (+) or free (−) base plane to be \( \pm e^{-i \omega t} \), where
\[ \phi' = \exp(i k' r), \quad k' = k \mathbf{k}', \quad k = 2\pi / \lambda, \]
\[ \hat{k}' = \hat{r}(\pi - \alpha, \beta), \]
with \( \hat{k}' \) as the direction of incidence. The corresponding wave reflected from a smooth plane at \( z = 0 \) is the image
\[ \phi = \exp(i k \mathbf{r}), \quad k = k \mathbf{k}, \quad \hat{k} = \hat{r}(\alpha, \beta), \]
with \( k \) as the direction of specular reflection. For brevity we define
\[ s = \hat{k} - \hat{r} \quad \text{and} \quad s' = \hat{k} - \hat{r}. \]
When convenient, we work with the direction cosines
\[ \mathbf{k} = \gamma_x \mathbf{x} + \gamma_y \mathbf{y} + \gamma_z \mathbf{z}, \]
with \( \gamma_x = \sin \alpha \cos \beta, \quad \gamma_y = \sin \alpha \sin \beta, \quad \text{and} \quad \gamma_z = \cos \alpha. \)

For an ellipsoid centered at the origin with principal semiaxes \( a, b, c \) along \( x, y, z \), we write the dyadic associated with the ellipsoid as
\[ \mathbf{E} = \frac{\mathbf{x} \mathbf{x}}{a^2} + \frac{\mathbf{y} \mathbf{y}}{b^2} + \frac{\mathbf{z} \mathbf{z}}{c^2}, \]
so that \( \mathbf{r} \mathbf{E} = \mathbf{r} \) is the equation of the ellipsoid. The use of \( \mathbf{E} \) and its reciprocal
\[ \mathbf{E}^{-1} = a^2 \mathbf{x} \mathbf{x} + b^2 \mathbf{y} \mathbf{y} + c^2 \mathbf{z} \mathbf{z}, \]
such that \( \mathbf{E}^{-1} \mathbf{E} = 1 \) is the identity, leads to simple forms for factors that arise in the development.

We let \( g(\mathbf{f}, \mathbf{k}) \) denote the scattering amplitude for \( \mathbf{f} \) incident on an ellipsoid at the phase origin. The scattering amplitude for the corresponding boss on a rigid or free base plane excited by \( \pm e^{i \omega t} \) follows by superposition\(^2\)
\[ f_+ (\mathbf{f}, \mathbf{k}) \mathbf{k} = g_s (\mathbf{f}, \mathbf{k}) \mathbf{E} \mathbf{k}. \]
The coherent reflected field, the ensemble averaged wave for the base plane plus bosses, has the form\(^2\)
\[ \phi(1 + Z)/(1 - Z), \quad Z = m f(\mathbf{k} \mathbf{k}')/k^2 \cos \alpha, \]
where $n$ is the average number of bosses in unit area. As before, we suppress subscripts $\pm$ where feasible.

The coherent power reflection coefficient equals

$$R = \frac{|1 + Z|}{|1 - Z|^2},$$

(10)

and the incoherent scattering in a direction $\mathbf{r}$ is specified by the differential scattering cross section per unit area:

$$\sigma(\mathbf{r}, \hat{k}) = \frac{(n/k^2) f(\mathbf{r}, \hat{k})/(1 - Z)^2}. $$

(11)

Energy conservation for lossless bosses is exhibited by (10) and (11) in that the total scattering from unit area, the integral of (11) over the observer's half-space of directions, satisfies

$$(1 - R) \cos \alpha = \int \sigma(\mathbf{r}, \hat{k}) d\Omega(\mathbf{r}).$$

(12)

Although more complete forms are available in terms of the statistical mechanics pair correlation function, these simple forms for sparse distributions appear appropriate for the applications in view. As discussed before, for some purposes a single scattering approximation may be better except near grazing incidence.

II. LARGE ELLIPSOIDAL BOSSES

For an ellipsoid whose semidiameters are large compared to wavelength, a Kirchhoff (tangent plane) approximation in the surface integral representation of $g$ suffices for the essentials. As before, for rigid (+) or free (−) surfaces, we obtain a geometrically reflected term $g_\pm$ and a diffracted (shadow forming) beam $g_d$:

$$g_\pm(\mathbf{r}, \hat{k}) = g_\pm(\mathbf{r}, \hat{k}) \pm g_i(\mathbf{r}, \hat{k}),$$

(13)

$$g_d = \pm \frac{k}{4\pi} \int_{1,2} e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{s})} ds(\mathbf{r}).$$

Here $\mathbf{p}$ is the radius vector to a point on the ellipsoid's surface and $\hat{\mathbf{v}}$ is the normal. The subscripts 1 and 2 indicate integration over the unirradiated and shadowed sides of the ellipsoid; the planar curve $\mathbf{k} \cdot \hat{\mathbf{v}} = 0$ corresponds to the shadow boundary.

The integral over the irradiated side may be evaluated by the method of stationary phase (except near the forward direction). Thus, in terms of the principal radii of curvature $\rho, \rho, \rho$; at the point of reflection $r$, we have

$$g(\mathbf{r}, \hat{k}) = i k e^{i\varphi} (\rho, \rho, \rho/2) - ik(abc/2b) De^{-ih},$$

$$P^2 = s \hat{\mathbf{E}} \cdot s - D = ss/\rho^2,$$

(14)

The form for $g$, exhibited in (14) is not valid near the forward direction ($\mathbf{r} \approx \hat{k}$) where the method of stationary phase breaks down. Near the forward direction we expand the integrand as a Taylor series and integrate the first two terms exactly to obtain

$$g(\mathbf{r}, \hat{k}) \sim (k^2abc/4\Gamma) Q(\mathbf{r}) = k \hat{k} \cdot \hat{\mathbf{E}} = \Gamma + \Delta,$$

$$\Delta = (\sin \beta)^2 b^2 + (\cos \beta)^2 a^2,$$

$$ss = 2(1 - \hat{k} \cdot \hat{k}).$$

Thus, although (14) need not vanish for $\mathbf{r} \approx \hat{k}$, the appropriate form (15) vanishes in the limit. In particular, for $\varphi = \beta$, and $\alpha = \pi/2 - \tau, \tau = 0$ (corresponding to near-grazing incidence and observation in the specular direction for the boss),

$$g(\mathbf{r}, \hat{k}) \sim k^2 \frac{abc}{\Delta} (1 - \Delta^2/\Delta^2).$$

(16)

The integral for $g_i$ may be evaluated exactly in terms of the Bessel function $J_n$. Using Green's theorem, we replace the surface of integration by the ellipsoid that encloses the shadowed side of the ellipsoid, i.e., by the disk defined by the shadow boundary $k \cdot \hat{\mathbf{v}} = 0$.

$$\int \sigma(\mathbf{r}, \hat{k}) d\Omega(\mathbf{r}) = \frac{X^2 + Y^2 + Z^2}{a^2 + b^2 + c^2} = 0.$$  

(17)

Thus, if $L_1$ and $L_2$ are the directed semidiameters of the disk, then

$$g_i(\mathbf{r}, \hat{k}) = -k^2 [2abc/2\Gamma](\mathbf{r} + \hat{k}) \cdot \mathbf{Q}_{ij}(U)/U.$$  

(18)

For the triaxial ellipsoid we write

$$L_i = L_i(\hat{\mathbf{k}} \cdot \hat{\mathbf{v}} + m, \hat{\mathbf{y}} + n, \hat{\mathbf{k}}),$$  

(19)

where the semidiameters are given by

$$L_i = \frac{B + (B^2 - 4AC)^{1/2}}{2A},$$  

$$A = \mathbf{k} \cdot \hat{\mathbf{E}} \cdot \mathbf{k},$$  

$$B = (a^2 + b^2 + c^2) - 1,$$

$$C = (abc/2b) (\mathbf{k} \cdot \hat{\mathbf{E}} \cdot \mathbf{k}).$$

Their orientations are expressed in terms of the direction cosines

$$\rho = \gamma (r^2 - b^2)(r^2 - c^2)/d_i,$$$$

$$m_i = \gamma (r^2 - a^2)(r^2 - c^2)/d_i,$$$$

$$n_i = \gamma (r^2 - a^2)(r^2 - b^2)/d_i,$$

with $r_i = L_i$, and

$$d_i = [\gamma (r^2 - b^2)(r^2 - c^2)]^2 + [\gamma (r^2 - a^2)(r^2 - c^2)]^2 + [\gamma (r^2 - a^2)(r^2 - b^2)]^2.$$  

(21)

In particular, for incidence in the $xz$ plane ($\beta = 0^\circ$, corresponding to broadside incidence at grazing if $b > a$), we obtain $L_1$ and $L_2$ from the above by interchanging $\hat{x}$ and $\hat{y}$, $a$ and $b$, as well as spin $\varphi$ and $\cos \varphi$. If $a = b$, then (22) corresponds to the spheroid with symmetry axis along $\hat{z}$. If $a = b = c$, then $U$ reduces to the form obtained earlier for the sphere,

$$U = ka[1 - (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}})^2]^{1/2}.$$  

(23)

Substituting (13), (14), and (18) into (8), we write $Z$ of (9) as
Z± = \(-\frac{we\Gamma}{2 \cos \alpha}\) 
\[\left(1 + \frac{\Delta^2 \sin^2 \alpha J_2(2ku \cos \alpha)}{k\Gamma^2u \cos \alpha} \right)^{1/2},\] 
\[\left(-\frac{ie^{-2k\cos \alpha}}{kc^2\Gamma}\right),\] 
\[u = \left(\begin{array}{c}
(L_x \hat{x})^2 + (L_y \hat{y})^2
\end{array}\right)^{1/2} = \left(\begin{array}{c}
(L_x \hat{x})^2 + (L_y \hat{y})^2
\end{array}\right)^{1/2},\] 
where \(w = \pi r a b\) (the packing fraction) is the fraction of the base plane covered by bosses.

### III. SPECIAL CASES

As discussed previously for hemispherical bosses, the first term of \(Z\) of (24) (the shadow term arising from the image wave) is the largest; the second (the shadow term of the incident wave) is negligible except near grazing. The third term (the incident wave reflected at the curved surface) indicates specular reflection at the tangent plane of the bosses. This term is significant in the reflection coefficient when \(we\Gamma/2 \cos \alpha \approx 1\). For near-grazing incidence we replace this last term by the alternate form of \(g\), obtained from (16).

Thus, up to moderately large values of \(\alpha\) and \(we\Gamma/2 \cos \alpha \neq 1\), we keep only the leading term of \(Z\) and use
\[R \approx \left(1 - we\Gamma/2 \cos \alpha \right)/(1 + we\Gamma/2 \cos \alpha).\] 
To this level of approximation, \(R\) vanishes at \(\alpha = \alpha_0\) with
\[\tan \alpha_0 = \left(2/we\Delta\right)\left(1 - w^2/4\right)^{1/2}.\] 
At normal incidence, (25) reduces to
\[R \approx \left(1 - w/2\right)/(1 + w/2).\] 
For \(we\Gamma/2 \cos \alpha \approx 1\), we replace (25) by
\[R \approx \left|1 - we\Gamma/2 \cos \alpha + iwe^{-2k\cos \alpha} / (2k\cos \alpha)\right|^2.\] 
For larger \(\alpha\), all the terms of \(Z\) of (24) are to be retained. As \(\alpha\) approaches grazing, we use the first two terms of \(Z\) of (24) and the alternate form of the third term obtained from (16). Consequently, in terms of the grazing angle \(\tau\), we obtain
\[R \approx 1 - 4\tau /we\Delta, \quad R \approx 1 - wk^2c^4\Delta \tau ; \quad \tau \approx 0.\]

The differential scattering cross sections are given by (11) in terms of \(f\) of (8) and \(Z\) of (24). For directions of observation not near the specular and \(\alpha\) not near grazing, we may neglect the diffracted wave contributions to \(f\) [i.e., \(g_s(\hat{r}, \hat{k})\) and \(g_s(\hat{r'}, \hat{k'})\)] and retain only the first term of \(Z\). Thus
\[\sigma_\pm = \frac{wabc^2}{4\pi} \left|D_{\pm} e^{-2k\cos \alpha} / (1 + we\Gamma/2 \cos \alpha)\right|^2,\] 
\[(P')^2 = s'\cdot E - s', \quad D' = s's'/(P')^2.\] 
The rapidly oscillating cross terms may be ignored except near grazing observation since they will not contribute significantly when integrated over a portion of the distribution or over \(k\hat{a}\) (with \(\hat{a}\) as the minimum semidiameter). Consequently, for both free and rigid surfaces
\[\sigma_\pm \approx \frac{wabc^2}{4\pi(1 + we\Gamma/2 \cos \alpha)^2} \left[D^2 + (D')^2\right],\] 
which drops the rapidly oscillatory terms and provides a baseline for various practical purposes.

In particular, for the back direction (\(\theta = \alpha, \varphi = \pi + \beta\)) for \(\alpha\) not near normal incidence,
\[\sigma_\pm (b) \approx \varepsilon_{\alpha} a b c^2 \left(\delta^2 \sin^2 \alpha + c^2 \cos^2 \alpha\right)^{-1} / (4\pi(1 + we\Gamma/2 \cos \alpha)^2),\] 
with \(\delta^2 = (a \cos \beta)^2 + (b \sin \beta)^2\).

In the forward (specular, i.e., \(\theta = \alpha, \varphi = \beta\)) direction, for \(\alpha\) not near grazing,
\[\sigma_\pm (f) \approx \frac{wabc^2}{4\pi(1 + we\Gamma/2 \cos \alpha)^2} \left[(a \cos \beta)^2 + (b \sin \beta)^2\right].\]

For near-grazing, the forward- and backscattering vanish as \(O(r^2)\) for the rigid (++) surface and as \(O(r^4)\) for the free (--):
we consider elongated ridgelike and vertical disklike bosses by using results for spheroids, as well as generalizations derived for triaxial ellipsoids. Plots versus $\alpha$ with azimuthal angle $\beta$ as the parameter (such that $\beta = 0^\circ$ corresponds to broadside incidence at grazing, and $\beta = 90^\circ$ to nose-on) exhibit the effect of surface anisotropy. For the triaxial case, we fix the boss volume (and height and base area) but vary the ratio of the base diameters to separate shape effects from volume effects.

For all cases, we take $k\bar{a} = 20$ (with $2\bar{a}$ as the smallest diameter) and $\omega = 0.2$ (and discount correlations). In the figures, curves for the rigid surfaces are shown solid and curves for the free surfaces are dotted. Curves obtained from Eqs. (25), (32), and (33) are shown dashed; these serve as baselines for the oscillations of $R$, $\sigma(b)$, and $\sigma(f)$, respectively, arising from interference of the wave components in the more complete expressions. Computations with the explicit approximations of (13) based on (14)-(18) suffice in general. However, in calculating $R$ we use the uniform integral approximation (13) for $g$, to avoid switching from (14) to (15) for $\alpha$ near grazing; computations based on $g_1$ of (14) gave unrealistically high values for $R$ near grazing for some values of the parameters.

Figures 1-3 correspond to isotropic distributions. Figure 1 compares $R$ for hemispheroids with fixed circular base.
FIG. 4. Graphs of $R$ for elongated ridgelike bosses $(a, b, c) = (1, 4, 1)$ with azimuthal angle $\beta$ as parameter. The higher curves for $\alpha$ not near $90^\circ$ correspond to the larger values of $\beta = 90^\circ, 45^\circ$, and $0^\circ$ (with $0^\circ$ as broadside incidence at grazing).

(of radius $a = 2$) and boss height $(c = 2, 3, 4)$ as the parameter. The associated backscattering and forward scattering are shown in Figs. 2 and 3. Scattering effects increase as the boss height (and scatterer volume) increases. To the present level of approximation, the $R$ baseline curves in Fig. 1 show zeros (shifted to smaller $\alpha$ with increasing $c$) corresponding to (26), and the forward-scattering baseline curves in Fig. 3 show maxima whose location and magnitude are determined by (34).

Plots for anisotropic distributions of aligned elongated ridgelike bosses or vertical disklike bosses, as well as triaxial generalizations, are shown in Figs. 4–12. The dependence on azimuthal angle is illustrated in Figs. 4–9, where we take $\beta$ as the parameter. The essentials are indicated by the simple approximations (25), (32), and (33) through their dependence on $\Delta(\beta)$ and $\delta(\beta)$. For ridgelike bosses $(a, b, c) = (1, 4, 1)$ as in Figs. 4–6, $\Delta$ decreases from 1 to 0 as $\beta$ increases from $0^\circ$ to $90^\circ$, while $\delta$ increases from 1 to 4. Thus, for moderate $\alpha$, $R$ of Fig. 4 increases with increasing $\beta$ as indicated by the baseline form (35). The trend is maintained at near grazing for the free surface and reversed for the rigid; see Eq. (29). The zero of $R$, is shifted to larger values of $\alpha$ with increasing $\beta$ as indicated by (26), as is the minimum of $R$. Similarly, the baseline curves for the back- and forward scattering in Figs. 5 and 6 are higher for $\beta = 0^\circ$ (broadside incidence) than for $\beta = 90^\circ$ (nose-on). Analogous plots of $R$, $\sigma(b)$, and $\sigma(f)$ for vertical disklike bosses $(a, b, c) = (1, 4, 4)$ are shown in Figs. 7–9.
FIG. 7. Plots of $R$ for vertical disklike bosses $(a,b,c) = (1,4,4)$ with azimuthal angle $\beta$ as the parameter. The higher curves for $\alpha$ not near $90^\circ$ correspond to the larger values of $\beta = 90^\circ$, $45^\circ$, and $0^\circ$.}

FIG. 8. The backscattering curves associated with Fig. 7. The top, center, and lower panels correspond to $\beta = 90^\circ$, $45^\circ$, and $0^\circ$, respectively.

FIG. 9. The forward-scattering curves associated with Fig. 7. The higher curves correspond to the smaller values of $\beta = 0^\circ$, $45^\circ$, and $90^\circ$.

FIG. 10. Graphs of $R$ for trizial bosses with height $c = 1$ and constant base area $ab = m^2$, with $t$ as the parameter, i.e., for fixed boss volume. The higher curves for $\alpha$ not near $90^\circ$ correspond to $t = 2$ and the lower curves to $t = 3$. 


R. J. Lucas and V. Twersky. Reflection and scattering of sound 2010
Figures 10–12 for \( c = 1 \) and \( ab = a^2t = 4 \) with \( t = 2,3 \) serve to separate shape and volume effects. The bosses for the two cases have the same volume and base area, but differ in the ratio of the base diameters \( a/b \).

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