OFFICE OF NAVAL RESEARCH
Contract N00014-86-K-0043
TECHNICAL REPORT No. 83

Propagators for Driven Coupled Harmonic Oscillators

by

Kyu-Hwang Yeon, Chung-In Um, Woo-Hyung Kahng and Thomas F. George

Prepared for Publication
in
Physical Review A

Departments of Chemistry and Physics
State University of New York at Buffalo
Buffalo, New York 14260

September 1988

Reproduction in whole or in part is permitted for any purpose of the
United States Government.

This document has been approved for public release and sale;
it is distribution is unlimited.
Propagators for Driven Coupled Harmonic Oscillators

Prepared for publication in Physical Review A

Propagators for coupled and driven coupled harmonic oscillators are evaluated exactly by the path-integral method. The propagators for coupled harmonic oscillators are used to obtain explicitly the energy expectation values.
Propagators for driven coupled harmonic oscillators

Kyu-Hwang Yeon
Department of Physics
Chungbuk National University
Cheong Ju, Chung Buk 360-763, Korea

Chung-In Um and Woo-Hyung Kahng
Department of Physics
College of Science
Korea University
Seoul 136-701 Korea

Thomas F. George *
Departments of Chemistry and Physics & Astronomy
239 Fronczak Hall
State University of New York at Buffalo
Buffalo, New York 14260

Propagators for coupled and driven coupled harmonic oscillators are evaluated exactly by the path-integral method. The propagators for coupled harmonic oscillators are used to obtain explicitly the energy expectation values.

PACS Nos. 02.90+p, 03.65.W, 03.65Db

* To whom correspondence should be addressed.
1. Introduction

Although the Feynman path-integral formulation offers a general approach for treating quantum-mechanical systems, only several problems can be solved exactly. Two of these are the driven harmonic oscillator with a quadratic Hamiltonian and the time-dependent damped driven harmonic oscillator. A number of situations such as superconducting quantum interference devices, quantum nondemolition measurements, magnetohydrodynamics, etc., can be described by driven coupled harmonic oscillators. Introducing the Caldirola-Kanai Hamiltonian, one can obtain the time-dependent Schroedinger equation for the damped harmonic oscillator. However, it has been a matter of debate as to whether or not this Schroedinger equation represents the quantum mechanical dissipative system. Some workers claim affirmation while others object to it. This problem has been critically reviewed by Greenberger and Cervero and Villaroel.

The purpose of this paper is to derive the propagator for a driven coupled harmonic oscillators (DCHO) system from our previous work for both coupled and coupled driven harmonic oscillators by means of the path-integral method. We introduce two harmonic oscillators that are coupled together with another spring. We review the classical case and construct the form of the propagator for DCHO, respectively, in Secs. 2 and 3. Section 4 gives the exact derivation of the propagator for the coupled harmonic oscillators (CHO), and in Sec. 5 we evaluate the exact propagator for DCHO by using the results obtained in Sec. 4. The energy expectation values of CHO are evaluated in Sec. 6, and finally we give results and discussion in Sec. 7.
2. Classical case

In this section we consider a system of two harmonic oscillators which are coupled together by means of another spring. We assume that the masses of the oscillators and three spring constants are all the same. Let the forces $f_1(t)$ and $f_2(t)$ exerted on the two oscillators and their displacements be $x_1$ and $x_2$. Then the Hamiltonian for DCHO can be written as

$$H = \frac{1}{2m} (p_1^2 + p_2^2) + \omega^2 (x_1^2 - x_1 x_2 + x_2^2) - f_1(t)x_1 - f_2(t)x_2 \quad (2.1)$$

where $\omega^2 = k/m$. Hamilton's equations of motion for Eq. (2.1) are

$$\dot{x}_1 = p_1/m \quad (2.2)$$
$$\dot{x}_2 = p_2/m \quad (2.3)$$
$$\dot{p}_1 = m\omega^2 (x_2 - 2x_1) + f_1(t) \quad (2.4)$$
$$\dot{p}_2 = m\omega^2 (x_1 - 2x_2) + f_2(t) \quad (2.5)$$

Equations (2.1)-(2.5) yield the Lagrangian,

$$L = (p_1\dot{x}_1 + p_2\dot{x}_2) - H$$

$$= \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) - m\omega^2 (x_1^2 - x_1 x_2 + x_2^2) + f_1(t)x_1 + f_2(t)x_2 \quad (2.6)$$

with the corresponding equations of motion
\[ x_1 + \omega^2 (2x_1 - x_2) = f_1(t)/m \] (2.7)

\[ x_2 + \omega^2 (2x_2 - x_1) = f_2(t)/m \] (2.8)

The classical solutions of Eqs. (2.7) and (2.8) are given by

\[
x_1(t) = A \sin(\omega t) + B \cos(\omega t) + C \sin(\sqrt{3}\omega t) + D \cos(\sqrt{3}\omega t)
+ \int_r^T dr \int d\nu e^{i\omega(2r-\nu-t)} \left[ f_1(\nu) + f_2(\nu) \right]
\] (2.9)

and

\[
x_2(t) = A \sin(\omega t) + B \cos(\omega t) - C \sin(\sqrt{3}\omega t) - D \cos(\sqrt{3}\omega t)
+ \int_r^T dr \int d\nu e^{i\omega(2r-\nu-t)} \left[ f_1(\nu) - f_2(\nu) \right]
\] (2.10)

3. Path integral of driven coupled harmonic oscillators

In the path-integral formulation, the solution of the Schroedinger equation is given as the path-dependent integral equations with propagator \( K \),

\[
\psi(x_1, x_2, t) = \int dx_1' dx_2' K(x_1, x_2, t; x_1', x_2', 0) \psi(x_1', x_2', 0)
\] (3.1)

which gives the wavefunction \( \psi(x_1, x_2, t) \) at time \( t \) in terms of the wave function \( \psi(x_1', x_2') \) at time \( t = 0 \). The propagator in Eq. (3.1) can be written by means of the Feynman path integral.
\[ K(x_1, x_2, t; x_1', x_2', 0) = \int_{(x_1', x_2', 0)}^{(x_1, x_2, t)} D(x(t)) \exp\left(\frac{1}{\hbar}S(x_1, x_2, x_1', x_2'; t)\right) \] (3.2)

where

\[ D(x(t)) = \lim_{N \to \infty} \frac{1}{A^{N-1}} \prod_{j=1}^{N} \left[ \frac{dx_j'}{dx_j} \right] \] (3.3)

and \( S(x_1, x_2, x_1', x_2'; t) \) is the action defined as the time integral over the Lagrangian \( L(x_1, x_2, x_1', x_2'; t) \) between \( t = t' \) and \( t = 0 \).

\[ S(x_1, x_2, x_1', x_2'; t) = \int_{0}^{t} dt \, L(x_1', x_2', x_1, x_2; t) \] (3.4)

In Eq. (3.3) \( A \) is the normalization factor given by

\[ A = [2\pi\hbar\epsilon/m^2]^{\frac{1}{2}}, \quad \epsilon = \lim_{N \to \infty} (t/N). \] (3.5)

Substituting Eq. (2.6) into Eq. (3.4), the action becomes

\[ S(x_1, x_2, x_1', x_2'; t) = S_c(x_1, x_2, x_1', x_2'; t) \]

\[ + \int_{0}^{t} dr \frac{M}{2} \left( \dot{y}_1^2(r) + \dot{y}_2^2(r) - 2\omega^2 \left( \dot{y}_1(r) y_2(r) - \dot{y}_2(r) y_1(r) \right) \right) \] (3.6)

where \( S_c \) is the classical action and \( y_i \) is the deviation of \( x_i(t) \) from its classical path \( x_{ci} \) given as...
Then the propagator [Eq. (3.2)] can be expressed as

\[
K(x_1,x_2,t; x'_1,x'_2,0) = F(t) e^{iS_{c}/\hbar}.
\]

Here, \( F(t) \) is the multiplicative function given in the form

\[
F(t) = \int_0^\infty Dz(t) \exp \left( \frac{im}{\hbar} \int_0^t \left( y_1^2 + y_2^2 - 2\omega^2(y_1 y_2 + y_2^2) \right) dt \right).
\]

It is easy to show that \( F(t) \) has the same form for CHO and DCHO. Therefore, the propagator depends only on the classical action in both cases. In Eq. (3.9), change the variables \( x_1 \pm x_2 \) into

\[
z_1 = \frac{1}{\sqrt{2}} (x_1 - x_2)
\]

\[
z_2 = \frac{1}{\sqrt{2}} (x_1 + x_2)
\]

we can reduce the condition \((y_1,y_2) = (0,0)\) to \((z_1,z_2) = (0,0)\). Applying Eqs. (3.10) and (3.11) to Eq. (3.9), the multiplicative function becomes

\[
F(t) = \int_0^\infty Dz(t) \exp \left( \frac{im}{\hbar} \int_0^t Dz(t) \left[ (z_1^2 - \omega^2 z_1^2) + (z_2^2 - 3\omega^2 z_2^2) \right] dt \right).
\]
In Eq. (3.12), $J$ becomes unity.

If the action is separated into the functionals with only same variables in the path integral, then this integral can be represented by the multiplication of path integrals with each variable. Therefore, Eq. (3.12) becomes

$$F(t) = F_1(t) F_2(t)$$
\[
- \left( \int_0^t Dz_1(t) \exp \left( \frac{im}{2iM} \int_0^t dt \left( \frac{z_1^2}{2M^2} - \omega^2 z_1^2 \right) \right) \right) \\
\times \left( \int_0^t Dz_2(t) \exp \left( \frac{im}{2iM} \int_0^t dt \left( \frac{z_2^2}{2M^2} - 3\omega^2 z_2^2 \right) \right) \right) .
\]
\quad (3.14)

Since \( F_1(t) \) and \( F_2(t) \) are the path integrals of the harmonic oscillator, the evaluation of Eq. (3.14) gives

\[
F(t) = \frac{m\omega}{2\pi iM} \left[ \frac{1}{\sin(\omega t) \sin(3\omega t)} \right]^{-\frac{1}{4}} e^{iS/M} .
\]
\quad (3.15)

Hence, the propagator of DCHO can be written as

\[
K(x_1, x_2, t; x'_1, x'_2, 0) = \frac{m\omega}{2\pi iM} \left[ \frac{1}{\sin(\omega t) \sin(3\omega t)} \right]^{-\frac{1}{4}} e^{iS/M} .
\]
\quad (3.16)

4. Propagator for the coupled harmonic oscillators

To evaluate the exact propagator expressed by Eq. (3.16), we should first obtain the propagator for CHO. The classical action of CHO is

\[
S_c = \int_0^t dr \left( \frac{\dot{\mathbf{x}}_c^2}{2} + \mathbf{x}_c^2 \right) - \frac{m\omega}{2}(\mathbf{x}_c^2 + \mathbf{x}_c^2 + \mathbf{x}_c^2 + \mathbf{x}_c^2) .
\]
\quad (4.1)

where \( \mathbf{x}_c \) and \( \dot{\mathbf{x}}_c \) are the classical path and velocity, respectively.

Integrating Eq. (4.1) over the time, we get

\[
S_c = \left. \frac{m\mathbf{x}_c^2}{2} \right|_{x_1, x_2, v_1, v_2, t}^t + \int_0^t dr \frac{m\omega^2}{2} \mathbf{x}_c^2 + \mathbf{x}_c^2 + \mathbf{x}_c^2 .
\]
\[- \int_0^t \text{d}r \frac{M}{2} x_{c2}(\dddot{x}_{c2} + \omega^2 (2x_{c2} - x_{c1})) \]
\[- \frac{M}{2} [x_{c1}(t)x'_{c1}(t) + x_{c2}(t)x'_{c2}(t) - x_{c1}(0)x'_{c1}(0) - x_{c2}(0)x'_{c2}(0)] \]  \hspace{1cm} (4.2)

Here the second and third terms become zero because of the equations of motion [see Eqs. (2.7) and (2.8)], given as

\[ \dddot{x}_1 + \omega^2 (2x_1 - x_2) = 0 \]  \hspace{1cm} (4.3)
\[ \dddot{x}_2 + \omega^2 (2x_2 - x_1) = 0 \]  \hspace{1cm} (4.4)

To obtain the exact expression of Eq. (4.2), we solve Eqs. (4.3) and (4.4) to obtain

\[ x_1 = x_1(t) - A \sin(\omega t) + B \cos(\omega t) + C \sin(\sqrt{3} \omega t) + D \cos(\sqrt{3} \omega t) \]  \hspace{1cm} (4.5)
\[ x_2 = x_2(t) - A \sin(\omega t) + B \cos(\omega t) - C \sin(\sqrt{3} \omega t) - D \cos(\sqrt{3} \omega t) \]  \hspace{1cm} (4.6)

and \( \dot{x}_1 \) and \( \dot{x}_2 \) are given, respectively, by

\[ \dot{x}_1 = \dot{x}_1(t) - \omega (A \cos(\omega t) - B \sin(\omega t) + \sqrt{3} C \sin(\sqrt{3} \omega t)) - \sqrt{3} D \sin(\sqrt{3} \omega t)) \]  \hspace{1cm} (4.7)
\[ \dot{x}_2 = \dot{x}_2(t) - \omega (A \cos(\omega t) - B \sin(\omega t) - \sqrt{3} C \cos(\sqrt{3} \omega t)) + \sqrt{3} D \sin(\sqrt{3} \omega t)) \]  \hspace{1cm} (4.8)
Equations (4.5)-(4.8) give

\begin{align*}
x_1' - x_1(0) &= B + D \\
x_2' - x_2(0) &= B - D \\
\dot{x}_1' - \dot{x}_1(0) &= \omega(A + \sqrt{3}C) \\
\dot{x}_2' - \dot{x}_2(0) &= \omega(A - \sqrt{3}C)
\end{align*}

The time-dependent constants \(A\), \(B\), \(D\) and \(D\) obtained from Eqs. (4.5) and (4.6), and Eqs. (4.9) and (4.10) can be expressed as

\begin{align*}
A &= \frac{1}{2} \sin(\omega t) \left\{ x_1 + x_2 - (x_1' + x_2') \cos(\omega t) \right\} \\
B &= \frac{1}{2} (x_1' + x_2') \\
C &= \frac{1}{2} \sin(\sqrt{3}\omega t) \left\{ x_1 - x_2 + (x_1' - x_2') \cos(\sqrt{3}\omega t) \right\} \\
D &= \frac{1}{2} (x_1' - x_2')
\end{align*}

Substitution of Eqs. (4.5)-(4.16) into (4.2) gives the classical action:

\[
S_c = \frac{m\omega}{4} \left\{ (x_1^2 + x_2^2 + x_1'^2 + x_2'^2) \left[ \cot(\omega t) + \sqrt{3} \cot(\sqrt{3}\omega t) \right] \right\} + 2(x_1'x_2 + x_1x_2') \left[ \cot(\omega t) - \sqrt{3} \cot(\sqrt{3}\omega t) \right]
\]
- 2(x_1 x'_1 + x_2 x'_2) [(1/sin(\omega t) + \sqrt{3}/\sin(\sqrt{3}\omega t))] \\
+ 2(x_1 x'_2 + x_2 x'_1) [-1/sin(\omega t) + (\sqrt{3}/\sin(\sqrt{3}\omega t))] . \tag{4.17}

Combining Eqs. (4.17) and (3.16), we obtain the propagator for CHO:

\begin{equation}
K(x_1, x_2, t; x'_1, x'_2, 0) = i\omega - \frac{i\omega}{2\pi i} [\sqrt{3}/\sin(\omega t) \sin(\sqrt{3}\omega t)]
\times \exp((i\omega/4)[(x_1^2 + x_2^2 + x_1'^2 + x_2'^2)\cot(\omega t) + \sqrt{3} \cot(\sqrt{3}\omega t)] \\
+ 2(x_1 x_2 + x_1' x_2')\cot(\omega t) - \sqrt{3} \cot(\sqrt{3}\omega t)] - 2(x_1 x'_1 + x_2 x'_2)[1/sin(\omega t)] \\
+ \sqrt{3}/\sin(\sqrt{3}\omega t)] + 2(x_1 x'_2 + x_2 x'_1) [-1/sin(\omega t) + \sqrt{3}/\sin(\sqrt{3}\omega t)] \}) . \tag{4.18}
\end{equation}

9. Propagator for driven coupled harmonic oscillators

When we set \( f_1(t) = f_2(t) = 0 \), DCHO reduces to CHO, whereby we can write the propagator for DCHO as

\begin{equation}
K(x_1, x_2, t; x'_1, x'_2, 0) = \exp[a(t)x_1^2 + b(t)x_1 x_2 + c(t)x_2^2 + d(t)x_1 \\
+ g(t)x_2 + h(t)] . \tag{5.1}
\end{equation}

Here \( a(t) \), \( b(t) \), \( c(t) \), \( d(t) \), \( f(t) \) and \( h(t) \) are time-dependent functions including \( x_1' \) and \( x_2' \), which need to be determined. Equation (5.1) must satisfy the Schroedinger equation.
\[ i \hbar \left( \partial K / \partial t \right) = - \mathbf{H} K \]  \hspace{2cm} (5.2)

Substitution of Eq. (5.1) into Eq. (5.2) gives the time-dependent coefficients

\[ a(t) = \frac{i \hbar}{2m} \left[ 4a^2(t) + c^2(t) \right] + \frac{m \omega^2}{i \hbar} \]  \hspace{2cm} (5.3)

\[ b(t) = \frac{i \hbar}{2m} \left[ 4b^2(t) + c^2(t) \right] + \frac{m \omega^2}{i \hbar} \]  \hspace{2cm} (5.4)

\[ c(t) = \frac{2i \hbar}{m} \left[ a(t)c(t) + b(t)c(t) \right] - \frac{m \omega^2}{i \hbar} \]  \hspace{2cm} (5.5)

\[ d(t) = \frac{i \hbar}{m} \left[ 2a(t)d(t) + c(t)g(t) \right] + \frac{i}{\hbar} f_1(t) \]  \hspace{2cm} (5.6)

\[ g(t) = \frac{i \hbar}{m} \left[ 2b(t)g(t) + c(t)d(t) \right] + \frac{i}{\hbar} f_2(t) \]  \hspace{2cm} (5.7)

\[ h(t) = \frac{i \hbar}{2m} \left[ d^2(t) + g^2(t) + 2a(t) + 2b(t) \right] \]  \hspace{2cm} (5.8)

Since Eqs. (5.3) and (5.4) have the same form, we get

\[ a(t) = b(t) \]  \hspace{2cm} (5.9)

Substituting Eq. (5.9) into Eq. (5.5) and changing the variables \( a \) and \( c \) into

\[ \eta = a + c/2 \]  \hspace{2cm} (5.10)

\[ \zeta = a - c/2 \]  \hspace{2cm} (5.11)
we obtain two ordinary differential equations:

\[
\dot{\eta} = \frac{2i\omega_m}{m} \eta^2 + \frac{\omega^2}{2i\omega_m} \tag{5.12}
\]

\[
\dot{\xi} = \frac{2i\omega_m}{m} \xi^2 + \frac{3\omega^2}{2i\omega_m} \tag{5.13}
\]

The solutions of Eqs. (5.12) and (5.13) are given by

\[
\eta = \frac{\text{Im} \omega_m}{2\omega} \cot(\omega t + \theta_1) \tag{5.14}
\]

\[
\xi = \frac{\sqrt{3}\text{Im} \omega_m}{2\omega} \cot(\sqrt{3}\omega t + \theta_2) \tag{5.15}
\]

where \(\theta_1\) and \(\theta_2\) are the constants to be determined. The time-dependent coefficients \(a(t)\), \(b(t)\) and \(c(t)\) obtained in comparison with Eqs. (5.10), (5.11), (5.14) and (5.15) are given as

\[
a(t) - b(t) = \frac{\text{Im} \omega_m}{4\omega} [\cot(\omega t + \theta_1) + \sqrt{3} \cot(\sqrt{3}\omega t + \theta_2)] \tag{5.16}
\]

\[
c(t) = \frac{\text{Im} \omega_m}{2\omega} [\cot(\omega t + \theta_1) - \sqrt{3} \cot(\sqrt{3}\omega t + \theta_2)] \tag{5.17}
\]

Equations (5.16) and (5.17) do not include the driven forces \(f_1(t)\) and \(f_2(t)\). Therefore, through setting \(f_1(t) = f_2(t) = 0\), Eqs. (5.16) and (5.17) do not change at all and should be equal to the coefficients of \(x_1^2\) and \(x_2^2\) in Eq. (4.18). Comparison of these two equations shows \(\theta_1\) and \(\theta_2\) to be zero.

Substituting Eq. (5.9) into Eqs. (5.6) and (5.7) and changing variables \(d\) and \(g\) into...
\[ \rho = d + g \] (5.18)

\[ \sigma = d - g \] (5.19)

we obtain the two differential equations

\[ \dot{\rho} = \frac{i k}{m} [2a(t) + c(t)] \rho + \frac{i}{k} [f_1(t) + f_2(t)] \] (5.20)

\[ \dot{\sigma} = \frac{i k}{m} [2a(t) + c(t)] \sigma + \frac{i}{k} [f_1(t) - f_2(t)] \] (5.21)

Combining Eqs. (5.20) and (5.21) with Eqs. (5.16) and (5.17), we obtain the solutions

\[ \rho = \left[1/\sin(\omega t)\right] \left( \int_0^T \frac{1}{k} \left[ f_1(r) + f_2(r) \right] \sin(\omega r) dr + \alpha \right) \] (5.22)

\[ \sigma = \left[1/\sin(\sqrt{3}\omega t)\right] \left( \int_0^T \frac{1}{k} \left[ f_1(r) - f_2(r) \right] \sin(\sqrt{3}\omega r) dr + \beta \right) \] (5.23)

where \( \alpha \) and \( \beta \) are constants to be determined. We can obtain the time-dependent coefficients \( d(t) \) and \( g(t) \) by substituting Eqs. (5.22) and (5.23) into Eqs. (5.18) and (5.19):

\[
\begin{align*}
    d(t) &= \left[1/2k\sin(\omega t)\right] \int_0^T dr \left[ f_1(r) + f_2(r) \right] \sin(\omega r) \\
    &\quad + \left[1/2k\sin(\sqrt{3}\omega t)\right] \int_0^T dr \left[ f_1(r) - f_2(r) \right] \sin(\sqrt{3}\omega r) \\
    &\quad + [\alpha/2\sin(\omega t)] + [\beta/2\sin(\sqrt{3}\omega t)]
\end{align*}
\] (5.24)
Substitution of Eqs. (5.16), (5.17), (5.24) and (5.25) into Eq. (5.8) yields

\[
g(t) = \left[ \frac{1}{2\sqrt{3}}\sin(\omega t) \right] \int_0^T dr \left[ f_1(r) + f_2(r) \right] \sin(\omega r) \\
- \left[ \frac{1}{2\sqrt{3}}\sin(\sqrt{3}\omega t) \right] \int_0^T dr \left[ f_1(r) - f_2(r) \right] \sin(\sqrt{3}\omega r) \\
+ \left[ \frac{a}{2\sin(\omega t)} \right] - \left[ \frac{\beta}{2\sin(\sqrt{3}\omega t)} \right] . \tag{5.25}
\]

Substitution of Eqs. (5.16), (5.17), (5.24) and (5.25) into Eq. (5.8) yields

\[
h(t) = - \frac{1}{4\omega} \left[ a^2 \cot(\omega t) + (\beta^2/\sqrt{3}) \cot(\sqrt{3}\omega t) \right] \\
- \left[ a/\sqrt{3}\omega\sin(\sqrt{3}\omega t) \right] \int_0^T dr \left[ f_1(r) + f_2(r) \right] \sin(\sqrt{3}\omega(t-r)) \\
- \left[ \beta/\sqrt{3}\omega\sin(\sqrt{3}\omega t) \right] \int_0^T dr \left[ f_1(r) - f_2(r) \right] \sin(\sqrt{3}\omega(t-r)) \\
+ \left[ 1/4\sqrt{3}\omega\sin(\sqrt{3}\omega t) \right] \int_0^T dr \int_0^T dv \left[ f_1(r) + f_2(r) \right] \\
\times [f_1(v) + f_2(v)] \sin[\omega(t-r)]\sin(\omega v) \\
+ \left[ 1/4\sqrt{3}\omega\sin(\sqrt{3}\omega t) \right] \int_0^T dr \int_0^T dv \left[ f_1(r) - f_2(r) \right] \\
\times [f_1(v) - f_2(v)]\sin[\sqrt{3}\omega(t-r)] \sin(\sqrt{3}\omega v) \ln[\sin(\omega t)] \\
\times \sin(\sqrt{3}\omega t) + \delta . \tag{5.26}
\]

Here, \( \delta \) is also a constant to be determined. When setting \( f_1(t) = f_2(t) = 0 \), Eqs. (5.24) and (5.25) should be reduced to the coefficients of \( x_1 \) and...
$x_2$, and Eq. (5.26) should also be reduced to the terms in the exponent in Eq. (4.18). Comparison between them gives the constants $\alpha$, $\beta$ and $\delta$:

$$\alpha = \frac{-\omega}{2\pi i \hbar} (x_1^r + x_2^r) ,$$  \hspace{1cm} (5.27)

$$\beta = \frac{-\omega}{2\pi i \hbar} (x_1^r - x_2^r) ,$$  \hspace{1cm} (5.28)

$$\delta = \ln\left(\frac{-\omega}{2\pi i \hbar}\right) .$$  \hspace{1cm} (5.29)

Substitution of the above results into Eq. (5.1) gives the propagator for DCHO:

$$K(x_1, x_2, t; x_1', x_2', 0) = \frac{-\omega}{2\pi i \hbar} \left\{ \sqrt{3} / \sin(\omega t) \sin(3\omega t) \right\}^{r_1}$$

$$\times \exp\left[ \frac{i \omega}{4\hbar} \left[ 2(x_1^2 + x_2^2 + x_1'^2 + x_2'^2)[\cot(\omega t) + \sqrt{3}\cot(3\omega t)] + 2(x_1 x_2 + x_1' x_2')[\cot(\omega t) - \sqrt{3}\cot(3\omega t)] - 2(x_1 x_1' + x_2 x_2') \frac{1}{\sin(\omega t)} + \sqrt{3} / \sin(3\omega t) \right]$$

$$+ 2(x_1 x_2' + x_1' x_2') \left[ -\frac{1}{\sin(\omega t)} + \frac{\sqrt{3}}{\sin(3\omega t)} \right]$$

$$+ \frac{2x_1}{\omega} \left[ \frac{1}{\sin(\omega t)} \right] \int_0^t \, dr \left[ f_1(r) + f_2(r) \right] \sin(\omega r)$$

$$+ \frac{1}{\sin(3\omega t)} \int_0^t \, dr \left[ f_1(r) - f_2(r) \right] \sin(3\omega r) \right\}$$
\[
\begin{align*}
+ \frac{2x_2}{m_\omega} \left[ \frac{1}{\sin(\omega t)} \right] & \int_0^\infty dr \left[ f_1(r) + f_2(r) \right] \sin(\omega r) \\
- \frac{1}{\sin(\sqrt{3}\omega t)} & \int_0^\infty dr \left[ f_1(r) - f_2(r) \right] \sin(\sqrt{3}\omega r) \\
+ \frac{4x_2^2}{m_\omega} \left[ \frac{1}{\sin(\omega t)} \right] & \int_0^\infty dr \left[ f_1(r) + f_2(r) \right] \sin[\omega(t-r)] \\
+ \frac{4x_2^2}{m_\omega} \left[ \frac{1}{\sin(\omega t)} \right] & \int_0^\infty dr \left[ f_1(r) + f_2(r) \right] \sin[\omega(t-r)] \\
- \frac{1}{\sin(\sqrt{3}\omega t)} & \int_0^\infty dr \left[ f_1(r) - f_2(r) \right] \sin(\sqrt{3}\omega r) \\
- \frac{1}{m^2\omega^2 \sin(\omega t)} & \int_0^\infty dr \int_0^\infty d\nu \left[ f_1(r) + f_2(r) \right] \left[ f_1(\nu) + f_2(\nu) \right] \\
\times \sin[\omega(t-r)] \sin(\omega \nu) \\
- \frac{1}{m^2\omega^2 \sin(\sqrt{3}\omega t)} & \int_0^\infty dr \int_0^\infty d\nu \left[ f_1(r) - f_2(r) \right] \left[ f_1(\nu) - f_2(\nu) \right] \\
\times \sin[\sqrt{3}\omega(t-r)] \sin(\sqrt{3}\omega \nu) \right] \right) .
\end{align*}
\]
6. Energy expectation values of coupled harmonic oscillators

The Hamiltonian of CHO is

\[ H = \frac{1}{2m} (p_1^2 + p_2^2) + m\omega^2 (x_1^2 + x_1 x_2 + x_2^2) \]  \hspace{1cm} (6.1)

Using Eqs. (3.1) and (3.2) with Eq. (6.1), we obtain the Schröedinger equation,

\[ i\hbar \frac{\partial}{\partial t} \psi(x_1, x_2, t) = H_{op} \psi(x_1, x_2, t) \]  \hspace{1cm} (6.2)

where \( H_{op} \) is the Hamiltonian operator in which the momentum \( p_1 \) is changed into \( p_1 = (\hbar/\iota)(\partial/\partial x_1) \). Since Eq. (6.2) can be separated into time and coordinate parts, we may write

\[ K(t) = e^{-iH_{op}t/\hbar} \]  \hspace{1cm} (6.3)

\[ H_{op} |l, n> = E_{ln} |l, n> \]  \hspace{1cm} (6.4)

Here the states \(|l, n>\) are the complete set with energy eigenvalues of \( H_{op} \). Since the function with states \(|l, n>\) can be expressed by

\[ \phi_{ln}(x_1, x_2) = \langle x_1, x_2 |l, n> \]  \hspace{1cm} (6.5)

the propagator at \( t > 0 \) becomes

\[ K(x_1, x_2; x_1', x_2', 0) = \langle x_1, x_2 | e^{-iH_{op}t/\hbar} |x_1', x_2'\rangle \]
\[ - \sum_{l} \sum_{n} \sum_{l'} \sum_{n'} <l',n'|H_{\text{op}}|l',n'> <l',n'|x_1',x_2'> \]

\[ = \sum_{l} \sum_{n} \phi_{2n}(x_1',x_2') e^{-iE_{2n}t/H} \phi_{2n}^*(x_1',x_2') \]  

Equation (6.6) should be the same as Eq. (4.18). Setting \( x_1' = x_1 \) and \( x_2' = x_2 \) in Eq. (4.18) and integrating over \( x_1 \) and \( x_2 \), we get

\[ \sum_{l} \sum_{n} \int dx_1dx_2 \phi_{2n}(x_1',x_2') e^{-iE_{2n}t/H} \phi_{2n}(x_1',x_2') = e^{-iE_{2n}t/H} \]  

and

\[ \int \int dx_1dx_2 \frac{\text{me}^{-i\omega t}}{2\pi iH} \left( \sqrt{\frac{1}{\sin(\omega t)\sin(\sqrt{3}\omega t)}} \right) \]

\[ \times \exp \left( \frac{i\omega t}{2H} \left[ (x_1 + x_2)^2 - \sqrt{3}(x_1 - x_2)^2 \right] \left[ \cot(\omega t) - 1/\sin(\omega t) \right] \right) \]

\[ = - \frac{1}{2} \left[ \sin(\omega t/2) \sin(\sqrt{3}/2\omega t) \right]^{-1} \]  

Hence, we have

\[ \sum_{l} \sum_{n} e^{-iE_{2n}t/H} = - \frac{1}{2} \left[ \sin(\omega t/2) \sin(\sqrt{3}/2\omega t) \right]^{-1} \]

\[ = \frac{e^{-i\omega t/2}}{(1 - e^{-i\omega t})} \left[ e^{-i\sqrt{3}\omega t/2} / (1 - e^{-i\sqrt{3}\omega t}) \right] \]
\[- \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \exp\{-i\omega[(\ell + \frac{1}{2}) + \sqrt{3}(n + \frac{1}{2})]\}\]  \hspace{1cm} (6.9)

Therefore the expectation values of CHO becomes

\[E_{\ell n} = [(\ell + \frac{1}{2}) + \sqrt{3}(n + \frac{1}{2})] \omega\]  \hspace{1cm} (6.10)

7. **Results and discussion**

In the previous sections we have obtained the exact propagators [Eqs. (4.18) and (5.30)] for CHO and DCHO by the path-integral method. The forms of the propagators are new. Setting \(f(t) = 0\), Eq. (5.30) is reduced to Eq. (4.18). Although DCHO is a nonconservative system, the quantum-mechanical problem for the momentum operator does not appear because the canonical momentum is equal to the kinetic momentum in our derivation.\(^{13}\)

Making use of Eq. (4.18), we have obtained the energy expectation values [Eq. (6.10)] for CHO, given by the sum of two energy expectation values corresponding to the quantum states of two oscillators. Even though we have not evaluated the wavefunction of CHO, we may easily surmise that the wavefunction will be given by the multiplication of two wavefunctions for two oscillators. In the case of DCHO, one cannot easily apply Eq. (5.20) to obtain the energy expectation values, since this equation cannot be expressed in the form of Eq. (6.6), and one should recognize that the energy operator is not equal to the Hamiltonian operator in a nonconservative system.\(^{9}\)

The evaluations for the wavefunctions, energy expectation values for CHO and DCHO, and propagator and other physical quantities for \(n\) coupled and
n driven coupled harmonic oscillators (arbitrary n) are in progress and will be reported in the near future.

Acknowledgments

This research was supported in part by a grant to Korea University from the BSRI Program, Ministry of Education 1987, Republic of Korea, and in part by the Office of Naval Research, the Air Force Office of Scientific Research (AFSC), under Contract F49620-86-C-0009, and the National Science Foundation under Grant CHE-8620274. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.
References


TECHNICAL REPORT DISTRIBUTION LIST, GEN

<table>
<thead>
<tr>
<th>No. Copies</th>
<th>Destination</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Office of Naval Research Attn: Code 1113 800 N. Quincy Street Arlington, Virginia 22217-5000</td>
</tr>
<tr>
<td>1</td>
<td>Dr. David Young Code 334 NORDA NSTL, Mississippi 39529</td>
</tr>
<tr>
<td>1</td>
<td>Naval Weapons Center Attn: Dr. Ron Atkins Chemistry Division China Lake, California 93555</td>
</tr>
<tr>
<td>1</td>
<td>Scientific Advisor Commandant of the Marine Corps Code RD-1 Washington, D.C. 20380</td>
</tr>
<tr>
<td>1</td>
<td>U.S. Army Research Office Attn: CRD-AA-IP P.O. Box 12211 Research Triangle Park, NC 27709</td>
</tr>
<tr>
<td>1</td>
<td>Mr. John Boyle Materials Branch Naval Ship Engineering Center Philadelphia, Pennsylvania 19112</td>
</tr>
<tr>
<td>1</td>
<td>Naval Ocean Systems Center Attn: Dr. S. Yamamoto Marine Sciences Division San Diego, California 91232</td>
</tr>
<tr>
<td>1</td>
<td>Dr. David L. Nelson Chemistry Division Office of Naval Research 800 North Quincy Street Arlington, Virginia 22217</td>
</tr>
</tbody>
</table>

1
ABSTRACTS DISTRIBUTION LIST, 056/625/629

Dr. J. E. Jensen
Hughes Research Laboratory
3011 Malibu Canyon Road
Malibu, California 90265

Dr. C. B. Harris
Department of Chemistry
University of California
Berkeley, California 94720

Dr. J. H. Weaver
Department of Chemical Engineering
and Materials Science
University of Minnesota
Minneapolis, Minnesota 55455

Dr. F. Kutzler
Department of Chemistry
Box 5055
Tennessee Technological University
Cookeville, Tennessee 38501

Dr. A. Reisman
Microelectronics Center of North Carolina
Research Triangle Park, North Carolina 27709

Dr. D. DiLella
Chemistry Department
George Washington University
Washington D.C. 20052

Dr. M. Grunze
Laboratory for Surface Science and Technology
University of Maine
Orono, Maine 04469

Dr. R. Reeves
Chemistry Department
Rensselaer Polytechnic Institute
Troy, New York 12181

Dr. J. Butler
Naval Research Laboratory
Code 6115
Washington D.C. 20375-5000

Dr. Steven M. George
Stanford University
Department of Chemistry
Stanford, CA 94305

Dr. L. Interante
Chemistry Department
Rensselaer Polytechnic Institute
Troy, New York 12181

Dr. Mark Johnson
Yale University
Department of Chemistry
New Haven, CT 06511-8118

Dr. Irvin Heard
Chemistry and Physics Department
Lincoln University
Lincoln University, Pennsylvania 19352

Dr. W. Knauber
Hughes Research Laboratory
3011 Malibu Canyon Road
Malibu, California 90265

Dr. K.J. Klaubunde
Department of Chemistry
Kansas State University
Manhattan, Kansas 66506
ABSTRACTS DISTRIBUTION LIST, 056/625/629

Dr. G. A. Somorjai
Department of Chemistry
University of California
Berkeley, California 94720

Dr. J. Murday
Naval Research Laboratory
Code 6170
Washington, D.C. 20375-5000

Dr. J. B. Hudson
Materials Division
Rensselaer Polytechnic Institute
Troy, New York 12181

Dr. Theodore E. Madey
Surface Chemistry Section
Department of Commerce
National Bureau of Standards
Washington, D.C. 20234

Dr. J. E. Demuch
IBM Corporation
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, New York 10598

Dr. M. G. Lagally
Department of Metallurgical and Mining Engineering
University of Wisconsin
Madison, Wisconsin 53706

Dr. R. P. Van Duyne
Chemistry Department
Northwestern University
Evanston, Illinois 60637

Dr. J. M. White
Department of Chemistry
University of Texas
Austin, Texas 78712

Dr. D. E. Harrison
Department of Physics
Naval Postgraduate School
Monterey, California 93940

Dr. R. L. Park
Director, Center of Materials Research
University of Maryland
College Park, Maryland 20742

Dr. W. T. Peria
Electrical Engineering Department
University of Minnesota
Minneapolis, Minnesota 55455

Dr. Keith H. Johnson
Department of Metallurgy and Materials Science
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Dr. S. Sibener
Department of Chemistry
Jes Franck Institute
5640 Ellis Avenue
Chicago, Illinois 60637

Dr. Arnold Green
Quantum Surface Dynamics Branch
Code 3817
Naval Weapons Center
China Lake, California 93555

Dr. A. Wold
Department of Chemistry
Brown University
Providence, Rhode Island 02912

Dr. S. L. Bernasek
Department of Chemistry
Princeton University
Princeton, New Jersey 08544

Dr. W. Kohn
Department of Physics
University of California, San Diego
La Jolla, California 92037
ABSTRACTS DISTRIBUTION LIST, 056/625/629

Dr. F. Carter
Code 6170
Naval Research Laboratory
Washington, D.C. 20375-5000

Dr. John T. Yates
Department of Chemistry
University of Pittsburgh
Pittsburgh, Pennsylvania 15260

Dr. Richard Colton
Code 6170
Naval Research Laboratory
Washington, D.C. 20375-5000

Dr. Richard Greene
Code 5230
Naval Research Laboratory
Washington, D.C. 20375-5000

Dr. Dan Pierce
National Bureau of Standards
Optical Physics Division
Washington, D.C. 20234

Dr. L. Kesmodel
Department of Physics
Indiana University
Bloomington, Indiana 47403

Dr. R. Stanley Williams
Department of Chemistry
University of California
Los Angeles, California 90024

Dr. K. C. Janda
University of Pittsburgh
Chemistry Building
Pittsburgh, PA 15260

Dr. R. P. Messmer
Materials Characterization Lab.
General Electric Company
Schenectady, New York 22217

Dr. E. A. Irene
Department of Chemistry
University of North Carolina
Chapel Hill, North Carolina 27514

Dr. Robert Gomer
Department of Chemistry
James Franck Institute
5640 Ellis Avenue
Chicago, Illinois 60637

Dr. Adam Heller
Bell Laboratories
Murray Hill, New Jersey 07974

Dr. Ronald Lee
R301
Naval Surface Weapons Center
White Oak
Silver Spring, Maryland 20910

Dr. Martin Fleischmann
Department of Chemistry
University of Southhampton
Southampton 509 5NH
UNITED KINGDOM

Dr. Paul Schoen
Code 6190
Naval Research Laboratory
Washington, D.C. 20375-5000

Dr. H. Tachikawa
Chemistry Department
Jackson State University
Jackson, Mississippi 39217

Dr. John W. Wilkins
Cornell University
Laboratory of Atomic and
Solid State Physics
Ithaca, New York 14853
ABSTRACTS DISTRIBUTION LIST, 056/625/629

Dr. R. G. Wallis  
Department of Physics  
University of California  
Irvine, California 92664

Dr. J. T. Keiser  
Department of Chemistry  
University of Richmond  
Richmond, Virginia 23173

Dr. D. Ramaker  
Chemistry Department  
George Washington University  
Washington, D.C. 20052

Dr. R. W. Plummer  
Department of Physics  
University of Pennsylvania  
Philadelphia, Pennsylvania 19104

Dr. J. C. Hemminger  
Chemistry Department  
University of California  
Irvine, California 92717

Dr. E. Yeager  
Department of Chemistry  
Case Western Reserve University  
Cleveland, Ohio 41106

Dr. T. F. George  
Chemistry Department  
University of Rochester  
Rochester, New York 14627

Dr. N. Winograd  
Department of Chemistry  
Pennsylvania State University  
University Park, Pennsylvania 16802

Dr. G. Rubloff  
IBM  
Thomas J. Watson Research Center  
P.O. Box 218  
Yorktown Heights, New York 10598

Dr. Roald Hoffmann  
Department of Chemistry  
Cornell University  
Ithaca, New York 14853

Dr. Horia Metiu  
Chemistry Department  
University of California  
Santa Barbara, California 93106

Dr. A. Steckl  
Department of Electrical and Systems Engineering  
Rensselaer Polytechnic Institute  
Troy, New York 12181

Dr. W. Goddard  
Department of Chemistry and Chemical Engineering  
California Institute of Technology  
Pasadena, California 91125

Dr. G. H. Morrison  
Department of Chemistry  
Cornell University  
Ithaca, New York 14853

Dr. P. Hansma  
Department of Physics  
University of California  
Santa Barbara, California 93106

Dr. J. Baldeschwieler  
Department of Chemistry and Chemical Engineering  
California Institute of Technology  
Pasadena, California 91125