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13. ABSTRACT (Maximum 200 words)

This report describes the research performed during two years of support by the AFOSR under grant number AFOSR-86-0136. The research falls into four distinct categories:

1) (i) A continuation of the PI's investigation of the theory of structure functions on continua.
2) (ii) The construction of a diffusion model for a system subject to continuous wear.
3) (iii) The introduction of criteria for reliability growth.
4) (iv) An investigation of the stability of stochastic models. Keywords: theoretical mathematics; mathematical models; computerized simulations; molecular interactions; (KT)

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THE THEORY OF STRUCTURE FUNCTIONS

Final Report to the Air Force Office of Scientific Research
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Grant Number: AFOSR-86-0136

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SUMMARY

This report describes the research performed during two years of support by the AFOSR under grant number AFOSR-86-0136. The research falls into four distinct categories:

(i) A continuation of the PI's investigation of the theory of structure functions on continua

(ii) The construction of a diffusion model for a system subject to continuous wear

(iii) The introduction of criteria for reliability growth

(iv) An investigation of the stability of stochastic models

The results obtained are described in detail in Sections 1-4 below.

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1. CONTINUUM STRUCTURE FUNCTIONS

A continuum structure function (CSF) is a mapping $\gamma:\Delta \to [0,1]$, where $\Delta = [0,1]^n$, which is nondecreasing in each argument and which satisfies $\gamma(0,\ldots,0) = 0$ and $\gamma(1,\ldots,1) = 1$. The theory of such functions generalizes traditional binary and multistate structure function theory, permitting more realistic and flexible modelling of systems subject to reliability growth, component degradation and partial availability. CSFs were introduced by Block and Savits [9] and Baxter [2] and subsequently studied in [3-7], [13-15].

Define $P_\alpha = \{x \mid \gamma(x) \geq \alpha$ whereas $\gamma(y) < \alpha$ for all $y < x\}$ where $y < x$ means that $y \leq x$ but that $y \neq x$ $(0 < \alpha \leq 1)$. Block and Savits [9] prove that if γ is right-continuous, i.e. if each $U_\alpha = \{x \mid \gamma(x) \geq \alpha\}$ is closed, then each P_α is nonempty and that the P_α's characterize γ by means of the decomposition

$$\gamma(x) = \int_0^1 \max_{y \in P_\alpha} \min_{1 \leq i \leq n} I_{\{x_i \geq y_i\}} \, d\alpha$$

where I_A denotes the indicator of A.

Structure functions for which each P_α is finite are of particular interest as we shall show. Intuitively, in two-dimensional space, if P_α is finite, each segment of ∂U_α should be parallel to one of the axes. This is, in fact, the case in n-dimensional space.

NOTATION $B_\varepsilon(x)$ denotes an open ball of radius ε centered at x

$U(x)$ denotes $\{y \mid y \geq x\}$

Theorem (Characterization of Finite P_α)

Suppose that U_α is closed $(0 < \alpha \leq 1)$. Then P_α is finite if and only if there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \cap U_\alpha = B_\varepsilon(x) \cap U(x)$ for all $x \varepsilon P_\alpha$.

The class of right-continuous CSFs for which each P_α is finite may be characterized as a natural generalization of the class of Natvig CSFs [3], the defining binary structure functions of the latter being replaced by multistate structure functions. In order to define this class of structure functions, it is necessary to introduce some terminology.

Definition

Let $S_i = \{0,1,\ldots,M_i\}$ for some non-negative integer M_i, $i = 1,2,\ldots,n$. A function $h: \Delta \to \mathop{X}_{i=1}^{n} S_i$ is said to be a reduction mapping if $h(x) = (h_1(x_1),\ldots,h_n(x_n))$ for all $x \varepsilon \Delta$ where $h_i:[0,1] \to S_i$ is a surjective, nondecreasing, right-continuous step function for $i = 1,2,\ldots,n$.

For each $z \varepsilon S_i$, define $h_i^{-1}(z) = \{x \varepsilon [0,1] \mid h_i(x) = z\}$; clearly, $\inf h_i^{-1}(z)$ exists and $h_i(\inf h_i^{-1}(z)) = z$. It can be shown that $\inf h^{-1}(z)$ exists, is unique and equals $(\inf h_1^{-1}(z_1),\ldots,\inf h_n^{-1}(z_n))$.

A mapping $\phi: \mathop{X}_{i=1}^{n} S_i \to \{0,1\ldots,M\}$, $M \geq 1$, which is

nondecreasing in each argument is said to be a <u>multistate structure function</u> (MSF). An MSF is said to be <u>coherent</u> if for any $i \varepsilon C$ and $j \geq 1$, there exists a $z \varepsilon \mathop{X}_{i=1}^{n} S_i$ such that $\phi((j-1)_i, z) < \phi(j_i, z)$ where (j_i, z) denotes $(z_1, \ldots, z_{i-1}, j, z_{i+1}, \ldots, z_n)$ [12].

For a reduction mapping $h: \Delta \to \mathop{X}_{i=1}^{n} S_i$, we write $S_h = \mathop{X}_{i=1}^{n} S_i$, the range of h, and $L_h = \{\inf h^{-1}(z) | z \varepsilon S_h\}$. We also write $x^A = \{x_i \varepsilon x | i \varepsilon A\}$, $h^A(x) = \{h_i(x_i) \varepsilon h(x) | i \varepsilon A\}$ and $S_h^A = \{z^A | z \varepsilon S_h\}$ where $A \subset C$.

<u>Definition</u>

Let $\{(\phi_\alpha, h_\alpha^{C_\alpha}),\ 0 < \alpha \leq 1\}$ be a collection of pairs of functions such that

(i) $h_\alpha : \Delta \to S_{h_\alpha}$ is a reduction mapping for each $\alpha \varepsilon (0,1]$

(ii) $\phi_\alpha : S_{h_\alpha}^{C_\alpha} \to \{0,1\}$ is a coherent MSF, the components of which are the elements of C_α for each $\alpha \varepsilon (0,1]$

(iii) $\bigcup_\alpha C_\alpha = \{1,2,\ldots,n\}$

(iv) For each $x \varepsilon \Delta$, $\phi_\alpha(h_\alpha^{C_\alpha}(x)) \geq \phi_\beta(h_\beta^{C_\beta}(x))$ whenever $\alpha < \beta$ $(0 < \alpha, \beta \leq 1)$.

The function $\gamma: \Delta \to [0,1]$ is said to be an <u>F-type CSF</u> if it satisfies the condition

$$\gamma(x) \geq \alpha \text{ iff } \phi_\alpha(h_\alpha^{C_\alpha}(x)) = 1 \quad (0 < \alpha \leq 1)$$

for all $x \varepsilon \Delta$.

Definition [3]

A CSF γ is said to be weakly coherent if

$$\sup_{x \varepsilon \Delta} [\gamma(1_i, x) - \gamma(0_i, x)] > 0.$$

Theorem

A weakly coherent CSF is of F-type if and only if it is right-continuous and each P_α is finite.

The importance of F-type CSFs is seen in the case where $X_1, \ldots, X_n$, the states of the n components, are independent random variables. A computationally tractable expression for $\phi(x) = P\{\gamma(X) \geq x\}$, the stochastic performance function, occurs only in certain special cases. If, however, P_x is finite, it can be shown that

$$\phi(x) = \sum_{j=1}^{N} \sum_{i=1}^{n} \overline{F}_i(y_i^{(j)}) - \sum_{j_1 < j_2} \sum \prod_{i=1}^{n} \overline{F}_i(y_i^{(j_1)} \vee y_i^{(j_2)})$$

$$+ \ldots + (-1)^{N-1} \prod_{i=1}^{n} \overline{F}_i(\max_{1 \leq j \leq N} y_i^{(j)})$$

where $\overline{F}_i(t) = P\{X_i \geq t\}$, $i = 1,2,\ldots,n$, so that $\phi(x)$ can be easily evaluated.

More is true. If γ is an arbitrary right-continuous CSF with stochastic performance function ϕ, there exists a sequence $\{\gamma_m\}$ of CSFs such that $\phi_m \to \phi$ pointwise as $m \to \infty$ where ϕ_m is the stochastic performance function corresponding to γ_m. Further, if ϕ is continuous, the convergence is uniform.

A condition under which ϕ is continuous is given by the following proposition.

Definition

Write $x \overset{*}{<} y$ if $x_i \leq y_i$ and $x_i < y_i$ if $y_i > 0$, $i = 1,2,\ldots,n$. A CSF γ is said to be strictly increasing if $\gamma(x) < \gamma(y)$ whenever $x \overset{*}{<} y$.

Proposition

If γ is a strictly increasing CSF and if X is an absolutely continuous random vector then ϕ is continuous.

Suppose, now, that the structure function changes with time, possibly reflecting improvements as the system is developed or changes in the use of the system. Let γ_t denote the CSF at time t. Suppose, further, that the states of the components comprise a stochastic process $\{X(t), t \geq 0\}$, perhaps reflecting the degradation of the components with use or the replacement of failed components.

Theorem

If $X(t) \xrightarrow{D} X$, an absolutely continuous random vector, as $t \to \infty$ and $\gamma_t \to \gamma$ pointwise on Δ as $t \to \infty$, then $\gamma_t(X(t)) \xrightarrow{D} \gamma(X)$ as $t \to \infty$.

The proof is by consideration of the set $E = \{x \in \Delta \mid \gamma_t(x_t) \not\to \gamma(x)$ for some class $\{x_t\}$ such that $\lim_{t\to\infty} x_t = x\}$. It can be shown that if $\gamma_t \to \gamma$ pointwise as $t \to \infty$, then E has Lebesgue measure zero and hence, if X is absolutely continuous, $PoX^{-1}(E) = 0$ so that, by Theorem 5.5 of Billingsley [8, p. 34], the result follows.

Consider, now, the class $\{\gamma_t'\}$ of CSFs where $\gamma_t':[0,1]\to[0,1]$ is defined by

$$\gamma_t'(x) = \begin{cases} 1 & \text{if } x \geq 1/2 + 1/t \\ 0 & \text{otherwise} \end{cases}$$

for $t \geq 2$ and the CSF $\gamma':[0,1]\to[0,1]$ defined by

$$\gamma'(x) = \begin{cases} 1 & \text{if } x \geq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, γ_t' does not converge pointwise to γ'. Since, however, the set of points (viz. $\{1/2\}$) at which convergence fails has Lebesgue measure zero, and hence $P \circ X^{-1}$ measure zero if X is absolutely continuous, it is reasonable to enquire whether there exists an alternative mode of convergence under which $\gamma_t'(X(t)) \xrightarrow{D} \gamma'(X)$ as $t\to\infty$. This motivates the following definition.

Let Λ be the group (under composition) of all homeomorphisms λ: $R^n \to R^n$ such that $\lambda(0,\ldots,0) = 0$ and $\lambda(1,\ldots,1) = 1$ and let d be the Euclidean metric on R^n. For any two right-continuous CSFs γ_1 and γ_2 we define

$$S(\gamma_1,\gamma_2) = \inf_{\lambda\varepsilon\Lambda}\left\{ \sup_{y\varepsilon R^n} |\gamma_1(y^*) - \gamma_2(\lambda(y)^*)| + \sup_{y\varepsilon R^n} d(y^*,\lambda(y)^*)\right\}$$

where y^* is the vector whose i^{th} component is $(y_i \vee 0)\wedge 1$.

It is easily verified that S is a metric on the space of right-continuous CSFs; we call it the <u>quasi-Skorohod</u> metric and we call the induced topology the <u>quasi-Skorohod</u> <u>topology</u> (QST).

Convergence in the QST is weaker than uniform convergence; it is neither weaker nor stronger than pointwise convergence.

Theorem

Suppose that $X(t) \xrightarrow{D} X$, an absolutely continuous random vector, as $t \to \infty$. Suppose, further that γ is a right-continuous CSF and that $\{\gamma_t\}$ is a class of right-continuous CSFs such that $\gamma_t \to \gamma$ in the QST as $t \to \infty$. Then $\gamma_t(X(t)) \xrightarrow{D} \gamma(X)$ as $t \to \infty$.

For the CSFs $\{\gamma_t'\}$ and γ' defined on page 7, it can be shown that $\gamma_t'(X(t)) \xrightarrow{D} \gamma'(X)$ as $t \to \infty$ whenever $X(t) \xrightarrow{D} X$, an absolutely continuous random vector, as $t \to \infty$.

Let δ_α denote the intersection of ∂U_α and $\{\alpha | 0 \leq \alpha \leq 1\}$, the diagonal of Δ. We say that δ_α is the <u>key vector</u> of U_α and we call the scalar δ_α the corresponding <u>key element</u>. Kim and Baxter [13] show that for any CSF γ, the key vector always exists, is unique and, if γ is continuous, $\gamma(\delta_\alpha) = \alpha$ for all $\alpha \varepsilon (0,1]$. Kim and Baxter [13] use the key element to define reliability importance when X is a random vector: they define the reliability importance of component i at level $\alpha \varepsilon (0,1]$ as

$$R_i(\alpha) = P\{\gamma(X) \geq \alpha | X_i \geq \delta_\alpha\} - P\{\gamma(X) \geq \alpha | X_i < \delta_\alpha\},$$

$i = 1,2,\ldots,n$. This generalizes Birnbaum's definition of reliability importance for the components of binary structure functions [1, Chapter 2] to the continuum case.

For any component i and any subset $A \subset \Delta$, define $A^i = \{x \varepsilon \Delta | (\cdot_i, x) = (\cdot_i, z)$ for some $z \varepsilon A\}$. Notice that $A \subset A^i$ and that $A = A^i$ if and only if whether or not $x \varepsilon A$ does not depend on the state of component i. Let μ denote Lebesgue measure on R^n.

Definition

Let γ be a CSF and let $\alpha \varepsilon [0,1]$. We say that component i is almost irrelevant to γ at level α if there exists a subset $E_\alpha \subset \Delta$ such that $\mu(E_\alpha^c) = 0$ and $U_\alpha \cap E_\alpha = (U_\alpha \cap E_\alpha)^i \cap E_\alpha$. Further, if component i is almost irrelevant to γ at level α for all $\alpha \varepsilon [0,1]$, we say that it is almost irrelevant to γ.

It can be shown that component i is almost irrelevant to γ if and only if there exists a CSF γ' such that $\gamma' = \gamma$ a.e.$[\mu]$ and $\sup_{x \varepsilon \Delta} [\gamma(1_i, x) - \gamma(0_i, x)] = 0$.

Before proceeding to a study of properties of $R_i(\alpha)$, it is convenient to deduce sufficient conditions under which $0 < P\{X_i \geq \delta_\alpha\} < 1$ for $0 < \alpha < 1$. It can be shown that the conditions that γ is continuous at $(0,\ldots,0)$ and $(1,\ldots,1)$ and that the support of each X_i is the unit interval are sufficient to ensure that $0 < P\{X_i \geq \delta_\alpha\} < 1$ for $0 < \alpha < 1$.

Theorem

Suppose that the CSF γ is continuous at $(0,\ldots,0)$ and $(1,\ldots,1)$ and that $X_1,\ldots,X_n$ are independent, absolutely continuous random variables, the support of each of which is the unit interval. Then for

all $\alpha\varepsilon(0,1)$ and each $i = 1,2,\ldots,n$, $R_i(\alpha) = 0$ if and only if component i is almost irrelevant to γ at level α.

Definition

A CSF γ is said to be strongly increasing if $\gamma(x) > \gamma(y)$ whenever $x_i > y_i$ for $i = 1,2,\ldots,n$.

Define a function $\delta:[0,1]\rightarrow[0,1]$ by $\delta(\alpha) = \delta_\alpha$ where δ_α is the key element of U_α. We call δ the key function of γ. It is easily seen that δ is well defined, nondecreasing and left-continuous, and is continuous if γ is strongly increasing.

Theorem

Let γ be a strongly increasing CSF which is continuous at $(0,\ldots,0)$ and $(1,\ldots,1)$ and suppose that $X_1,\ldots,X_n$ are independent, absolutely continuous random variables, the support of each of which is the unit interval. then $R_i(\alpha)$ is continuous on $(0,1)$ for $i = 1,2,\ldots,n$.

It is of interest to determine when one component is uniformly more important than another, i.e. when $R_i(\alpha) \geq R_j(\alpha)$ for all α, c.f. Theorem 2.1 of Natvig [17].

Definition

Let γ be a CSF and let δ_α denote the key element of U_α. We say that component i is connected in series (parallel) to the remainder of the components if, for all $\alpha\varepsilon[0,1]$, $x\varepsilon U_\alpha \Rightarrow (\Leftarrow) x_i \geq \delta_\alpha$.

In the special case when γ is a binary structure function, this definition reduces to the usual definitions of being connected in series and in parallel.

Theorem

Let γ be a CSF which is continuous at $(0,\ldots,0)$ and $(1,\ldots,1)$ and suppose that $X_1,\ldots,X_n$ are random variables each with support the unit interval. If component i is connected in series to the remainder of the components and if $X_i \overset{st}{\leq} X_j$, then $R_i(\alpha) \geq R_j(\alpha)$ for all $\alpha\varepsilon(0,1)$, $j \neq i$.

Notice that it was not necessary to assume that $X_1,\ldots,X_n$ are mutually independent.

Theorem

Let γ be a CSF which is continuous at $(0,\ldots,0)$ and $(1,\ldots,1)$ and let $X_1,\ldots,X_n$ be independent random variables, each with support the unit interval. If component i is connected in parallel to the remainder of the components and if $X_i \overset{st}{\geq} X_j$, then $R_i(\alpha) \geq R_j(\alpha)$ for all $\alpha\varepsilon(0,1)$, $j \neq i$.

The algorithm previously introduced to calculate $\phi(\alpha)$ can easily be modified to evaluate $R_i(\alpha)$.

2. A DIFFUSION MODEL

This section introduces a model for a system whose state changes continuously with time, perhaps reflecting continuous wear. It is assumed that the state of the system is initially $x_0>0$ and thereafter follows Brownian motion with negative drift and an absorbing barrier at the origin. It is further assumed that the state of the system is increased by a repairman who arrives according to a Poisson process of rate $\lambda>0$; if the state of the system when the repairman arrives exceeds a threshold α $(0\leq\alpha<x_0)$, no action is taken, otherwise the repairman instantaneously increases the state by a random amount Y where $Y\geq\alpha$ almost surely.

Let X(t) denote the state of the system at time t, $t\geq 0$, let $F(x,t) = P\{X(t)\leq x\}$ denote the distribution function of X(t) and suppose that the corresponding density $f(x,t) = \partial F(x,t)/\partial x$ exists for $x>0$. Since $\{X(t), t\geq 0\}$ is a diffusion process with jump discontinuities, it is a special case of the model analyzed by Feller [11] and hence, from equation (27) of Feller [11], we obtain the following integro-differential equation (valid for $x>0$):

$$(*)\quad \frac{\partial}{\partial t}f(x,t) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}f(x,t) - \mu\frac{\partial}{\partial x}f(x,t) - \lambda I_{\{x\leq\alpha\}}f(x,t)$$

$$+ \lambda\int_\alpha^x I_{\{x-z\leq\alpha\}}f(x-z,t)g(z)dz + \lambda F(0,t)g(x)$$

where μ and σ^2 are the parameters of the Brownian motion $(\mu<0,\sigma>0)$ and g is the density of Y.

Feller [11] has obtained the general solution to integro-differential equations of the form (*), but his solution is extremely difficult to apply, and hence a different approach was adopted: an explicit formula for $F(x,t)$, $0 \leq x \leq \alpha$, was obtained by a purely probabilistic argument which does not make use of (*), and these results and (*) were used to obtain an expression for the Laplace transform of $f(x,t)$. The distribution function of $X(t)$ is

$$F(x,t) = \int_0^t B(x,t-u)e^{-\lambda(t-u)}dH(u) \qquad (0 \leq x \leq \alpha)$$

where

$$1-B(x,t) = \int_x^\infty \frac{1}{\sigma\sqrt{2\pi t}}\left[\exp\left\{-\frac{(z-\alpha-\mu t)^2}{2\sigma^2 t}\right\} - \exp\left\{\frac{-2\mu\alpha}{\sigma^2} - \frac{(z+\alpha-\mu t)^2}{2\sigma^2 t}\right\}\right]dz \qquad (x \geq 0)$$

and where

$$H(t) = W_{x_0,\alpha}(t) + \sum_{n=1}^{\infty} W_{x_0,\alpha} * K^{(n)}(t)$$

where the asterisk denotes Stieltjes convolution and the superscript denotes n-fold recursive Stieltjes convolution. An expression for K is given by the following formula:

$$K(t) = \int_0^t \int_\alpha^\infty W_{u,\alpha}(s)\lambda e^{-\lambda(t-s)}dV(u)ds$$

where

$$V(x) = \int_0^\infty \int_\alpha^x B((x-y)\wedge\alpha,t)\lambda e^{\lambda t} g(y)dydt$$

$$+ \int_0^\infty [B(x,t) - B(\alpha,t)]\lambda e^{-\lambda t} dt$$

and where

$$W_{a,b}(t) = \int_0^t \frac{a-b}{\sigma\sqrt{2\pi x^3}} \exp\{-(a-b+\mu x)^2/2\sigma^2 x\}dx.$$

An expression for $F(x,t)$, $x>\alpha$, cannot be obtained by an analogous argument. One may, however, solve (*) to yield an expression for the Laplace transform of $f(x,t)$. This is uninformative and awkward to deal with but, for the stationary distribution, viz. $F(x) = \lim_{t\to\infty} F(x,t)$, an explicit expression may be obtained. This may be differentiated to yield (admittedly clumsy) formulae for the stationary mean and variance.

The preceding analysis can readily be extended to the case where the repairman arrives according to a stationary renewal process in which the distribution function of the inter-renewal times is Q, say.

Define $T_0 = \inf\{t|X(t) = 0\}$, the first passage time to state 0. We deduce expressions for the distribution function of T_0 and for $E(T_0)$ and we derive the limiting distribution of T_0 as $\alpha\to\infty$.

Let p be the probability that when a repair is performed, the system is at state 0. Clearly, defining $P(t) = \int_0^t \bar{Q}(u)du/\int_0^\infty \bar{Q}(u)du$,

$$p = \int_0^\infty B(0,t)dP(t)/\int_0^\infty B(\alpha,t)dP(t)$$

Observe that T_0 satisfies the following relation:

$$T_0 \overset{D}{=} S_{x_0} + \sum_{i=1}^{n} Y_i \overset{D}{=} S_{x_0} + \sum_{i=1}^{n} S_{Y_i} \text{ with probability } p(1-p)^n$$

for $n = 0,1,2,\ldots$ where $Y_i \overset{D}{=} Y$ for all i, where S_a is the first passage time from $a>0$ to 0 in Brownian motion and where $\overset{D}{=}$ denotes equality in distribution. It is readily seen that

$$P\{S_Y \leq t\} = \int_0^\infty W_{Y,0}(t)dG(y) = D(t), \text{ say,}$$

and hence the distribution function of T_0, $L(t)$ say, is given by

$$L(t) = \sum_{n=0}^{\infty} W_{x_0,0} * D^{(n)}(t)p(1-p)^n$$

where $D^{(0)}(t)$ is the Heaviside function and $W_{a,0}$ is the distribution function of S_a. Now

$$E(T_0) = \sum_{n=0}^{\infty} [E(S_{x_0}) + nE(S_Y)]p(1-p)^n.$$

Since $E(S_{a-b}) = (b-a)/\mu$, and hence

$$E(S_Y) = \int_0^\infty E(S_y)dG(y) = -m/\mu,$$

it is easily seen that

$$E(T_0) = -[px_0 + m(1-p)]/p\mu$$

where $m = E(Y)$.

Theorem

As $\alpha \to \infty$, $T_0/E(T_0)$ converges in distribution to a unit exponential variate.

Now we deduce an expression for $\Pi_x(t_1,t_2) = P\{X(t) > x \text{ for all } t \varepsilon [t_1,t_2]\}$.

Consider, firstly, the case $x \geq \alpha$. The first passage time from $X(t_1)$ to x is equal in distribution to $S_{X(t_1)-x}$. Hence, if $W_{a,b}$ denotes the distribution of S_{a-b}, the first passage time from a to b in Brownian motion with parameters μ and σ^2 $(a > b)$

$$\Pi_x(t_1,t_2) = \int_x^\infty \overline{W}_{y,x}(t_2-t_1)d_yF(y,t_1).$$

Consider, now, the case $x < \alpha$. Let $\{Z_x(t),\ t \geq 0\}$ denote Brownian motion with parameters μ and σ^2, an absorbing barrier at x and initial condition $Z_x(0) = \alpha$. Let $B_x(y,t) = P\{Z_x(t) \leq y\}$ denote the distribution function of $Z_x(t)$ and let $T_{y,x}$ be the first passage time from y to x in the process $\{X(t),\ t \geq 0\}$. It can be shown that the distribution function of $T_{y,x}$, $L_{y,x}(t)$ say, is given by

$$L_{y,x}(t) = \sum_{n=0}^{\infty} W_{y,x} * D^{(n)}(t) p_x (1-p_x)^n.$$

where $p_x = \int_0^\infty B_x(x,t)dP(t)/\int_0^\infty B_x(\alpha,t)dP(t)$. Hence,

$$\Pi_x(t_1,t_2) = \int_x^\infty \overline{L}_{y,x}(t_2-t_1)d_yF(y,t_1).$$

It is henceforth assumed that the process $\{X(t), t \geq 0\}$ is stationary, and that the repairman arrives according to a Poisson process of rate λ. Let C_1 denote the cost per visit of the repairman, let C_2 denote the cost of a repair (this may depend on the distribution of Y) and let C_3 denote the cost per unit time of the system being in state 0. We calculate $C(\lambda)$, the average cost per unit time over an infinite horizon for a given arrival rate λ. To do so, it is convenient to define the following random variables:

N_1 = number of visits by the repairman in a cycle

N_2 = number of repairs performed in a cycle

T_1 = cumulative time spent in state 0 in a cycle

where, by a cycle, we mean the interval between two successive instants at which the state of the system first crosses the threshold α from above following a visit by the repairman (recall that the sequence of such instants comprises an embedded renewal process). The duration of a generic interval is denoted T^*. It is clear that $N_2 = I_{\{Z(T) \leq \alpha\}}$ and that $T_1 = \int_0^T I_{\{Z(t)=0\}} dt$ where T is an exponential variate with parameter λ and $\{Z(t), t \geq 0\}$ denotes Brownian motion with parameters μ and σ^2, an absorbing barrier at the origin and initial condition $Z(0) = \alpha$.

It is easily seen that $E(N_1) = \lambda E(T^*)$. Further, $E(N_2) = P\{Z(T) \leq \alpha\}$ and it can be shown that $E(T_1) = e^{-\alpha\theta_1(\lambda)}/\lambda$ where $\theta_1(\lambda) = [\mu+(\mu+2\lambda\sigma^2)^{1/2}]/\sigma^2$.

Let $\widetilde{C}(\lambda)$ denote the average cost per cycle for a given arrival rate λ. By the foregoing,

$$\tilde{C}(\lambda) = C_1 \lambda E(T^*) + C_2 q(\lambda) + C_3 e^{-\alpha\theta_1(\lambda)}/\lambda.$$

Then $C(\lambda) = \tilde{C}(\lambda)/E(T^*)$ and, since $E(T^*) = e^{-\alpha\theta_1(\lambda)}/\lambda - mq(\lambda)/\mu$, it follows that

$$C(\lambda) = C_1 \lambda + \frac{C_2 \lambda q(\lambda) + C_3 e^{-\alpha\theta_1(\lambda)}}{e^{-\alpha\theta_1(\lambda)} - \lambda m q(\lambda)/\mu},$$

where

$$q(\lambda) = \frac{\theta_2(\lambda) - \theta_1(\lambda)\exp\{\alpha[\theta_2(\lambda)-\theta_1(\lambda)]\}}{\theta_2(\lambda) - \theta_1(\lambda)}$$

and where $\theta_2(\lambda) = [\mu-(\mu+2\lambda\alpha^2)^{1/2}]/\sigma^2$, for $\lambda>0$; by continuity, we define $C(0) = C_3$.

We seek to minimize $C(\lambda)$ by varying λ.

Firstly, we consider the case $C_1 = 0$.

<u>Theorem</u>

If $C_2 \geq -mC_3/\mu$, then $C(\lambda)$ achieves its minimum value, C_3, at $\lambda=0$, otherwise $C(\lambda)$ decreases monotonically to $-\mu C_2/m$ as $\lambda\to\infty$.

Now we consider the general case.

<u>Theorem</u>

If $C_1 + C_2 \geq -mC_3/\mu$, then $C(\lambda)$ achieves its minimum value, C_3, at $\lambda=0$, otherwise there exists a unique λ^* $(0<\lambda^*<\infty)$ which minimizes $C(\lambda)$.

For the case $\alpha=0$, we can obtain an explicit expression for λ^*.

<u>Theorem</u>

Suppose $\alpha=0$. If $C_1 + C_2 \geq -mC_3/\mu$, then $C(\lambda)$ achieves its minimum value, C_3, at $\lambda=0$, otherwise $C(\lambda)$ is minimized at

$$\lambda^* = \frac{C_1 - \sqrt{-C_1(C_2+mC_3/\mu)}}{mC_1/\mu} .$$

3. RELIABILITY GROWTH

Let $\{N(t),\ t\geq 0\}$ be a stochastic point process; we identify the points of $\{N(t),\ t\geq 0\}$ with the failures of a system. Our concern is with systems which exhibit reliability growth, i.e. systems whose reliability increases in time. We model such systems by stochastic point processes, the points of which become "scarcer" according to various criteria introduced in this section. Define the intensity function $h(t) = \lim_{\delta\to 0+} P\{N(t+\delta) - N(t)>0\}\delta^{-1}$ and say that $\{N(t),\ t\geq 0\}$ is orderly if $P\{N(t+\delta) - N(t)>1\} = o(\delta)$ for all $t\geq 0$. Let $H(t) = E[N(t)]$ denote the expected number of failures in $(0,t]$, let X_n denote the time from the $(n-1)^{th}$ failure to the nth failure $(n\geq 2)$, X_1 being the time to the first failure, and define the forward recurrence time at t as $\gamma(t) = \sum_{i=1}^{N(t)+1} X_i - t$.

The following four criteria for reliability growth are posited:

C1 $N(t+s) - N(t)$ is stochastically decreasing in t for all $s>0$

C2 $\gamma(t)$ is stochastically increasing in t

C3 X_n is stochastically increasing in n

C4 $H(t)/t$ is nonincreasing in t.

The relationships between these criteria are as in the following diagram; no other implications hold.

$$\begin{tikzcd}
 & C1 \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
C2 \arrow[rr, Rightarrow, "*"] \arrow[dr, Rightarrow] & & C4 \\
 & C3 &
\end{tikzcd}$$

*for orderly processes only.

By way of illustration, these criteria are applied to three models: the initial faults model, the nonhomogeneous Poisson process (NHPP) and the renewal process.

The Initial Faults Model

Suppose that there are initially N faults in the system, each of which will eventually give rise to a failure. The times until the failures actually occur are assumed to be independent random variables, each with distribution function F, so that $N(t) \sim Bin(N,F(t))$. The sequence of times at which a failure occurs is distributed as the order statistics corresponding to N independent observations of a random variable with distribution function F. It can be shown that a sufficient condition for C1, C2 or C3 to hold is that F is DFR, and it is obvious that C4 holds if $F(t)/t$ is nonincreasing in t.

The Nonhomogeneous Poisson Process

Let $\{N(t), t \geq 0\}$ be an NHPP with hazard function $\lambda(t) = dE[N(t)]/dt$. A sufficient condition for C1, C2 or C3 to hold is that λ is nonincreasing, i.e. that F, the distribution function of the time to the first failure, is DFR. Further, C4 holds if F is DFRA.

The Renewal Process

Brown [10] proves that a renewal process in which the distribution of the inter-renewal times is DFR satisfies C1, C2 and C4. Trivially, <u>any</u> renewal process satisfies C3.

Theorem

Criteria C1, C2 and C4 are preserved by the formation of series systems whereas criterion C3 is not.

4. STABILITY THEORY

No stochastic model is a true representation of an actual physical phenomenon: assumptions such as independence, equality in distribution, exponentiality etc. are seldom satisfied in practice. It is therefore of interest to investigate the stability of such models, i.e. to determine how a violation of the assumptions affects the properties of the model. Let $\mathcal{U}$ denote the class of all "input" to a stochastic model and let $\mathcal{V}$ denote the corresponding class of "output". Thus, for example, in a GI/G/1 queue, $U \varepsilon \mathcal{U}$ would comprise the arrival process and the sequence of service times whereas $V \varepsilon \mathcal{V}$ would be, for example, the sequence of waiting times. Let $\mathcal{U}^*$ be the class of "perturbed" "input" and let $\mathcal{V}^*$ be the corresponding "output" class. Then a stability analysis would be to determine conditions under which

$$\rho(U,U^*)<\varepsilon \Rightarrow \rho(V,V^*) < \delta(\varepsilon)$$

where $\varepsilon>0$, $\delta(\varepsilon)>0$ and ρ is a probability metric, typically the uniform metric, on the space of distribution functions (for simplicity, we write $\rho(X,Y)$ to denote the distance between the distribution functions of X and Y).

We now present two stability analyses: one of a characterization of the bivariate Marshall-Olkin distribution and one of the estimation of the mean of the (univariate) exponential distribution.

Recall that if $G(x,y) = P\{X_1>x,X_2>y\}$ denotes the bivariate survivor function of (X_1,X_2), the Marshall-Olkin distribution is

defined by

$$G(x,y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)\}$$

for $\lambda_1, \lambda_2 > 0$, $\lambda_{12} \geq 0$ and $x,y \geq 0$ (e.g. Marshall and Olkin [16]).

Let B denote the class of all bivariate survivor functions of pairs of nonnegative random variables. For $G \varepsilon B$, define the hazard vector $(h_1(t), h_2(t)) = \nabla[-\log G(t,t)]$, assuming that this exists, and write $H_1(x,y) = -\partial \log G(x,y)/\partial x$, $H_2(x,y) = -\partial \log G(x,y)/\partial y$ and $c = H_1(0,0) + H_2(0,0)$. The corresponding bivariate Marshall-Olkin distribution is defined as

$$\tilde{G}(x,y) = \begin{cases} \exp\{-cy - h_1(0)(x-y)\} & \text{if } x \geq y \\ \exp\{-cx - h_2(0)(y-x)\} & \text{if } x \leq y, \end{cases}$$

i.e. $\tilde{G}$ is the bivariate survivor function of the Marshall-Olkin distribution with $\lambda_1 = c - h_2(0)$, $\lambda_2 = c - h_1(0)$ and $\lambda_{12} = h_1(0) + h_2(0) - c$. Notice that if (X_1, X_2) has survivor function $\tilde{G}$, then $E(X_i) = 1/h_i(0)$ and $E(X_i^2) = 2/[h_i(0)]^2$ $(i=1,2)$ and $F(X_1 X_2) = [1/h_1(0) + 1/h_2(0)]/c$.

<u>Definition</u>

Suppose that $G \varepsilon B$, the survivor function of (X_1, X_2), satisfies the inequalities

(i) $G(x+t, y+t) \leq G(x,y)G(t,t)$ for all $x,y,t \geq 0$

(ii) $G_i(x+t) \leq G_i(x)G_i(t)$ for all $x,t \geq 0$ where G_i is the survivor function of X_i $(i=1,2)$.

Then G is said to be <u>weakly bivariate NBU</u> (WBNBU). Further, G is said to be WBNBU* if G is WBNBU and if the partial derivatives $\partial G(x,y)/\partial x$ and $\partial G(x,y)/\partial y$ exist.

Define

$$\beta(G) = \{[1/h_1(0) - E(X_1)] + [1/h_2(0) - E(X_2)]$$

$$+ [2E(X_1) - h_1(0)E(X_1^2)] + [2E(X_2) - h_2(0)E(X_2^2)]$$

$$+ 4[E(X_1) + E(X_2) - cE(X_1X_2)\}/c.$$

<u>Theorem</u>

If G is WBNBU*, then G is Marshall-Olkin if and only if $\beta(G) = 0$.

We now quantify the stability of this characterization by deriving a bound on the uniform distance between an arbitrary WBNBU* survivor function G and the corresponding Marshall-Olkin survivor function $\tilde{G}$.

<u>Theorem</u>

$\rho(G,\tilde{G}) \leq (1+c^2e^c)[\beta(G)]^{1/3}$ for all $G \varepsilon B$.

Suppose that Y is exponentially distributed with probability density function (p.d.f.)

$$f_Y(x) = \frac{1}{\mu} e^{-x/\mu} \qquad (x \geq 0; \mu > 0).$$

To perform hypothesis tests on μ, one makes use of the fact that, if $Y_1, Y_2, \ldots, Y_n$ are n independent, identically distributed random variables, each with p.d.f. f_Y, then $2 \sum_{i=1}^{n} Y_i/\mu \sim \chi^2_{2n}$. In practice, the assumption of exponentiality is only an approximation; it is therefore of interest to enquire how well the χ^2_{2n} distribution approximates that of $2 \sum_{i=1}^{n} X_i/\mu$ where $X_1, X_2, \ldots, X_n$ are n independent, absolutely continuous, identically distributed random variables with common distribution function F_X and mean μ and $F_X \in \mathcal{F}$, the class of all possible perturbations of the exponential distribution. If is the union of the class of HNBUE distributions and the class of HNWUE distributions, a simple bound on the uniform distance $\rho_{\mathcal{F}}$ between $Z_n = 2 \sum_{i=1}^{n} Y_i/\mu$ and $W_n = 2 \sum_{i=1}^{n} X_i/\mu$ can be obtained without making any assumptions concerning the mechanism generating the perturbation. It can be shown that

$$\rho(W_n, Z_n) \leq \frac{3}{2^{1/3}} M_n^{2/3} |1-(\sigma/\mu)^2|^{1/3}$$

where $\sigma^2 = \mathrm{var}(X)$ and

$$M_n = \frac{\sqrt{n}(n-1)^{n-1} e^{-(n-1)}}{2(n-1)!} .$$

Thus, if X is HNBUE or HNWUE and if the coefficient of variation of X is close to unity, then the distribution of $2 \sum_{i=1}^{n} X_i/\mu$ is uniformly close to that of the χ^2_{2n} distribution. In particular, the sampling distribution of the usual test statistic for hypothesis

tests concerning the scale parameter of the exponential distribution is robust with respect to moderate departures from exponentiality provided that the perturbed random variable is HNBUE or HNWUE.

Suppose, now, that no assumptions concerning $\mathcal{F}$ are made. It is necessary to make an assumption concerning the mechanism perturbing the exponential variate Y. Three possible cases are considered:

(i) Mixing

Suppose that an exponential distribution is "contaminated" with an arbitrary non-negative random variable with distribution function H, ie.

$$\overline{F}_X(t) = (1-\varepsilon)e^{-t/\lambda} + \varepsilon\,\overline{H}(t)$$

where λ is chosen so that $\mu = (1-\varepsilon)\lambda + \varepsilon\alpha$ and where $a = \int_0^\infty \overline{H}(t)dt$. It is assumed that $\varepsilon > 0$ is small. It can be shown that $\rho(W_n, Z_n) \leq O(\varepsilon^{1/3})$.

(ii) Additive Error

Suppose, now, that $X \overset{D}{=} Y_\lambda + Z_\varepsilon$ where Y_λ is an exponential variate with mean λ and Z_ε is an arbitrary random variable; λ is chosen so that $\mu = \lambda + E(Z_\varepsilon)$. It can be shown that

$$\rho(W_n, Z_n) \leq (1 + 1/\delta)[\delta b(\delta)]^{1/(1+\delta)}\ [2E(|Z_\varepsilon|)]^{\delta/(1+\delta)}$$

where $b(\delta) = E(|X|^{2+\delta}) + E(|Y_\lambda|^{2+\delta})$ $(\delta > 0)$ is assumed to be finite.

(iii) Right Censoring

Lastly, suppose that $X = \min(Y_\lambda, N)$ where N is a non-negative random variable independent of Y_λ and where λ is chosen such that $E(X) = \mu$. It can be shown that

$$\rho(W_n, Z_n) \leq |\lambda^2 - \mu^2| + \varepsilon\lambda^2 + 2\varepsilon$$

where $\varepsilon = P\{N \leq \eta\}$ and where η is the solution to the equation

$$P\{N \leq \eta\} = \gamma e^{-\eta/\gamma}(\eta + \gamma),$$

writing $\gamma = \max(\lambda, \mu)$.

In each of the above cases, it is clear that a small perturbation of Y (i.e. $\rho(X,Y)$ is small) yields a sampling distribution of $2\sum_{i=1}^{n} X_i/\mu$ which is uniformly close to that of the χ^2_{2n} distribution.

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APPENDIX

Papers in Technical Journals

"Structure Functions with Finite Minimal Vector Sets" (with S.M. Lee), Journal of Applied Probability, Vol. 26, No. 1 (to appear).

"A Weak Convergence Theorem for Continuum Structure Functions" (with S.M. Lee). Submitted for publication.

"Further Properties of Reliability Importance for Continuum Structure Functions" (with S.M. Lee). Submitted for publication.

"An Inventory with Constant Demand and Poisson Restocking" (with E.Y. Lee), Probability in the Engineering and Informational Sciences, Vol. 1, 203-210, 1987.

"A Diffusion Model for a System Subject to Continuous Wear" (with E.Y. Lee), Probability in the Engineering and Informational Sciences, Vol. 1, 405-416, 1987.

"Optimal Control of a Model for a System Subject to Continuous Wear" (with E.Y. Lee), Probability in the Engineering and Information Sciences, Vol. 2, No. 3 (to appear).
"Further Properties of a Model for a System Subject to Continuous Wear" (with E.Y. Lee). Submitted for publication.

"Some Criteria for Reliability Growth", Microelectronics and Reliability (to appear).

"The Stability of a Characterization of the Bivariate Marshall-Olkin Distribution" (with S. T. Rachev). Submitted for publication.

"A Note on the Stability of the Estimation of the Exponential Distribution" (with S.T. Rachev). Submitted for publication.

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(i) Doctoral Students

Seung Min Lee: "Continuum Structure Functions: Finite Minimal Vector Sets, Weak Convergence and Reliability Importance". Graduated with Ph.D. in December 1987.

Eui Yong Lee: "A Diffusion Model for a System Subject to Continuous Wear". Graduated with Ph.D. in May 1988.

Gregory Chlouverakis: In preparation

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Papers Presented at Conferences

Operations Research Society of America/Institute of Management Sciences Joint National Meeting, New Orleans, May 1987.

Papers Presented: "On the Theory of Cannibalization"
"Continuum Structure Functions: Axiomatization and Reliability Importance"

Conference on Stochastic Processes and Their Applications, Stanford University, August 1987.

Paper Presented: "The Stochastic Performance of Continuum Structure Functions"

Operations Research Society of America/Institute of Management Sciences, Joint National Meeting, St. Louis, October 1987.

Papers Presented: "Structure Functions with Finite Minimal Vector Sets"
"A Diffusion Model for a System Subject to Continuous Wear"
"On Reliability Growth Analysis"

Operations Research Society of America/Institute of Management Sciences, Joint National Meeting, Washington, D.C., April 1988.

Papers Presented: "Further Properties of Reliability Importance for Continuum Structure Functions"
"Further Analysis of a Model for a System Subject to Continuous Wear"
"A Model for a Repairman who Induces Faults"