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Heat conduction with freezing or thawing

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$$\frac{d^2T}{d\eta^2} + 2\left(\eta + \frac{1}{\eta}\right)\frac{dT}{d\eta} = 0$$

$$\eta\frac{d^2T}{d\eta^2} + (1+\eta)\frac{dT}{d\eta} = 0$$

$$\left(X + \frac{k_1}{h}\right)^2 = 2\alpha_1 S_{Te} t + \left(\frac{k_1}{h}\right)^2$$

$$-k_1\frac{\partial T_1}{\partial x} + k_2\frac{\partial T_2}{\partial x} = \pm\rho\ell\frac{dX}{dt};\ x = X$$

$$\rho_o C\frac{\partial T}{\partial t} = \mathrm{div}(k\Delta T) + c_{p2}\rho_o\left[D_{\theta\ell}\nabla\theta + D_{T\ell}\nabla T\right]\nabla T$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{d\theta_2}{dt} - T_o\frac{dS}{dt} + T_f\frac{dX}{dt} + \alpha_2\frac{\partial T_2(X}{\partial x}$$

$$= Q_i$$

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$$\frac{\exp\left(-\gamma^2\right)}{\mathrm{erf}\,\gamma} - \frac{k_{21}\sqrt{\alpha_{12}}\left(T_o - T_f\right)e^{-\alpha_{12}\gamma^2}}{\left(T_f - T_s\right)\mathrm{erfc}\left(\gamma\sqrt{\alpha_{12}}\right)} = \frac{\ell\gamma\sqrt{\pi}}{c_1\left(T_f - T_s\right)}$$

$$\frac{T}{t}\,dx$$

$$\frac{\left(T_f - T_a\right)}{\dfrac{\ln\left(\frac{r_n}{R}\right)}{k_f} + \dfrac{\ln\frac{r_o}{r_i}}{k_p} + \dfrac{\ln\frac{r_i}{r_o}}{k_i} + \dfrac{1}{r_i h_o}} + LR\frac{dR}{dt} = 0$$

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1}\left[\frac{\partial T_1}{\partial t} + \left(1 - \frac{\rho_2}{\rho_1}\right)\frac{dX}{dt}\frac{\partial T_1}{\partial x}\right]$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2\theta}{\partial x_1^2} = \frac{\partial\theta}{\partial\tau} + \left(\frac{\rho_2}{\rho} - 1\right)S_T\frac{\partial\theta}{\partial x_1}\left(\frac{\partial\theta}{\partial x_1}\right)_\xi$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{d\theta_1}{dt} - T_f\frac{dx}{dt} - \alpha_1\left[\frac{\partial T_1(X,t)}{\partial x} - \frac{\partial T_1(0,t)}{\partial x}\right] = 0$$

$$\mathrm{div}\left(\bar{k}\,\mathrm{grad}\,T\right) + q_r + q_g = \rho c_p\frac{DT}{Dt} - \mu\phi - T\beta\frac{Dp}{Dt}$$

$$-k\frac{\partial T(R,t)}{\partial r} = \rho\ell\frac{dR}{dt}$$

$$\rho_o C\frac{\partial T}{\partial t} = \mathrm{div}(k\Delta T) - \rho_o\ell\frac{\partial\theta_2}{\partial t}$$

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial z^2} = \frac{1}{\kappa_1}\frac{\partial T_1}{\partial t}$$

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$$-k_1\left(\frac{\partial T_2}{\partial x}\right)^2 + k_2\left(\frac{\partial T_2}{\partial x}\right)\frac{\partial T_1}{\partial x} = \pm\rho\ell\alpha_1\frac{\partial^2 T_1}{\partial x^2};\ x = X$$

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Heat conduction with freezing or thawing

Virgil J. Lunardini

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Freezing of water or melting of ice are phenomena that underlie many important scientific and engineering studies of cold regions. Mathematical methods of treating these phase-change heat transfer problems are critical to understanding and dealing with the problems that freeze-thaw causes. While convection may be an important heat transfer mode, it can often be neglected without significant error. This report deals only with problems for which conduction is the basic heat transfer mode or for which the solutions can be obtained in terms of conduction-like problems. Where possible, exact solutions are presented, but since these are quite limited for phase-change problems, approximate solutions are examined in some detail. The approximate methods are 1) the perturbation method, which leads to quasi-stationary techniques, 2) the heat balance integral method, and 3) Biot's variational principle. The theory associated with these methods is discussed in the appendixes. The available exact solutions are derived and explained. Graphical solutions are then developed for practical applications to widely occurring problems. The approximate solutions are used to generate design curves—such as those for phase-change depth, temperature, and heat flow vs time. The results are presented so as to be easily accessible to practicing engineers without recourse to elaborate calculations. This is especially true for application to soil systems.

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PREFACE

This monograph was prepared by Dr. Virgil J. Lunardini, Mechanical Engineer, Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory. This study was conducted under DA Project 4A161101A91D, In-House Laboratory Independent Research, Work Unit 71462, Heat Transfer with Freezing or Thawing.

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CHAPTER 1. BASIC EQUATIONS

The problem of phase change occurs quite regularly in engineering problems dealing with permafrost and seasonally frozen ground, thermal storage systems for solar energy, the freezing of food or biological material, and the solidification or melting of metals. If the system is near the phase change temperature, a small change in the thermal regime may cause significant melting or solidification.

During a melting or freezing process, the system will be divided into regions separated by a phase change interface (or region), which is usually at the phase change (fusion) temperature. In general, and certainly for soil systems including permafrost, the thermal properties of the frozen and unfrozen regions are different, but are not strong functions of temperature for each individual phase. The transient nature of certain quantities, such as the surface temperature, the boundary heat fluxes, and the location of the phase change interface is of particular interest. The concepts of solid and liquid relate to the thermodynamic state of the water (or other liquids) contained in the pores of a system, if a porous medium is being considered.

In this monograph, the first chapter develops the important basic equations and presents some justification for ignoring the effects of convection.

Chapters 2-4 deal with problems that can be expressed in plane (Cartesian) coordinates. These problems are further divided into boundary conditions of temperature, prescribed heat flux, and surface convection.

Chapter 5 examines some plane geometries involving three-dimensional freezing or thawing. The quasi-steady method allows a comprehensive set of design graphs to be presented for the phase-change location as a function of time.

Chapters 6 and 7 deal with problems in the cylindrical and spherical coordinate systems, respectively. For cylinders, design curves are presented for phase change around horizontal cylinders.

No attempt has been made to include the extensive literature of numerical solutions to phase-change conduction problems; this is due to

space limits rather than a judgment on the value of numerical methods. In any event a thorough understanding of the basic physics of the problems must precede the application of numerical methods.

Many of the applications have been directed at water/ice soil-systems, but it should be clear that the basic solutions can be applied to such diverse areas as metallurgy, biological systems, solar latent heat methods, and preservation of food.

1.1 THE NATURE OF THE THERMODYNAMIC SYSTEM

Before any equations or physical laws are discussed, the thermodynamic system being considered must be defined. In most cases it is possible to consider a material as a continuum, on a macroscopic basis (i.e. on a size level above that of atomic dimensions). Thus we can consider the properties of a substance at a "point" by considering a finite volume of material, large enough to contain sufficient atoms for an average property to have meaning, yet small enough so that the concept of a mathematical point is valid. This is the usual concept of a continuous material with thermodynamic properties defined for a given spatial reference frame.

A porous material differs from a continuous material in that its structure is quite nonhomogeneous when viewed from the usual macroscopic level. Thus a material may consist of a framework or skeleton of solid material enclosing numerous voids. These voids may be filled, or partially filled, with fluids or other solids. An important example is a soil system consisting of a mineral skeleton whose voids may contain air, water, water vapor, ice, hydrocarbons, or various solute solutions.

The nonhomogeneous structure of a porous material can be considered explicitly or the material can be idealized as a continuum. In the latter case we consider a finite volume of matter, considerably larger than the molecular scale, but still small enough to qualify as a mathematical point. Over this volume the properties of the different constituents can be spatially averaged to arrive at an average property of the porous system valid at a point located at the center of the volume. The concept is analogous to the definition of a macroscopic property of a substance made up of atoms and molecules, but the size scale of the averaging volume is larger. This approach is valid as long as the size of the volume is small enough that the point values of the properties have some meaning.

When porous systems are to be considered, the basic equations for a continuum will be considered valid, with the average properties of the porous material used. The energy equation for a continuum will be considered below, followed by a discussion of its applicability to porous materials.

1.2 GENERAL ENERGY EQUATION FOR A CONTINUUM

With fundamental laws, it is possible to write a general equation for the temperature of a body at any point and time. A thermodynamic analysis requires that we select the boundaries of the system we wish to investigate and account for all the energy crossing the boundaries and generated within the system. The net flow of energy into such a system in a given time interval must be exactly equal to the change in the stored energy in the system.

Consider an element of volume dV having side lengths dx, dy, and dz, as shown in Figure 1.1. The faces forming the boundaries of this volume are fixed in space; however, these are not physical boundaries and both energy and mass may cross them.

The first law of thermodynamics applied to this system may be written (see Nomenclature at end of chapter) in the form

$$Q - W + (E_i - E_o) + Q_g = \Delta E_s \tag{1.1}$$

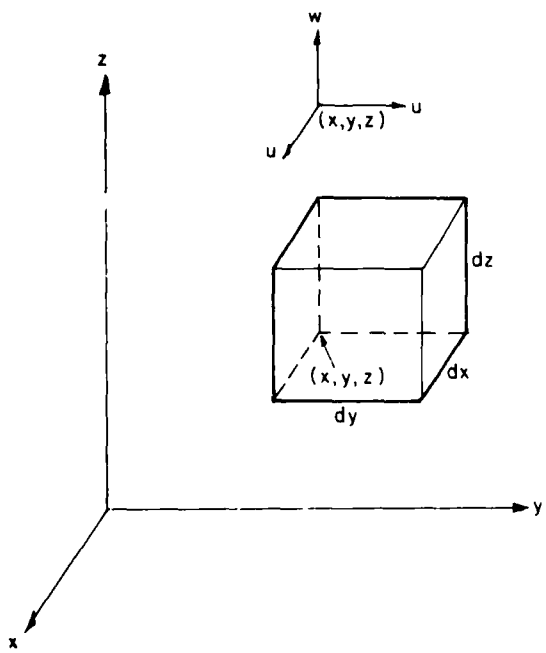


Figure 1.1 Thermodynamic system for general energy equation.

where

Q = the net heat transferred across the boundaries

W = the net mechanical work across the boundaries

E_i, E_o = the energies, in all forms, carried into and out of the system, with the mass flow

Q_g = the energy developed within the system by electrical, chemical, magnetic, nuclear, etc., means

ΔE_s = the change in stored energy.

The time interval over which our observations will be made is dt. The face perpendicular to the x axis and closest to the origin passes through the point x, y, z, and the velocity components of the medium at this point are u, v, and w, respectively.

1.2.1 Energy Transfer without Mass Transfer

Energy transfer across the boundaries in the form of heat may take place by conduction and by radiation. The energy transfer by conduction is computed by applying Fourier's equation to each of the three directions. The net energy into the system will be the sum of the energies into the system in each of the three directions. Fourier's law (or Biot's law) of conduction for the x-direction is

$$q_x = -k_x A \frac{\partial T}{\partial x} \quad . \tag{1.2}$$

This law states that the rate of energy flow in the direction of the temperature drop q_x is proportional to the area A and the temperature gradient $\partial T/\partial x$. The proportionality coefficient k_x is the thermal conductivity associated with the x direction. The negative sign is a convention so that heat flow in the direction of decreasing temperature is positive.

If Fourier's equation is applied to the face perpendicular to the x-axis and passing through the point x, y, z, the energy transfer by conduction Q_{cx} in the time dt is

$$dQ_{cx} = -k_x \, dy \, dz \frac{\partial T}{\partial x} dt \quad .$$

The energy transfer by conduction in the x direction at x + dx is the conduction at x plus the change in conduction in the distance dx:

$$dQ_{c(x + dx)} = \left[-k_x \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left(-k_x \frac{\partial T}{\partial x} \right) dx \right] dy \, dz \, dt \quad .$$

The net conduction of energy into the system in the x direction is the difference between these two values:

$$dQ_{xc} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) dV\, dt \ .$$

Similar equations may be written for dQ_{yc} and dQ_{zc} and the net energy transferred into the system by conduction is

$$dQ_c = \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right)\right] dV\, dt \ . \tag{1.3}$$

The energy transfer by the radiation process Q_r will depend on the "radiation geometry," the emissivities and temperatures of the surroundings and the material within the boundaries. This complex function will be written as

$$dQ_r = q_r\, dV\, dt \tag{1.4}$$

where q_r is the net radiant energy conducted into the system per unit volume, per unit time.

In addition to energy transfer in the form of heat, mechanical work not associated with mass transfer may also occur. This work is due to friction caused by the shearing or tangential stresses only, since the normal stresses cannot move the faces of the chosen system and, therefore, can do no work. The frictional work will be a function of the shearing stresses, the velocities, and the viscosity. Relationships involving these variables are necessary to write explicit expressions for the frictional work. We express this term in the general form

$$W = q_f\, dV\, dt \tag{1.5}$$

where q_f is the frictional energy per unit volume and per unit time. The derivation of q_f can be found in standard texts, i.e. Kreith (1973), and is

$$q_f = -\left\{\frac{\rho}{2}\frac{Ds^2}{Dt} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y} + w\frac{\partial P}{\partial z}\right\} - \mu\phi \tag{1.6}$$

The dissipation function ϕ is

$$\phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] - \frac{2}{3}\nabla^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 \tag{1.7}$$

where P = pressure

ρ = density

μ = viscosity

$s = |\underline{s}|$, fluid speed

$\underline{s}$ = velocity relative to x, y, z frame

$D/Dt = (\partial/\partial t) + s.\nabla$ is the total or substantive derivative

$\nabla = \text{div } \underline{s} = (\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z)$

u,v,w = components of velocity in x, y, z directions.

1.2.2 Energy Transfer Associated with Mass Flow

The energy entering the system with the mass flow, is made up of three parts: 1) the microscopic energies called the internal energy, 2) the macroscopic energies resulting from flow velocities and gravitational potentials, and 3) the flow energy needed to move the mass into the system.

The internal energy is due to the microscopic kinetic energy resulting from the motions of the atoms and the potential energy associated with the force fields existing between the atoms. The symbol U indicates the internal energy per unit of mass. The macroscopic energy consists of kinetic and potential energy terms. The kinetic energy is that due to the gross velocity of the aggregate of material involved in the flow and is designated, for unit mass, by the symbols $C^2/2$, where C is the flow velocity.

The macroscopic potential energy is that caused by external force fields, such as gravity. The flow energy, resulting from the work done in moving the mass, is expressed as the product of the pressure p and the specific volume v_s or pv_s. The sum of these energy terms is the total enthalpy designated by the symbol H:

$$H = U + \frac{C^2}{2} + pv_s + \frac{gZ}{g_o} \quad . \tag{1.8}$$

The energy flow into the control volume, in the x direction, is

$$E_{ix} = \rho u \, dy \, dz \, H_x \, dt \ .$$

The energy flow out of the control volume is

$$E_{ox} = \left[\rho u \, dy \, dz \, H_x + \frac{\partial}{\partial x} (\rho u H_x) \, dx \, dy \, dz\right] dt \ .$$

The net inflow of energy due to mass flow in all three directions is

$$E_i - E_o = -\left[\frac{\partial}{\partial x} (\rho u H_x) + \frac{\partial}{\partial y} (\rho v H_y) + \frac{\partial}{\partial z} (\rho w H_z\right] dV \, dt \ . \tag{1.9}$$

The value of H depends on the type of material and the flow velocities. For example, if the material is a perfect gas, and if we assume that

the macroscopic potential energy changes may be neglected, the total enthalpy is

$$H_x = c_p T + \frac{u^2}{2}$$

where c_p is the specific heat at constant pressure.

1.2.3 Energy Generated

Energy may be released or absorbed by electrical, nuclear, magnetic, or chemical means. For example, an electric conductor subjected to a voltage potential, as well as temperature potentials, will generate energy that will be proportional to the square of the current flow and the electrical resistance. Similarly, if a chemical reaction is involved, the energy released or absorbed will depend upon the type of reaction and the reaction rate. The latent heat of fusion, while it is actually associated with changes in the internal energy of the substance, can often be expressed as an energy generation term. The generation term becomes

$$Q = q_g dV\, dt \tag{1.10}$$

where q_g is the energy released per unit volume and per unit time.

1.2.4 Stored Energy

The energy stored within the volume of material is made up of internal energy and macroscopic kinetic and potential energy terms. The macroscopic kinetic energy, due to the velocity, is $s^2/2$ for unit mass. The changes in macroscopic potential energy are assumed to be so small that they may be neglected. The change in stored energy ΔE_s may be written as

$$\Delta E_s = \frac{\partial}{\partial t}\left[\rho\left(U+\frac{s^2}{2}\right)\right] dV\, dt \tag{1.11}$$

where $s^2 = u^2 + v^2 + w^2$.

If the material is a perfect gas, the value of U may be taken as $U = c_v T$ + constant where c_v is the specific heat at constant volume. The general energy equation may now be written as

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) + q_r - q_f + q_g$$
$$= \frac{\partial}{\partial x}(\rho u H_x) + \frac{\partial}{\partial y}(\rho v H_y) + \frac{\partial}{\partial z}(\rho w H_z) + \frac{\partial}{\partial t}\left(\rho U + \frac{\rho s^2}{2}\right). \tag{1.12}$$

After some manipulation, this can be rewritten

$$\text{div}\ (\bar{\bar{k}}\ \text{grad}\ T) + q_r + q_g = \rho c_p \frac{DT}{Dt} - T\beta \frac{Dp}{Dt} - \mu\phi \tag{1.13}$$

where $\beta = -(1/\rho)(\partial\rho/\partial T)_p$ is the coefficient of thermal expansion, and k is the thermal conductivity tensor.

Special cases of eq 1.13 occur for gases and liquids. For a perfect gas, $\beta T = 1$, and

$$\text{div}\ (\bar{\bar{k}}\ \text{grad}\ T) + q_r + q_g = \rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} - \mu\phi\ . \tag{1.14}$$

The incompressible fluid has $\beta = 0$, and

$$\text{div}\ (\bar{\bar{k}}\ \text{grad}\ T) + q_r + q_g = \rho c_p \frac{DT}{Dt} - \mu\phi\ . \tag{1.15}$$

For flow through permeable materials, such as soils, the viscous dissipation is often negligible. If the material is also isotropic, then eq 1.15 can be written as

$$\text{div}\ (\bar{\bar{k}}\ \text{grad}\ T) + q_r + q_g = \rho c_p \frac{DT}{Dt}\ . \tag{1.16}$$

Finally, if the thermal conductivity is constant,

$$k\nabla^2 T + q_r + q_g = \rho c_p \frac{DT}{Dt} \tag{1.16a}$$

where ∇^2 is the usual Laplacian operator.

1.2.5 Conduction Equation

Equation 1.13 is the most general form of the energy equation. It must be simplified if solutions are to be found. For a solid where the material is not in motion relative to itself and all the velocities are zero, the dissipation term and the pressure term may be dropped, and

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t}.$$

Equation 1.13 is then

$$\text{div}\ (\bar{\bar{k}}\ \text{grad}\ T) + q_r + q_g = \rho c_p \frac{\partial T}{\partial t}\ . \tag{1.17}$$

We assume that the material is orthotropic and that the radiation term will be handled as a boundary condition. Then eq 1.17 becomes the general conduction equation:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) + q_g = \rho c \frac{\partial T}{\partial t} \tag{1.18}$$

This can be further simplified if all the properties are assumed constant.

Then

$$\nabla^2 T + \frac{q_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1.19}$$

where $\alpha = k/\rho c$ is the thermal diffusivity.

For cylindrical coordinates

$$\nabla^2 T = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial T}{\partial z}\right)\right] . \tag{1.19a}$$

For spherical coordinates

$$\nabla^2 T = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right] . \tag{1.19b}$$

One often begins with eq 1.19 in solving conduction problems. No subscript is used on the specific heat, since $c_p \simeq c_v$ for solids or liquids.

For convective systems with liquid flows, the energy equation is often written as

$$\nabla^2 T + \frac{q_g}{k} = \frac{1}{\alpha}\left[\frac{\partial T}{\partial t} + \underline{s} \cdot \nabla T\right] . \tag{1.20}$$

These equations can be simplified for particular situations. A common, one-dimensional problem is that of a region initially liquid at a temperature T_o, which has its surface temperature suddenly dropped to $T_s < T_f$ for $t > 0$. The material freezes at T_f and the solidification front is denoted by $X(t)$. The process of thawing is obviously similar. Consider the case of melting. For the liquid region (region 1), convection is possible and eq 1.15 is

$$\frac{\partial}{\partial x}\left(k_1\frac{\partial T_1}{\partial x}\right) = \rho_1 c_1\left[\frac{\partial T_1}{\partial t} + u\frac{\partial T_1}{\partial x}\right] . \tag{1.21}$$

The solid phase may be assumed stationary and impervious to liquid flow; therefore eq 1.18 is appropriate:

$$\frac{\partial}{\partial x}\left(k_2\frac{\partial T_2}{\partial x}\right) = \rho_2 c_2\frac{\partial T_2}{\partial t} . \tag{1.22}$$

If the bulk velocity u in the liquid is due only to the volume change during melting, caused by the density differences of the two phases, a simple equation results. Consider a volume of solid $A\Delta X$, which melts in the time Δt. The mass of the material in the volume that changes phase is constant but the volume change of the solid can be different than that of the liquid:

$$A\ u\ \Delta t = V_2 - V_1 = m\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) .$$

Then the velocity induced in the liquid is

$$u = \left(1 - \frac{\rho_2}{\rho_1}\right) \frac{dX}{dt} . \tag{1.23}$$

The energy equation for the liquid phase is then

$$\frac{\partial}{\partial x}\left(k_1 \frac{\partial T_1}{\partial x}\right) = \rho_1 c_1 \left[\frac{\partial T_1}{\partial t} + \left(1 - \frac{\rho_2}{\rho_1}\right) \frac{dX}{dt} \frac{\partial T_1}{\partial x}\right] . \tag{1.24}$$

If the thermal conductivity is constant, than eq 1.22 and 1.24 are

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \left[\frac{\partial T_1}{\partial t} + \left(1 - \frac{\rho_2}{\rho_1}\right) \frac{dX}{dt} \frac{\partial T_1}{\partial x}\right] \tag{1.25}$$

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2} \frac{\partial T_2}{\partial t} . \tag{1.26}$$

These are the constant property forms of the energy equation.

1.2.6 Variable Thermal Conductivity Transformations

While the constant conductivity assumption is not unreasonable, two cases of variable conductivity can be handled without too much difficulty.

Case I. Temperature variation of thermal conductivity

A new temperature can be defined by the Kirchoff transformation

$$\chi = \int_0^T k(T')\ dT' . \tag{1.27}$$

Equation 1.22 can then be transformed to

$$\frac{\partial^2 \chi_2}{\partial x^2} = \frac{\rho_2 c_2}{k_2(T)} \frac{\partial \chi_2}{\partial t} = \frac{1}{\alpha_2(\chi_2)} \frac{\partial \chi_2}{\partial t} . \tag{1.28}$$

This is of the same form as the constant property equation and can be treated similarly.

Case II. Spatial variation of thermal conductivity.

For this case a new space variable is defined

$$x_1 = \int_0^x \frac{dx'}{k(x')}$$

The transformation of eq 1.22 is now

$$\frac{\partial^2 T_2}{\partial x_1^2} = k_2(x_1)\ \rho_2 c_2 \frac{\partial T_2}{\partial t} = \frac{1}{\alpha(x_1)} \frac{\partial T_2}{\partial t}$$

This equation can also be considered in the form of the constant property equation, with a new thermal diffusivity. In either of these cases, the "thermal diffusivity" may be a function of the independent variables.

Often, the density of the frozen and unfrozen phases are assumed to be identical; thus no convection is caused by density difference and the problem is one of pure conduction. The basic equations are then

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t} \qquad 0 \leq x < X(t) \tag{1.29}$$

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2} \frac{\partial T_2}{\partial t} \qquad x > X(t) \tag{1.30}$$

where the thermal diffusivity α may be a variable, and

$$T_1(X,t) = T_2(X,t) = T_f \tag{1.30a}$$

$$T_1(x,0) = T_2(x,0) = T_o \quad . \tag{1.30b}$$

Equation 1.30a describes the natural condition that the frozen and unfrozen layers are at the fusion temperature at the boundary where phase change is occurring.

1.3 Energy Balance at the Phase Change Interface

At the interface between the phases, energy will be released or absorbed as the material freezes or thaws, respectively. The conservation of energy applied to the mass of the volume $A\Delta X$, which undergoes phase change during Δt, is

$$(q_1 - q_2)\ \Delta t + q\Delta t - W = \Delta E\ .$$

In this relation q_1 and q_2 are conduction heat transfers while q may represent other heat flows. The work associated with the volume change of the mass is W. The energy balance is

$$(-k_1 \frac{\partial T_1}{\partial x} + k_2 \frac{\partial T_2}{\partial x})\ A\Delta t + p\ A\Delta x \left(1 - \frac{\rho_2}{\rho_1}\right) + qA\Delta t = m\ (h^f - h^i)\ .$$

The enthalpy change from the solid to the liquid phase, $h^f - h^i$, is the

latent heat ℓ:

$$-k_1 \frac{\partial T_1}{\partial x} + k_2 \frac{\partial T_2}{\partial x} + p \left(1 - \frac{\rho_2}{\rho_1}\right) \frac{dX}{dt} + q = \rho_2 \ell \frac{dX}{dt} . \tag{1.31a}$$

A similar relation holds for the solidification process:

$$-k_1 \frac{\partial T_1}{\partial x} + k_2 \frac{\partial T_2}{\partial x} + p \left(1 - \frac{\rho_1}{\rho_2}\right) \frac{dX}{dt} + q = -\rho_1 \ell \frac{dX}{dt} . \tag{1.31b}$$

The work term in eq 1.31 may be neglected if the system pressure p is much less than 30,000 atmospheres. Thus this term is normally not considered and

$$-k_1 \frac{\partial T_1}{\partial x} + k_2 \frac{\partial T_2}{\partial x} = \pm \rho \ell \frac{dX}{dt} \quad ; \; x = X \tag{1.32}$$

where the upper sign is for melting and the lower sign is for freezing. Equation 1.32 is also valid for cylindrical and spherical coordinates if x is replaced by r.

The derivation assumes that the interface motion is in the positive direction of the space coordinate. If this is reversed, as for solidification of a cylinder of water, then the signs for the latent heat are reversed in eq 1.32.

1.4 NONLINEARITY OF SOLIDIFICATION PROBLEMS

The phase change introduces a basic nonlinearity into the boundary conditions of the problem. If we consider eq 1.30a, the temperature at the phase change interface is

$$T(x,t)_{x=X} = T_f$$

The differential of this equation is zero since T_f is a constant:

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial t} dt = 0 .$$

Now if this is evaluated at x = X:

$$\left(\frac{\partial T}{\partial x}\right) \frac{dX}{dt} = - \frac{\partial T}{\partial t} .$$

Combining this result with eq 1-29 and 1-32 produces a boundary condition in the form

$$-k_1 \left(\frac{\partial T_2}{\partial x}\right)^2 + k_2 \left(\frac{\partial T_2}{\partial x}\right) \frac{\partial T_1}{\partial x} = \pm \rho \ell \alpha_1 \frac{\partial^2 T_1}{\partial x^2} \quad ; x = X \tag{1.33}$$

where the upper sign is again for melting. The nonlinear nature of the problem is quite obvious in this form.

The nonlinearity can be explicitly expressed in the differential equation for the energy. Consider a melting problem with no density change. The following system of equations will be considered:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{1.34}$$

$$T(X,t) = T_f \tag{1.35a}$$

$$T(x,0) = T_o \tag{1.35b}$$

$$T(0,t) = T_s \tag{1.35c}$$

$$\rho \ell \frac{dX}{dt} = -k \frac{\partial T}{\partial x} (X,t) \ . \tag{1.35d}$$

The phase change interface will be immobilized by using a new variable called the Landau transformation,

$$\zeta = \frac{x}{X} \ . \tag{1.36}$$

The equation is nondimensionalized by using the following variables

$$\nu = \frac{T - T_o}{T_s - T_f} \qquad \tau = \alpha t / X_o^2 \qquad \xi = \frac{X}{X_o} \qquad S_T = \frac{c}{\ell} (T_s - T_f) \ .$$

Equations 1.34 and 1.35 are

$$\frac{\partial^2 \nu}{\partial \zeta^2} = \xi^2 \frac{\partial \nu}{\partial \tau} + \zeta \, S_T \frac{\partial \nu}{\partial \zeta} (1,\tau) \frac{\partial \nu}{\partial \zeta} \tag{1.37}$$

$$\nu(\zeta,o) = 0 \tag{1.37a}$$

$$\nu(o,\tau) = \frac{T_s - T_o}{T_s - T_f} \tag{1.37b}$$

$$\nu(1,\tau) = \frac{T_f - T_o}{T_s - T_f} \tag{1.37c}$$

$$\frac{d\xi}{d\tau} = \frac{-S_T}{\xi} \frac{\partial \nu}{\partial \zeta} (1,\tau) \ . \tag{1.37d}$$

The domain of the phase change is now limited to (0,1) and the nonlinearity is clear in eq 1.37

This inherent nonlinearity, along with the unknown motion of the phase change interface, has limited the exact solutions of phase change problems to a mere handful of cases with particularly simple geometries and boundary conditions.

1.5 HEAT AND MASS FLOW IN POROUS MATERIALS

A porous material, such as a soil, is often represented as a continuum, as already noted. However, it is useful to examine some heat and mass flow relations that may occur within the pores or voids of the material. We direct our attention to systems with water although other liquids may be present.

1.5.1 Conservation of Mass and Energy

A porous medium can be considered a matrix of solids, which creates a solid skeleton with voids between the solid particles as seen in Figure 1.2. The void space can contain water in the vapor, liquid, and solid phases and noncondensable gases such as air. We will consider these four constituents bound within the solid matrix and the solid matrix itself to make up the porous medium.

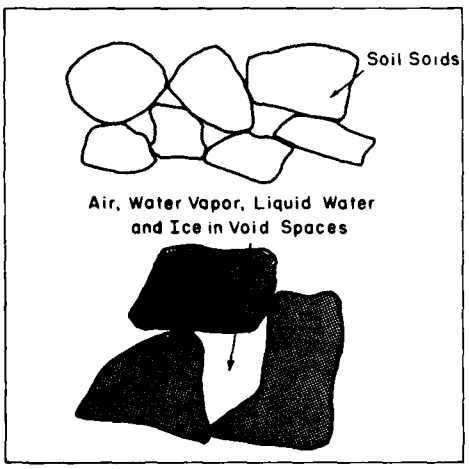


Figure 1.2 Schematic of a porous medium such as soil.

In a given volume V of the material, m_i will denote the mass of component i. The subscripts 1-4 will denote vapor, water, ice, and air respectively. The mass ratio of each bound constituent is

$$\theta_i = \frac{m_i}{m_o} \tag{1.38}$$

where m_o is the mass of the solids and the solid density is defined as $\rho_o = m_o/V$.

The total bound mass is

$$\theta = \sum_{i=1}^{4} \theta_i \quad . \tag{1.39}$$

Since the masses of water vapor and air are negligible, θ can be

approximated by

$$\theta = \theta_2 + \theta_3 \ . \tag{1.40}$$

Conservation of mass and energy must hold for each of the constituents.

The mass flux rate for a component is defined as j_i. We are concerned mainly with the water liquid and vapor fluxes. Then the conservation of mass requires

$$\frac{\partial(\rho_o \theta_i)}{\partial t} = -\text{div } j_i + I_i \ . \tag{1.41}$$

where I_i denotes the amount of component i that is created or destroyed within a differential volume (in particular, by phase change). Since

$$\sum_{i=1}^{4} I_i = 0 \tag{1.42}$$

it follows that

$$\frac{\partial \rho_o \theta}{\partial t} = -\text{div}\left(\sum_{i=1}^{4} j_i\right) \ . \tag{1.43}$$

For the porous medium we will assume that conduction and enthalpy flows will be the energy flux terms. From eq 1.16 it follows that

$$-\text{div } q = \sum_{i=1}^{4} \text{div } (j_i h_i) + \sum_{i=0}^{4} \frac{\partial(\rho_o \theta_i h_i)}{\partial t} \tag{1.44}$$

or

$$\frac{\partial \sum_{i=0}^{4} \rho_o \theta_i h_i}{\partial t} = -\text{div } (q + \sum_{1}^{4} j_i h_i) \tag{1.45}$$

where q is the conduction heat transfer and h_i the enthalpy of each component.

The specific heat at constant pressure is

$$c_{pk} = \frac{\partial h_k}{\partial T} \ .$$

Then

$$\frac{\partial(\rho_o \theta_i h_i)}{\partial t} = h_i \frac{\partial \rho_o \theta_i}{\partial t} + \rho_o \theta_i c_{pi} \frac{\partial T}{\partial t} \tag{1.46}$$

$$\sum_{i=0}^{4} \rho_o \theta_i c_{pi} \frac{\partial T}{\partial t} + \sum_{i=1}^{4} h_i \frac{\partial \rho_o \theta_i}{\partial t} = -\text{div}(q + \sum_{i=1}^{4} j_i h_i) \ . \tag{1.47}$$

From eq 1.41,

$$\sum_{i=1}^{4} h_i \frac{\partial(\rho_o \theta_i)}{\partial t} = - \sum_{i=1}^{4} h_i \text{ div } j_i + \sum_{i=1}^{4} h_i I_i$$

$$\sum_{i=0}^{4} \rho_o \theta_i c_{pi} \frac{\partial T}{\partial t} = \sum_{i=1}^{4} h_i \text{ div } j_i - \sum_{i=1}^{4} h_i I_i - \text{div } (q + \sum_{i=1}^{4} j_i h_i) \ . \quad (1.48)$$

Now

$$\sum \text{div } j_i h_i = \sum j_i \text{ div } h_i + h_i \text{ div } j_i = \sum j_i \ c_{pi} \ \nabla T + \sum h_i \text{ div } j_i \ . \quad (1.49)$$

Thus

$$\rho_o C \frac{\partial T}{\partial t} = -\text{div } q - \sum_{i=1}^{4} (h_i I_i + c_{pi} \ j_i \ \nabla T) \quad (1.50)$$

where the weighted average specific heat for the porous system is

$$C = \sum_{i=0}^{4} \theta_i c_{pi} \ . \quad (1.51)$$

From Fourier's law

$$q = -k \ \nabla T \quad (1.52)$$

and

$$\rho_o C \frac{\partial T}{\partial t} = \text{div } (k \ \nabla T) - \sum_{i=1}^{3} h_i I_i - \sum_{i=1}^{3} c_{pi} \ j_i \ \nabla T \ . \quad (1.53)$$

The convective heat transfer term, in eq 1.53, is often ignored if the mass flux includes only diffusion. This is not acceptable if water fluxes such as filtration or groundwater are present (Luikov 1964, 1975).

1.5.2 Mass Fluxes

The flux of a gas (in particular, water vapor) is governed by Fick's law

$$j_i = j_v = - \frac{Da}{RT} \nabla p_v \quad (1.54)$$

where

j_v = mass flux, lbm/(s-ft^2)

Da = diffusion coefficient of water in air, ft^2/s

R = gas constant, (ft-lbf)/(lbm°R)

T = absolute temperature, °R

p_v = partial pressure of water vapor, lbf/ft^2.

If the partial pressure is a function of moisture content θ and temperature, then

$$\nabla p_v = \left(\frac{\partial p_v}{\partial \theta}\right)_T \nabla\theta + \left(\frac{\partial p_v}{\partial T}\right)_\theta \nabla T \ . \tag{1.55}$$

The relative humidity η is related to the vapor partial pressure by

$$p_v = p_s \eta \tag{1.56}$$

where p_s is saturation vapor pressure, a function of temperature only.

The vapor pressure is a function of water content and temperature if the water content is less than the maximum moisture due to sorption θ_s. For $\theta > \theta_s$ (moist body), the relative humidity is unity and the vapor pressure is a function of temperature only. Then

$$j_v = -\frac{Da}{RT}\left[p_s \left(\frac{\partial \eta}{\partial \theta}\right)_T \nabla\theta + \eta \left(\frac{\partial p_s}{\partial T}\right)_\theta \nabla T\right] \ . \tag{1.57}$$

This equation is often written as

$$\frac{j_v}{\rho_w} = -D'_{\theta V} \nabla\theta - D'_{TV} \nabla T \tag{1.58}$$

or

$$\frac{j_v}{\rho_o} = -D_{\theta V} \nabla\theta - D_{TV} \nabla T \tag{1.59}$$

where $D_{\theta V}$ and D_{TV} are modified to account for the decrease in diffusion due to the pore geometry and water content using a factor ε. The flux of mass can carry with it a flux of both sensible and latent heat. It follows from eq 1.57, that

$$D_{\theta V} = \frac{\varepsilon_\theta Da}{\rho_o RT} \ p_s \left(\frac{\partial \eta}{\partial \theta}\right)_T \tag{1.60}$$

$$D_{TV} = \frac{\varepsilon_T Da}{\rho_o RT} \eta \left(\frac{\partial p_s}{\partial T}\right)_\theta \ . \tag{1.61}$$

For a moist body, $D_{\theta V} = 0$.

Capillary water transport is controlled by the Darcy equation, for low flow velocities:

$$j_2 = j_\ell = -K_\ell \rho_w \nabla\Phi \tag{1.62}$$

where K_ℓ is the hydraulic conductivity of the medium and Φ the total potential for water transport.

The moisture potential for a soil system can be expressed as

$$\Phi = \psi + \rho_w g z \qquad (1.63)$$

where ψ = moisture potential, a function of temperature and water content and $\rho_w g z$ = gravitational potential energy. Examples of moisture potentials are shown in Figure 1.3. Equation 1.62 can then be written as

$$\frac{j_\ell}{\rho_w} = -K_\ell \left[\frac{1}{\rho_w g} \left\{ \left(\frac{\partial \psi}{\partial T}\right)_\theta \nabla T + \left(\frac{\partial \psi}{\partial \theta}\right)_T \nabla \theta \right\} + \nabla z \right] \qquad (1.64)$$

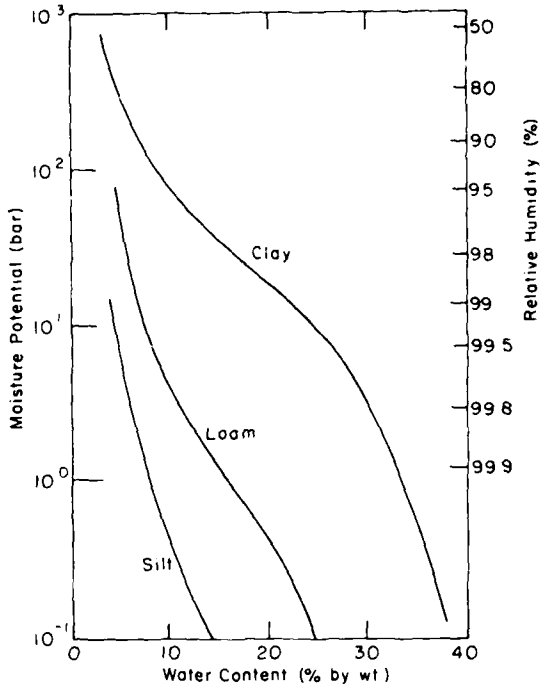


Figure 1.3 Suction and relative humidity of different soils (Johansen 1975).

or

$$\frac{j_\ell}{\rho_o} = -K_\ell \left[\frac{1}{\rho_o g} \left\{ \left(\frac{\partial \psi}{\partial \theta}\right)_T \nabla \theta + \left(\frac{\partial \psi}{\partial T}\right)_\theta \nabla T \right\} + \nabla z \right] \qquad (1.65)$$

$$\frac{j_\ell}{\rho_w} = -D'_{\theta\ell} \nabla \theta - D'_{T\ell} \nabla T - K_\ell \nabla z \qquad .66)$$

$$\frac{j_\ell}{\rho_o} = -D_{\theta\ell} \nabla \theta - D_{T\ell} \nabla T - K_\ell \frac{\rho_w}{\rho_o} \nabla z \qquad (1.67)$$

where

$$D'_{\theta\ell} = \frac{K_\ell}{\rho_w g} \left(\frac{\partial \psi}{\partial \theta}\right)_T \qquad D_{\theta\ell} = \frac{K_\ell}{\rho_o g} \left(\frac{\partial \psi}{\partial \theta}\right)_T$$

$$D'_{T\ell} = \frac{K_\ell}{\rho_w g} \left(\frac{\partial \psi}{\partial T}\right)_\theta \qquad D_{T\ell} = \frac{K_\ell}{\rho_o g} \left(\frac{\partial \psi}{\partial T}\right)_\theta$$

If the gravitational potential is ignored, or $\nabla z = 0$,

$$\frac{j_\ell}{\rho_o} = -D_{\theta\ell} \nabla \theta - D_{T\ell} \nabla T \,. \qquad (1.68)$$

The total flux of moisture can be written as

$$\frac{j}{\rho_o} = -D_\theta \nabla\theta - D_T \nabla T \tag{1.69}$$

or

$$\frac{j}{\rho_o} = - D_\theta [\nabla\theta + \delta \ \nabla T] \tag{1.70}$$

where $D_\theta = D_{\theta V} + D_{\theta \ell}$ (ft^2/s)

$D_T = D_{TV} + D_{T\ell}$ $(lbmw\ ft^2)/(lbms\text{-}s\text{-}°F))$

$\delta = D_T/D_\theta$ $(lbmw/(lbms\,°F))$

lbmw = mass of water

lbms = mass of soil.

The overall conservation of mass, eq 1.43, is then

$$\frac{1}{D_\theta} \frac{\partial\theta}{\partial t} = \nabla^2\theta + \delta \ \nabla^2 T \quad . \tag{1.71}$$

1.5.3 Energy Equation for Unfrozen Medium

If the medium remains above the solidification temperature of the moisture, then no ice can be present. From the relations developed, the energy equation is

$$\rho_o C \frac{\partial T}{\partial t} = \text{div}\ (k\nabla T) + \ell_e \ I_2 - (c_{p1} j_1 + c_{p2} \ j_2)\nabla T \tag{1.72}$$

where ℓ_e is the latent heat of evaporation.

If $j_1 = 0$, (no vapor flux), then $I_2 = I_1 = 0$, and

$$\rho_o C \frac{\partial T}{\partial t} = \text{div}\ (k\nabla T) + c_{p2} \ \rho_o [D_{\theta\ell} \ \nabla\theta + D_{T\ell} \ \nabla T] \nabla T \ . \tag{1.73}$$

If convection is negligible, then the usual energy equation governs:

$$\rho_o C \frac{\partial T}{\partial t} = \text{div}\ (k\nabla T) \ . \tag{1.74}$$

1.5.4 Energy Equation for Frozen Porous Medium

We shall assume negligible vapor flux and convective heat transfer. Then

$$\theta = \theta_2 + \theta_3 \tag{1.75}$$

$$I_2 = \rho_o \frac{\partial\theta_2}{\partial t} - \text{div}\ j_2 \tag{1.76}$$

$$\rho_o \frac{\partial\theta}{\partial t} = \text{div}\ j_2 \tag{1.77}$$

$$\rho_o C \frac{\partial T}{\partial t} = \text{div}\ (k\nabla T) + \rho_o \ell \left[\frac{\partial \theta_3}{\partial t}\right] \quad . \tag{1.78}$$

If melting/freezing occurs but there is no liquid flux,

$$\rho_o C \frac{\partial T}{\partial t} = \text{div}\ (k\nabla T) - \rho_o \ell \frac{\partial \theta_2}{\partial t} \quad . \tag{1.79}$$

1.5.5 Moisture Flows in Soils

A qualitative picture of the moisture flow coefficients, for soils, can be obtained from Figure 1.4. In this figure, the relative humidity varies slowly with water content until the soil is quite dry (thus $D_{\theta V}$ is negligible for wet soils) increases to a maximum for dry soils, and disappears at zero water content. D_{TV} decreases as the water content increases due to increased liquid water and also decreases as θ decreases because of the low relative humidity. A maximum value occurs at a low water content where liquid continuity is low. The moisture potential increases as θ decreases but is large for all water contents. The hydraulic conductivity tends to decrease as water content drops and thus $D_{\theta\ell}$ is relatively constant for wet soils and drops abruptly as the soil dries out due to low water continuity. The variation of moisture potential with temperature is relatively weak and thus $D_{T\ell}$ is modest for soil systems.

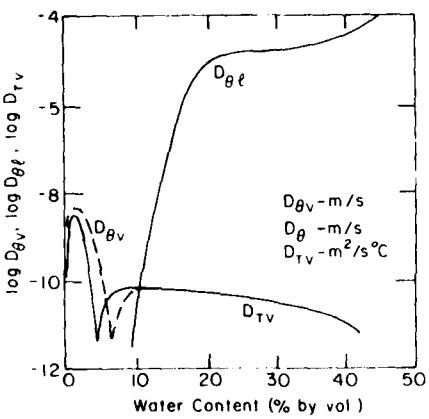


Figure 1.4. Moisture diffusivities shown qualitatively (Johansen 1975).

In addition to the coupled heat and mass flows described above, heat flows result from radiation and free convection in the voids. Radiation in the voids is a function of the temperature levels, pore geometry, and water content. The radiation effect decreases rapidly with decreasing pore size, increasing water content, and decreasing temperature. Thus, radiation will be most prominent for low saturation, large pore size, and high temperatures. The radiation heat flow can be written as

$$q_r = -h_r \nabla T \tag{1.80}$$

where h_r is a radiation coefficient, which is a function of pore size, water content, and absolute temperature.

Free convection in porous media will be initiated if the Rayleigh number exceeds a critical value. The Rayleigh number R_a for a layer of porous material containing a fluid is

$$R_a = \frac{\beta g \rho d K \Delta T}{\alpha \mu} \tag{1.81}$$

where

d = layer thickness

K = permeability

ΔT = temperature difference

α = thermal diffusivity.

As the pore size decreases, a larger temperature difference is needed to initiate convective motion.

The free convection can be calculated from

$$q_c = -a \left(\frac{\Delta T}{d}\right)^b d \nabla T = -h_c \nabla T \tag{1.82}$$

where a and b denote coefficients for a particular convective system. Free convection rapidly decreases in importance as the pore size decreases. For dry systems, air convection will predominate, while highly saturated media will experience convection in water.

1.6 CONDUCTION IN POROUS MEDIA

It has been known for many years that, notwithstanding the mass and heat flux relations just derived, pure conduction may be an excellent assumption for porous media (Porkhaev 1959, Martynov 1959).

The total heat flux that might occur in a porous medium is

$$q_{TOT} = -\left(k + \rho_w h_v D'_{TV} + \rho_w h_\ell D'_{T\ell} + h_c + h_r\right) \nabla T$$

$$-\left(\rho_w h_v D'_{\theta V} + \rho_w h_\ell D'_{\theta \ell}\right) \nabla \theta \ . \tag{1.83}$$

An effective thermal conductivity can be defined to account for that part of the non-conduction heat flux which is expressed in terms of the temperature gradient:

$$k_e = k + \rho_w h_v D'_{TV} + \rho_w h_\ell D'_{T\ell} + h_c + h_r \ . \tag{1.84}$$

If the second term on the right-hand side of eq 1.78 is negligible, then pure conduction is an excellent assumption, with the thermal conductivity

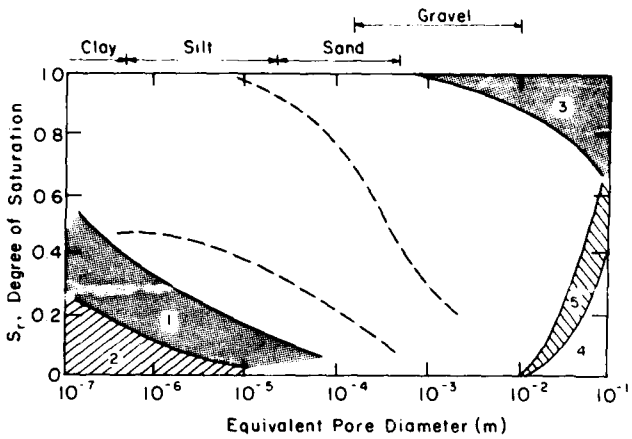


Figure 1.5. Regions of predominant influence of the various heat transfer mechanisms in relation to soil grain size and degree of saturation. Expected variations in degree of saturation under field conditions lie within region bounded by dashed lines (Johansen 1975). 1--thermal redistribution of moisture, 2--vapor diffusion due to moisture gradients, 3--free convection in water, 4--free convection to air, 5--heat radiation.

and other properties of the medium altered to account for the radiation and moisture effects.

Clearly, such an assumption is not possible if a significant heat flow from bulk water movement occurs. Figure 1.5 is a qualitative attempt to outline the important heat flow regimes in a soil system (Johansen 1975). Note that for nearly all practical cases, pure conduction will predominate or can be corrected with the use of an effective thermal conductivity. The figure is strictly valid only for a thawed soil. If a freeze interface exists, then the flow of soil moisture can be significantly increased and convection may need to be considered explicitly.

For many applications, a soil system can be reasonably treated as a continuum and the temperature field evaluated on the basis of conduction only. The effects of moisture can be incorporated into the soil properties.

NOMENCLATURE FOR CHAPTER 1.

A	area
c_p, c_v	specific heat at constant pressure, constant volume
C	flow velocity
d	thickness of permeable layer
D	diffusion coefficient
E_i, E_o	energy flow across boundaries
E_s	stored energy in system
g	gravitational acceleration
g_o	conversion factor for systems of units
h_r	radiation coefficient

h_c	coefficient of convection
h^f, h^i	final and initial specific enthalpies
h_i	specific enthalpy of ith component in porous body
$h_\ell h_v$	enthalpy of liquid and vapor
H	enthalpy
I	source/sink of component in porous body
k	thermal conductivity
$\bar{\bar{k}}$	thermal conductivity tensor
K	permeability of porous layer
K_ℓ	hydraulic conductivity
ℓ	latent heat of fusion
j	mass flux of water
j_i	mass flux rate of ith component of porous body
m	mass
m_i	mass per unit volume of ith component of porous system
p_s	saturation pressure of water vapor
p_v	partial pressure of water vapor
P	pressure
q	heat transfer rate
q_g, q_f, q_r	generated, frictional, and radiant energy per unit volume, per unit time
Q	energy in form of heat
Q_g	energy generation in system
R	gas constant
R_a	Rayleigh number
s	$\lvert \underline{s} \rvert$ fluid speed
$\underline{s}$	velocity relative to x, y, z frame
S_r	degree of saturation of soil
S_T	$\frac{c}{\ell}(T_s - T_f)$, Stefan number
t	time
T	temperature
T_o, T_f, T_s	initial, fusion, and surface temperatures
u, v, w	components of velocity in x, y, z directions
U	internal energy
v_s	specific volume

V	volume
W	work
x, y, z	Cartesian coordinates
x_i	transformed coordinates
X	phase change interface position
Z	location of fluid in gravitational field.
α	thermal diffusivity
β	coefficient of thermal expansion
δ	D_T/D_θ
∇	div $\underline{s}$
∇A	gradient of scalar quantity A.
ϵ	diffusion correction factor
ζ	x/X, Landau transformation
η	relative humidity
θ	moisture content
μ	viscosity
ν	$(T - T_o)/(T_s - T_f)$
ξ	X/X_o
ρ	density
τ	$\alpha t/X_o^2$
ϕ	frictional dissipation function
Φ	total potential for water transport
χ	transformed temperature
ψ	moisture potential

<u>Subscripts</u>

ℓ - liquid

v - vapor

w - water

1, 2 - thawed and frozen regions

0,1,2,3,4 - solid, vapor, liquid, ice, air components in a porous medium

CHAPTER 2. PLANE PROBLEMS WITH TEMPERATURE BOUNDARY CONDITIONS

As noted in Chapter 1, the principal mode of energy transfer is often diffusion or molecular conduction. In the following chapters only those problems that can be formulated in terms of pure conduction will be considered, and useful solutions for plane, cylindrical, and spherical geometries will be examined. These can include pure substances, alloys, mixtures, and some cases of porous media.

Ideally, exact solutions of important engineering problems of freezing and thawing are sought. However, because of the nonlinearity of the phase change system, there are very few complete analytic solutions. Thus, approximate solutions will be sought for those problems that have not been solved exactly. Several approximate methods have been widely used, including quasi-steady methods, the heat balance integral concept, and variational methods. Discussions of these methods are given in Appendices A-D.

This chapter will look at plane problems for which the temperature is specified as a boundary condition. Chapters 3 and 4 will consider convection and heat flux boundary conditions.

2.1 NEUMANN PROBLEM AND VARIATIONS

Systems with plane interfaces occur frequently in engineering design. The first and still the most comprehensive exact solution method is by Neumann (c 1860), generalized in Carslaw and Jaeger (1959). It is instructive to examine, in detail, this classic problem and its solution.

2.1.1 Exact Similarity Solution.

Initially, a semi-infinite region, as shown in Figure 2.1, is at a constant temperature T_o and the temperature of the surface is suddenly dropped to T_s and held constant (step change at surface). Initially the medium is assumed to be in a liquid state, i.e. $T_o > T_f$, where

T_o = initial temperature

T_f = fusion temperature of medium

T_s = surface temperature.

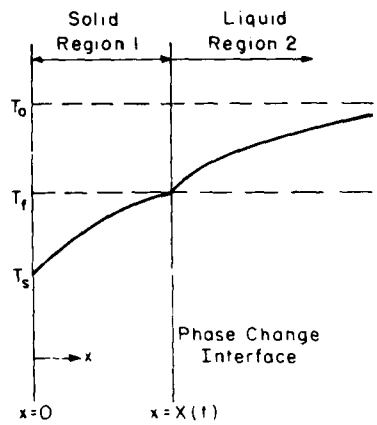


Figure 2.1. Temperature distribution in partially frozen medium.

The Neumann problem can then be formulated as

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} \tag{2.1}$$

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} \tag{2.2}$$

$$\lim_{x \to \infty} T_2 = T_o \tag{2.3a}$$

$$T_1\ (0,\ t) = T_s\ . \tag{2.3b}$$

The location of the freezing interface is a function of time. The temperatures at this location, of the liquid and solid, are both equal to the fusion temperature:

$$T_1(X,t) = T_2(X,t) = T_f \tag{2.3c}$$

where T_f is the freezing point of the medium.

The energy balance at the phase change interface is

$$k_1 \frac{\partial T_1}{\partial x} - k_2 \frac{\partial T_2}{\partial x} = \rho \ell \frac{dX}{dt}\ ;\ x = X \tag{2.3d}$$

where ℓ is latent heat of fusion on a mass basis.

A solution to this problem is obtainable by using a well-known similarity transformation. Let

$$\eta = \frac{x}{2\sqrt{\alpha_1 t}}\ . \tag{2.4}$$

Equation 2.1 then becomes

$$\frac{d^2 T_1}{d\eta^2} + 2\eta \frac{dT_1}{d\eta} = 0 \tag{2.5}$$

The first integral of eq 2.5 can be obtained immediately:

$$\frac{dT_1}{d\eta} = A_1 e^{-\eta^2} \quad .$$

A second integration gives a formal solution for the temperature:

$$T_1 = A_1 \int_0^\eta e^{-\beta^2} d\beta \quad . \tag{2.6}$$

The error function is defined as

$$\text{erf}\ \eta = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\beta^2}\ d\beta \quad .$$

The error function has been numerically evaluated and tabulated by Carslaw and Jaeger (1959) (see App. D). Thus a formal solution to the conduction equation is available, if the similarity transformation is valid. This will be the case if the differential equation and all the boundary and initial conditions can be expressed in terms of the single, independent variable η, which is true for the Neumann problem. However, as will be seen, this similarity solution is valid only for problems where the phase change interface moves proportionally to $\sqrt{t}$. This precludes the use of the similarity transformation for many interesting problems such as convection/radiation boundary conditions, variable initial temperature, variable surface temperature, etc.

A solution of eq 2.1 and 2.3b is then

$$T_1 = T_s + A_1\ \text{erf}\ \frac{x}{2\sqrt{\alpha_1 t}} \quad . \tag{2.7}$$

It also follows that

$$T_2 = T_o - B_1 \left[1 - \text{erf}\ \frac{x}{2\sqrt{\alpha_2 t}}\right] = T_o - B_1\ \text{erfc}\ \frac{x}{2\sqrt{\alpha_2 t}} \quad . \tag{2.8}$$

The notation used in eq 2.8 is erfc η = 1 - erf η.

Then, using eq 2.7 and 2.8 in eq 2.3c leads to

$$T_s + A_1 \text{erf}\ \frac{X}{2\sqrt{\alpha_1 t}} = T_o - B_1 \text{erfc}\ \frac{X}{2\sqrt{\alpha_2 t}} = T_f \quad . \tag{2.9}$$

Now, since eq 2.9 must be satisfied for all values of time, then X must be proportional to $\sqrt{t}$, for then

$$T_s + A_1 \text{erf}\ \frac{a}{2\sqrt{\alpha_1}} = T_o - B_1 \text{erfc}\ \frac{a}{2\sqrt{\alpha_2}} = T_f = \text{constant} \quad .$$

where it is assumed that $X \approx a\sqrt{t}$. Thus, we may write

$$X = 2\gamma\sqrt{\alpha_1 t} \tag{2.10}$$

where γ is a constant. Now, from eq 2.7 and 2.8,

$$\frac{\partial T_1}{\partial x} = A_1 \frac{1}{\sqrt{\pi\alpha_1 t}} \exp\left(-\frac{x^2}{4\alpha_1 t}\right)$$

$$\frac{\partial T_2}{\partial x} = B_1 \frac{1}{\sqrt{\pi\alpha_2 t}} \exp\left(-\frac{x^2}{4\alpha_2 t}\right) \quad .$$

Thus from eq 2.3d and 2.10,

$$k_1 A_1 e^{-\gamma^2} - k_2 B_1\sqrt{\alpha_{12}} \exp(-\alpha_{12}\gamma^2) = \alpha_1 \rho \ell \gamma\sqrt{\pi}$$

where α_{12} denotes the ratio of α_1 to α_2.

Then using eq 2.9 and 2.10,

$$T_s + A_1 \operatorname{erf} \gamma = T_o - B_1 \operatorname{erfc} \gamma\sqrt{\alpha_{12}} = T_f$$

$$A_1 = \frac{T_f - T_s}{\operatorname{erf} \gamma} \qquad B_1 = \frac{(T_o - T_f)}{\operatorname{erfc}(\gamma\sqrt{\alpha_{12}})}$$

and finally, the equation for the constant γ is

$$\frac{\exp(-\gamma^2)}{\operatorname{erf} \gamma} - \frac{k_{21}\sqrt{\alpha_{12}}\,(T_o - T_f)e^{-\alpha_{12}\gamma^2}}{(T_f - T_s)\operatorname{erfc}(\gamma\sqrt{\alpha_{12}})} = \frac{\ell\gamma\sqrt{\pi}}{c_1(T_f - T_s)} \quad . \tag{2.11}$$

The temperatures are now given by

$$T_1 = T_s + \frac{(T_f - T_s)}{\operatorname{erf} \gamma} \operatorname{erf} \frac{x}{2\sqrt{\alpha_1 t}} \tag{2.12}$$

$$T_2 = T_o - \frac{(T_o - T_f)}{\operatorname{erfc}(\gamma\sqrt{\alpha_{12}})} \operatorname{erfc} \frac{x}{2\sqrt{\alpha_2 t}} \quad . \tag{2.13}$$

For various properties and conditions, the value of γ may be numerically evaluated; this has been done, as given in Table 2.1, for water with the following properties:

(ice)	(liquid water)
$k_1 = 0.0053$ cal/s-cm-°C	$k_2 = 0.00144$
$\alpha_1 = 0.0115$ cm^2/s	$\alpha_2 = 0.00144$
$c_1 = 0.501$ cal/g-°C	$c_2 = 1.0$
$\ell = 79.71$ cal/g	

Table 2.1 Value of γ, eq 2.11, for water (Carslaw and Jaeger 1959).

$(T_f - T_s)$	$(T_o - T_f)$°C					
°C	0	1	2	3	4	5
1	0.056	0.054	0.053	0.051	0.050	0.049
2	0.079	0.077	0.076	0.074	0.073	0.071
3	0.097	0.095	0.093	0.091	0.090	0.088
4	0.111	0.110	0.108	0.106	0.104	0.103
5	0.124	0.123	0.121	0.119	0.117	0.115

The solution is also valid for the thaw case if the meaning of the property subscripts is interchanged; i.e. region 1 is now thawed.

2.1.2 Heat Balance Integral Solution

A simple, but useful, approximate solution to the Neumann problem can be obtained by the use of the heat balance integral method. The system of equations, derived in Appendix B, for the freeze problem, is

$$\frac{d\theta_1}{dt} - T_f \frac{dX}{dt} - \alpha_1 \left[\frac{\partial T_1(X,t)}{\partial x} - \frac{\partial T_1(0,t)}{\partial x}\right] = 0 \tag{2.14}$$

$$\frac{d\theta_2}{dt} - T_o \frac{d\delta}{dt} + T_f \frac{dX}{dt} + \alpha_2 \frac{\partial T_2(X,t)}{\partial x} = 0 \ . \tag{2.15}$$

where subscripts 1, 2 denote the solid and liquid regions and δ = temperature penetration depth.

$$\theta_1 = \int_0^X T_1(x,t)dx \tag{2.16}$$

$$\theta_2 = \int_X^\delta T_2(x,t)dx \tag{2.17}$$

$$T_1(0,t) = T_s \tag{2.18a}$$

$$T_1(X,t) = T_f \tag{2.18b}$$

$$T_2(\delta,t) = T_o \tag{2.18c}$$

$$\frac{\partial T_2\ (\delta,t)}{\partial x} = 0 \tag{2.18d}$$

$$T_2(X,t) = T_f \tag{2.18e}$$

$$k_1 \frac{\partial T_1(X,t)}{\partial x} - k_2 \frac{\partial T_2(X,t)}{\partial x} = \rho_1 \ell \frac{dX}{dt} \tag{2.19}$$

$$-k_1 \left[\frac{\partial T_1(X,t)}{\partial x}\right]^2 + k_2 \frac{\partial T_1(X,t)}{\partial x} \frac{\partial T_2(X,t)}{\partial x} = \rho_1 \ell \alpha_1 \frac{\partial^2 T_1(X,t)}{\partial x^2} \tag{2.20}$$

Assume that the solutions for the phase-change interface X and the thermal penetration depth δ are

$$X = 2\gamma \sqrt{\alpha_1 t} \tag{2.21}$$

$$\delta = 2\beta \sqrt{\alpha_2 t} \quad . \tag{2.22}$$

A linear approximation for the temperature in region 1, satisfying eq 2.18a,b, is

$$T_1 = T_s + (T_f - T_s) \frac{x}{X} \quad . \tag{2.23}$$

A quadratic approximation for T_2 which satisfies eq (2.18 c,d,e) is

$$T_2 = T_f + 2\Delta T \frac{x-X}{\delta-X} - \Delta T \frac{(x-X)^2}{(\delta-X)^2} \tag{2.24}$$

where

$$\Delta T = T_o - T_f \ .$$

The key to the solution is to recognize, from the exact solution, that δ/X should not vary with time, although it is a function of ϕ, S_T, and the property values. Thus, let $\delta/X = b(\phi, S_T, \text{property values})$, where

$$S_T = \frac{c_1}{\ell} (T_f - T_s)$$

$$\phi = \frac{T_o - T_f}{T_f - T_s} \quad .$$

The ratio of the sensible to latent heat is defined as the Stefan number, S_T. This dimensionless parameter will be used frequently in solidification and melting problems. Equation 2.24 is then

$$T_2 = T_f + \frac{2\Delta T}{(b-1)} \frac{x-X}{X} - \frac{\Delta T(x-X)^2}{(b-1)^2 X^2} \quad . \tag{2.25}$$

Then, with eq 2.25, eq 2.15 is

$$(T_f + \frac{2}{3}\Delta T)(b - 1)\frac{dX}{dt} - T_o b\frac{dX}{dt} + \frac{2\alpha_2 \Delta T}{(b-1)X} + T_f\frac{dX}{dt} = 0 \quad . \qquad (2.26)$$

By use of eq 2.21, the value of b can be obtained from eq 2.26:

$$b = \sqrt{2.25 + \frac{3\alpha_{2\perp}}{\gamma^2}} - \frac{1}{2} \quad . \qquad (2.27)$$

Equation 2.16 for $\theta_\perp$ is

$$\theta_\perp = \frac{T_s + T_f}{2} X \quad . \qquad (2.28)$$

In eq 2.14, the term $\partial T_\perp(X,t)/\partial x$ is evaluated using eq 2.19 and eq 2.23 and 2.25. This leads to the following differential equation for X:

$$\left(\frac{1}{2} + \frac{1}{S_T}\right)\frac{dX}{dt} = \frac{\alpha_\perp}{X}\left(1 - \frac{2k_{2\perp}\phi}{b-1}\right) \quad . \qquad (2.29)$$

The solution is straightforward and can be written as

$$X = 2\gamma\sqrt{\alpha_\perp t}$$

with

$$\gamma^2 = \frac{-b_\perp - \sqrt{b_\perp^2 - 4a\, S_T^2}}{2a} \qquad (2.30)$$

$$a = \left(S_T + 2 + \frac{2k_{2\perp}\,\phi S_T}{\alpha_{2\perp}}\right)(S_T + 2)$$

$$b_\perp = -2\, S_T\left(S_T + 2 + \frac{k_{2\perp}\,\phi\, S_T}{\alpha_{2\perp}}\right) - \frac{4}{3}\frac{(k_{2\perp}\,\phi\, S_T)^2}{\alpha_{2\perp}} \quad .$$

As often occurs with the heat balance integral method, a more complicated approximation for $T_\perp$ does not significantly improve the accuracy. A quadratic approximation $T_\perp$ that satisfies eq 2.18a,b and 2.20 is

$$T_\perp = T_f + \frac{px(x-X)}{X^2} - \frac{\ell S_T}{c_\perp}\frac{(x-X)^2}{X^2} \qquad (2.31)$$

$$p = \frac{\ell}{c_\perp} + \frac{k_{2\perp}\Delta T}{(b-1)} + \sqrt{\left(\frac{\ell}{c_\perp} - \frac{k_{2\perp}\Delta T}{b-1}\right)^2 + 2\left(\frac{\ell}{c_\perp}\right)^2 S_T} \quad .$$

The equation for X is now

$$\left(\frac{p}{6} + \frac{\ell}{3}\frac{S_T}{c_1} + \frac{\ell}{c_1}\right)\frac{dX}{dt} = \frac{\alpha_1}{X}\left[-p + \frac{2\ell}{c_1} S_T - \frac{2k_{21}\Delta T}{b-1}\right] \quad .$$

The solution to this equation is

$$\gamma^2 = \frac{3(1 + 2S_T - 3F_2 - F_1}{5 + 2\,S_T + F_2 + F_1} \tag{2.32}$$

$$F_1 = \sqrt{(1 - F_2)^2 + 2S_T}$$

$$F_2 = \frac{k_{21}\,\phi\,S_T}{b-1} \quad .$$

The approximate equations for γ, eq 2.32 and 2.30, can be compared to exact values, eq 2.11, given by Carslaw and Jaeger (1959), for water-ice systems. The values are shown in Table 2.2.

Clearly, both approximate equations yield good results for small values of S_T. Sparrow et al. (1978) have calculated exact values of γ for $k_{21} = \alpha_{21} = 1$. These comparisons are shown in Table 2.3. Typical soil ratios are also compared in Table 2.4. These calculations indicate that the heat balance integral gives a very good approximate equation for γ over a wide range of S_T, ϕ, and property values. Equation 2.30 should be within 5% for all practical cases for soil systems.

Nixon and McRoberts (1973) found a semi-empirical relation for γ, with

Table 2.2 Accuracy of approximate γ-values for ice-water system from (from Lunardini and Varotta 1981).

		γ				
S_T	ϕ	Exact	eq 2.30	Error %, eq 2.30	eq 2.32	Error %, eq 2.32
0.0063	0	0.056*	0.0559	-0.2	---	---
0.0063	5	0.049*	0.0486	-0.8	---	---
0.0314	0	0.124*	0.1243	+0.3	---	---
0.0314	1	0.115*	0.1156	0.5	---	---
0.1	0	0.2200	0.2181	-0.9	0.2232	1.5
0.1	5	0.1176	0.1190	1.2	0.1200	2.0
1	0	0.6201	0.5774	-6.9	0.6600	6.4
1	5	0.1449	0.1479	2.1	0.1497	3.3

($k_{21} = 0.2717$, $\alpha_{21} = 0.1252$)
*Values from Carslaw and Jaeger (1959).

Table 2.3 Comparison of γ-values, $\alpha_{21} = k_{21} = 1.0$ (from Lunardini and Varotta 1981).

		γ				
S_T	ϕ	Exact	eq 2.30	Error %, eq 2.30	eq 2.32	Error %, eq 2.32
0.0058	0	0.0538	0.0538	0	0.0539	0.2
0.0058	5	0.0459	0.0458	-0.2	0.0458	-0.2
0.1	4	0.1278	0.1276	-0.2	0.1290	0.9
1	0	0.6201	0.5774	-6.9	0.6600	6.4
1	4	0.1704	0.1706	0.1	0.1737	1.9

Table 2 4. Accuracy of approximate γ-values, $k_{21} = 0.51$, $\alpha_{21} = 0.3355$ (from Lunardini and Varotta 1981).

		γ				
S_T	ϕ	Exact	eq 2.30	Error %, eq 2.30	eq 2.32	Error %, eq 2.32
0.0058	0	0.0538	0.0538	0	0.0539	0.2
0.0058	5	0.0468	0.0465	0	0.0465	0
0.03	0	0.1219	0.1215	-0.30	0.1225	0.5
0.03	5	0.0860	0.0861	0.1	0.0865	0.6
0.1	0	0.2200	0.2181	-0.9	0.2232	1.5
0.1	5	0.1168	0.1174	0.5	0.1185	1.5
1	0	0.6201	0.5774	-6.9	0.6600	6.4
1	5	0.1460	0.1477	1.2	0.1496	2.5

$\phi = 0$, as

$$\gamma = \left(1 - \frac{S_T}{8}\right)\sqrt{\frac{S_T}{2}} \quad . \tag{2.33}$$

In the limit as $\phi \to 0$, eq 2.30 reduces to

$$\gamma = \sqrt{\frac{S_T}{2 + S_T}} \tag{2.34}$$

and eq 2.32 is

$$\gamma = \sqrt{\frac{3(1 + 2\,S_T - \sqrt{1 + 2\,S_T})}{5 + 2\,S_T + \sqrt{1 + 2\,S_T}}} \quad . \tag{2.35}$$

The quasi-steady Stefan solution is shown (App. A) to be

$$\gamma = \sqrt{\frac{S_T}{2}} \quad . \tag{2.36}$$

All of the above equations are quite close, and accurate, for small S_T values, but eq 2.35 gives the best overall accuracy.

2.1.3 Stefan Problem

Problems of conduction with phase change are sometimes generally classed as Stefan problems or moving boundary problems. However, in the Neumann problem, if the medium is initially at the phase-change temperature only one phase will be present. This special case of the Neumann problem is also often referred to as the Stefan solution, following the original work of Stefan (1891).

If the initial temperature of the liquid is T_f, then eq 2.11 is

$$\gamma e^{\gamma^2} \operatorname{erf} \gamma = \frac{c_1 \ (T_f - T_s)}{\ell\sqrt{\pi}} \quad . \tag{2.37}$$

Now, when x is small, the error function may be approximated as

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \ x^{2n+1}}{(2n+1) \ n!} = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} + \ldots\right) \ .$$

Thus, if γ is small

$$\operatorname{erf} \gamma \simeq \frac{2}{\sqrt{\pi}} \gamma \tag{2.38}$$

and eq 2.37 is

$$\gamma \frac{2}{\sqrt{\pi}} \gamma = \frac{c_1 (T_f - T_s)}{\ell\sqrt{\pi}}$$

or

$$\gamma^2 = \frac{c_1 (T_f - T_s)}{2\ell} \tag{2.39}$$

and

$$X = \sqrt{\frac{2k_1 (T_f - T_s) t}{\rho_1 \ell}} \quad . \tag{2.40}$$

Introducing the Stefan number gives the phase change depth as

$$X = \sqrt{2 \ S_T \ \alpha t} \tag{2.41}$$

and

$$\gamma = \sqrt{S_T/2} \quad . \tag{2.42}$$

Equations 2.39 and 2.40 were presented by Stefan (1891) who used the quasi-steady approximation discussed in Appendix A. The Stefan solution is not an exact solution in the sense of the Neumann solution. For rocks or metals, $S_T = 2.0$, but for water this value is very small, about 0.03, if $T_f - T_s = 5°C$. Therefore, eq 2.38 is an acceptable approximation for water, and for many soil systems. These values of γ compare favorably with the exact values of Table 2.1 for water at $(T_o - T_f) = 0$, as can be noted below:

$(T_f - T_s)°C$	1	2	3	4	5
γ (eq 2.42)	0.054	0.076	0.094	0.108	0.121

2.1.4 Modified Berggren Equation

One of the most common problems facing engineers in the cold climate regions of the world is the need to estimate the depth of freeze or thaw of a soil system. The Neumann solution is widely used for soil freezing estimates but special names have been given to it that can be confusing. Berggren (1943) was apparently the first to actually apply the Neumann solution to soil phase change problems. Aldrich and Paynter (1953) later used the Stefan form of the phase change solution to arrive at the modified Berggren equation. Equation 2.10 can be changed to the Stefan form as

$$X = \lambda\sqrt{(2k_f/L)(T_f - T_s)t} \tag{2.43}$$

where λ, which replaces γ, can be determined from the exact solution. Stefan (1891) originally solved a similar problem for the growth of sea ice when the sensible heat to latent heat ratio was small and the water was at the freezing temperature. Equation 2.43 reduces to the Stefan equation, eq 2.40, when λ equals one; hence the terminology "Stefan form."

The surface temperature of a soil system does not normally remain constant during the freeze season and the surface index I_s is often used:

$$X = \lambda\sqrt{(2k_f/L)I_s} \quad . \tag{2.44}$$

The surface index is defined as

$$I_s = \int_0^{\varepsilon} [T_f - T_s(t')]\, dt' = (T_f - \overline{T}_s)\varepsilon \qquad (2.45)$$

where ε is the length of the freeze season.

Thus an average constant surface temperature $\overline{T}_s$ can be calculated for the season and be used in eq 2.44, if I_s is known. Unfortunately, the surface index is rarely available for a location; however, the air temperature index I_f is usually tabulated and I_s can be replaced with the n-factor, defined as

$$n = I_s/I_f \text{ or } I_s/I_t. \qquad (2.46)$$

The quantity n is the relation between the air index and the surface index. A procedure for obtaining a value at a given site is given by Lunardini (1978).

Finally, the modified Berggren equation is written as

$$X = \lambda\sqrt{2k_f I_f n)/L}. \qquad (2.47)$$

Berg and Aitken (1973), among many others, have shown that the modified Berggren equation gives good results for seasonal phase change depths even if the surface temperature varies with time. The coefficient λ can be found by equating eq 2.10 and 2.43, leading to the following equation:

$$\frac{e^{-\lambda^2\mu}}{\text{erf }(\lambda\sqrt{\mu})} - \frac{p\phi e^{-q\lambda^2\mu}}{\text{erfc }(r\lambda\sqrt{\mu})} = \frac{\lambda}{2}\sqrt{\pi/\mu}. \qquad (2.48)$$

The parameters ϕ and μ take into account the soil temperatures, specific heat and latent heat. The parameter μ is one-half of the Stefan number which is the ratio of the sensible heat and latent heat for a soil system. For small values of μ and ϕ, it can be expected that λ will be nearly one and eq 2.47 will reduce to the Stefan equation.

The quantities p, q and r are ratios of the thermal properties of the soil system for the frozen and thawed states:

$$p = (k_t/k_f)\sqrt{\alpha_f/\alpha_t}$$

$$q = \alpha_f/\alpha_t$$

$$r = \sqrt{\alpha_f/\alpha_t}.$$

These relations are for the freezing case. Aldrich and Paynter (1953) used the relations ϕ and $\mu = (C_f/L)(T_f - T_s)$(the Stefan number) and noted that calculations with typical soil properties indicated $\alpha_f/\alpha_t \approx 1.0$, $C_t/C_f \approx 1$, and thus $k_t/k_f \approx 1.0$. They then solved eq 2.48 with $p = q = r = 1$ and obtained a widely used graph for λ (see Sanger 1969). Actually, this procedure is valid only when the water content of a soil is zero. Nixon and McRoberts (1973) made a parametric study of eq 2.11, but presented a graph of λ valid only for $r = \sqrt{\alpha_{ft}} = 1.43$, which was said to represent moist soils.

2.1.4.1 Soil Thermal Properties

It is clear that p, q and r will vary for different soil systems, but a relatively simple procedure can be used to generate these functions for any soil. Gold and Lachenbruch (1973) noted that the weighted, geometric mean for the thermal conductivity of a mixture gives results that are often as good as those from more complicated methods. The thermal conductivity of a soil can then be expressed as

$$k = (k_s)^{x_s} (k_\ell)^{x_\ell} (k_g)^{x_g} \tag{2.49}$$

where k_s, k_ℓ and k_g are, respectively, the thermal conductivities of the solid, liquid, and gaseous phases; x_s, x_ℓ and x_g are the respective volume fractions. For soil systems, the thermal conductivity of the solids and gases will not vary significantly as phase change occurs (Kersten 1949). There will be only a small error if it is assumed that the frozen state contains only ice with no unfrozen water. Thus the ratio of the frozen to unfrozen conductivities of the soil mixture can be related to the thermal conductivity of ice and water as follows:

$$k_t/k_f = (k_w/k_i)^{x_\ell} \tag{2.50}$$

where k_w, k_i are the thermal conductivities of water and ice, respectively. The volumetric specific heat for the system may be expressed for the thawed and frozen states as follows:

$$C_t = C_{st}(1-x_\ell) + C_w x_\ell \tag{2.51}$$

$$C_f = C_{sf}(1-x_\ell) + C_i x_\ell \tag{2.52}$$

where C_{st} and C_{sf} are specific heats of unfrozen and frozen soil solids, respectively. The neglect of the gas phase is insignificant since the density of the gas (air) is low. It is fortunate that the specific heats of different soil solids and ice are all similar in magnitude. For example, the volumetric specific heat of organic solids is about 2300 kJ/m^3-K, for mineral solids it is 1760, and for ice it is 1920 (Lunardini 1971). The properties of the frozen materials are evaluated at 25°F (269 K) while the thawed values are at 40°F (277.4 K). If one assumes that the values for the solids, except for ice, change little through the phase change, then

$$C_t/C_f = 1 + [(C_w/C_i) - 1]\, x_\ell \tag{2.53}$$

or

$$C_t/C_f = 1 + 1.023\, x_\ell \,.$$

The property values to use in eq 2.48 can then be expressed as simple functions of the soil water content:

$$q = R^{x_\ell}\,(1 + 1.023\, x_\ell)$$

$$r = \sqrt{q}$$

$$p = r/R^{x_\ell}.$$

R, the ratio of the thermal conductivity of ice to water, is about 3.89. Lunardini (1980) solved eq 2.48 numerically to find the roots, which are the values of λ. Figures 2.2 through 2.4 give the values of λ to use in eq 2.47 for the freezing case.

2.1.4.2 Thaw Case

It might appear that the same relations could be used for either the thawing or the freezing case. This, however, is not true. In the thawing case, the medium is initially frozen at T_o and energy must be conducted through the thawed layer from the phase change interface. Since the thermal conductivity of the thawed region is considerably less than that of

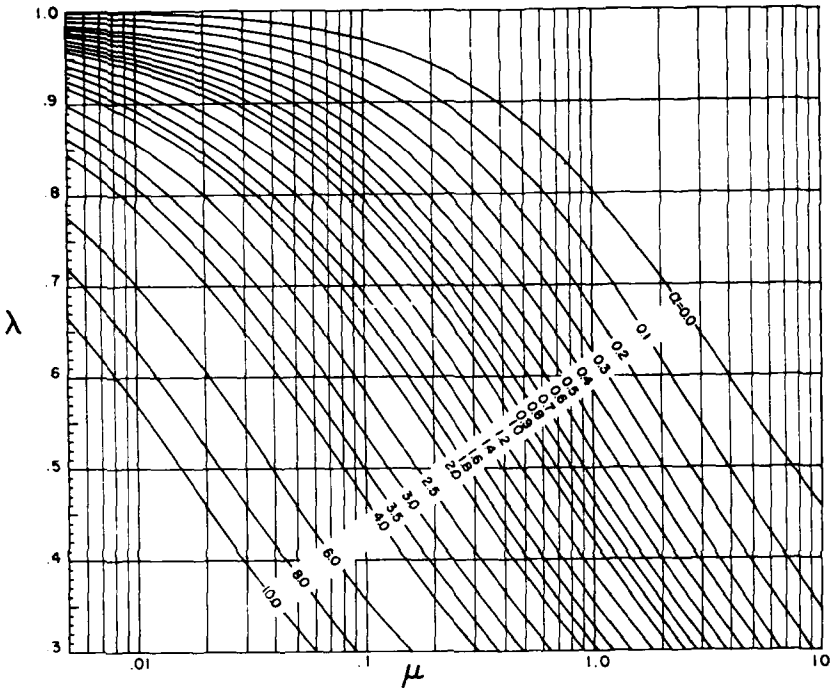


Figure 2.2. Freeze or thaw, Neumann equation parameter, $x_\ell = 0.0$.

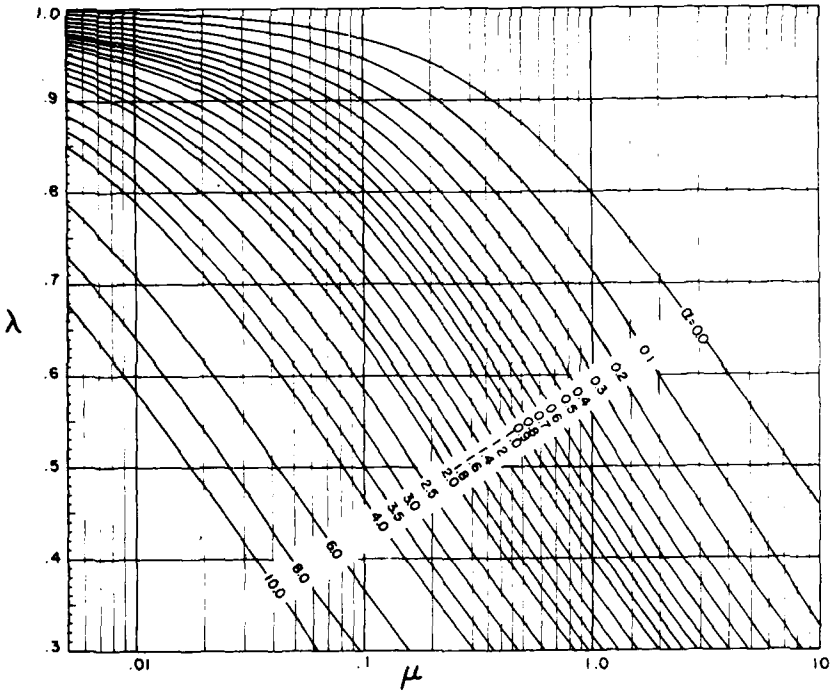


Figure 2.3. Freezing case, Newmann equation parameter $x_\ell = 0.4$.

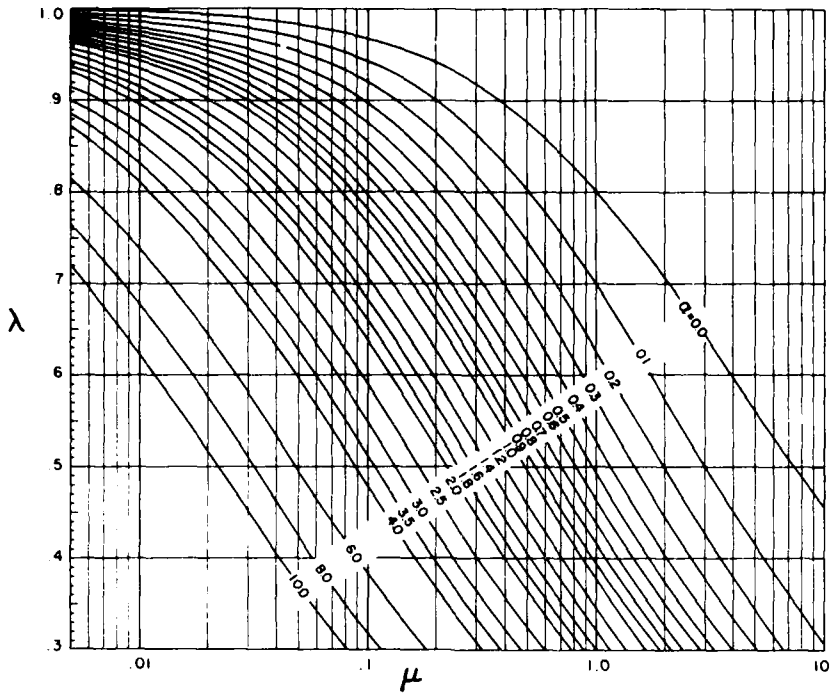


Figure 2.4. Freezing case, Newmann equation parameter, $x_\ell = 1.0$.

the frozen region, the heat flow will be reduced, even with the same temperature gradient. However, the general form of the equation will be the same after making appropriate changes for the property values. The thaw depth is expressed as

$$X = \lambda\sqrt{(2k_t I_t n)/L} \tag{2.54}$$

where λ is again given by eq 2.48 but $\mu = (C_t/2L)(T_s - T_f)$. The functions p, q and r all change because of the property changes of the thawed and frozen states:

$$q = 1/[R^{x_\ell}(1 + 1.023\ x_\ell)].$$

$$r = \sqrt{q}.$$

$$p = r\ R^{x_\ell}.$$

The λ values for thawing are now given in Figures 2.5-2.7. Notice that when $x_\ell = 0$, the λ values are the same for freezing or thawing, which is the Aldrich and Paynter (1953) case and was used for the Sanger (1969)

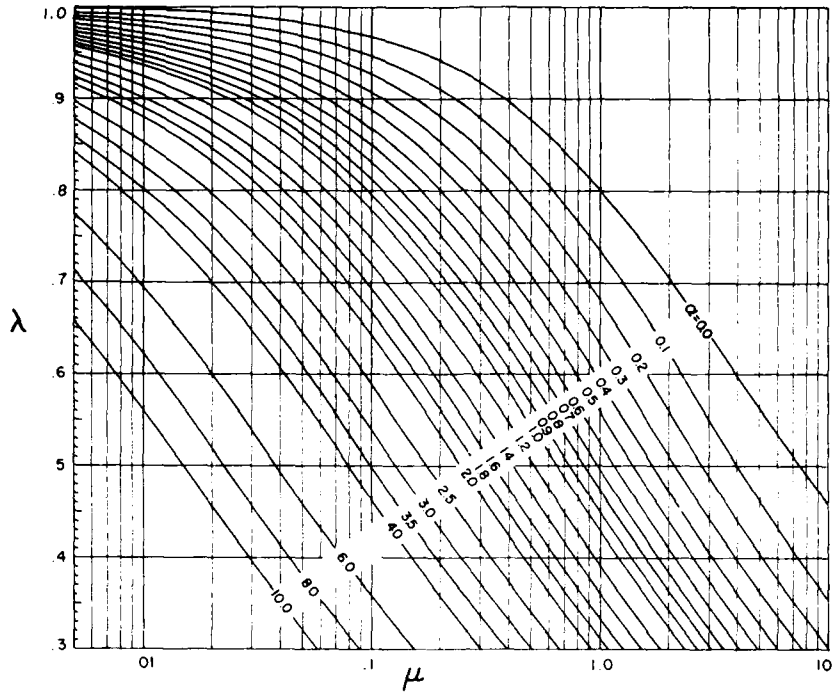


Figure 2.5. Thawing case, Newmann equation parameter $x_\ell = 0.4$.

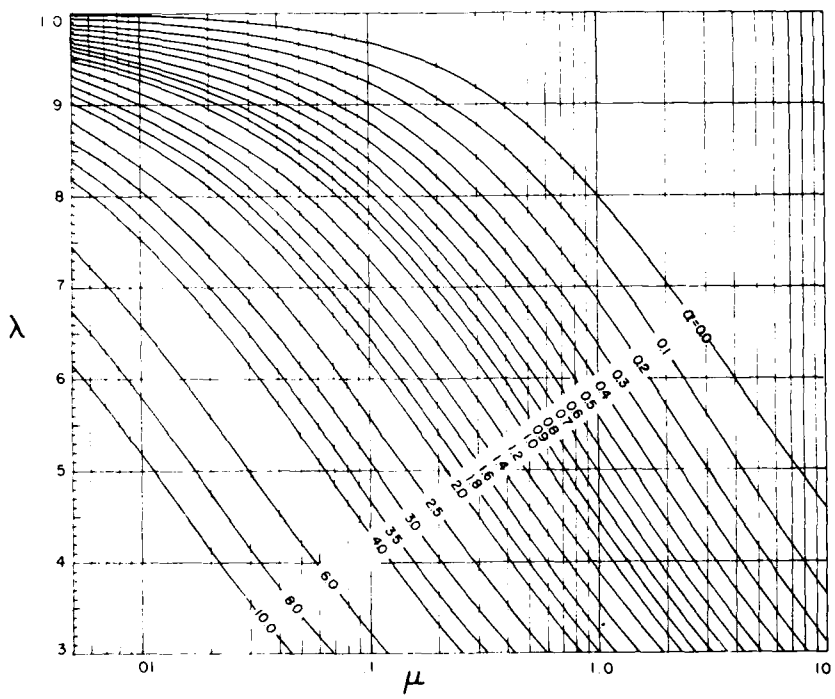


Figure 2.6. Thawing case, Newmann equation parameter, $x_\ell = 0.8$.

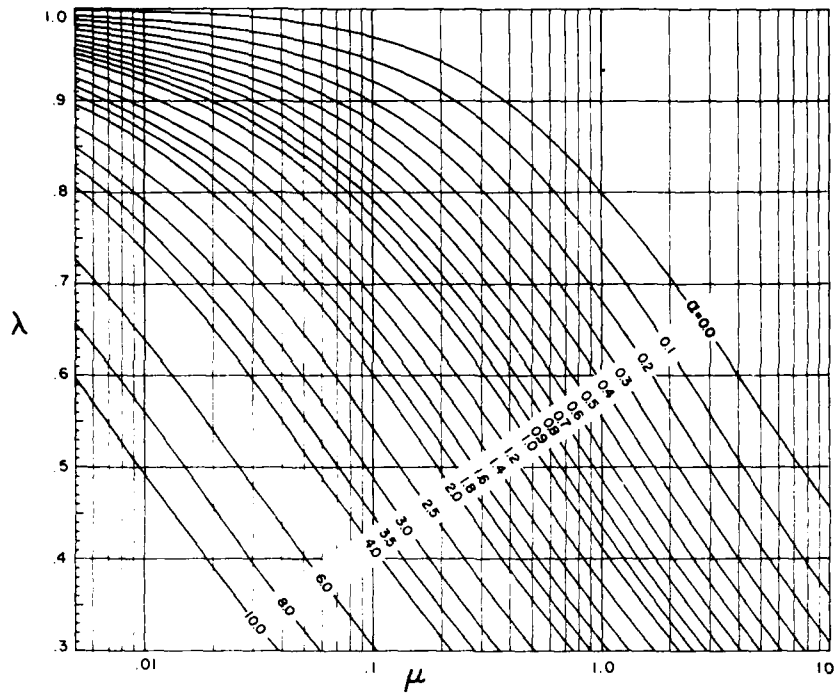


Figure 2.7. Thawing case, Newmann equation parameter, $x_\ell = 1.0$.

graph. The charts for λ are for the exact solution of the Neumann problem with property values typical of soil systems.

2.1.5 Constant Phase Change Rate

Stefan (1891) gives a solution for the case of a constant heat flux at the phase change interface. The equations are

$$\frac{\partial^2 T_l}{\partial x^2} = \frac{1}{\alpha_l}\frac{\partial T_l}{\partial t} \tag{2.55}$$

$$T(x,0) = T(X,t) = T_f$$

$$k_l \frac{\partial T_l}{\partial x} = \rho_l \ell \frac{dX}{dt}$$

$$X(0) = 0$$

$$\frac{\partial T_l}{\partial x} = G \qquad x = X$$

If the heat flux at the phase change interface is constant, then the interface moves at constant velocity and an exact solution can be found using the similarity transformation approach.

Let

$$\eta = \frac{V}{\alpha_l}(x - Vt)$$

where V is the constant velocity of the interface. It is easy to show that a formal solution of the energy equation is

$$T_l = A\, e^{-\eta} .$$

If we consider a semi-infinite solid initially at the fusion temperature, what surface temperature must be prescribed to have the phase change interface move at constant velocity? The solution follows immediately from the above equations:

$$T_l\ (x,t) = T_f + \frac{\ell}{c_l}\left[1 - e^{\frac{-V}{\alpha_l}(x-X)}\right] \tag{2.56}$$

$$X = Vt = \frac{k_l G t}{\rho_l \ell} . \tag{2.56a}$$

The required surface temperature is

$$T_l(0,t) = T_f + \frac{\ell}{c_l}\left(1 - e^{\frac{V^2 t}{\alpha_l}}\right) .$$

As noted by Stefan (1891), the surface temperature of an ice layer may drop rapidly to a constant value. During the transient phase the ice layer will grow linearly with time and then grow as the square root of time.

This is an inverse problem wherein applied boundary conditions must be found to obtain a given phase change interface motion. Rubinsky and Shitzer (1978) give a general solution as follows:

$$T_l(x,t) = T_f + \sum_{n=0}^{\infty} a_n(X)\,\frac{(x - X)^n}{n!} \tag{2.57}$$

$$a_o = 0$$

$$a_1 = \frac{\partial T_l(X,t)}{\partial x}$$

$$a_n = \frac{A(X,t)}{\alpha}\left[\frac{d\, a_{n-2}}{dX} - a_{n-1}\right]$$

$$A(X,t) \equiv \frac{dX}{dt} \quad .$$

The temperature gradient at the interface is obtained from the usual interface energy relation:

$$k_1 \frac{\partial T_1(X,t)}{\partial x} - Q_i = \rho \ell \frac{dX}{dt}$$

where Q_i is the heat flux supplied to interface from the original phase. The Stefan problem posed here has $Q_i \equiv 0$ and

$$\frac{dX}{dt} = V = A = \text{constant}$$

$$\frac{\partial T_1}{\partial x}(X,t) = \frac{\rho \ell}{k_1} V = \text{constant}.$$

Substituting these relations into eq 2.57 yields an infinite series that reduces to eq 2.56.

The problem does not have significant practical value since it requires that a variable surface temperature be imposed upon the solid to maintain melting at a constant rate.

2.1.6 Problems for the Finite Slab

Most of the solutions associated with a finite thickness are similar to those of the semi-infinite region, with minor adjustments. The problems may be adapted to the freezing of water of finite depth, ice formation on solid surfaces, etc.

2.1.6.1 Exact Single Phase Solution

A slab of thickness D with the temperature of the liquid initially at the fusion temperature T_f is considered here. At $t = 0$, the surface temperature drops to T_s and is held there. The surface at $x = D$ is effectively insulated since the liquid temperature has a constant value T_f.

Ruoff (1958) presented an exact solution for this problem by means of a similarity transformation. The results are identical to those of the Neumann solution for the semi-infinite solid.

The equations for the phase change interface depth γ and the temperatures in the solid are identical to eq 2.10, 2.11 and 2.12, respectively, for the semi-infinite solid, until $X = D$, when the slab is completely frozen.

2.1.6.2 Insulated Lower Surface

There are no exact solutions for the finite slab when the initial temperature is different from the fusion temperature. Cho and Sunderland (1969) present an approximate solution for the case of a slab insulated at the nonisothermal surface. The problem (see Fig. 2.8) is described as

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} \tag{2.58}$$

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} \tag{2.59}$$

$$T_2(x,0) = T_o \tag{2.60a}$$

$$T_1(0,t) = T_s \tag{2.60b}$$

$$\frac{\partial\ T_2(D,t)}{\partial x} = 0 \tag{2.60c}$$

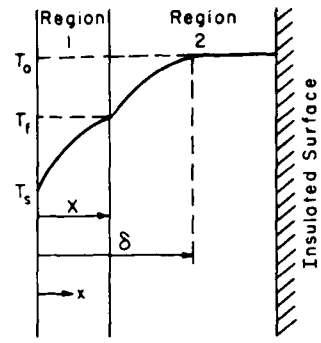


Figure 2.8. Two-phase finite slab.

$$T_1(X,t) = T_2(X,t) = T_f \tag{2.60d}$$

$$-k_1 \frac{\partial T_1(X,t)}{\partial x} + k_2 \frac{\partial T_2(X,t)}{\partial x} = -\rho_1 \ell \frac{dX}{dt} \quad . \tag{2.60e}$$

The assumption of a thermal penetration depth δ gives rise to two new boundary conditions:

$$\frac{\partial T_2(\delta,t)}{\partial x} = 0 \tag{2.60f}$$

$$T_2(\delta,t) = T_o \quad , \tag{2.60g}$$

For the solid region, the exact solution for the semi-infinite solid, eq 27, is applicable:

$$\frac{T_1 - T_s}{T_f - T_s} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\operatorname{erf}\gamma} \tag{2.61}$$

$$X = 2\,\gamma\sqrt{\alpha_1 t} \quad . \tag{2.62}$$

In the liquid region, the heat balance integral will be used:

$$\frac{d\theta_2}{dt} - T_o \frac{d\delta}{dt} + T_f \frac{dX}{dt} + \alpha_2 \frac{\partial T_2(X,t)}{\partial x} = 0 \tag{2.63}$$

where

$$\theta_2 = \int_X^\delta T_2 \, dx \ . \tag{2.64}$$

Assume a quartic temperature for the liquid region. Then, with eq 2.60d,f,g,

$$T_2 = T_o - (T_o - T_f) \left(\frac{\delta - x}{\delta - X}\right)^4 \ . \tag{2.65}$$

Assume that the thermal penetration depth is

$$\delta = 2\beta\sqrt{\alpha_1 t} \ . \tag{2.66}$$

Equations 2.62-2.66 then lead to

$$\beta - \gamma = \frac{5}{2} \left[- \gamma + \sqrt{\gamma^2 + \frac{8}{5} \alpha_{21}}\right] \ . \tag{2.67}$$

Then using eq 2.61e

$$\frac{e^{-\gamma^2}}{\text{erf } \gamma} - \frac{2(T_o - T_f)k_2\sqrt{\pi}}{(T_f - T_s)\, k_1(\beta - \gamma)} = \frac{\ell\gamma\sqrt{\pi}}{c_1\, (T_f - T_s)} \ . \tag{2.68}$$

Equations 2.67 and 2.68 may be easily solved for γ and β. When the temperature penetration reaches the far wall, $\delta = D$, then the insulated wall temperature will start to decrease. The times when $\delta = D$ and when $X = D$ are given by

$$t_1 = \frac{D^2}{4\alpha_1\beta^2} \tag{2.69}$$

$$t_2 = \frac{D^2}{4\alpha_1\gamma^2} \ . \tag{2.70}$$

The insulated wall temperature for $t_1 < t < t_2$ may be estimated by assuming

$$T_2(x,t) = T_w - (T_w - T_f) \left(\frac{D - x}{D - X}\right)^4$$

where $T_w = T_2\ (D,t)$ is the temperature of the insulated wall when $t_1 < t < t_2$. Using the heat balance integral in region 2 again, a differential equation for the transient temperature T_w may be generated:

$$(D - X)\frac{d}{dt}(T_w - T_f) + \left(\frac{5\alpha_2}{D - X} - \frac{dX}{dt}\right)(T_w - T_f) = 0 \ .$$

The solution to this equation is

$$\frac{T_2(D,t) - T_f}{T_o - T_f} = \left(\frac{a}{b}\right)^{-(1+c)} e^{c\left(\frac{1}{b} - \frac{1}{a}\right)} \qquad t_1 < t < t_2 \qquad (2.71)$$

where

$$a = 1 - \frac{2\gamma}{D}\sqrt{\alpha_1 t}$$

$$b = 1 - \frac{\gamma}{\beta}$$

$$c = \frac{5}{2}\gamma^2 \frac{\alpha_2}{\alpha_1} \ .$$

When the time is greater than t_2 the entire liquid region has frozen and the problem reduces to that of a single phase.

Cho and Sunderland (1969) also give a solution to the above problem if two distinct phase change temperatures exist, with two different latent heats. This solution will not be discussed here.

2.2 NEUMANN PROBLEM WITH VARIABLE PROPERTIES

2.2.1 Solid/Liquid Density Difference

If the density of the solid and liquid phases differ, as is usual, a solution can be found with the similarity technique. This problem actually involves convection in the frozen phase, due to the motion of the solid, but it can be formulated as a conduction problem.

If the density ratio is not one, then the problem can be posed according to Figure 2.9. Reference frame x_1 is attached to the free surface of medium 1 and moves as medium 1 expands, assuming that $\rho_1/\rho_2 < 1$, while reference frame x_2 is fixed at the location of the original free surface. The thickness of the solidified material between the original free surface location and the phase change interface is X_2. The solution to this problem follows from the usual similarity transformation and has been given by Crank (1975), and Carslaw and Jaeger (1959):

$$\frac{\partial T_1}{\partial t} = \alpha_1 \frac{\partial^2 T_1}{\partial x_1^2} \qquad 0 < x_1 < X_2 \qquad (2.72)$$

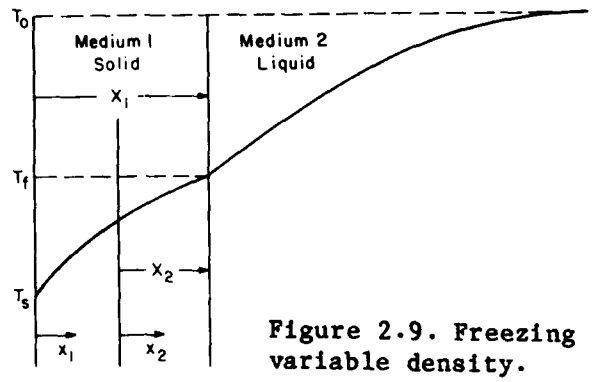


Figure 2.9. Freezing in a medium with variable density.

$$T_1\ (0,t) = T_s \tag{2.72a}$$

$$T_1\ (X_1,t) = T_f \tag{2.72b}$$

$$\frac{\partial T_2}{\partial t} = \alpha_2 \frac{\partial^2 T_2}{\partial x_2^2} \qquad X_2 < x_2 < \infty \tag{2.73}$$

$$T_2\ (X_2,t) = T_f \tag{2.73a}$$

$$T_2\ (x_2,0) = T_o \tag{2.73b}$$

$$\lim_{x_2 \to \infty} T_2\ (x_2,t) = T_o \tag{2.73c}$$

$$k_1 \frac{\partial T_1(X_1,t)}{\partial x_1} - k_2 \frac{\partial T_2(X_2,t)}{\partial x_2} = \rho_2 \ell \frac{dX_2}{dt} = \rho_1 \ell \frac{dX_1}{dt} \quad . \tag{2.73d}$$

The solution is

$$T_1 = T_s + (T_f - T_s) \frac{\mathrm{erf}(x_1/2\sqrt{\alpha_1 t})}{\mathrm{erf}\gamma} \qquad 0 < x_1 < X_1 \tag{2.74}$$

$$T_2 = T_o - (T_o - T_f) \frac{\mathrm{erfc}(x_2/2\sqrt{\alpha_2 t})}{\mathrm{erfc}(\rho_{12}\sqrt{\alpha_{12}})} \qquad X_2 < x_2 < \infty \tag{2.75}$$

$$X_1 = 2\gamma\sqrt{\alpha_1 t} \quad . \tag{2.76}$$

The parameter γ is obtained from the following transcendental equation:

$$\frac{k_{21}\sqrt{\alpha_{12}}\phi\exp(-\rho^2{}_{12}\alpha_{12}\gamma^2)}{\mathrm{erfc}(\gamma\sqrt{\alpha_{12}\rho_{12}})} - \frac{e^{-\gamma^2}}{\mathrm{erf}\gamma} + \frac{\gamma\sqrt{\pi}}{S_T} = 0 \quad . \tag{2.77}$$

The solution of eq 2.77 requires a numerical procedure, and the number of parameters makes a graphical presentation of the solution impractical. An approximate solution to the problem has been found, with the heat balance integral method, which yields an expression for γ that does not require numerical solution (Lunardini 1983a).

The approximate relation for γ is

$$\gamma^2 = \frac{b_1 - \sqrt{b^2{}_1 - 4aS_T{}^2}}{2a} \tag{2.78}$$

where

$$b_1 = 2S_T\left[2 + S_T + \frac{\phi k_{21}\rho_{12}S_T}{\alpha_{21}}\right] + \frac{4}{3}\frac{(\phi k_{21}S_T)^2}{\alpha_{21}}$$

$$a = (2 + S_T)\left[2 + S_T + 2S_T\frac{\phi k_{21}\rho_{12}}{\alpha_{21}}\right] \quad .$$

This approximate solution will be in error by less than 3% for most practical problems.

The effect of the density variation, on the Neumann solution, will be small, unless the density ratio is quite small, as can be seen from Figure 2.10.

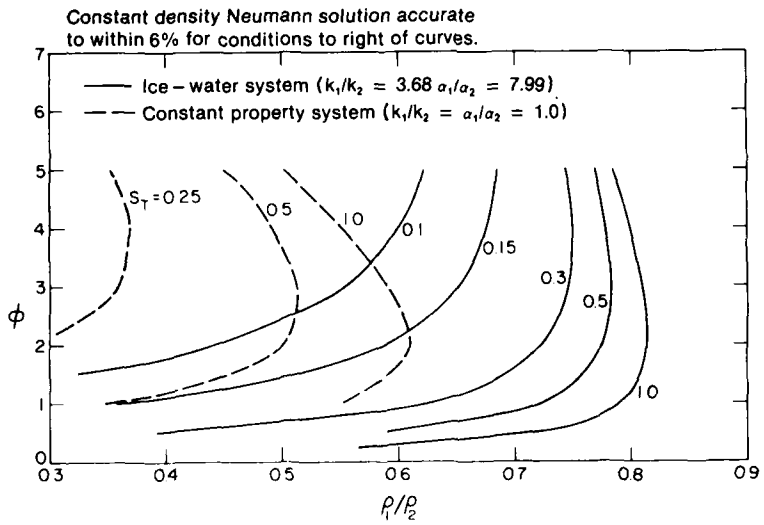


Figure 2.10. Range of validity of constant density solution for 6% accuracy.

The equations are valid for the thaw case except that region 1 is now thawed and the total thaw depth is given by $\rho_{12}X_1$.

2.2.2 Variable Thermal Conductivity

Cho and Sunderland (1974) have extended the Neumann problem when the thermal conductivity varies linearly with temperature. The equations are

$$C_1 \frac{\partial T_1}{\partial t} = \frac{\partial}{\partial x}\left(k_1 \frac{\partial T_1}{\partial x}\right) \qquad 0 < x < X(t) \tag{2.79}$$

$$C_2 \frac{\partial T_2}{\partial t} + (\rho_2 - \rho_1)\, c_2 \frac{dX}{dt}\frac{\partial T_2}{\partial x} = \frac{\partial}{\partial x}\left(k_2 \frac{\partial T_2}{\partial x}\right) \qquad x > X(t)\ . \tag{2.80}$$

The thermal diffusivities are

$$\alpha_1 = \alpha_s \left(1 + \beta_1 \frac{T_1 - T_s}{T_o - T_s}\right) \tag{2.81}$$

$$\alpha_2 = \alpha_L \left(1 + \beta_2 \frac{T_2 - T_s}{T_o - T_s}\right) \tag{2.82}$$

where α_s and α_L are the values of the thermal diffusivity of the solid and the liquid phases, respectively, at T_s. The liquid phase does not exist at T_s and α_L is obtained by extrapolation. The initial conditions are

$$T_2 = T_o > T_f, \qquad x > 0 \tag{2.83a}$$

and

$$X(0) = 0\ . \tag{2.83b}$$

The boundary condition at the free surface, $x = 0$, is

$$T_1\ (0,t) = T_s < T_f, \qquad t > 0 \tag{2.83c}$$

At the moving fusion front, $x = X(t)$, two additional conditions must be satisfied:

$$T_1 = T_2 = T_f \tag{2.83d}$$

and

$$k_1 \frac{\partial T_1}{\partial x} - k_2 \frac{\partial T_2}{\partial x} = \rho_1 \ell \frac{dX}{dt}\ . \tag{2.83e}$$

A modified error function is defined as follows. Consider a second-order nonlinear ordinary differential equation

$$\frac{d}{d\eta}\left\{(1 + \beta\theta)\frac{d\theta}{d\eta}\right\} + 2\eta\frac{d\theta}{d\eta} = 0 \tag{2.84}$$

with the boundary conditions

$$\theta\ (0) = 0$$

$$\theta\ (\infty) = 1\ .$$

If the solution is designated as $\phi_\beta\ (\eta)$, which can be called the "modified error function," then by definition

$$\frac{d}{d\eta}\left\{(1 + \beta\phi_\beta)\frac{d\phi_\beta}{d\eta}\right\} + 2\eta\frac{d\phi_\beta}{d\eta} = 0 \tag{2.85}$$

$$\phi_\beta(0) = 0$$

$$\phi_\beta(\infty) = 1$$

The function $\phi_\beta(\eta)$ can be found by a numerical technique such as the Runge-Kutta method. Note that when $\beta = 0$, eq 2.84 becomes linear, and

$$\phi_o(\eta) = \text{erf}\ (\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-z^2}\,dz$$

When the thermal conductivity varies linearly with temperature, and the property of the modified error function $\phi_\beta\ (\eta)$ is used, the solutions of eq 2.79-2.83 are found to be

$$T_1 = T_s + (T_f - T_s)\frac{\phi_\delta(x/2\sqrt{\alpha_s t})}{\phi_\delta(\gamma)} \tag{2.86}$$

and

$$T_2 = T_o - (T_o - T_f)\frac{1 - \phi_\varepsilon\dfrac{x}{2\sqrt{\alpha_s\alpha_{21}}\ (1+\beta_2 a)} + \dfrac{(\frac{1}{\rho_{12}} - 1)\gamma}{\sqrt{\alpha_{12}}(1+\beta_2 a)}}{1 - \phi_\varepsilon(\zeta)}\ . \tag{2.87}$$

The constants γ, δ, ε and a are determined from

$$\delta\phi_\delta(\gamma) = \beta_1\ \theta_f \tag{2.88}$$

$$\varepsilon = \frac{\beta_2(1 - a)}{1 + \beta_2 a} \tag{2.89}$$

$$\phi_\varepsilon(\zeta) = \frac{\theta_f - a}{1 - a} \tag{2.90}$$

and

$$\frac{\phi_\delta'(\gamma)}{\lambda\phi_\delta(\gamma)} - \frac{1-\theta_f}{\theta_f}\,\frac{C_{21}}{1+\beta_2 a}\,\frac{\phi_\varepsilon'(\zeta)}{\zeta\{1-\phi_\varepsilon(\zeta)\}} = 2h/\theta_f \tag{2.91}$$

where

$$C_{21} = C_2/C_1$$

$$\zeta = \gamma/\rho_{21}\sqrt{(1+\beta_2 a)\alpha_{21}}$$

$$h = \frac{\rho_1\,\ell\alpha_s}{k_f\,(T_o - T_s)}$$

k_f = thermal conductivity of solid at T_f

$$\theta_f = \frac{T_f - T_s}{T_o - T_s}\ .$$

The phase-change location is given by

$$X = 2\gamma\sqrt{\alpha_s t}\ . \tag{2.92}$$

The function can be evaluated with Figures 2.11-2.14.

The effect of variable conductivity on the phase change rate is small unless β_1 is large and h is small (the Stefan number is large).

Pedroso and Domoto (1973) have presented a perturbation solution for the Stefan problem if the thermal properties are variable. The original equations are

$$\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) = \rho c\,\frac{\partial T}{\partial t} \tag{2.93}$$

$$T(0,t) = T_s \tag{2.93a}$$

$$T(X,t) = T_f \tag{2.93b}$$

$$T(x,0) = T_f \tag{2.93c}$$

$$\rho\ell\,\frac{dX}{dt} = k\,\frac{\partial T(X,t)}{\partial x}\ . \tag{2.93d}$$

The following transformations are defined:

$$\bar{k}\,(T_f - T_s) = \int_{T_s}^{T_f} k\,dy \qquad \bar{c}(T_f - T_s) = \int_{T_s}^{T_f} c\,dy$$

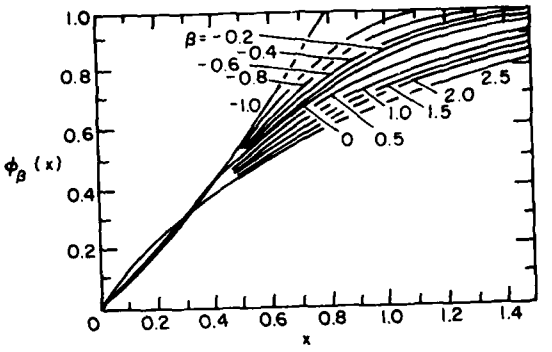


Figure 2.11. Graph of $\phi_\beta(x)$.

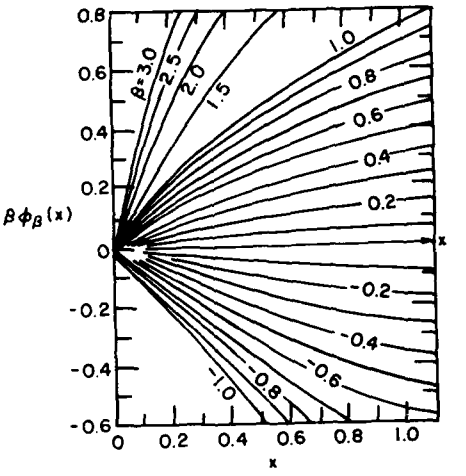


Figure 2.12. Graph of $\beta\phi_\beta(x)$.

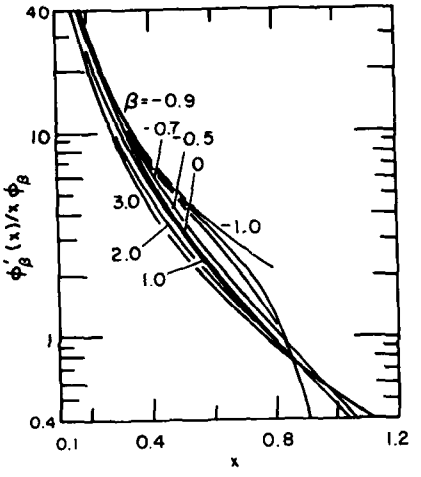


Figure 2.13. Graph of $\phi_\beta'(x)/x\phi_\beta$.

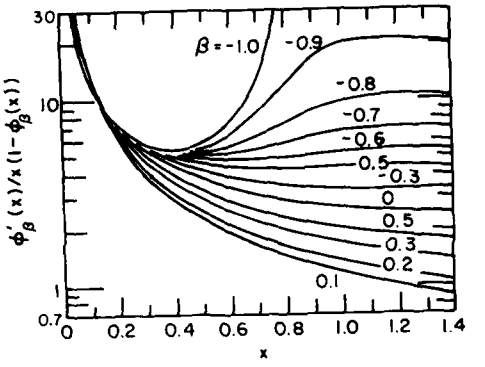


Figure 2.14. Graph of $\phi_\beta'(x)/[x\{1-\phi_\beta(x)\}]$.

$$t' = \frac{\bar{k}(T_f - T_s)}{\rho \ell}\, t$$

$$\theta\bar{k}(T_f - T_s) = \int_{T_s}^{T} k\, dy$$

$$f(\theta) = \frac{c(\theta)}{k(\theta)}\,\frac{\bar{k}}{\bar{c}}$$

$$\eta_1 = \frac{x}{X} \quad .$$

Note that t' is not dimensionless. The equations are then

$$\theta'' + \overline{S}_T \, f \, \theta'(1) \, \eta_1 \theta' = 0 \tag{2.94}$$

$$\theta(0) = 0 \qquad \theta(1) = 1 \qquad X \frac{dX}{dt} = \theta'(1) \qquad X(0) = 0$$

where $\theta' = \frac{d\theta}{d\eta_1}$.

The perturbation parameter is the Stefan number:

$$\overline{S}_T = \frac{\overline{c}(T_f - T_s)}{\ell} \quad .$$

The temperature and function f are expanded as

$$\theta = \sum_{n=0}^{N} \overline{S}_T{}^n \, \theta_n \tag{2.95}$$

$$f = f(\theta_o) + \overline{S}_T \, f'(\theta_o) \, \theta_1 + \frac{\overline{S}_T^2}{2} \left[f''(\theta_o)\theta_1{}^2 + 2f'(\theta_o)\theta_2 \right] + \ldots \quad . \tag{2.96}$$

General integral solutions are given for arbitrary functions of f (Pedroso and Domoto 1973).

For the case where $f = a + b\theta$ the interface position for the first four terms of the expansion is

$$X^2 = \frac{2(\tau_2 - \overline{S}_T \tau_3)}{\tau_o \tau_2 + \overline{S}_T \, (\tau_1 \tau_2 - \tau_o \tau_3) + \overline{S}_T{}^2 (\tau_2{}^2 - \tau_1 \tau_3)} \, \tau \tag{2.97}$$

where

$$\tau_o = 1$$

$$\tau_1 = \frac{a}{3} + \frac{b}{4}$$

$$\tau_2 = -\left(\frac{2}{45} a^2 + \frac{ab}{18} + \frac{b^2}{56}\right)$$

$$\tau_3 = \frac{16}{95} a^3 + \frac{67}{2160} a^2 b + \frac{29}{1512} ab^2 + \frac{63}{15680} b^3 \quad .$$

This solution agrees very well with a numerical solution of eq 2.94. The perturbation technique presented here is not applicable to two-phase problems. Difficulties are encountered in obtaining a zeroth-order solution or in obtaining a uniformly convergent perturbation solution.

2.2.3 Variable Latent Heat

For many materials the latent heat is a fixed quantity or varies weakly with the thermodynamic state. For soil systems, however, the latent heat is directly proportional to the water content, and it follows that latent heat may also vary since the water content can vary spatially. We will ignore that all the water in a soil system need not change phase.

For simplicity, based on data for clay soils, Lock (1969) assumed that the water content decays exponentially from the value at the soil surface. The equations for a freezing system, initially at the freezing temperature, are

$$\frac{\partial^2\theta}{\partial y^2} = S_T \frac{\partial\theta}{\partial\tau} \tag{2.98}$$

$$\theta(0,\tau) = -1$$

$$\theta(\xi,\tau) = 0$$

$$\frac{\partial\theta}{\partial y}(\xi,\tau) = f(\xi)\frac{d\xi}{d\tau}$$

$$\xi(0) = 0$$

$$\ell = \ell_o f(y) = \ell_o e^{-y}$$

where

$$\theta = \frac{T_1 - T_f}{\theta_c} \qquad \xi = X\sqrt{\frac{\rho\ell_o}{k\theta_c t_c}}$$

$$\tau^* = \frac{t}{t_c} = \frac{2\pi t}{P} \qquad \theta_c = \frac{\pi}{2}(T_f - \overline{T}_s)$$

$$y = x\sqrt{\frac{\rho\ell_o}{k\theta_c t_c}}$$

$\overline{T}_s$ = average value of T_s when $T_s < T_f$

P = period of surface oscillation.

The zeroth-order, quasi-steady equation for the phase change interface is

$$\frac{d\xi}{d\tau^*} = \frac{1}{\xi f(\xi)} \ . \tag{2.99}$$

The solution to this equation depends upon the water content function f.

If the water content is constant, then the Stefan solution results:

$$\tau^* = \frac{\xi^2}{2} \tag{2.100}$$

With an exponential water content function

$$\tau^* = 1 - (1+\xi)e^{-\xi} \ . \tag{2.101}$$

Finally, if the average value of the latent heat between the surface and the freezing depth is used:

$$\tau^* = \frac{\xi}{2}\,(1-e^{-\xi}) \tag{2.102}$$

From these results it can be noted that the time to freeze a layer $\xi = 1$, for an exponential decay, will give a value that is about 50% that of the constant latent heat solution and 85% that for an average latent heat assumption. Some caution should therefore be used in applying the constant or average water content solutions.

2.3 NEUMANN PROBLEM WITH VARIABLE TEMPERATURES

2.3.1 Exact Solution

An extension of the similarity method was used by Tao (1978) to obtain an exact solution with arbitrary surface and initial conditions. The equations, for a freeze problem, are

$$\alpha_1 \frac{\partial^2\theta_1}{\partial x^2} = \frac{\partial\theta_1}{\partial t} \qquad 0 < x < X(t) \tag{2.103}$$

$$\alpha_2 \frac{\partial^2\theta_2}{\partial x^2} = \frac{\partial\theta_2}{\partial t} \qquad X(t) \leq x \tag{2.104}$$

$$\theta_1(0,t) = U(t) \tag{2.105a}$$

$$\theta_2(x,0) = V(x) \tag{2.105b}$$

$$\theta_1(X,t) = \theta_2(X,t) = \theta_f \tag{2.105c}$$

$$k_1 \frac{\partial\theta_1(X,t)}{\partial x} - k_2 \frac{\partial\theta_2(X,t)}{\partial x} = \rho\ell \frac{dX}{dt} \tag{2.105d}$$

where

$$\theta_i(x,t) = T_i - T_s, \ i = 1,2$$

$$U(t) = T(0,t) - T_s$$

$V(x) = T_o(x) - T_s$

T_s = initial value of the surface temperature = $T_o(0)$

$T_o(x)$ = initial temperature distribution in the liquid

The surface and initial temperatures are defined by

$$U(t) = \sum_{n=1}^{\infty} u_n \, t^n/n! \tag{2.106}$$

$$u_n = \left.\frac{d^n U}{dt^n}\right)_{t=0} \tag{2.106a}$$

$$V(x) = \sum_{n=0}^{\infty} v_n \, (x/\sqrt{\alpha_2})^n/n! \tag{2.107}$$

$$v_n = \sqrt{\alpha_2} \left(\frac{d^n V}{dx^n}\right)_{x=0} \tag{2.107a}$$

Solutions to eq 2.103 and 2.104 are given by

$$\theta_1 = \sum_{n=1} u_n (4t)^n \, G_{2n}(\eta_1) + \sum_{n=0} a_n (4t)^{n/2} \, F_n(\eta_1) \tag{2.108}$$

$$\theta_2 = \sum_{n=0} (4t)^{n/2} \left[u_n G_n(\eta_2) + b_n \, i^n\text{erfc}\, \eta_2\right] \tag{2.109}$$

where $\eta_i = x/(2\sqrt{\alpha_i t})$.

The functions G_n, F_n, and i^nerfc, which are in the error function family, are defined in Appendix D.

Let

$$X(t) = 2\sqrt{\alpha_i} \sum_{n=o}^{\infty} c_n \sqrt{t^{n+1}} \quad . \tag{2.110}$$

These relations satisfy eq. 2.103-2.105b. The following set of equations for a_n, b_n, c_n are obtained from eq 2.105c and d.

$$\sum_{1}^{[N/2]} u_n A_{2n}{}^N(1) + \sum_{0}^{N} a_n A_n{}^N(1) - \sum_{0}^{[N/3]} a_{2n} B_{2n}{}^N(1) = 0 \tag{2.111a}$$

$$\sum_{1}^{N} v_n A_n{}^N(\omega) + \sum_{0}^{N} b_n B_n{}^N(\omega) = 0 \tag{2.111b}$$

$$\sum_{1}^{[N/2]} u_n P_{2n-1}^{N}(1) + \sum_{0}^{N} a_n P_{n-1}^{N}(1) + \sum_{0}^{[N/2]} a_{2n} Q_{2n-1}^{N}(1)$$

$$-\omega(k_2/k_1)\left[\sum_{1}^{N} v_n P_{n-1}^{N}(\omega) - \sum_{0}^{N} b_n Q_{n-1}^{N}(\omega)\right]$$

$$= (\rho \ell \alpha_1/k_1)(N+1)c_N. \tag{2.111c}$$

From this set of algebraic equations, the coefficients a_n, b_n and c_n, may be determined, step by step, starting from N = 1:

$$A_n^N(\omega) = 2^n \sum_{r=1}^{\mu_1} \omega^r G_{n-r}(\omega c_o) \sum_{\alpha_i} \frac{1}{\alpha_1!\ \alpha_2!\ \ldots\ \alpha_\mu!} c_1^{\alpha_1} c_2^{\alpha_2} \ldots c_\mu^{\alpha_\mu}, \tag{2.112}$$

$$\mu_1 = \min\ (N-n, n)$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_\mu = r$$

$$\alpha_1 + 2\alpha_2 + \ldots + \mu\alpha_\mu = N - n.$$

$$B_n^N(\omega) = 2^n \sum_{r=0}^{N-n} (-\omega)^r\, i^{n-r} \operatorname{erfc}(\omega c_o) \sum_{\alpha_i} \frac{1}{\alpha_1!\alpha_2!\ldots\alpha_\mu!} c_1^{\alpha_1} c_2^{\alpha_2} \ldots c_\mu^{\alpha_\mu} \tag{2.113}$$

$$P_n^N(\omega) = 2^n \sum_{r=0}^{\mu_2} \omega^r G_{n-r}(\omega c_o) \sum_{\beta_i} \frac{1}{\beta_1!\beta_2!\ldots\beta_\mu!} c_1^{\beta_1} c_2^{\beta_2} \ldots c_\mu^{\beta_\mu} \tag{2.114}$$

$$\mu = N-n-1$$

$$\mu_2 = \min\ (\mu, n)$$

$$\beta_1 + \ldots \beta_\mu = r$$

$$\beta_1 + \ldots \mu\beta_\mu = \mu$$

$$Q_n^N(\omega) = 2^n \sum_{r=0}^{N-n-1} (-\omega)^r i^{n-r} \operatorname{erfc}(\omega C_o) \sum_{\beta_i} \frac{1}{\beta_1!\beta_2!\ldots\beta_\mu!} c_1^{\beta_1} c_2^{\beta_2} \ldots c_\mu^{\beta_\mu} \tag{2.115}$$

$$\beta_1 + \beta_2 + \ldots + \beta_\mu = r,$$

$$\beta_1 + 2\beta_2 + \ldots + \mu\beta_\mu = N - n - 1.$$

The values of a_o, b_o, c_o, for N = 0, are

$$a_o = \theta_f/\text{erf}\ (c_o), \qquad b_o = (\theta_f - v_o)/\text{erfc}\ (\omega c_o)$$

and c_o satisfies

$$a_o\Phi_1(c_o) + b_o\omega(k_{21})\Phi_1(\omega c_o) = (2\rho\ell\alpha_1/k_1)c_o. \tag{2.116}$$

with

$$(-1)^{m+1}\ \Phi_m\ (\eta) = i^{-m}\ \text{erfc}\ \eta\ , \qquad m > 0\ . \tag{2.117}$$

The quantity c_o is exactly equal to γ of eq 2.10.

2.3.2 Variable Initial Temperature, Approximate Solution

The solution of Tao (1978) extended the similarity technique of Neumann to a semi-infinite slab with arbitrary initial temperature. Unfortunately this exact solution is such that numerical computations are extremely difficult because of transient functions that require an increasing number of series terms as time increases. Tao's solution is perhaps best used to verify the accuracy of approximate and numerical solutions or for short time solutions.

The heat balance integral technique solves the energy equation on average over a space volume, instead of at each point of space. A modification of the integral method utilizing a single integration over an entire, nonconstant property volume has yielded accurate solutions (Yuen 1980, Lunardini 1981a, 1982, 1983b).

This section presents an approximate solution to the modified Neumann problem for which a linear initial temperature distribution exists. Such an initial temperature is common for soil systems with a geothermal temperature gradient.

Figure 2.15 shows the case of a slab of material with an initial temperature distribution (G could represent a geothermal gradient). At zero time the surface temperature drops to T_s and freezing commences.

The governing equations are

$$\alpha_1 \frac{\partial^2 T_1}{\partial x^2} = \frac{\partial T_1}{\partial t} \qquad 0 \leq x \leq X \tag{2.118}$$

$$T_1(X,t) = T_f \tag{2.118a}$$

$$T_1(0,t) = T_s \tag{2.118b}$$

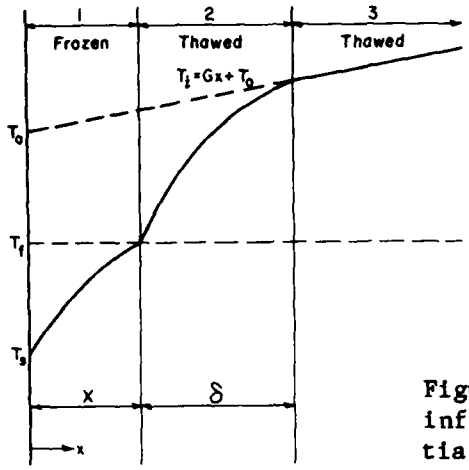


Figure 2.15. Freezing of a semi-infinite region with linear initial temperature.

$$\alpha_2 \frac{\partial^2 T_2}{\partial x^2} = \frac{\partial T_2}{\partial t} \qquad X \leq x \leq X + \delta \tag{2.119}$$

$$T_2(X,t) = T_f \tag{2.119a}$$

$$\frac{\partial T_2(X + \delta, t)}{\partial x} = G \tag{2.119b}$$

$$T_2(X + \delta, t) = G(X + \delta) + T_o \quad . \tag{2.119c}$$

The integration of the energy equations over the region $0 \leq x \leq X + \delta$, detailed by Lunardini (1981a), is

$$\frac{d}{dt}\left[\rho_1 c_1 \theta_1 + \rho_2 c_2 \theta_2 - \rho_1 \ell X + (\rho_2 c_2 - \rho_1 c_1) T_f X\right.$$

$$\left. - \rho_2 c_2 (X + \delta)\left\{T_o + \frac{G}{2}(X + \delta)\right\}\right] = -k_1 \frac{\partial T_1(0,t)}{\partial x} + k_2 G \tag{2.120}$$

$$\theta_1 = \int_o^X T_1(x,t)dx \tag{2.121a}$$

$$\theta_2 = \int_X^{X+\delta} T_2(x,t)dx \quad . \tag{2.121b}$$

The energy balance at the freezing front can be written as two equations, Lunardini (1981b):

$$-k_1 \left[\frac{\partial T_1(X,t)}{\partial x}\right]^2 + k_2 \frac{\partial T_2(X,t)}{\partial x} \frac{\partial T_1(X,t)}{\partial x} = \rho_1 \ell \alpha_1 \frac{\partial^2 T_1(X,t)}{\partial x^2} \tag{2.122a}$$

$$-k_1 \frac{\partial T_1(X,t)}{\partial x} \frac{\partial T_2(X,t)}{\partial x} + k_2 \left[\frac{\partial T_2(X,t)}{\partial x}\right]^2 = \rho_2 \ell \alpha_2 \frac{\partial^2 T_2(X,t)}{\partial x^2} \quad . \quad (2.122b)$$

Quadratic temperature profiles in regions 1 and 2 which satisfy the boundary conditions are

$$T_1 = T_f + (T_f - T_s) \frac{x - X}{X^2} \left[\frac{x}{g} - x + X\right] \quad (2.123)$$

where

$$g = \frac{\alpha_{12} (\Delta T + GX)X}{\delta [G(\delta+2X) + 2\Delta T]} + 1$$

$$T_2 = T_f + [G(\delta+2X) + 2\Delta T] \frac{x-X}{\delta} - (GX + \Delta T) \frac{(x-X)^2}{\delta^2} \quad . \quad (2.124)$$

In general, the simplest temperature profiles that will satisfy the boundary conditions should be chosen. The accuracy of the method usually increases as the order of a polynomial temperature choice increases up to some polynomial order (see App. B). However, the use of high-order polynomials (third and higher) is often not justified since a small increase in accuracy requires significantly more computational effort. Equation 2.122b can be used to find a relation between X and δ. In nondimensional form, this is

$$\frac{b^2 [\sigma(b + 2) + 2]}{\alpha_{21}(1 + \sigma) + b[\sigma(b + 2) + 2]} - k_{21} \phi[\sigma(b + 2) + 2]$$

$$= \frac{2\alpha_{21} (1 + \sigma)}{S_T[\sigma(b + 2) + 2]} \quad (2.125)$$

where $b = \delta/X$.

Equation 2.120, the energy integral equation, can now be written nondimensionally as

$$\tau = \int_0^\sigma K \, d\sigma \quad (2.126)$$

where

$$\tau = \alpha_1 \left(\frac{G}{\Delta T}\right)^2 t$$

$$K = \frac{\frac{1}{3} + \frac{1}{S_T} + c_{21}\phi \left[1 + \sigma + \frac{b+\sigma b'}{3}\right] + \frac{1}{6g} \left(1 - \frac{\sigma g'}{g}\right)}{\frac{1}{\sigma} \left(2 - \frac{1}{g}\right) - k_{21}\phi} \quad . \quad (2.127)$$

The derivatives of b and g can be found by simultaneously solving the following equations:

$$b' = \frac{db}{d\sigma} = \frac{\dfrac{k_{21}\phi\alpha_{21}}{2\beta} - \dfrac{b}{2}\left[\dfrac{4(g-1)}{S_T} - \dfrac{1}{g^2}\right]g'}{\dfrac{2(g-1)^2}{S_T} - 1 + \dfrac{1}{g}} \tag{2.128}$$

$$g' = \frac{\alpha_{21}}{b[\sigma(b+2)+2]}\left[1 - \frac{1+\sigma}{b[\sigma(b+2)+2]}\left\{2b'[\sigma(b+1)+1] + b(b+2)\right\}\right] . \tag{2.129}$$

The problem has now been reduced to a simple numerical quadrature of eq 2.126 using the auxiliary relations of eq 2.127-2.129.

Unlike the Neumann solution, the frozen zone for the general case reaches a steady-state value. At steady state, the net heat flux at the phase change interface will be zero so that $(dX/dt)_{t=\infty} = 0$. Then the temperature in region 1 is

$$T_{1\infty} = \frac{\Delta Tx}{\phi X_\infty} + T_s \quad . \tag{2.130}$$

At the solidification interface

$$k_1 \frac{\partial T_{1\infty}}{\partial x} = k_2 G \ . \tag{2.131}$$

Thus

$$X_\infty = \frac{\Delta T}{k_{21}\phi G} \tag{2.132}$$

or

$$\sigma_\infty = \frac{1}{k_{21}\phi} \quad . \tag{2.133}$$

Equation 2.126 was solved numerically using Simpson's rule (Lunardini 1984). The results are presented in Figures 2.16-2.19, as the ratio of σ to the frozen depth for the Neumann case, σ_N. It is possible to present the results for soil systems, quite efficiently, since the property ratios can be described as functions of the soil water content x_ℓ (Lunardini and Varotta 1981). The property ratios used are given in Table 2.5.

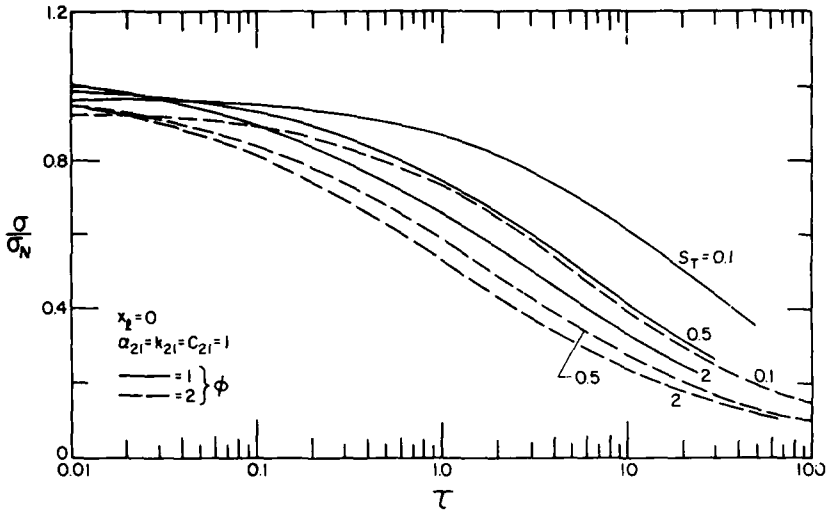


Figure 2.16. Ratio of freeze depth to that of Neumann solution, $x_\ell = 0.0$.

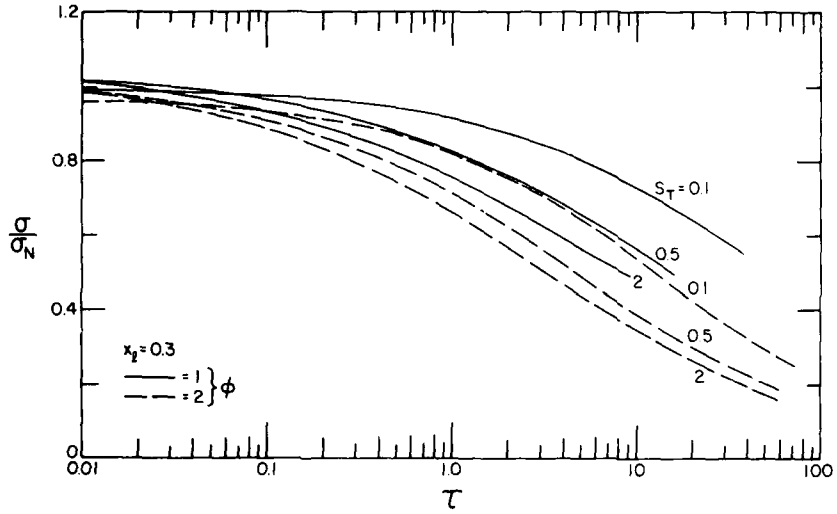


Figure 2.17. Ratio of freeze depth to that of Neumann solution, $x_\ell = 0.3$.

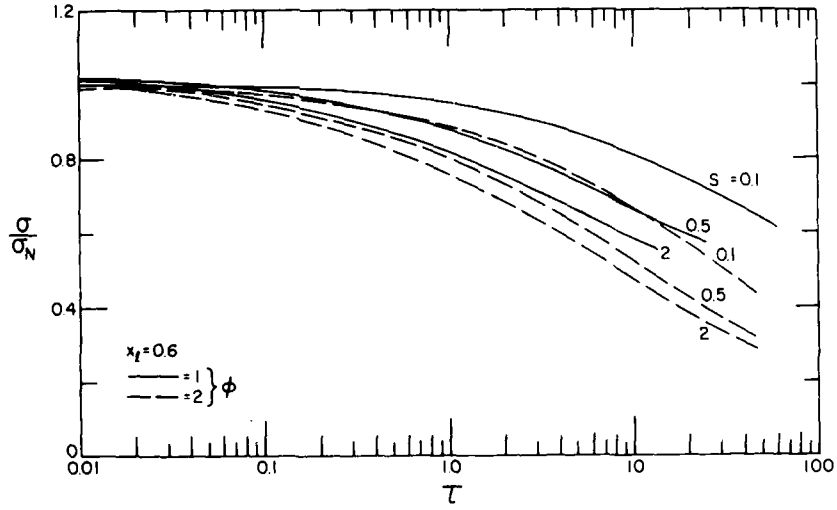


Figure 2.18. Ratio of freeze depth to that of Neumann solution, $x_\ell = 0.6$.

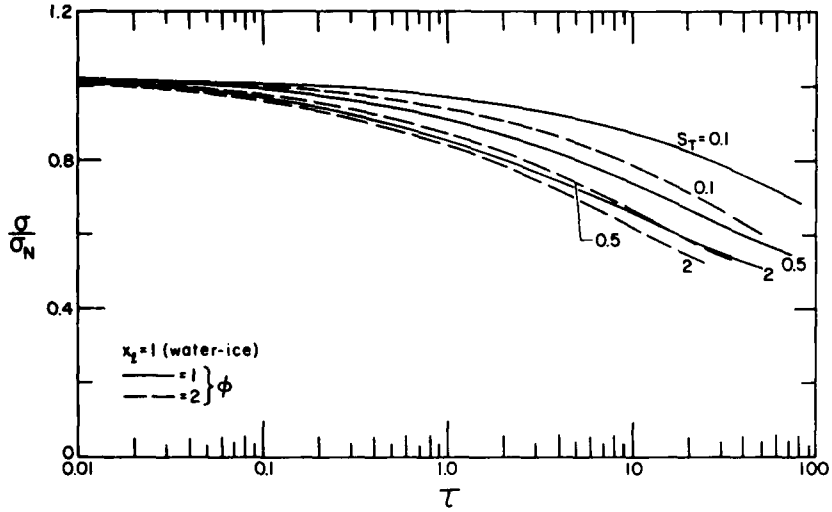


Figure 2.19. Ratio of freeze depth to that of Neumann solution, x_ℓ = 1.0, ice-water system.

Table 2.5 Frozen and thawed properties ratios, for soils, used with Figures 2.16 through 2.19.

x_ℓ	$k_{12} = k_f/k_t$	$\alpha_{12} = \alpha_f/\alpha_t$
0	1	1
0.3	1.5031	1.9642
0.6	2.2593	3.6470
1.0	3.8895	7.8685

As was expected the ratio σ/σ_N is close to 1 during the early growth, since the solution starts exactly as does the Neumann solution. As time goes on, the solidification rate for the general case decreases much more rapidly than that for the Neumann solution. Under natural conditions there will be a temperature gradient for a soil system - the geothermal gradient. The effect of this gradient on the accuracy of the constant temperature solution widely used is of interest for design in cold or arctic regions. For soil systems the Neumann solution will be quite acceptable even if the time is measured in years since τ is still quite small; if G is large, then it will be necessary to use eq 2.126 with appropriate property ratios. These can be interpolated from Figures 2.16-2.19 if the property ratios fall within the ranges on the plots. The effect of the initial temperature gradient can be compensated for, at later times, by using a temperature parameter ϕ_N with the Neumann solution as

$$\phi_N = \phi + \frac{1}{2k_{21}} \quad . \tag{2.134}$$

The results are also valid for the melting case if the initial temperature decreases linearly with depth. The more interesting case of a melting system with a temperature profile that increases with depth (typical geothermal profile) is only valid until $X + \delta = \Delta T/G$. After this time, melting will occur at both surfaces of the thawed zone and the solution will not be valid.

2.3.3 Sinusoidal Surface Temperature

The Neumann solution is for a surface temperature that instanteously changes to a fixed value at the beginning of phase change. Practical surface temperatures will rarely be of this form and it is of some interest to examine the relation between the Neumann solution and a sinusoidal surface temperature. This variable surface temperature problem cannot be solved exactly; however, an acceptable quasi-steady solution can be found.

Consider a system that is initially at the fusion value and then undergoes a sinusoidal surface temperature variation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.135}$$

$$T(X,t) = T_f \tag{2.135a}$$

$$T(0,t) = T_s(t) = T_f + \theta_c \sin\left(\frac{2\pi t}{P}\right) \tag{2.135b}$$

$$k\,\frac{\partial T(X,t)}{\partial x} = \pm\,\rho\ell\,\frac{dX}{dt}\ . \tag{2.135c}$$

The sign in eq 2.135c denotes the thawing (-) or the freezing (+) situation. The equations can be nondimensionalized by using

$$\theta = \frac{T-T_f}{\theta_c} \qquad \tau^* = \frac{t}{t_c} = \frac{2\pi t}{P} \qquad S_{Ta} = \frac{c\theta_c}{\ell}$$

$$y = \left(\frac{\rho\ell}{k\theta_c t_c}\right)^{\frac{1}{2}} x \qquad \xi_1 = \left(\frac{\rho\ell}{k\theta_c t_c}\right)^{\frac{1}{2}} X$$

where

$$\theta_c = \frac{\pi}{2}\,(T_f - \overline{T}_s)$$

$\overline{T}_s$ = average value of T_s when $T_s < T_f$.

The equations to solve are

$$\frac{\partial^2\theta}{\partial y^2} = S_{Ta} \frac{\partial\theta}{\partial\tau^*} \tag{2.136}$$

$$\theta\ (\xi_1, \tau^*) = 0 \tag{2.136a}$$

$$\theta\ (0, \tau^*) = \theta_s(\tau^*) = -\sin\tau^* \tag{2.136b}$$

$$\frac{d\xi_1}{d\tau^*} = \pm \frac{\partial\theta\ (\xi_1, \tau^*)}{\partial y} \tag{2.136c}$$

$$\xi_1(0) = 0\ . \tag{2.136d}$$

The solution will be obtained for the first-order quasi-steady equations. These equations are described in Appendix A. The solution here will start with freezing an initially thawed medium and then thawing the frozen system. The equations for each of the phase changes are found, using the properties of the frozen and thawed material. The effect due to the difference between the maximum thaw and freeze depths will not be considered. This will not be a serious error if the Stefan numbers are less than one.

The solutions are straightforward (freeze starts at zero time) and are

$$\theta = \theta_s(\tau)\ (1 - \frac{y}{2 \sin \frac{\tau^*}{2}}) +$$

$$\frac{S_{Ta}}{2}\ y\ (y - 2 \sin \frac{\tau^*}{2})\ [2 \sin^2 \frac{\tau^*}{2} - 1 - \frac{1}{6} \sin \frac{\tau^*}{2}(y + 2 \sin \frac{\tau^*}{2})] \tag{2.137}$$

$$\xi_1 = 2 \sin \frac{\tau^*}{2} - \frac{S_{Ta}}{3}\ [\frac{2}{3} + 2 \cos \frac{\tau^*}{2} - \frac{8}{3} \cos^3 \frac{\tau^*}{2}]\ . \tag{2.138}$$

Equations 2.137 and 2.138 can then be used for $0 \leq \tau^* \leq \pi$ with the properties of the frozen and thawed material.

Lock et al. (1969) give a similar solution for the phase change

$$\xi_1 = 2 \sin \frac{\tau^*}{2} - \frac{S_{Ta}}{3} \sin \frac{\tau^*}{2} \sin\tau^*\ . \tag{2.139}$$

However, this relation does not appear to be as accurate as eq 2.138, although the values are not too different.

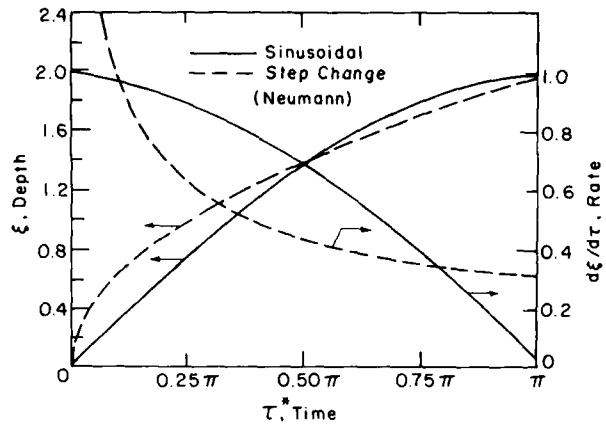


Figure 2.20. Freezing of semi-infinite medium with sinusoidal and step change surface temperatures.

Seban (1971) noted that the even simpler Stefan equation gave acceptable results and also that the effect of convection due to density variation, in water, is small but may be important for some systems (see Yen 1968).

The same procedure can also be used for other surface temperature variations. If the surface temperature is not symmetric about the fusion temperature the results are more complicated, but follow in the same way.

Equation 2.139 can be compared to the step change of surface temperature, the Neumann solution. Using the approximation for γ given by eq 2.34, with a constant surface temperature equivalent to the sinusoidal temperature, gives the step change solution as

$$\xi_1 = 2\sqrt{\frac{\tau^*}{\pi + S_{Ta}}} \tag{2.140}$$

Figure 2.20 shows the freeze depths and the freezing rates for S_{Ta} = 0.1. Notice that the total freeze depths, for the two surface temperature cases, are within 1% but the freeze depths and, especially the freezing rates, differ considerably at intermediate times. Thus the Neumann solution will model a variable surface temperature if the total phase change depth is desired but will not give comparable intermediate values for the phase change depth or rate.

2.4 MELTING TEMPERATURE RANGE

Not all materials exhibit a fixed phase change temperature. For soils, rocks, metal alloys, etc., melting will occur over a temperature range.

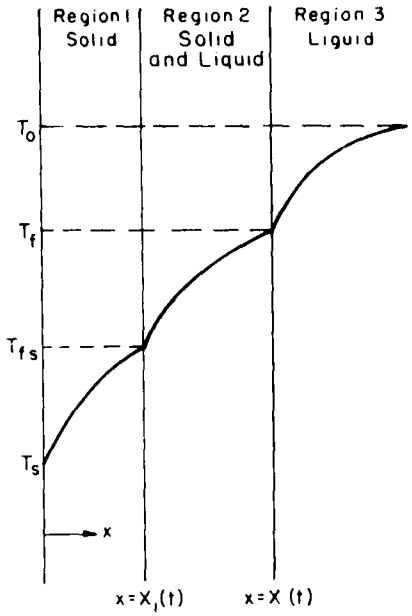


Figure 2.21. Solidification with a mixed phase region.

2.4.1 Metals and Alloys

Cho and Sunderland (1969) give an exact solution for the case of a binary eutectic mixture. The method is outlined here. Tien and Geiger (1967) give an approximate solution when the mixture is initially at the liquidus temperature. The system is described in terms of three regions shown in Figure 2.21. A completely solidified region, an initially liquid region, and a region with both solid and liquid phases are formed by isothermal planes at the liquidus (T_f) and solidus (T_{fs}) temperatures. The liquid begins to freeze at T_f and freezing is complete when the solid front reaches the volume of interest. Equilibrium freezing occurs if the element is completely frozen just as the solid front reaches it. Normal nonequilibrium freezing is such that the element still has a liquid fraction, which then freezes isothermally at T_{fs} before the solid front moves on. The properties of the solid and liquid are assumed not to vary with temperature and volume changes are negligible. The solid fraction distribution, in the freezing zone, is linear with distance. It has a value of zero at the liquidus front and f at the solidus front. For binary eutectic mixtures, f_e is the solid fraction at the eutectic composition. Tien and Geiger (1967) have shown that the linear assumption is not important for the phase change process.

2.4.1.1 Exact Solution

The solid fraction is given by

$$f_s = f_e\left(1 - \frac{x - X_1}{X - X_1}\right) \qquad X_1 < x < X \ . \tag{2.141}$$

The energy liberated per unit volume when a volume of liquid changes to a solid is

$$E_\ell = \frac{\ell\, dm_s}{V} = \frac{\ell}{V}\, d\,(m\, f_s) = \frac{\ell}{V}\left[m\,\frac{df_s}{dt} + f_s\,\frac{dm}{dt}\right]$$

$$E_\ell = \rho_2 \ell\, \frac{df_s}{dt} \ . \tag{2.142}$$

This will represent a uniformly distributed heat source in the freezing zone:

$$E_\ell = \frac{\rho_2 \ell\, f_e}{(X - X_1)^2}\left[(X - x)\,\frac{dX_1}{dt} + (x - X_1)\,\frac{dX}{dt}\right] \ .$$

The equations for this problem are

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1}\,\frac{\partial T_1}{\partial t} \qquad 0 < x < X_1 \tag{2.143}$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\rho_2 \ell}{k_2}\,\frac{df_s}{dt} = \frac{1}{\alpha_2}\,\frac{\partial T_2}{\partial t} \qquad X_1 < x < X \tag{2.144}$$

$$\frac{\partial^2 T_3}{\partial x^2} = \frac{1}{\alpha_3}\,\frac{\partial T_3}{\partial t} \qquad x > X \tag{2.145}$$

$$T_3\,(x,0) = T_o \tag{2.146a}$$

$$T_3\,(\infty,t) = T_o \tag{2.146b}$$

$$T_1\,(0,t) = T_s \tag{2.146c}$$

$$T_1 = T_2 = T_{fs} \qquad x = X_1 \tag{2.146d}$$

$$T_2 = T_3 = T_f \qquad x = X \ . \tag{2.146e}$$

At the solidus front, the remaining liquid will solidify and the energy balance is

$$k_1\,\frac{\partial T_1}{\partial x} = k_2\,\frac{\partial T_2}{\partial x} + \rho_1 \ell\,(1 - f_e)\,\frac{dX_1}{dt} \qquad x = X_1 \ . \tag{2.146f}$$

At the liquidus front the solid fraction is zero and the phase change for the freezing zone has been accounted for by the energy generation term. Then, only equality of heat fluxes is required:

$$k_2 \frac{\partial T_2}{\partial x} = k_3 \frac{\partial T_3}{\partial x} \qquad x = X \ . \tag{2.146g}$$

The solution is an adaptation of the Neumann method except that there are two parameters defined as

$$X_1 = 2\,\psi\,\sqrt{\alpha_1 t} \tag{2.147}$$

$$X = 2\,\gamma\,\sqrt{\alpha_1 t} \ . \tag{2.148}$$

The parameters ψ and γ must be found by the simultaneous solution of the following equations:

$$\frac{e^{-\psi^2}}{\operatorname{erf}\psi} + \frac{\sqrt{\pi}\ k_{21}\ f_e \ell}{2\ Cp_2\ (T_{fs} - T_s)\ (\gamma - \psi)} - \frac{k_{21}}{\sqrt{\alpha_{21}}}\ \frac{(T_f - T_{fs} + \frac{\ell f_e}{Cp_2})\ e^{-\psi^2\alpha_{12}}}{(T_f - T_s)\ (\operatorname{erf}\gamma\sqrt{\alpha_{12}} - \operatorname{erf}\psi\sqrt{\alpha_{12}})}$$

$$= \frac{\sqrt{\pi}\ (1 - f_e)\ \ell\psi}{Cp_1\ (T_{fs} - T_s)} \tag{2.149}$$

$$\frac{(T_f - T_{fs} + \ell\ f_e/Cp_2)e^{-\frac{\gamma^2}{\alpha_{21}}}}{(\operatorname{erf}\gamma\sqrt{\alpha_{12}} - \operatorname{erf}\psi\sqrt{\alpha_{12}})} - \frac{\sqrt{\pi\alpha_{21}}\ \ \ell f_e}{2\ Cp_2\ (\gamma - \psi)} = (T_o - T_f)k_{32}\ \frac{\sqrt{\alpha_{23}}\ e^{-\gamma^2\alpha_{13}}}{\operatorname{erfc}\gamma\sqrt{\alpha_{13}}} \ . \tag{2.150}$$

The temperatures are given by

$$\frac{T_1 - T_s}{T_{fs} - T_s} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\operatorname{erf}\psi} \tag{2.151}$$

$$\frac{T_2 - T_s}{T_{fs} - T_s} = 1 - \frac{\ell f_e\left(\frac{x}{2\sqrt{\alpha_1 t}} - \psi\right)}{Cp_2\ (T_{fs} - T_s)\ (\gamma - \psi)}$$

$$+ \frac{(T_f - T_{fs} + \ell f_e/Cp_2)}{(T_{fs} - T_s)}\ \frac{[\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right) - \operatorname{erf}\psi\sqrt{\alpha_{12}}]}{\operatorname{erf}\gamma\sqrt{\alpha_{12}} - \operatorname{erf}\psi\sqrt{\alpha_{12}}} \tag{2.152}$$

$$T_3 = T_o - (T_o - T_f) \frac{\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha_3 t}}\right)}{\operatorname{erfc} \gamma\sqrt{a_{13}}} \quad . \tag{2.153}$$

2.4.1.2 Approximate Solution

Tien and Geiger (1967) presented an interesting approximate solution to the preceding problem. The initial liquid temperature is at the liquidus temperature, thus there are only two regions to consider:

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t} \qquad 0 < x < X_1 \tag{2.154}$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\rho_2 \ell}{k_2} \frac{df_s}{dt} = \frac{1}{\alpha_2} \frac{\partial T_2}{\partial t} \qquad X_1 < x < X \tag{2.155}$$

$$T_1(0,t) = T_s \tag{2.156a}$$

$$T_1(X,t) = T_2(X,t) = T_{fs} \tag{2.156b}$$

$$T_2(X,t) = T_f \tag{2.156c}$$

$$k_1 \frac{\partial T_1}{\partial x} = k_2 \frac{\partial T_2}{\partial x} + \rho_1 \ell (1-f_e) \frac{dX_1}{dt} \quad ; \; x = X_1 \tag{2.156d}$$

$$\frac{\partial T_2}{\partial x} = 0 \qquad x = X \tag{2.156e}$$

$$\frac{df_s}{dt} = f_e \frac{d}{dt} \left(1 - \frac{x - X_1}{\Delta X}\right) \tag{2.156f}$$

where

$$\Delta X = X - X_1 \quad . \tag{2.156g}$$

The solution in the solid region is given by eq 2.151:

$$\frac{T_1 - T_s}{T_{fs} - T_s} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\operatorname{erf} \psi}$$

where, as previously,

$$X_1 = 2\psi \sqrt{\alpha_1 t} \quad .$$

In the mixed phase region, the heat balance integral method is used. The

heat balance equation is given by

$$\frac{1}{\alpha_2}\left[\frac{d\theta_2}{dt} - T_2\ (X,t)\ \frac{dX}{dt} + T_2\ (X_1,t)\ \frac{dX_1}{dt}\right] = \frac{\partial T_2\ (X,t)}{\partial x}$$

$$- \frac{\partial T_2\ (X_1,t)}{\partial x} + \frac{\rho_2 \ell f_e}{k_2}\left(\frac{1}{2}\frac{d\Delta X}{dt} + \frac{dX_1}{dt}\right) \qquad (2.157)$$

with

$$\theta_2 = \int_{X_1}^{X} T_2\ (x,t)\ dx\ . \qquad (2.158)$$

A quadratic temperature profile is assumed, which satisfies eq 2.156b,c,e

$$T_2 = T_{fs} + \frac{2\Delta T_f}{\Delta X}\ (x - X_1) - \frac{\Delta T_f}{(\Delta X)^2}\ (x - X_1)^2$$

where $\Delta T_f = (T_f - T_{fs})$.

Equation 2.158 is then

$$\left(1 + \frac{\ell}{c_2}\frac{f_e}{\Delta T_f}\right)\frac{dX_1}{dt} + \left(\frac{1}{3} + \frac{1}{2}\frac{\ell f_e}{c_2 \Delta T_f}\right)\frac{d\Delta X}{dt} = \frac{2\alpha_2}{\Delta X}\ . \qquad (2.159)$$

With eq 2.151 the boundary condition eq 2.156d yields a second differential equation:

$$\frac{(T_{fs} - T_s)}{\sqrt{\pi\ \alpha_1 t}}\ \frac{e^{-\left(\frac{X_1}{2\sqrt{\alpha_1 t}}\right)^2}}{\text{erf}\ \psi} = \frac{2k_2 \Delta T_f}{\Delta X} + \rho_2 \ell\ (1 - f_e)\ \frac{dX_1}{dt}\ . \qquad (2.160)$$

The following equations will be used

$$X_1 = 2\ \psi\sqrt{\alpha_1 t}$$

$$\Delta X = 2\psi_1\sqrt{\alpha_2 t}\ . \qquad (2.161)$$

Inserting eq 2.147 and 2.161 into eq 2.159 and 2.160 yields the equation for ψ and ψ_1, the solutions of which are

$$\psi_1 = \frac{\sqrt{\alpha_{12}}}{k_{12}\ \frac{(T_{fs} - T_s)}{\Delta T_f}\ \frac{e^{-\psi^2}}{\sqrt{\pi}\ \text{erf}\ \psi} - \frac{\alpha_{12}\ell(1 - f_e)\psi}{c_2 \Delta T_f}} \qquad (2.162)$$

$$B_1\psi + \frac{B_4}{B_2 \dfrac{e^{-\psi^2}}{\text{erf } \psi} - B_3\psi} = B_2 \frac{e^{-\psi^2}}{\text{erf } \psi} - B_3\psi \qquad (2.163)$$

where

$$B_1 = 1 + \frac{\ell f_e}{c_2 \Delta T_f}$$

$$B_2 = k_{12} \left(\frac{T_{sf} - T_s}{\Delta T_f}\right) \frac{\alpha_{21}}{\sqrt{\pi}}$$

$$B_3 = \frac{\ell (1 - f_e)}{c_2 \Delta T_f}$$

$$B_4 = \left(\frac{1}{3} + \frac{\ell f_e}{2c_2 \Delta T_f}\right) \alpha_{21} \ .$$

2.4.2 Soils

While many materials undergo phase change at a fixed temperature, soil systems exhibit a definite zone of phase change (Lunardini 1981b). The variation of unfrozen water with temperature causes the soil system to freeze or thaw over a finite temperature range. Exact and approximate solutions will be described for conduction phase change of plane layers of soil with water contents that vary linearly, quadratically, and exponentially with temperature. The temperature and phase change depths are found to vary significantly from those of the constant-temperature or Neumann problem.

At any temperature below the normal freezing point, unfrozen water, ice and soil solids will be in an equilibrium state. Figure 2.22 shows the unfrozen water ξ as a function of temperature for a typical soil. At

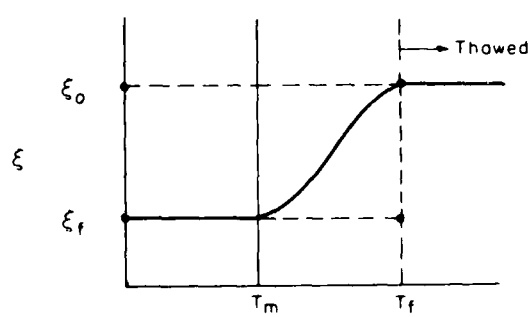


Figure 2.22. Unfrozen water vs temperature.

T_f all of the water is in the liquid form, while at T_m the free water is all frozen. There may be a residual amount of bound water that will remain unfrozen even at very low temperature, denoted by ξ_f. We will assume that for $T < T_m$, unfrozen water may exist but no phase change will occur. The region $T_m \leq T \leq T_f$ is called the zone of phase change or the mushy zone. In this region water will solidify to ice and unfrozen water and ice will coexist. The form of the ξ function for soils can be expressed in various functional relations. The simplest relation is a linear one:

$$\xi = \xi_o + \frac{\Delta\xi}{\Delta T_m} (T - T_f) \tag{2.164}$$

where $\Delta\xi = \xi_o - \xi_f$ and $\Delta T_m = T_f - T_m$.

A function that more closely approximates the soil water data is an exponential form:

$$\xi = a_1 e^{b_1 T} + d_1 \tag{2.165}$$

Another functional relation, which can closely model the data, and is easy to manipulate analytically is a quadratic form:

$$\xi = \xi_o + a_2 (T - T_f) + b_2 (T - T_f)^2 \tag{2.166}$$

$$a_2 = \frac{2\,\Delta\xi}{\Delta T_m}$$

$$b_2 = \frac{\Delta\xi}{\Delta T_m^{\,2}} .$$

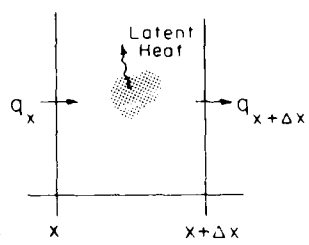


Figure 2.23. Heat flow in mushy zone.

If ξ_o, ξ_f and ΔT_m are the same for all of these functions, then the mean unfrozen water slope, $d\xi/dT$, will be identical.

Consider a small volume of material within the mushy zone as shown in Figure 2.23. Energy will be conducted in and out of the volume and latent heat will be released during solidification. Thus the problem is one of conduction with a distributed energy source. The energy released due to solidification of a mass of water Δm_w is as follows:

$$E_\ell \equiv -\ell\, \Delta m_w = -\ell\, \gamma_d\, \Delta\xi\, A\, \Delta x . \tag{2.167}$$

The energy equation then becomes

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\delta x}\right) - \ell\, \gamma_d \frac{\partial \xi}{\partial t} = \frac{\partial CT}{\partial t} . \tag{2.168}$$

The thermal conductivity and the specific heat are functions of the unfrozen water and may be represented by

$$k = k_f + (k_t - k_f)\frac{\xi}{\xi_o} \tag{2.169}$$

$$C = C_f + (C_t - C_f)\frac{\xi}{\xi_o} \tag{2.170}$$

where k_f, k_t are fully frozen and fully thawed thermal conductivities, and C_f, C_t fully frozen and fully thawed specific heats, respectively. Clearly these properties are functions of the particular form of the unfrozen water function (Frivick 1980).

2.4.2.1 Exact Solution for Linear Unfrozen Water Function

If ξ varies linearly with temperature, then an exact solution may be found. Although this will be a poor representation of a real soil system, it will constitute a valuable check for approximate solution methods. Assume that k and C vary from one region to another but are constant for each region.

The most general case will be a problem with three regions as shown in Figure 2.24. The equations for the three regions are

$$\frac{\partial^2 T_i}{\partial x^2} = \frac{1}{\alpha_i}\frac{\partial T_i}{\partial t} \tag{2.171}$$

$$T_i\ (0,t) = T_s \tag{2.171a}$$

$$T_i\ (X_i,t) = T_m \tag{2.171b}$$

$$\frac{\partial T_i\ (X_i)}{\partial x} = k_{21}\frac{\partial T_2\ (X_i)}{\partial x} \tag{2.171c}$$

$$X_i = 2\,\eta\,\sqrt{\alpha_i\ t}\ . \tag{2.172}$$

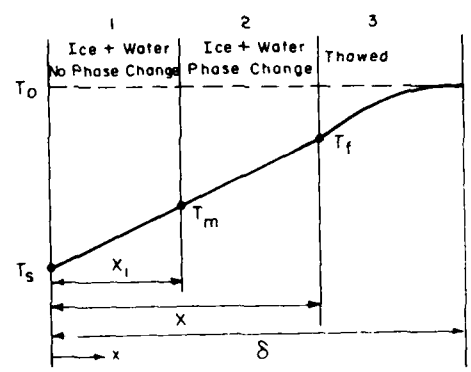


Figure 2.24. Geometry for solidification with a phase change zone.

The energy equation for region 2, the phase change region, can be written in the same form as eq 2.171 since ξ is a linear function of T_2, and it follows from eq 2.167 that

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_4} \frac{\partial T_2}{\partial t} \tag{2.173}$$

$$T_2 (X_1, t) = T_m \tag{2.173a}$$

$$T_2 (X, t) = T_f \tag{2.173b}$$

$$\frac{\partial T_2 (X)}{\partial x} = k_{32} \frac{\partial T_3 (X)}{\partial x} \tag{2.173c}$$

$$X = 2 \gamma \sqrt{\alpha_4 t} \tag{2.174}$$

$$\alpha_2 / \alpha_4 = 1 + \frac{C_{32} \phi_1}{S_{Tm}} \tag{2.175}$$

$$\frac{\partial^2 T_3}{\partial x^2} = \frac{1}{\alpha_3} \frac{\partial T_3}{\partial t} \tag{2.176}$$

$$\lim_{x \to \infty} T_3 (x,t) = T_o \tag{2.176a}$$

$$T_3 (x,0) = T_o \tag{2.176b}$$

$$T_3 (X) = T_f \tag{2.176c}$$

where

$$S_{Tm} = \frac{C_3 (T_f - T_s)}{\gamma_d \ell \Delta \xi}$$

$$\phi = \frac{T_o - T_f}{T_f - T_s}$$

$$\phi_1 = \frac{T_f - T_s}{T_f - T_m} \quad .$$

The solution to these equations follows as a similarity solution:

$$\frac{T_1 - T_s}{T_m - T_s} = \frac{\text{erf } (x/2\sqrt{\alpha_1 t})}{\text{erf } \psi} \tag{2.177}$$

$$\frac{T_2 - T_f}{\Delta T_m} = \frac{\text{erf } (x/2\sqrt{\alpha_4 t}) - \text{erf } \gamma}{\text{erf } \gamma - \text{erf } (\sqrt{\alpha_{14}}\ \psi)} \tag{2.178}$$

$$\frac{T_3 - T_o}{T_o - T_f} = \frac{-\text{erfc } (x/2\sqrt{\alpha_3\ t})}{\text{erfc } \left(\sqrt{\alpha_{43}}\ \gamma\right)} \quad . \tag{2.179}$$

The unknowns ψ and γ are found by using eq 2.178c, 2.154c:

$$\frac{(T_m - T_s)\ e^{-\eta^2}\ (1 - \alpha_{14})}{\Delta\ T_m} = \frac{k_{21}\ \sqrt{\alpha_{14}}\ \text{erf } \eta}{\text{erf } \gamma - \text{erf } \left(\sqrt{\alpha_{14}}\ \psi\right)} \tag{2.180}$$

$$\frac{\Delta T_m\ k_{23}}{T_o - T_f}\ \sqrt{\alpha_{34}}\ e^{-\gamma^2}\ (1 - \alpha_{43}) = \frac{\text{erf } \gamma - \text{erf } \left(\sqrt{\alpha_{14}}\ \psi\right)}{\text{erfc } \left(\sqrt{\alpha_{43}}\ \gamma\right)} \quad . \tag{2.181}$$

Equations 2.180 and 2.181 can now be solved simultaneously for ψ and γ, which completes the exact solution. Lunardini (1985) compared the solution to the Neumann solution for specific cases.

We let

$T_o = 4°C \qquad T_s = -6 \qquad T_f = 0°C$

$k_1 = 0.00828$ cal/s-cm °C $\quad \alpha_1 = 180.6546$ cm^2/hr

$\xi_o = 0.20\ \frac{g_w}{g_s} \qquad \gamma_d = 1.68\ g_s/cm^3$

$k_2 = 0.00703 \qquad \alpha_2 = 153.3818$

$\xi_f = 0.0782 \qquad \ell = 80$ cal/g_w

$k_3 = 0.00578 \qquad \alpha_3 = 126.1091 \qquad \Delta\xi = 0.1218$

$C_1 = C_2 = C_3 = 0.165$ cal/cm^3 °C

$S_T = 0.0605.$

Table 2.6 shows the effect of changing the width of the phase change zone by varying T_m.

Figure 2.25 shows the temperature profiles for the gradual and instantaneous latent heat cases after 24 hours. The Neumann solution has a temperature that always exceeds the gradual case and is significantly different within the zone of phase change. For a linear water content

Table 2.6 Effect of phase change temperature, T_m, on solidification of a soil (from Lunardini 1985).

Case	T_m	ψ	γ	X* (cm)	X_1* (cm)	$\Delta X = X - X_1$
1	- 4	0.0617	1.395	33.33	8.13	25.2
2	- 2	0.1135	1.6614	28.34	14.95	13.39
3	- 1	0.1376	2.062	25.0	18.12	6.88
4	-.5	0.14922	2.6965	23.27	19.65	3.52
5	-.1	0.1571	5.058	21.41	20.69	0.72
Neumann	0	0.1606	---	21.15	21.15	0

*For t = 24 hours.

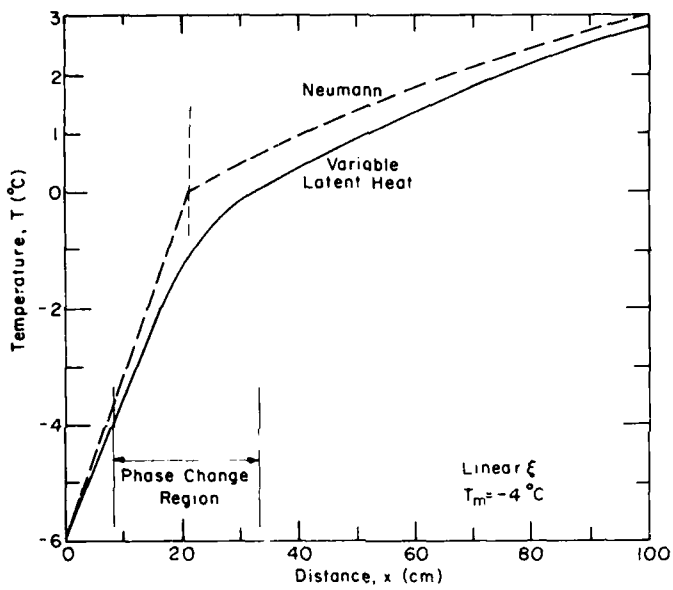


Figure 2.25. Temperature in three zones, 24 hours after initiation of freeze.

function, the Neumann solution can be quite different than the actual solution.

2.4.2.2 Heat Balance Integral Approximations

Approximate solutions for several unfrozen water content functions may be obtained with the heat balance integral method. In Figure 2.26, the governing equations for the two-zone problem, with $T_s \geq T_m$, are

$$\alpha_4 \left[\frac{\partial T_2(X)}{\partial x} - \frac{\partial T_2(o)}{\partial x}\right] = \frac{d}{dt} \int_0^X T_2 \, dx - T_f \frac{dX}{dt} \qquad (2.181)$$

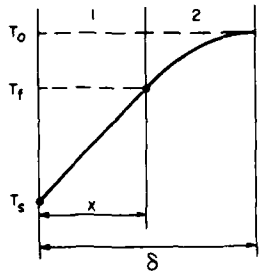


Figure 2.26. Two-zone geometry for heat balance integral.

$$-\alpha_3 \frac{\partial T_3(X)}{\partial x} = \frac{d}{dt} \int_X^{\delta} T_3 \, dx - T_o \frac{d\delta}{dt} + T_f \frac{dX}{dt} \quad (2.182)$$

The boundary conditions are

$$T_2(0,t) = T_s \quad (2.183a)$$

$$T_2(X) = T_3(X) = T_f \quad (2.183b)$$

$$\frac{\partial T_2(X)}{\partial x} = k_{32} \frac{\partial T_3(X)}{\partial x} \quad (2.183c)$$

$$T_3(\delta,t) = T_o \quad (2.184a)$$

$$\frac{\partial T_3(\delta,t)}{\partial x} = 0 \quad . \quad (2.184b)$$

Quadratic temperature profiles which satisfy the boundary conditions are given by

$$T_2 = T_f + b\,(X-x) + c\,(X-x)^2 \quad (2.185)$$

$$T_3 = T_o + (T_f - T_o) \frac{(\delta - x)^2}{(\delta - X)^2} \quad (2.186)$$

where

$$c\,X^2 = (T_s - T_f) - b\,X$$

$$b = -2 \frac{k_{32}\,(T_o - T_f)}{(\delta - X)} \quad .$$

Using

$$\delta - X = B\,x \quad (2.187)$$

the solution for γ follows directly as

$$3\,\alpha_{43}\,(B - 2\,k_{32}\,\phi)\,(1 + \frac{B}{3}) - 1 - \frac{\phi\,k_{32}}{B} = 0 \quad (2.188)$$

$$\gamma^2 = \frac{1}{B(1 + \frac{B}{3})\,\alpha_{43}} \quad (2.189)$$

$$X = 2\,\gamma\sqrt{\alpha_4\,t} \quad . \quad (2.190)$$

For the same example problem as before, with $T_m = -4°C$, the heat balance integral (HBI) solution may be compared to the exact solution.

Let

$$T_o = 4°C \qquad T_s = T_m = -4 \quad T_f = 0$$

$$k_2 = 0.00703 \text{ cal/s-cm -°C} \quad C_2 = C_3 = 0.165 \text{ cal/cm}^3\text{-°C}$$

$$k_3 = 0.00578.$$

Then

Exact solution $\gamma = 1.23645$

HBI solution, eq 2.189 $\gamma = 1.29942.$

The heat balance integral solution is within 5% of the exact solution. This accuracy is typical of the heat balance integral method and gives us confidence to apply the method to problems for which exact solutions are unavailable.

Exponential Unfrozen Water Function. If the unfrozen water has the exponential form discussed earlier, the equations for the two regions are

$$k_2 \frac{\partial^2 T_2}{\partial x^2} = \frac{\partial}{\partial t} (C_2 T_2 + \ell \gamma \xi) \tag{2.191}$$

$$\frac{\partial^2 T_3}{\partial x^2} = \frac{1}{\alpha_3} \frac{\partial T_3}{\partial t} . \tag{2.192}$$

The heat balance integral equation for region 2 is

$$k_2 \left[\frac{\partial T_2(X)}{\partial x} - \frac{\partial T_2(o)}{\partial x}\right] = \frac{d}{dt} \int_0^X (C_2 T_2 + \ell \gamma_d \xi) \, dx$$

$$- (C_2 T_f + \ell \gamma_d \xi_o) \frac{dX}{dt} \tag{2.193}$$

while eq 2.182 for region 3 is still valid.

Unfortunately the exponential form of eq 2.165 can only be integrated if T_2 is a linear function of x. For use in the ξ function only, we shall assume that

$$T_2 = T_f - \frac{(T_f - T_s)}{X} (X-x) . \tag{2.185a}$$

Then

$$\xi = a_2 e^{b_2 (X-x)} + d_1 \tag{2.185b}$$

where

$$a_2 = \xi_o - d_1$$

$$b_2 = \frac{b_1 (T_s - T_f)}{X} \quad .$$

For use in eq 2.193 and 2.182 we assume the usual quadratic forms for the temperatures. With these relations we can find the differential equations for X and δ:

$$2\,\alpha_2 \left[\frac{1}{X} - \frac{2k_{32}\,\phi}{\delta - X}\right] = \frac{d}{dt}\left\{\frac{k_{32}\,\phi}{3(\delta - X)}\,X^2 + \frac{X}{3}\right.$$

$$\left. - \frac{1}{C_{23}S_T\Delta\xi}\left[\frac{\xi_s - \xi_o}{\ln R} + d_1 - \xi_o\right] X\right\} \tag{2.194}$$

$$\frac{2\,\alpha_3}{\delta - X} = \frac{1}{3}\,\frac{d}{dt}\,(\delta + 2X) \tag{2.195}$$

where

$$R = \frac{\xi_s - d_1}{\xi_o - d_1}$$

$$\xi_s = a_1\,e^{b_1 T_s} + d_1 \quad .$$

With the relation $\delta - X = BX$, eq 2.174 and 2.175 become

$$-\alpha_{23}\left(1 - \frac{2\,k_{32}\,\phi}{B}\right) B\,\left(\frac{B}{3} + 1\right) + \frac{k_{32}\,\phi}{3B} + \frac{1}{3} =$$

$$\frac{1}{C_{23}\,S_T\,\Delta\xi}\left[\frac{\xi_s - \xi_o}{\ln R} + d_1 - \xi_o\right] \quad . \tag{2.196}$$

Since

$$X = 2\,\gamma\,\sqrt{\alpha_4 t} \tag{2.197}$$

then

$$\gamma^2 = \frac{1}{B\,\left(\frac{B}{3} + 1\right)\,\alpha_{43}} \quad . \tag{2.198}$$

This solution can be compared to the previous cases. The water content

function will have the same limiting values as for the linear case. Then

$a_1 = 0.1278$

$b_1 = 0.763$

$d_1 = 0.0722$

Equations 2.196 and 2.198 give $\gamma = 1.0246$.

The exponential solution is 21% less than the linear case. Thus the form of the unfrozen water content function will be significant. Since the solution of this section is somewhat suspect because of the use of a linear temperature profile for the water content, a solution with a quadratic ξ function will be examined.

<u>Quadratic Unfrozen Water Relation</u>. A quadratic relation can represent the unfrozen water content with acceptable accuracy. The relation can be given as

$$\xi = \xi_o + a_3 (T - T_f) + b_3 (T - T_f)^2$$

where $\quad a_3 = \dfrac{2 \Delta \xi}{\Delta T_m} \qquad b_3 = \dfrac{\Delta \xi}{(\Delta T_m)^2}$.

The heat balance integral forms of the energy equations are

$$k_2 \left[\frac{\partial T_2(X)}{\partial x} - \frac{\partial T_2(0)}{\partial x}\right] = \frac{d}{dt} \int_0^X (C_2 T_2 + \ell \gamma_d \xi)\, dx$$

$$- (C_2 T_f + \ell \gamma_d \xi_o) \frac{dX}{dt} \tag{2.199}$$

$$-\alpha_3 \frac{\partial T_3(X)}{\partial x} = \frac{d}{dt} \int_X^\delta T_3\, dx - T_o \frac{d\delta}{dt} + T_f \frac{dX}{dt} \quad . \tag{2.200}$$

Substitution of eq 2.185 and 2.186 into eq 2.199 and 2.200 leads to

$$2\, C_{32}\, \phi \left(1 - \frac{1}{P_1}\right) \left(3 + \frac{2\, k_{32}\, \phi}{P_1}\right) + \frac{P_1}{2} + 1$$

$$+ \frac{C_{32}\, \phi}{S_T} \left[2 + P_1 - \frac{\phi}{5} \left(\frac{P_1}{2} + \frac{3}{2}\, P_1 + 3\right]\right. = 0 \tag{2.201}$$

where $\quad P_1 = \dfrac{2\, k_{32}\, \phi}{B}$.

As before, $X = 2\gamma\sqrt{\alpha_4 t}$.

Table 2.7. Effect of ξ function, $T_o = 4°C$, $T_s = T_m = -4°C$, $T_f = 0°C$, $k_2 = 0.00703$, $C = 0.165$, $\xi_o = 0.2$, $\Delta\xi = 0.1218$, $k_{32} = 0.82219$, $C_{32} = 1.0$ (Lunardini 1985)

Solution	γ	Difference from linear ξ (%)	X (t = 24 hrs) (cm)
Exact, linear ξ	1.2365	- 4.8	29.54
H.B.I. - linear ξ	1.2994	--	31.04
H.B.I. - exponential ξ	1.0246	-21.1	24.48
H.B.I - quadratic ξ	1.1561	-11.0	27.62
Neumann (ξ, step function)	0.7846	-39.6	18.74

Then

$$\gamma^2 = \frac{P_1 \alpha_{34}}{2 k_{32} \phi \left(\frac{2 k_{32} \phi}{3 P_1} + 1\right)} \tag{2.202}$$

The solutions may be compared as shown in Table 2.7.

Nakano and Brown (1971) solved a version of this problem numerically. Their results showed a significant effect of a freezing zone on the temperature profile but the magnitudes of the differences did not follow those shown here. The reason for the difference cannot be explained since the numerical computation method was not given in adequate detail.

Lunardini (1985) solved the above cases with variable thermal properties and showed that the property effect could be accommodated with the constant property solution.

2.4.3 Finite, Insulated Slab

An approximate solution to the finite, insulated slab problem is given by Cho and Sunderland (1969). The method follows closely the results of Section 2.4.1 with the geometry noted in Figure 2.27. The equations, following previous notation, are

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t} \qquad 0 < x < X_1 \tag{2.203}$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\rho_2 \ell}{k_2} \frac{df_s}{dt} = \frac{1}{\alpha_2} \frac{\partial T_2}{\partial t} \qquad X_1 < x < X \tag{2.204}$$

$$\frac{\partial^2 T_3}{\partial x^2} = \frac{1}{\alpha_3} \frac{\partial T_3}{\partial t} \qquad X < x < D \tag{2.205}$$

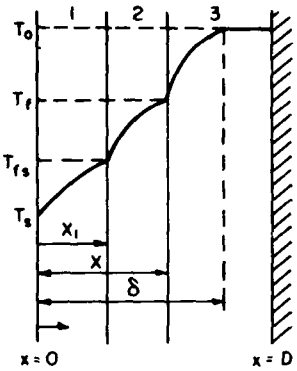


Figure 2.27. Phase change regions for finite slab.

$$T_3(x,0) = T_o \tag{2.206a}$$

$$\frac{\partial T_3(D,t)}{\partial x} = 0 \tag{2.206b}$$

$$T_1(0,t) = T_s \tag{2.206c}$$

$$T_1(X_1,t) = T_2(X_1,t) = T_{fS} \tag{2.206d}$$

$$T_2(X,t) = T_3(X,t) = T_f \tag{2.206e}$$

$$k_1 \frac{\partial T_1}{\partial x}(X_1,t) = k_2 \frac{\partial T_2(X_1,t)}{\partial x} + \rho_1 \ell\,(1-f_e)\frac{dX_1}{dt} \tag{2.206f}$$

$$k_2 \frac{\partial T_2(X,t)}{\partial x} = k_3 \frac{\partial T_3(X,t)}{\partial x} \tag{2.206g}$$

$$\frac{\partial T_3(\delta,t)}{\partial x} = 0 \tag{2.206h}$$

$$T_3(\delta,t) = T_o \tag{2.206i}$$

The usual assumptions, as follow, are used:

$$X_1 = 2\psi\sqrt{\alpha_1 t}$$

$$X = 2\gamma\sqrt{\alpha_1 t}$$

$$\delta = 2\beta\sqrt{\alpha_1 t} \quad .$$

The exact solutions for regions 1 and 2 are given by

$$\frac{T_1 - T_s}{T_{fs} - T_s} = \frac{\operatorname{erf} \dfrac{x}{2\sqrt{\alpha_1 t}}}{\operatorname{erf} \psi} \tag{2.207}$$

$$\frac{T_2 - T_s}{T_{fs} - T_s} = 1 - \frac{\ell f_e \left(\dfrac{x}{2\sqrt{\alpha_1 t}} - \psi\right)}{c_{p2}\,(T_{fs} - T_s)(\gamma - \psi)}$$

$$+ \frac{T_f - T_{fs} + \ell f_e / c_{p2}}{(T_{fs} - T_s)} \; \frac{\left(\operatorname{erf} \dfrac{x}{2\sqrt{\alpha_2 t}} - \operatorname{erf} \psi\sqrt{\alpha_{12}}\right)}{\operatorname{erf} \gamma\sqrt{\alpha_{12}} - \operatorname{erf} \psi\sqrt{\alpha_{12}}} \; . \tag{2.208}$$

The heat balance integral for region 3 is again

$$\frac{d\theta_3}{dt} - T_o \frac{d\delta}{dt} + T_f \frac{dX}{dt} + \alpha_3 \frac{\partial T_3(X,t)}{\partial x} = 0 \tag{2.209}$$

$$\theta_3 = \int_X^{\delta} T_3 \, dx \; .$$

The fourth-power temperature for region 3 is

$$T_3 = T_o - (T_o - T_f) \left(\frac{\delta - x}{\delta - X}\right)^4 \; .$$

Then from eq 2.209,

$$\beta - \gamma = \frac{5}{2} \left[-\gamma + \sqrt{\gamma^2 + \frac{8}{5}\alpha_{31}}\right] \tag{2.210}$$

The equation for ψ follows directly from eq 2.206f

$$\frac{e^{-\psi^2}}{\operatorname{erf} \psi} + \frac{k_{21} \, f_e \ell \sqrt{\pi}}{2c_{p2}(T_{fs} - T_s)(\gamma - \psi)} - \frac{k_{21} \left(T_f - T_{fs} + \dfrac{\ell f_e}{c_{p2}}\right) e^{-\psi^2 \alpha_{12}}}{\sqrt{\alpha_{21}} \, (T_f - T_s)(\operatorname{erf} \gamma\sqrt{\alpha_{12}} - \operatorname{erf} \psi\sqrt{\alpha_{12}}}$$

$$= \frac{\sqrt{\pi} \, (1 - f_e)\ell\psi}{c_{p1} \, (T_{fs} - T_s)} \; . \tag{2.211}$$

Equation 2.206g leads to

$$\frac{-\ell f_e}{c_{p2} \, (\gamma - \psi)} + \frac{2\left(T_f - T_{fs} + \dfrac{\ell f_e}{c_{p2}}\right)\sqrt{\alpha_{12}} \; e^{-\gamma^2 \alpha_{12}}}{\sqrt{\pi} \, (\operatorname{erf} \eta\sqrt{\alpha_{12}} - \operatorname{erf} \psi\sqrt{\alpha_{12}})} = \frac{4k_{32} \, (T_o - T_f)}{\beta - \gamma} \; . \tag{2.212}$$

The simultaneous solution of eq 2.210-2.212 will give ψ, γ, β. The temperature of the slab at x = D follows exactly as in Section 2.1.6.2:

$$\frac{T_3(D,t) - T_f}{T_o - T_f} = \left(\frac{a}{b}\right)^{-(1+c)} e^{c\left(\frac{1}{b} - \frac{1}{a}\right)} \qquad t_1 < t < t_2 \qquad (2.213)$$

where

$$a = 1 - \frac{2\gamma}{D}\sqrt{\alpha_1 t}$$

$$b = 1 - \frac{\gamma}{\beta}$$

$$c = \frac{5}{2}\alpha_{31}\gamma^2 \quad .$$

2.5 SUBCOOLED LIQUID - FRAZIL ICE

A problem of some practical interest relates to the freezing of a liquid, initially below its fusion temperature, as shown in Figure 2.28. The liquid is in a metastable state and phase change results in the release of latent heat which warms the liquid to its normal fusion point. The solid phase remains at the freezing temperature. This can be related physically to the formation of frazil ice from water that supercools due to turbulence. The problem mathematically is

$$T_1 = T_f \qquad (2.214)$$

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} \qquad (2.215)$$

$$\lim_{x\to\infty} T_2\ (x,t) = T_o$$

$$T_2\ (X,t) = T_f$$

$$k_2 \frac{\partial T_2(X,t)}{\partial x} = -\rho_2 \ell \frac{dX}{dt} \quad .$$

The solution follows directly as

$$T_2 = T_o + \frac{(T_f - T_o)\ \mathrm{erfc}\ \left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{\mathrm{erfc}\ \gamma} \qquad (2.216)$$

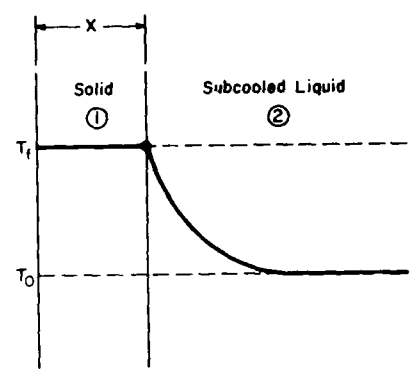


Figure 2.28. Geometry for subcooled liquid.

$$X = 2\gamma \sqrt{\alpha_2 t} \tag{2.217}$$

$$\gamma \operatorname{erfc}(\gamma)\, e^{\gamma^2} = \frac{c_2 (T_f - T_o)}{\ell \sqrt{\pi}} \quad . \tag{2.218}$$

2.6 SOLIDIFICATION IN CONTACT WITH COLD WALL

The freezing of liquid in contact with a cold wall is of importance in casting metals and in the intrusion of magmas during geological processes. At time $t = 0$, a semi-infinite liquid at T_o is brought into contact with a semi-infinite cold wall at T_c. A solid phase forms instantly and grows with time, while the temperature between the cold wall, and solid phase remains constant (see Fig. 2.29). The density of the solid and liquid phases are assumed equal:

$$\frac{\partial^2 T_w}{\partial x^2} = \frac{1}{\alpha_w} \frac{\partial T_w}{\partial t} \qquad x \leq 0 \tag{2.219}$$

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_2}{\partial t} \qquad 0 \leq x \leq X(t) \tag{2.220}$$

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{\alpha_2} \frac{\partial T_2}{\partial t} \qquad x \geq X(t) \tag{2.221}$$

$$\left.\begin{aligned} T_1 &= T_w \\ k_1 \frac{\partial T_1}{\partial x} &= k_w \frac{\partial T_w}{\partial x} \end{aligned}\right\} \qquad x = 0$$

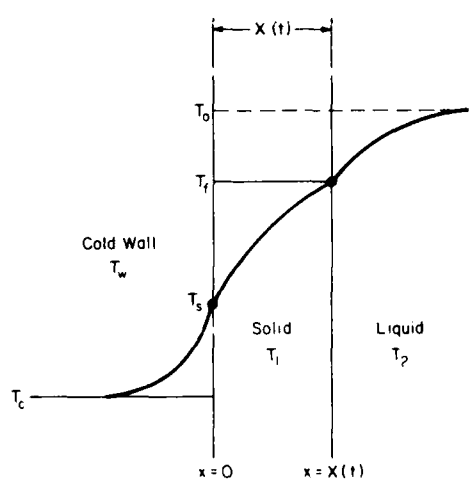


Figure 2.29. Freezing against a cold wall.

$$\lim_{x \to -\infty} T_w = T_c$$

$$T_w(x,0) = T_c$$

$$\left.\begin{aligned} &T_1 = T_2 = T_f \\ &k_1 \frac{\partial T_1}{\partial x} - k_2 \frac{\partial T_2}{\partial x} = \rho_2 \ell \frac{dX}{dt} \end{aligned}\right\} \quad x = X$$

$$\lim_{x \to \infty} T_2 = T_o$$

$$T_2(x,0) = T_o \ .$$

The solution follows from the usual similarity transformation:

$$\frac{T_2 - T_f}{T_o - T_f} = \frac{1 - \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha_2 t}}\right)}{\operatorname{erfc}\left(\gamma \sqrt{\alpha_{12}}\right)} \tag{2.222}$$

$$\frac{T_w - T_c}{T_s - T_c} = \operatorname{erfc}\left(\frac{-x}{2\sqrt{\alpha_w t}}\right) \tag{2.223}$$

$$\frac{T_1 - T_s}{T_f - T_s} = \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha_1 t}}\right)}{\operatorname{erf} \gamma} \tag{2.224}$$

$$X = 2\gamma \sqrt{\alpha_1 t} \ . \tag{2.225}$$

The energy balance at the interface yields the equation for γ:

$$\frac{e^{-\gamma^2}}{\operatorname{erf} \gamma} - \frac{k_{21} \sqrt{\alpha_{12}}\, \phi e^{-\gamma^2 \alpha_{12}}}{\operatorname{erfc}\left(\gamma \sqrt{\alpha_{12}}\right)} = \frac{\gamma \sqrt{\pi}}{S_T} \ . \tag{2.226}$$

The interface temperature between the cold wall and the frozen liquid is

$$T_s = \frac{T_f + T_c\, k_{w1} \sqrt{\alpha_{1w}} \operatorname{erf} \gamma}{1 + k_{w1} \sqrt{\alpha_{1w}} \operatorname{erf} \gamma} \ . \tag{2.227}$$

Since the interface temperature is constant, the solution will be identical

to the Neumann solution with the same surface temperature. The results are also valid for melt near a hot wall if $S_T = c_1(T_s - T_f)/\ell$ and region 1 is thawed or liquid.

2.7 THAW WITH CONSOLIDATION OF MELTED MEDIUM

An interesting problem arises because some porous media, such as soils, can become more compact after thawing. A certain amount of the water in the frozen soil is forced out as the solid particles settle and the thawed soil becomes denser. The situation is similar to the variable density problem of Section 2.2.1. A further complication arises if the densities of the solid and liquid phases are different. Consider a frozen soil system, saturated with water, as shown in Figure 2.30. The initial temperature is T_o and the surface temperature suddenly drops to T_s, which is below the fusion temperature. As thaw progresses, pore water will be forced out of the thawed material due to the thaw strain, which proceeds such that the interface between the pore water and the thawed region moves at the velocity

$$u_2 = A' \frac{dX_3}{dt} \tag{2.228}$$

where A' is soil strain due to thawing and X_3 total depth of thawed material. A velocity is induced in the pore water by the density difference of ice and water. The water surface moves at the velocity

$$u_1 = (1 - \rho_{iw}) \frac{dX_3}{dt} \tag{2.229}$$

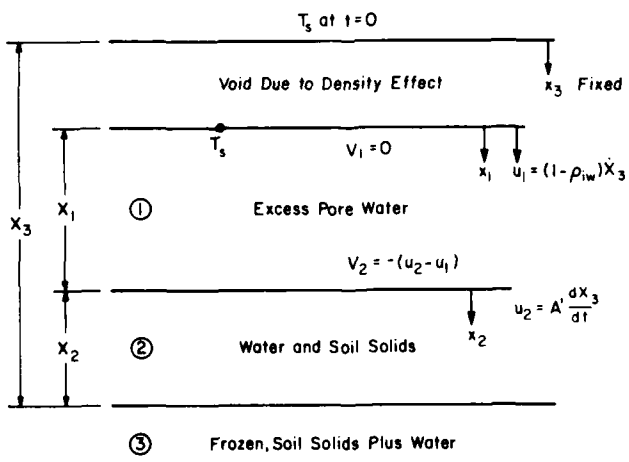


Figure 2.30. Geometry for thaw settlement.

where ρ_{iw} is the ratio of density of ice to density of water. Three coordinate frames will be used:

1. x_3 - stationary coordinate frame at original surface of frozen soil.
2. x_1 - moving frame attached to the upper surface of the excess pore water layer.
3. x_2 - moving coordinate fixed to the moving interface between the water and the thawed material; the soil solids are stationary relative to x_2.

The basic equations for the pore water T_1, the thawed compacted soil T_2, and the frozen unconsolidated soil T_3, are given below:

$$\frac{\partial^2 T_1}{\partial x_1^2} = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} \qquad 0 \leq x_1 \leq X_1 \tag{2.230}$$

$$T_1(0,t) = T_s$$

$$k_1 \frac{\partial T_1}{\partial x_1}(X_1,t) = k_2 \frac{\partial T_2}{\partial x_2}(0,t)$$

$$T_1(X_1,t) = T_2(0,t)$$

$$\alpha_2 \frac{\partial^2 T_2}{\partial x_2^2} - \frac{\rho_w c_w}{\rho_2 c_2} v_2 \frac{\partial T_2}{\partial x_2} = \frac{\partial T_2}{\partial t} \qquad 0 \leq x_2 \leq X_2 \quad . \tag{2.231}$$

The pore water velocity, relative to frame x_2, is

$$v_2 = -(u_2 - u_1) = -A\frac{dX_3}{dt} \tag{2.232}$$

where $A = A' - 1 + \rho_{iw}$.

A similarity solution for this region will only work when the thaw is proportional to the square root of time:

$$T_2\ (X_2,t) = T_f$$

$$-k_2 \frac{\partial T_2}{\partial x_2}(X_2,t) + k_3 \frac{\partial T_3}{\partial x_3}(X_3,t) = \rho_{3\ell}\frac{dX_3}{dt} \tag{2.233}$$

$$\frac{\partial^2 T_3}{\partial x_3^2} = \frac{1}{\alpha_3}\frac{\partial T_3}{\partial x_3} \qquad X_3 \leq x_3 \leq \infty \tag{2.234}$$

$$T_3(X_3,t) = T_f$$

$$T_3(x_3,0) = T_o$$

$$\lim_{x_3 \to \infty} T_3(x_3,t) = T_o \quad .$$

The following relations will be useful:

$$X_1 = AX_3$$

$$X_2 = (1 - A')X_3$$

The solution to the set of equations follows from a similarity transformation:

$$T_1 = T_s + B_1 \operatorname{erf} \frac{x_1}{2\sqrt{\alpha_1 t}} \tag{2.235}$$

$$T_2 = A_2 + B_2 \operatorname{erf} \left(\frac{x_2}{2\sqrt{\alpha_2 t}} + K\gamma\right) \tag{2.236}$$

$$T_3 = T_o + \frac{(T_f - T_o) \operatorname{erfc} \left(\frac{x_3}{2\sqrt{\alpha_3 t}}\right)}{\operatorname{erfc} (\gamma\sqrt{\alpha_{23}})} \tag{2.237}$$

$$X_3 = 2\gamma \sqrt{\alpha_2 t}$$

$$A_2 = T_s + p_3 (T_f - T_s)$$

$$B_2 = \frac{T_f - T_s}{p_1 + p_2}$$

$$B_1 = p_4 (T_f - T_s)$$

$$p_1 = k_{21}\sqrt{\alpha_{12}} \; e^{\gamma^2(A^2\alpha_{21} - K^2)} \operatorname{erf} (A\,\gamma\sqrt{\alpha_{21}}) - \operatorname{erf} K\gamma$$

$$p_2 = \text{erf}\ [\gamma(K + 1 - A')]$$

$$p_3 = \frac{p_1}{p_1 + p_2}$$

$$p_4 = \frac{k_{21}\sqrt{\alpha_{12}}\ e^{\gamma^2(A^2\alpha_{21} - K^2)}}{p_1 + p_2}$$

$$\frac{e^{-\gamma^2(K + 1 - A')^2}}{p_1 + p_2} - \frac{k_{32}\sqrt{\alpha_{23}}\ \phi\ e^{-\gamma^2\alpha_{23}}}{\text{erfc}\ (\gamma\sqrt{\alpha_{23}})} = \frac{\sqrt{\pi}\ \gamma\ \rho_{32}}{S_T} \tag{2.238}$$

where

$$K = \frac{\rho_w c_w}{\rho_2 c_2} A \quad .$$

Nixon (1975) examined the case of a soil initially at the fusion temperature in which the excess pore water vanished instantaneously as it was formed. However, in that study, the interface boundary condition was applied inappropriately, leading to erroneous results. Physically, the vanishing of the water layer is equivalent to an infinite thermal diffusivity for region 1. The equations then reduce to

$$T_2 = T_s + \frac{(T_f - T_s)\left[\text{erf}\ \left(\frac{x_2}{2\sqrt{\alpha_2 t}} + K\gamma\right) - \text{erf}\ K\gamma\right]}{\text{erf}\ [\gamma(K + 1 - A')] - \text{erf}\ K\gamma} \tag{2.239}$$

$$\frac{e^{-\gamma^2(K + 1 - A')^2}}{\text{erf}\ [\gamma(K + 1 - A')] - \text{erf}\ K\gamma} - \frac{k_{32}\sqrt{23}\ \phi\ e^{-\gamma^2\alpha_{23}}}{\text{erfc}\ \gamma\sqrt{\alpha_{23}}} = \frac{\sqrt{\pi}\ \gamma\ \rho_{32}}{S_T} \quad . \tag{2.240}$$

The equation for the frozen zone does not change.

If $A' = 0$ and $\rho_{iw} = 1.0$, the above equations reduce to the familiar Neumann problem.

Table 2.8 lists some calculated values for the thaw parameter. The thaw depth is controlled by the thermal resistance of the thawed layer and the excess water rather than the convection. As the soil consolidates, the conductive resistance to heat flow decreases and this effect is augmented if the excess water layer is neglected. With no water layer, the thaw rate

Table 2.8. Effect of convection and conductive resistance on thaw of strained soil, $S_T = 1.0$, $\phi = 0.0$.

A'	ρ_{iw}	K	Excess water layer	γ	γ (Nixon 1975)	γ (Neumann Solution)
0.35	1.0	0.5	Absent	0.7384	0.5771	0.6203
0.35	1.0	0.5	Present	0.6078	---	0.6203
0.35	0.92	0.386	Present	0.6413	---	0.6203
0.50	1.0	0.714	Absent	0.8276	0.5618	0.6203
0.75	1.0	1.071	Absent	1.4119	0.5386	0.6203
0.75	0.92	0.957	Present	0.6388	---	0.6203
0.90	1.0	1.286	Absent	1.7821	0.5248	0.6203
0.90	0.92	1.171	Present	0.6450	---	0.6203
1.0	1.0	1.429	Absent	---	0.5152	0.6203

Property ratios are unity except as noted.

exceeds the Neumann case by 18%. For the same case with a water layer, the thaw lags the Neumann case by 2%. This shows the actual minor effect of the convection. With the Nixon (1975) result the thaw lags the Neumann case by 7%. This is unreasonable since the thermal resistance of the system is less than that for the Neumann problem, and thus the thaw should progress at a more rapid rate.

The conclusion of Nixon (1975) that convection has little effect is correct. Of course the convection for this problem is minor and should not be confused with convective flows which are independent of thaw strains or density differences.

Figures 2.31-2.34 are solutions for typical soil systems where x_1 is the volumetric water fraction of the media.

2.8 Freeze of a Flowing Fluid

The freezing of a river with convection from the water to the ice or the solidification of a fluid flowing in contact with a cold wall at constant temperature can be estimated as follows. As shown in Figure 2.35 the fluid contacts a cold surface at constant temperature T_s. The surface could be a thin wall with zero heat capacity. For a wall of finite capacity a solution is given in the next chapter. Energy flows from a warm fluid at constant temperature T_i and with a constant convective coefficient h. The problem is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{2.241}$$

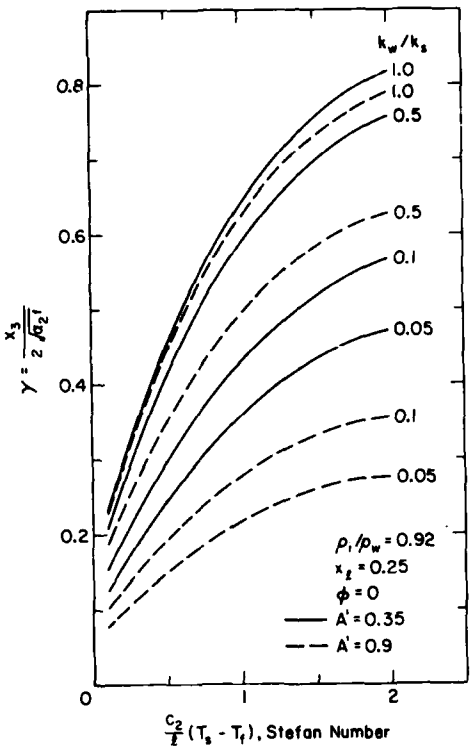


Figure 2.31. Phase change parameter for thaw settlement, x_ℓ = 0.25.

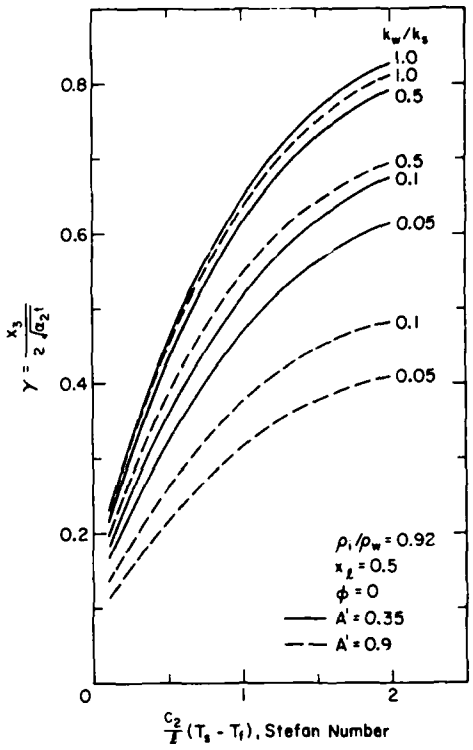


Figure 2.32. Phase change parameter for thaw settlement, x_ℓ = 0.5.

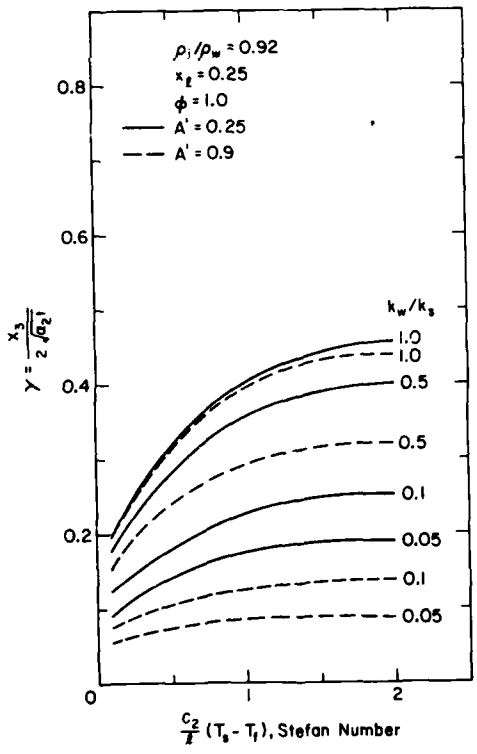


Figure 2.31. Phase change parameter for thaw settlement, x_ℓ = 0.25; ϕ = 1.0.

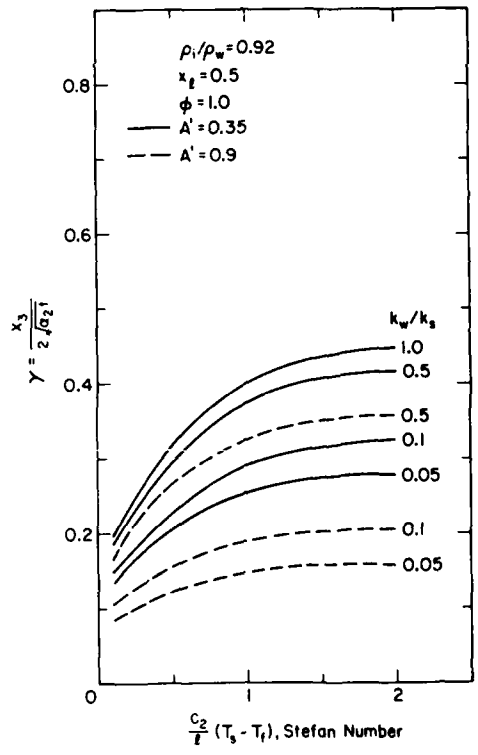


Figure 2.34. Phase change parameter for thaw settlement, x_ℓ = 0.5; ϕ = 1.0.

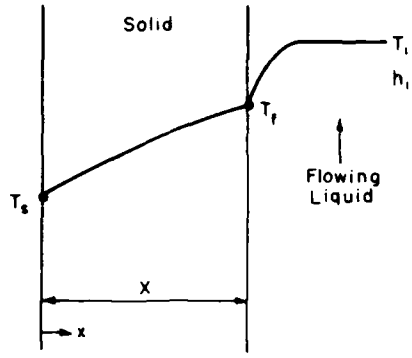


Figure 2.35. Freezing with convective heat flow from liquid region.

$$\rho \ell \frac{dX}{dt} = k \frac{\partial T(X,t)}{\partial x} - h_l \ (T_l - T_f) \tag{2.242}$$

$$T(0,t) = T_s \tag{2.243}$$

$$T(X,t) = T_f \ . \tag{2.244}$$

The steady state solution is

$$X_s = \frac{k}{h_l} \left(\frac{T_f - T_s}{T_l - T_f}\right) \ . \tag{2.245}$$

The equation can be nondimensionalized as

$$\frac{\partial^2 U}{\partial \zeta^2} = S_T \frac{\partial U}{\partial \tau'} \tag{2.246}$$

$$\frac{dS}{d\tau'} = \frac{\partial U(S,\tau')}{\partial \zeta} - 1 \tag{2.247}$$

$$U(0,\tau') = 0 \tag{2.248}$$

$$U(S,\tau') = 1 \tag{2.249}$$

where

$$U = \frac{T - T_s}{T_f - T_s} \qquad \tau' = \frac{h_l (T_l - T_f) t}{\rho \ell \ X_s} \qquad S = \frac{X}{X_s} \qquad \zeta = \frac{x}{X_s} \ .$$

A perturbation method will be used with an expansion of the rate of change of the phase-change interface; thus the time variable will be replaced by S. Then eq 2.246 becomes

$$\frac{\partial^2 U(\zeta,S)}{\partial \zeta^2} = S_T g \frac{\partial U(\zeta,S)}{\partial S} \tag{2.250}$$

$$g = \frac{dS}{d\tau'} \quad . \tag{2.251}$$

Now the temperature and the phase change interface speed are expanded in asymptotic series with the Stefan number as a parameter:

$$U(\zeta,S) = \sum_{i=0}^{N} S_T^{\,i}\, U_i\,(\zeta,S) \tag{2.252}$$

$$g(\tau') = \sum_{i=0}^{N} S_T^{\,i}\, g_i(\tau') \quad . \tag{2.253}$$

Substituting these equations into eq 2.247-2.250 yields the following sets of equations:

$$\frac{\partial^2 U_i}{\partial \zeta^2} = \begin{cases} 0 & i = 0 \\ \sum_{j=1}^{i} g_{j-1} \dfrac{\partial U_{i-j}}{\partial S} & i > 0 \end{cases} \tag{2.254}$$

$$g_i = \begin{cases} \dfrac{\partial U_i(S,S)}{\partial \zeta} - 1 & i = 0 \\ \dfrac{\partial U_i(S)}{\partial \zeta} & i > 0 \end{cases} \tag{2.255}$$

$$U_i(0,S) = 0 \tag{2.256}$$

$$U_i(S,S) = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \quad . \end{cases} \tag{2.257}$$

The phase change interface can be found by inverting eq 2.253 and integrating term by term to obtain

$$\tau' = \sum_{i=0}^{N} S_T^{\,i}\, \tau_i \quad . \tag{2.258}$$

The relations for τ_i as functions of g_i are given in Appendix A. The solutions for the first few cases are as follows:

$$U_o = \frac{\zeta}{S} \tag{2.259a}$$

$$g_o = \frac{1-S}{S} \tag{2.259b}$$

$$\tau_o = -\,[S + \ln(1-S)] \tag{2.259c}$$

$$U_1 = \frac{-(1-S)}{6S^3} (\zeta^2 - S^2)\zeta \tag{2.260a}$$

$$g_1 = -\frac{1}{3} g_o \tag{2.260b}$$

$$\tau_1 = \frac{1}{3} \tau_o \tag{2.260c}$$

$$U_2 = \frac{g_o}{12S^2} \left[\left(\frac{3-2S}{S^2}\right) \frac{\zeta^5}{10} + \frac{\zeta^3}{3} - \frac{S^2}{5} \left(\frac{19}{6} - S\right)\zeta \right] \tag{2.261a}$$

$$g_2 = \frac{g_o}{45} (7-3S) \tag{2.261b}$$

$$\tau_2 = -\frac{1}{90} \left[3S^2 + 2S + 2\ln (1-S) \right] \tag{2.261c}$$

$$\tau_3 = -\frac{1}{1890} \left[8S^3 - 21S^2 - 10S + 10 \ln (1-S) \right] . \tag{2.262}$$

Seeniraj and Bose (1982) developed this method and noted that the interface position S compared very well to analytical results of Savino and Siegel (1969) and numerical results of Beaubouef and Chapman (1967).

Lapadula and Mueller (1966) used Biot's method to obtain the following result:

$$\tau' = -\frac{2 S_T^2 + 10 S_T + 15}{5(3 + S_T)} [S + \ln (1-S)] . \tag{2.263}$$

This agrees well with the perturbation solution if $S_T < 1.0$.

NOMENCLATURE - CHAPTER 2

A'	thaw strain of soil
b	δ/x
B	$\delta/x-1$
c	specific heat
C	ρc, heat capacity or volumetric specific heat
c_{12}	c_1/c_2
D	thickness of slab
E_ℓ	energy liberated during solidification
f_e	solid fraction at eutectic
f_s	solid fraction

g	$dS/d\tau$, dimensionless phase change interface rate of movement
G	temperature gradient
h_i	surface coefficient of convection
I_f, I_t	index for freeze or thaw
I_s	surface index
k	thermal conductivity
k_{i2}	k_i/k_2
K	$\frac{\rho_w c_w}{\rho_2 c_2}(A' - 1 + \rho_{iw})$
ℓ	latent heat of fusion
L	$\rho\ell$, volumetric latent heat
m	mass
n	ratio of surface to air index
q	heat flux rate
p,q,r	property ratios
p	period
R	k_i/k_w
S	X/X_s
S_T	$\frac{c_1}{\ell}(T_f - T_s)$ freeze $\frac{c_1}{\ell}(T_s - T_f)$ thaw
S_{Tm}	$\frac{C_3 (T_f - T_s)}{\gamma_d \ell \Delta \xi}$
t	time
T	temperature
T_c	initial temperature of cold wall
T_f, T_o, T_s	fusion, initial, and surface temperature
T_{fs}	solidus temperature
T_m	lowest temperature for freeze, mushy zone
T_w	wall temperature, finite slab
T_i	temperature of flowing fluid
U	$\frac{T - T_s}{T_f - T_s}$ dimensionless temperature
v	pore water velocity

V	volume
x	Cartesian coordinate
x_i	volumetric fractions of soil solids (s), liquids (ℓ), gases (g)
X	phase-change depth
X_s	steady state phase-change depth
α	thermal diffusivity
α_{12}	α_1/α_2
β	$\delta/(2\sqrt{\alpha t})$ thermal penetration depth parameter
γ	$X/(2\sqrt{\alpha t})$ phase-change depth parameter
δ	thermal penetration depth
ΔT	$(T_o - T_f)$
ΔT_m	$T_f - T_m$
ΔT_f	$T_f - T_{fs}$
ΔX	$X - X_1$
ε	length of season
ζ	x/X_s
η	$x/(2\sqrt{\alpha t})$
η_1	x/X
θ	integrated temperature or temperature
λ	phase-change parameter $= \gamma\sqrt{2/S_T}$
μ	$S_T/2$
ξ	ratio of unfrozen water to soil solid mass
ξ_o, ξ_f, ξ_s	values of ξ at T_f, T_m, T_s
ρ	density
ρ_{12}	ρ_1/ρ_2
σ	$G\ X/\Delta T$
τ	$\alpha_1 \left(\frac{G}{\Delta T}\right)^2 t$, dimensionless time
τ'	$\dfrac{h_1(T_1 - T_f)t}{\rho \ell X_s}$
τ^*	$\dfrac{2\pi t}{P}$
ϕ	$(T_o - T_f)/(T_f - T_s)$ superheat parameter

ϕ_1	$(T_f-T_s)/\Delta T_m$
ϕ_2	$T_f/\Delta T_m$
ϕ_m	$(T_o-T_f)/\Delta T_m$
ψ	$X_1/(2\sqrt{\alpha_1 t})$ phase-change depth parameter

Subscripts

f	frozen
g	gas
i	ice
ℓ	liquid
s	solid
t	thawed
w	water or wall
1,2,3	regions of material
∞	steady-state

CHAPTER 3. PLANE PROBLEMS WITH CONVECTION (RADIATION) AT FREE SURFACE

The thermal boundary condition at the free surface of a body can be specified in terms of temperature or energy flow. The energy flow can further be characterized as a specified heat flux or a heat flow dependent upon the ambient conditions and the surface temperature itself. The most widely encountered boundary condition for a solid is that for which energy flows between the solid and an ambient fluid. The heat flux at the surface can be specified as

$$-k \frac{\partial T(0,t)}{\partial x_1} = h\ [T_a(t) - T(0,t)] \tag{3.1}$$

where h (see Nomenclature the end of this chapter) is the surface conductance, which is not a thermophysical property of the ambient fluid since it depends upon the state of the fluid motion. Surface conductance is usually specified or obtained from suitable correlations.

The surface boundary condition often includes radiation effects. Consider a wall, in contact with a fluid, which exchanges both radiation and a convective heat transfer with the surroundings. The surface energy balance can be written as

$$-k \frac{\partial T(0,t)}{\partial x_1} = h_c[T_a - T(0,t)] + \sigma'\varepsilon'F(T_a^4 - T(0,t)^4) \tag{3.2}$$

where

σ' = Stefan-Boltzmann constant

ε' = emissivity

F = radiation shape factor

h_c = surface coefficient or film conductance, for convection.

If the ambient temperature is not too different than the surface temperature, then eq 3.2 will reduce to eq 3.1 where

$$h = h_c + 4\ \sigma'\varepsilon'F\ T_a^3\ . \tag{3.3}$$

Therefore the nonlinear boundary condition given by eq 3.2 can be linearized and combined with the usual linear, convective condition. Thus the boundary condition of eq 3.1 is often referred to as a "radiation boundary condition." Radiation does not exist in purely convective problems and h is simply h_c.

When h is very large, eq 3.1 reduces to

$$T(0,t) \equiv T_a$$

which is the type of boundary condition discussed in Chapter 2.

Finally if h $[T_a - T(0,t)]$ is a specified function of time, then eq 3.1 reduces to that for a specified surface heat flux. This type of boundary condition will be considered in Chapter 4.

3.1 SINGLE PHASE PROBLEMS

Problems with only a single phase (solid or liquid) that experiences temperature changes are much simpler mathematically than two-phase situations.

3.1.1 Exact Solution for Semi-infinite Medium

Consider a semi-infinite solid, initially at the fusion temperature, which starts to melt due to convection or a heat flux imposed on the free surface at time zero. If we refer to Figure 3.1, the problem can be formulated as

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial\tau} \qquad (3.4)$$

$$\theta(\sigma,\tau) = 0 \qquad (3.4a)$$

$$\frac{\partial\theta}{\partial x}(\sigma,\tau) = -\frac{1}{S_{Te}}\frac{d\sigma}{d\tau} \qquad (3.4b)$$

$$\frac{\partial\theta}{\partial x}(0,\tau) = -1 + \varepsilon\,\theta(0,\tau) \qquad (3.4c)$$

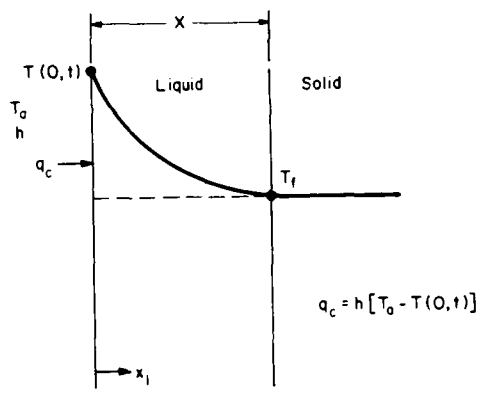


Figure 3.1. Melt of a semi-infinite medium with surface convection.

where

$$\theta = \frac{T - T_f}{T_a - T_f} \qquad x = \frac{x_1}{d} \qquad \sigma = \frac{X}{d} \qquad d = \frac{k}{h}$$

$$\tau = \frac{\alpha t}{d^2} \qquad S_{Te} = \frac{c(T_a - T_f)}{\ell} = \text{Stefan number.}$$

Note that when $\varepsilon = 0$, a constant surface heat flux exists:

$$-k\,\frac{\partial T(0,t)}{\partial x_1} = (T_a - T_f)\,h = \text{constant.}$$

If $\varepsilon = 1$, the general convective heat transfer boundary condition applies:

$$-k \frac{\partial T(0,t)}{\partial x_1} = h[T_a - T(0,t)] \ .$$

Lozano and Reemsten (1981) show that the following equation is a solution to eq 3.4:

$$\theta = \frac{1}{S_{Te}} \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n [x - \sigma]^{2n}}{\partial \tau^n} \ . \tag{3.5}$$

The convective boundary condition is used to find

$$\sigma = \sum_{p=1}^{\infty} \frac{S^{(p)}}{p!} \tau^p \tag{3.6}$$

$$S^{(p)} = -\sum_{n=1}^{p-1} \frac{1}{(2n+1)!} \sum_{\alpha \varepsilon \ \Gamma(2n+1,n+p)} [\alpha] \ S^{(\alpha)}$$

$$+ \frac{\beta}{2n!} \sum_{\alpha \varepsilon \ \Gamma(2n,n-1+p)} [\alpha] \ S^{(\alpha)} \ . \tag{3.7}$$

For specific cases, eq 3.5-3.7 can be quantitatively evaluated, but the numerical work is at best tedious.

For a constant surface heat flux ($\varepsilon = 0$) and $S_{Te} = 0.2$, Lozano and Reemsten (1981) obtained the following expression for the first four terms of the interface position:

$$\sigma = 0.2\tau - 0.004\tau^2 + 0.0002667\tau^3 - 0.272 \times 10^{-4}\tau^4 \tag{3.8}$$

The location of the interface vs time, using the first 50 terms of eq 3.6, is given in Table 3.1. Equation 3.8 is compared to other constant heat flux solutions in the next chapter:

In general

$$\sigma = S_{Te} \ \tau \left[1 - \frac{1}{2} S_{Te}^{\ 2} \ \tau + \frac{5}{6} S_{Te}^{\ 4} \ \tau^2 - \frac{17}{8} S_{Te}^{\ 6} \ \tau^3 + \frac{827}{120} S_{Te}^{\ 8} \ \tau^4 + \ . \ . \right] \ . \tag{3.8a}$$

The solution presented here is exact, but it is perhaps best used to check approximate methods, due to the time-consuming

Table 3.1 Position of the solid/liquid interface, constant heat flux, exact solution, ($\varepsilon = 0$), $S_{Te} = 0.2$.

τ	$\sigma_{50}(\tau)$
0.0	0
0.1	0.019960 263981
0.2	0.039842 090909
0.3	0.059646 987880
0.4	0.079376 404411
0.5	0.099031 735838
0.6	0.118614 326441
0.8	0.157566 424063
1.0	0.196242 534396
1.2	0.234651 840602
1.5	0.291784 196041
1.8	0.348362 261688
2.0	0.385783 461461

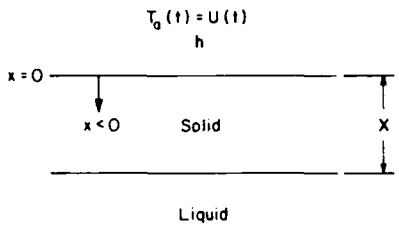


Figure 3.2. Freezing of a medium with convection and initial temperature at fusion value.

numerical reduction necessary. We shall use the results of this section to compare approximate solutions.

Westphal (1967) considered freezing of a semi-infinite medium with convection, letting the ambient temperature be a function of time, as shown in Figure 3.2. He obtained an exact solution using infinite series. The problem can be expressed as

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.9}$$

$$T(X,t) = T_f \tag{3.10}$$

$$k \frac{\partial T}{\partial x} = \rho \ell \frac{dX}{dt} \; ; \; x = X \tag{3.11}$$

$$k \frac{\partial T}{\partial x} = h[T - T_a(t)]; \qquad x = 0 \tag{3.12}$$

$$X(0) = 0. \tag{3.13}$$

A formal solution to eq 3.9, attributed to Portnov (1962), which does not satisfy any boundary conditions is

$$T(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp\left[-\gamma^2 \left(\frac{x}{X} - y\right)^2\right] \gamma \, \Phi_1(Xy) \, dy$$

$$+ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} \exp\left[-\gamma^2 \left(\frac{x}{X} - y\right)^2\right] \gamma \, \Phi_2(Xy) \, dy \tag{3.14}$$

where

$$\mu = 2\sqrt{\alpha t}$$

$$\gamma = \frac{X}{\mu} \quad .$$

The functions Φ_1 Φ_2 X are found from eq 3.10-3 13, expanding the functions in infinite series

$$\Phi_1(\omega) = \sum_0^\infty \phi_{1n}\,\omega^n \tag{3.15a}$$

$$\Phi_2(\omega) = \sum_0^\infty \phi_{2n}\omega^n \tag{3.15b}$$

$$X(\mu) = \sum_1^\infty X_n\,\mu^n \tag{3.15c}$$

$$U(\mu) = \sum_0^\infty U_0\mu^n \quad . \tag{3.15d}$$

Three equations are found by substituting eq 3.14 into eq 3.10-3.12:

$$\sqrt{\pi}\,\mu hU = 2k\left[\int_0^\infty \beta\,\exp(-\beta)^2\,\Phi_1\,(\beta\mu)\,d\beta + \int_{-\infty}^0 \beta\,\exp\,(-\beta)^2\,\Phi_2\,(\beta\mu)\,d\beta\right]$$

$$+\,h\mu\left[\int_0^\infty \exp\,(-\beta)^2\,\Phi_1\,(\beta\mu)\,d\beta + \int_{-\infty}^0 \exp\,(-\beta)^2\,\Phi_2\,(\beta\mu)\,d\beta\right] \tag{3.16}$$

$$T_f = \frac{1}{\sqrt{\pi}}\left[\int_{-\mu}^\infty \exp\,(-\beta)^2\,\Phi_1\,(\beta\mu + X)\,d\beta\right.$$

$$\left. + \int_{-\infty}^{-\mu} \exp\,(-\beta)^2\,\Phi_2\,(\beta\mu + X)\,d\beta\right] \tag{3.17}$$

$$2\,\alpha\rho\ell\,\frac{dX}{d\mu} = \frac{2k}{\sqrt{\pi}}\left[\int_{-\mu}^\infty \beta\,\exp\,(-\beta)^2\,\Phi_1\,(\beta\mu + X)\,d\beta\right.$$

$$\left. + \int_{-\infty}^{-\mu} \beta\,\exp\,(-\beta)^2\,\Phi_2\,(\beta\mu + X)\,d\beta\right] \tag{3.18}$$

where

$\beta = -\gamma(1 - y)$

Substituting eq 3.15 into eq 3.16-3.18, after differentiating and letting $\mu \to 0$, leads to a set of equations for the coefficients ϕ_{1n}, ϕ_{2n}, X_n.

Westphal (1967) gives a solution for the case when the ambient temperature is constant and water is the medium. Table 3.2 shows the

Table 3.2. Phase-change depth vs time, convection with constant ambient temperature, $S_{Te} = 2/35$.

$\sigma = \frac{h}{k} X$	Time ($\mu \frac{h}{k}$) Exact (Westphal 1967)	Quasi-steady (eq 3.24)	Heat balance integral (Eq 3.31)
0.003565	0.5	0.500	0.501
0.01418	1.0	1.000	1.002
0.03161	1.5	1.499	1.503
0.05562	2.0	2.000	2.004
0.08543	2.5	2.497	2.505
0.1208	3.0	2.994	3.005
0.1611	3.5	3.491	3.504
0.2056	4.0	3.984	4.004
0.2534	4.5	4.471	4.503
0.3028	5.0	4.940	5.002

interface vs time for $(T_f - T_a) = 10°C$, $S_{Te} = c(T_f - T_a)/\ell = 2/35$. Also shown are approximate solutions which will be discussed later.

The results shown in Table 3.2 are in close agreement (less than 0.7% difference) with the solution given by eq 3.8 for small values of time. This is expected since the Stefan number is small and the surface heat flux is relatively constant at early times.

The solution method is closely allied to the concept used by Tao (1978) to solve the extended Neumann problem (see Section 2.3.1).

3.1.2 Analog Solution

There are no exact solutions for this problem when the initial temperature is different than the fusion temperature and the ambient temperature can vary. Kreith and Romie (1955) used an electrical analog to obtain a solution, for the freeze case, when the initial temperature was at the fusion value, which is the problem described in Section 3.1. At the surface of the solid, the conduction will equal the convection heat transfer:

$$k_1 \frac{\partial T_1}{\partial x} = h(T - T_a), \qquad x = 0 . \tag{3.19}$$

The analog solutions for the surface temperature and the depth of freeze are given in Figures 3.3 and 3.4. These figures can also be used

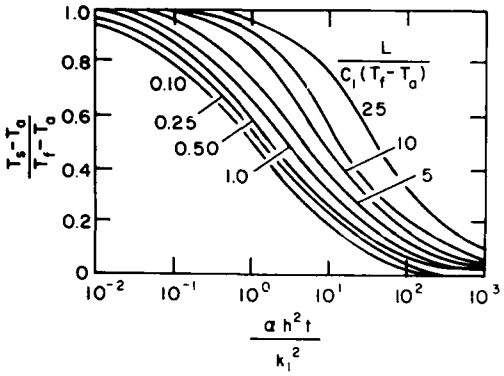


Figure 3.3. Surface temperature during solidification of a semi-infinite medium with convection boundary condition at the surface (from Kreith and Romie 1955).

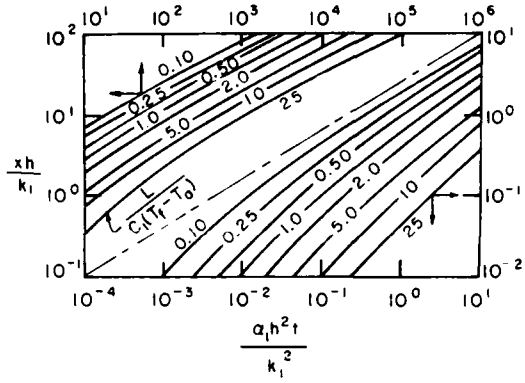


Figure 3.4. Depth of solidification for semi-infinite region subjected to convection at the surface (from Kreith and Romie 1955).

for melting if the properties of the thawed material are used and if the latent heat is taken as negative so that $\ell/c_l\ (T_f - T_a)$ will be positive.

The depth of freeze for this problem agrees within 10% with the Neumann solution ($T_o = T_f$, $T_s = T_a$) if X h/k is greater than one. The heat flow to the surface "sees" two thermal resistances: a conductive resistance that increases with time as the frozen layer grows and a convective resistance that is constant. Thus, after a certain time, the relative effect of the surface resistance is approaching zero and the solidification proceeds essentially as in the constant surface temperature case.

3.1.3 Quasi-Steady Approximation

The freeze problem of interest here reduces to the following for the zeroth quasi-steady approximation:

$$\frac{\partial^2 T_l}{\partial x^2} = 0 \tag{3.20}$$

$$T_l(X,t) = T_f \tag{3.20a}$$

$$T_l(x,0) = T_f \tag{3.20b}$$

$$X(0) = 0 \tag{3.20c}$$

$$k_l \frac{\partial T_l}{\partial x} = \rho_l \ell \frac{dX}{dt}\ ; \qquad x = X \tag{3.20d}$$

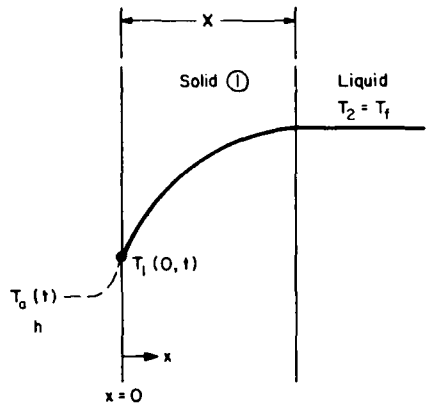


Figure 3.5. Freeze of a semi-infinite medium with convection and one phase.

$$k_1 \frac{\partial T_1}{\partial x} = h[T - T_a(t)] \; ; \qquad x = 0 \quad . \tag{3.20e}$$

The liquid region remains at the fusion temperature, as noted in Figure 3.5. The solution to eq 3.20 and 3.20e is

$$T = a(x + \frac{k_1}{h}) + T_a(t) \; . \tag{3.21}$$

Then from eq 3.20a and 3.20d

$$\frac{dX}{dt} = \frac{k_1}{\rho_1 \ell} \frac{(T_f - T_a)}{(X + \frac{k_1}{h})} \quad . \tag{3.22}$$

The solution to this equation which satisfies eq 3.20c is

$$p_1^2 = (X + \frac{k_1}{h})^2 = \frac{2k_1}{\rho_1 \ell} \int_0^t [T_f - T_a(t)]\, dt + (\frac{k_1}{h})^2 \; . \tag{3.23}$$

If the surface coefficient h is a function of time, then the equation for the phase change interface is

$$\frac{dp_1^2}{dt} = \frac{2k_1}{\rho_1 \ell} (T_f - T_a) - 2p_1 \frac{k_1}{h^2} \frac{dh}{dt}$$

Unfortunately, this equation cannot be simply integrated. Equation 3.23 was first reported by London and Seban (1943), assuming that $T_a(t)$ was constant. The equation was given in its present form by Foss and Fan (1972) and was used for surface temperature calculations during freezing and thawing by Lunardini (1978a, 1978b).

If T_a is constant, then eq 3.23 is

$$(X + \frac{k_1}{h})^2 = \frac{2k_1}{\rho_1 \ell}(T_f - T_a)t + (\frac{k_1}{h})^2 = 2\alpha_1 S_{Te} t + (\frac{k_1}{h})^2 \tag{3.24}$$

and the surface temperature (valid also for $T_a = T_a(t)$) is

$$T_s = T_a + \frac{T_f - T_a}{p_1} \frac{k_1}{h} \tag{3.25}$$

where

$$S_{Te} = \frac{c_1(T_f - T_a)}{\ell} \quad .$$

Equation 3.20b cannot be strictly satisfied since there is no time coordinate in the energy equation. However, eq 3.25 does give $T_s(0) = T_f$ and the temperature in region 1 is only valid for $t > 0$. Equations 3.24 and 3.25 agree very well with Figures 3.3 and 3.4, the analog solutions of Kreith and Romie (1955), particularly for small Stefan numbers as is expected. Comparison with the exact solution of Table 3.2 confirms the accuracy of this approximation for small Stefan numbers.

3.1.4 Heat Balance Integral Approximation

Goodman (1958) has solved this problem using the integral method. The labor involved in the integral method tends to far exceed that of the quasi-steady approximation. The equations for the freeze problem are

$$\frac{d\theta}{dt} - T_f \frac{dX}{dt} = \alpha_1 [\frac{\rho_1 \ell}{k_1} \frac{dX}{dt} - \frac{\partial T(0,t)}{\partial x}] \tag{3.26}$$

$$\theta = \int_0^X T(x,t)dx \tag{3.27}$$

$$T(X,t) = T_f \tag{3.28}$$

$$k_1 \frac{\partial T(0,t)}{\partial x} = h[T(0,t) - T_a] \tag{3.29}$$

$$k_1 [\frac{\partial T(X,t)}{\partial x}]^2 = \rho_1 \ell \alpha_1 \frac{\partial^2 T_1(X,t)}{\partial x^2} \quad . \tag{3.30}$$

The boundary conditions are used to find the coefficients of the quadratic temperature approximation:

$$T = T_f + a(x-X) + b(x-X)^2$$

$$a = \frac{-(1+\sigma) + \sqrt{(1+\sigma)^2 + (\beta-1)\,\sigma(\sigma+2)}}{\frac{k}{2h\Delta T}\,(\beta-1)\,\sigma(\sigma+2)}$$

$$b = -\frac{1}{4}\,\frac{(\beta-1)}{\Delta T}\,a^2$$

where

$$\tau S_{Te} = \frac{h^2(T_f - T_a)t}{k_1\rho_1\ell}$$

$$\Delta T = T_f - T_a$$

$$\sigma = \frac{h}{k_1}\,X$$

$$\beta = 1 + 2\,S_{Te}$$

$$S_{Te}\tau = \frac{1}{12\beta}\Big\{[(1+2\beta) + (2+\beta)\sigma][1 + \beta\sigma(2+\sigma)]^{\frac{1}{2}}$$

$$- \frac{2(\beta-1)}{\sqrt{\beta}}\ln\frac{[1+\beta\sigma(2+\sigma)]^{\frac{1}{2}} + [(1+\sigma)\beta]^{\frac{1}{2}}}{1+\sqrt{\beta}} - 4\beta(\beta-1)\ln\frac{-1+\beta(2+\sigma) + [1+\beta\sigma(2+\sigma)]^{\frac{1}{2}}}{2\beta}$$

$$+ (\beta^2+5\beta)\frac{\sigma^2}{2} + 2(\beta^2+4\beta-2)\sigma - (1+2\beta)\Big\} \quad . \qquad (3.31)$$

The surface temperature is given by

$$\frac{T_f - T(0,t)}{T_f - T_a} = \frac{(\beta-1)\sigma^2 + 2(\beta-2)\sigma - 2 + 2[1 + \beta\sigma(2+\sigma)]^{\frac{1}{2}}}{(\beta-1)(2+\sigma)^2} \qquad (3.32)$$

Equations 3.31 and 3.32 are plotted as Figures 3.6 and 3.7. Comparison of these figures with Figures 3.3 and 3.4 shows that the integral approximation is quite close to the analog solution.

The heat balance integral approximation is compared to the exact solution of Westphal (1967) in Table 3.2 The agreement for this case of low Stefan number (S_{Te} = 2/35) is remarkable.

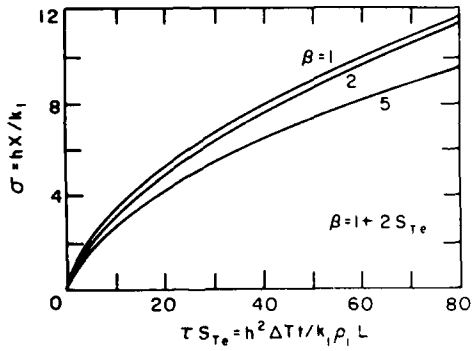


Figure 3.6. Thickness of melt vs time, for aerodynamic heating or radiation boundary condition (eq 3.31).

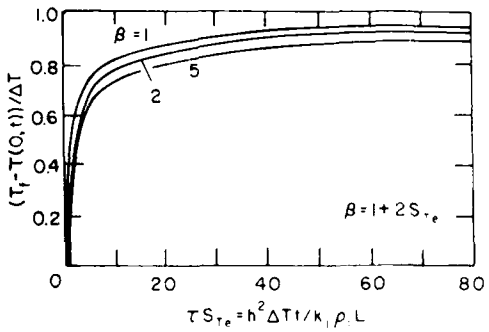


Figure 3.7. Surface temperature vs time, for aerodynamic heating or radiation boundary condition (eq 3.32).

It is interesting to note that if the Stefan number is zero ($\beta = 1$), then the integral solution for the phase-change interface and the surface temperature are

$$S_{Te}\tau = \frac{\sigma^2}{2} + \sigma \tag{3.33}$$

$$\frac{T_f - T(0,t)}{\Delta T} = \frac{\sigma}{1+\sigma} \quad . \tag{3.34}$$

Replacing τ and σ by the usual variables, t and X, leads to

$$(X + \frac{k_1}{h})^2 = \frac{2k_1 \Delta T}{\rho_1 \ell} t + (\frac{k_1}{h})^2$$

$$T(0,t) = T_a + \frac{\Delta T}{(X + \frac{k_1}{h})} \frac{k_1}{h} \quad .$$

These are identical to eq 3 24 and 3.25 of the quasi-steady approximation.

Cho and Sunderland (1981) presented an approximate method of solving this problem for the single phase case. They assumed that the temperature profile was of the same form as for the case of the non-melting problem. Their results agree very well with eq 3.31, but they note that the zero-subcooling solution should be a good approximation to the subcooling problem. This is not true, as will be shown in Section 3.2. The subcooling has a very significant effect upon the rate of phase change and may be ignored only at the risk of serious error.

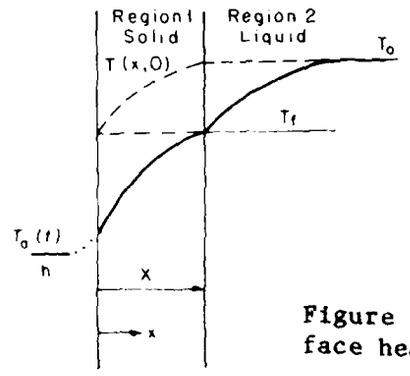


Figure 3.8. Freezing with surface heat flow.

3.1.5 Constant Heat Flux From Liquid Region

A problem with significance in terms of the freezing or melting of ice layers over bodies of water can be formulated if the heat flux from the material changing phase is assumed to be constant

3.1.5.1 Quasi-Steady Solution

Foss and Fan (1974) solved the problem using the quasi-steady method. The initial temperature distribution is not known, but the initial air temperature and the surface temperature of the water are at the freezing temperature (see Fig. 3.8). At t = 0 the air temperature drops below freezing and may then vary with time. The equations are

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t} = 0 \tag{3.35}$$

$$k_1 \frac{\partial T_1(0,t)}{\partial x} = h\,[T_1(0,t) - T_a(t)] \tag{3.35a}$$

$$T_1(0,0) = T_f \tag{3.35b}$$

$$T_a(0) = T_f \tag{3.35c}$$

$$T_1(X,t) = T_f \tag{3.35d}$$

$$k_1 \frac{\partial T_1(X,t)}{\partial x} - k_2 \frac{\partial T_2(X,t)}{\partial x} = \rho_1 \ell \frac{dX}{dt} \quad . \tag{3.35e}$$

At the solid/liquid interface the heat flux from the liquid is assumed to be constant; therefore,

$$k_2 \frac{\partial T_2(X,t)}{\partial x} = q_w \quad . \tag{3.35f}$$

This assumption reduces the problem to a single phase case. The solution to eq 3.35 with boundary conditions (eq 3.35a,d) is

$$T_1 = \frac{(T_f + \frac{h\ T_a X}{k_1})}{1 + \frac{h\ X}{k_1}} (1 + \frac{h\ x}{k_1}) - \frac{h}{k_1} T_a x \quad . \tag{3.36}$$

The differential equation for X is obtained from eq 3.35e using eq 3.36

$$\rho_1 \ell \frac{dX}{dt} = \frac{h\ (T_f - T_a)}{1 + \frac{hX}{k_1}} - q_w \quad . \tag{3.37}$$

The integration of this equation depends upon the functional form of the time variation of the ambient temperature.

3.1.5.2 Constant Ambient Temperature

For a constant ambient temperature eq 3.37 can be easily integrated. The solution is

$$\frac{h\ q_w}{k_1 \rho_1 \ell}\ t = \frac{h\ (T_f - T_a)}{q_w} \ln \left\{ \frac{h\ (T_f - T_a) - q_w}{h\ (T_f - T_a) - q_w(1 + \frac{h\ X}{k_1})} \right\} - \frac{h\ X}{k_1} \tag{3.38}$$

or

$$S_{Te}\ \tau = \frac{1}{q_w{}^{*2}} \ln \left[\frac{1 - q_w{}^*}{1 - q_w{}^*(1+\sigma)} \right] - \frac{\sigma}{q^*{}_w} \tag{3.39}$$

where

$$q_w{}^* = \frac{q_w}{h\ (T_f - T_a)}$$

$$S_{Te} = \frac{c_1 (T_f - T_a)}{\ell}$$

$$\sigma = \frac{h\ X}{k_1}$$

The steady-state value of X can be found by letting dX/dt = 0 in eq 3.37 Then

$$X_{max} = k_1 \left[\frac{T_f - T_a}{q_w} - \frac{1}{h} \right] \tag{3.40}$$

$$\sigma_{max} = \frac{S_{Te}}{q_w{}^*} - 1 \quad . \tag{3.41}$$

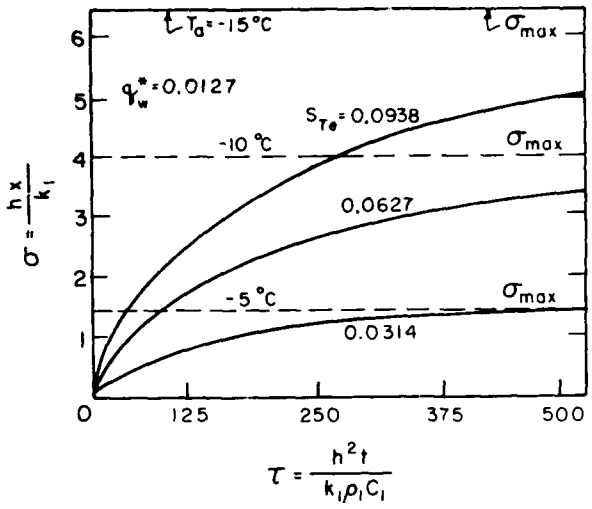


Figure 3.9. Phase change depth for a constant ambient temperature (eq 3.39).

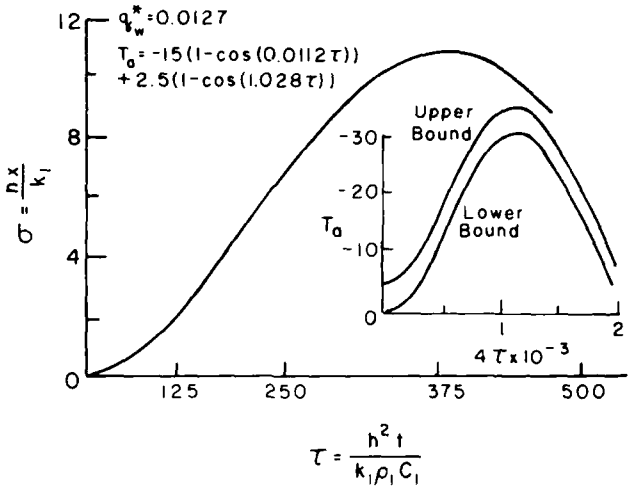


Figure 3.10. Phase change depth for a sinusoidal ambient temperature.

The transient and steady-state values of the phase-change depth are plotted in Figure 3.9 for specific values of q_w^* and T_a.

3.1.5.3 Sinusoidal ambient temperature

If the ambient temperature varies with time, a numerical solution of eq 3.37 will normally be necessary. Foss and Fan (1974) present a solution for a particular yearly sinusoid, with daily fluctuations included, and a particular value of q_w^*, as shown in Figure 3.10.

3.1.5.4 Convection in the Liquid Region, Perturbation Method

The growth of ice at the surface of a river or in latent heat storage systems can be mathematically described by freezing of a liquid flowing over a cold wall or in contact with a cold ambient. The temperature and properties of the fluid and the ambient are constant. The problem (see Fig. 3.11) is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.42}$$

$$\rho \ell \frac{dX}{dt} = k \frac{\partial T}{\partial x}(X,t) - h_l (T_l - T_f) \tag{3.43}$$

$$k \frac{\partial T(0,t)}{\partial x} = h_o [T(0,t) - T_a] \tag{3.44}$$

$$T(X,t) = T_f \tag{3.45}$$

where

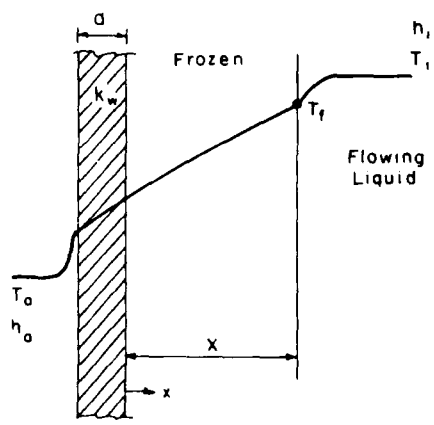


Figure 3.11. Freeze of a fluid flowing over a cold surface.

$$\frac{1}{h_o} = \frac{a}{k_w} + \frac{1}{h} \quad . \tag{3.46}$$

Clearly this problem is identical to that of Section 3.1.5 if $q_w = h_i\ (T_i - T_f)$. Seeniraj and Bose (1982) used a perturbation method to find solutions valid for small Stefan numbers ($S_{Te} < 1.0$).

A steady state solution will occur when the thermal resistance of the ambient equals the combined resistance of the solid layer and the fluid. This is

$$X_s = k\left[\frac{T_f - T_a}{h_i(T_i - T_f)} - \frac{1}{h_o}\right] \quad . \tag{3.47}$$

In nondimensional form, the equations are

$$\frac{\partial^2 u}{\partial \xi^2} = S_{Te}\ \frac{\partial u}{\partial \tau^*} \tag{3.48}$$

$$\frac{dS}{d\tau^*} = \frac{\partial u(S,\tau^*)}{\partial \xi} - \frac{B_i}{B_i+1} \tag{3.49}$$

$$\frac{\partial u(0,\tau^*)}{\partial \xi} = B_i\ u(0,\tau^*) \tag{3.50}$$

$$u(S,\tau^*) = 1 \tag{3.51}$$

where

$$\xi = \frac{x}{X_s} \qquad S = \frac{X}{X_s} \qquad u = \frac{T - T_a}{T_f - T_a} \qquad B_i = \frac{h_o\ X_s}{k} \qquad \tau^* = \frac{k\ (T_f - T_a)t}{\rho \ell\ X_s^2} \quad .$$

The perturbation method uses an expansion of the rate of change of the phase change interface (see App. A). Thus we will replace the time variable in eq 3.48 by S, assuming that S is a monotonic function of τ. Then

$$\frac{\partial^2 u(\xi,S)}{\partial \xi^2} = S_{Te}\ g\ \frac{\partial u(\xi,S)}{\partial S} \tag{3.52}$$

$$g = \frac{dS}{d\tau^*} \quad . \tag{3.53}$$

Now we expand the temperature $u(\xi,S)$ and the speed of the phase change interface g in asymptotic series with the Stefan number as the parameter. This will be valid for $S_{Te} < 1.0$:

$$u(\xi,S) = u_o(\xi,S) + S_{Te}\, u_1(\xi,S) + S_{Te}^{\,2}\, u_2(\xi,S) + \ldots \tag{3.54}$$

$$g = g_o(S) + S_{Te}\, g_1(S) + S_{Te}^{\,2}\, g_2(S) + \ldots \tag{3.55}$$

Substituting eq 3.54 and 3.55 into eq 3.49-52 yields

$$\frac{\partial^2 u_i}{\partial \xi^2} = \begin{cases} 0 & i = 0 \\ \sum_{j=1}^{i} g_{j-1} \dfrac{\partial u_{i-j}}{\partial S} & i > 0 \end{cases} \tag{3.56}$$

$$u_i(S) = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases} \tag{3.57}$$

$$\frac{\partial u_i(0)}{\partial \xi} = B_i\, u_i(0) \tag{3.58}$$

$$g_i = \begin{cases} \dfrac{\partial u_i(S)}{\partial \xi} - \dfrac{B_i}{B_i+1} & i = 0 \\ \dfrac{\partial u_i(S)}{\partial \xi} & i > 0 \end{cases} \quad . \tag{3.59}$$

The phase change interface can be found by inverting eq 3.54 and integrating term by term to obtain

$$\tau^* = \tau_o + \tau_1\, S_{Te} + \tau_2\, S_{Te}^{\,2} + \ldots \tag{3.60}$$

The relations for τ_i as functions of g_i are given in Appendix A.

Using this perturbation technique allows simple mathematical techniques to yield the following solutions:

$$u_o = \frac{B_i\, \xi+1}{1+B_i\, S} \tag{3.61}$$

$$g_o = \frac{B_i^2(1-S)}{(1+B_i)(1+B_i\, S)} \tag{3.62}$$

$$\tau_o = -\left(\frac{B_i+1}{B_i}\right)\left[S + \frac{B_i+1}{B_i} \ln\,(1-S)\right] \tag{3.63}$$

$$u_1 = -\frac{B_i^3(1-S)}{6(1+B_i)(1+B_i S)^4}\left[\xi^2(B_i\xi+3)(1+B_i S)-S^2(B_i S+3)(B_i\xi+1)\right] \tag{3.64}$$

$$g_1 = \frac{-B_i^{\,3}(1-S)S}{3(1+B_i)(1+B_iS)^4}\,[B_i^{\,2}S^2+3B_iS+3]$$

$$\tau_1 = -\frac{1}{3B_i}\Big[S(1+B_i) + \frac{\ln\,(1+B_iS)}{B_i(1+B_i)} + \frac{S}{(1+B_iS)} + \frac{B_i^{\,2}+3B_i+3}{1+B_i}\ln\,(1-S)\Big] \quad (3.66)$$

The surface heat flux is given by

$$\frac{qX_s}{k\,(T_f - T_a)} = B_i \sum_{i=0}^{N} S_{Te}^{\;i}\, u_i(0,S)\ . \quad (3.67)$$

The zeroth solution (i=0) is identical to the quasi-steady solutions of eq 3.36 and 3.38, as expected.

The results for the surface temperature agreed well with the data of Savino and Siegel (1967) for $B_i = 1.35$ and $S_{Te} = 0.232$.

If the surface coefficient h is very large then the problem reduces to that of a cold wall held at a constant surface temperature T_a. The same solution can then be used for this case if $h_o \equiv k_w/a$.

If, additionally, the wall thickness is zero, then the Biot number is infinite and the problem reduces to that of the constant surface temperature case discussed in Section 2.8. This is also true if the Biot number is very large.

The perturbation solution derived here is identical to that derived by Huang and Shih (1975). Their method also included the use of a Landau transformation and they noted that for $B_i < 1.0$, the zeroth order solution is acceptable for $S_{Te} < 0.1$ while the first-order solution is good for $S_{Te} < 1.0$. The additional complexity of the Landau transformation does not seem warranted for this problem.

3.1.6 Freezing of a Finite Slab

The freeze (or thaw) of a finite slab of material is of interest for energy storage problems. Consider the slab - shown in Figure 3.12 - which is initially at the freezing point and is suddenly immersed in an ambient fluid with a variable temperature. The mathematical description of the problem is the same for both sides of the slab but the ambient conditions can differ on each side. At x = 0, the equations are as follows, while at x = a, the form of the equations is the same but the properties can be different and are denoted by primes:

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} \qquad (3.68)$$

$$k \frac{\partial T(0,t)}{\partial x} = h[T(0,t) - T_a(t)] \qquad (3.68a)$$

$$T(X,t) = T_f \qquad (3.68b)$$

$$\rho \ell \frac{dX}{dt} = k \frac{\partial T(X,t)}{\partial x} \qquad (3.68c)$$

$$X(0) = 0 \quad . \qquad (3.68d)$$

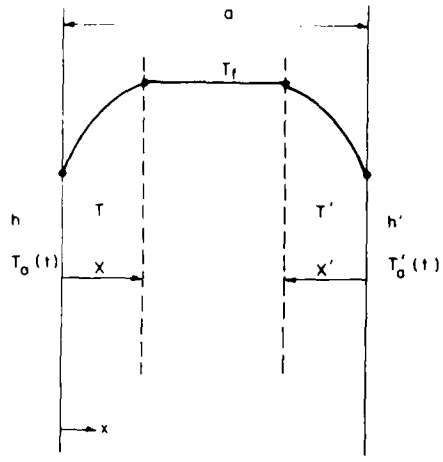


Figure 3.12. Freezing of a finite slab with convective cooling.

The ambient temperature is always below freezing but it varies sinusoidally as

$$T_a(t) = T_m + D \sin \Omega t \quad . \qquad (3.68e)$$

Gutman (1986) considered a perturbation solution of this problem. Nondimensional equations can be written as

$$\frac{\partial^2 v}{\partial \zeta^2} = S_T \frac{\partial v}{\partial \tau_1} \qquad (3.69)$$

$$\frac{\partial v(0,\tau_1)}{\partial \zeta} = v(0,\tau_1) - 1 + p \sin \omega\tau_1 \qquad (3.69a)$$

$$v(\sigma,\tau_1) = 0 \qquad (3.69b)$$

$$\frac{d\sigma}{d\tau_1} = \frac{-\partial v(\sigma,\tau_1)}{\partial \zeta} \qquad (3.69c)$$

where

$$v = \frac{T_f - T}{T_f - T_m} \qquad \zeta = \frac{xh}{k} \qquad \sigma = \frac{Xh}{k} \qquad \tau_1 = \frac{\alpha S_T h^2 t}{k^2}$$

$$S_T = \frac{c(T_f - T_m)}{\ell} \qquad p = \frac{D}{T_f - T_m} \qquad \omega = \frac{\rho \ell\ k\ \Omega}{h^2(T_f - T_m)} \quad .$$

The temperature and phase change interface are expanded in a power series using S_T and p as small parameters:

$$v = v_o + S_T v_1 + p\ v_2 + \ldots \qquad (3.70)$$

$$\sigma = \sigma_o + S_T \sigma_1 + p\ \sigma_2 + \ldots \quad . \qquad (3.71)$$

The system of equations is

$$\frac{\partial^2 v_i}{\partial \zeta^2} = \begin{cases} 0 & i = 0,2 \\ \dfrac{\partial v_o}{\partial \tau_1} & i = 1 \end{cases} \tag{3.72}$$

$$\frac{\partial v_i(0,\tau_1)}{\partial \zeta} = \begin{cases} v_o(0,\tau_1) - 1 & i = 0 \\ v_1(0,\tau_1) & i = 1 \\ v_2(0,\tau_1) + \sin \omega\tau_1 & i = 2 \end{cases} \tag{3.73}$$

$$v_i(\sigma_o) = \begin{cases} 0 & i = 0 \\ -\sigma_i \dfrac{\partial v_o(\sigma_o,\tau_1)}{\partial \zeta} & i = 1,2 \end{cases} \tag{3.74}$$

$$\frac{d\sigma_i}{d\tau_1} = -\frac{\partial v_i(\sigma_o,\tau_1)}{\partial \zeta} \tag{3.75}$$

$$\sigma_i\ (0) = 0\ . \tag{3.76}$$

Functions evaluated at σ are expanded in Taylor series about $\sigma = \sigma_o$ as discussed in Appendix A. The solutions are

$$v_o = \frac{\sigma_o - \zeta}{1 + \sigma_o} \tag{3.77}$$

$$\sigma_o = \sqrt{1 + 2\ \tau_1} - 1 \tag{3.77a}$$

$$v_1 = \frac{1}{6(1 + \sigma_o)^3}\left[\zeta^2(\zeta + 3) - \frac{2\sigma_o^{\ 2}(\sigma_o+3)(\zeta+1)}{1+\sigma_o}\right] \tag{3.78}$$

$$\sigma_1 = \frac{-\sigma_o^{\ 2}(\sigma_o+3)}{6(1+\sigma_o)^2} \tag{3.78a}$$

$$v_2 = \frac{[\zeta-\sigma_o]}{(1+\sigma_o)} \sin \omega\ \tau_1 - \frac{(1 - \cos \omega\ \tau_1)(\zeta+1)}{\omega(1+\sigma_o)} \tag{3.79}$$

$$\sigma_2 = \frac{\cos \omega\ \tau_1 - 1}{\omega(1+\sigma_o)}\ . \tag{3.79a}$$

The time for the slab to solidify completely τ_f can be found from

$$\sigma(\tau_f) + \nu\ \sigma'(\tau_f) = B_o \tag{3.80}$$

where

$$\nu = h/h' \qquad B_o = \frac{h\ a}{k}\ .$$

If τ_f is expanded as

$$\tau_f = \tau_{fo} + S_T \tau_{f1} + p\, \tau_{f2} \tag{3.81}$$

then

$$\tau_{fo} = \frac{1}{2(1-\eta)} \left[(1+\eta)A^2 - (1-\nu^2)(1-\eta) - 2A^2\sqrt{\eta} + \frac{(1-\eta)(\nu^2-\eta)}{A^2} \right] \tag{3.82a}$$

where

$$A = B_o + \nu + 1$$

$$\tau_{f1} = -\psi[\sigma_1(\tau_{fo}) + \nu\, \eta\, \sigma_1(\tau_{fo}\, \eta/\nu^2)] \tag{3.82b}$$

$$\tau_{f2} = \frac{\psi(1-\cos\omega\tau_{fo})}{\omega(1+\sigma_o)} \tag{3.82c}$$

$$\psi = [(1+\sigma_o)^{-1/2} + \eta\, (\nu^2+2\eta\, \tau_{fo})^{-1/2}]^{-1} \quad . \tag{3.83}$$

For the case of a constant ambient temperature, the solution compares very well with the exact results in Table 3.2

The case of a slab insulated at x = a is obtained if $\nu = \eta = 0$. (This is also the case of a semi-infinite medium.) The solidification time is

$$\tau_f = \frac{B_o\ (B_o+2)}{2} + \frac{B_o^{\ 2}(B_o+3)}{6(B_o+1)}\ S_T + (1-\cos\omega\tau_{fo})p/\omega \tag{3.84}$$

3.2 TWO-PHASE PROBLEMS

When the medium is initially at a different value than the fusion temperature, the problem is considerably more complicated. There is no exact solution for this case but Lunardini (1981, 1982) has obtained an approximate solution.

3.2.1 Heat Balance Integral Approximation

The geometry of the problem is shown in Figure 3.13. Using the heat balance integral with collocation, the overall energy balance for the volume of interest is

$$\frac{d}{dt}\, [\rho_1 c_1 \theta_1 + \rho_2 c_2 \theta_2 + \rho_1 \ell X + (\rho_2 c_2 - \rho_1 c_1) T_f X - \rho_2 c_2 T_0 (X+\delta)] = -k_1 \frac{\partial T_1(0,t)}{dx} \ . \tag{3.85}$$

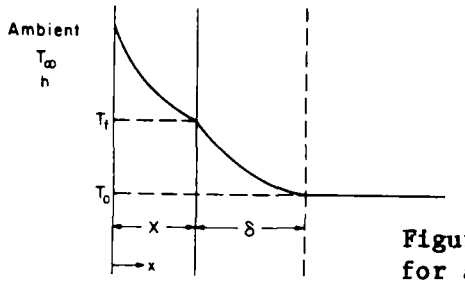


Figure 3.13 Surface convection for a semi-infinite body

where

$$\theta_1 = \int_0^X T_1(x,t)dx$$

$$\theta_2 = \int_X^{X+\delta} T_2(x,t)dx \quad .$$

The term $(\rho_2 c_2 - \rho_1 c_1)\, T_f dX/dt$ is the net sensible flux of enthalpy at the phase change interface due to the sudden jump in the specific heats of the frozen and thawed volumes. The retention of the sensible enthalpy term gives better numerical comparisons to the exact solution.

The energy balance, at the phase change interface, can be written as collocation equations (see App. B):

$$-k_1 \frac{\partial T_1(X,t)}{\partial x} + k_2 \frac{\partial T_2(X,t)}{\partial x} =$$

$$-\rho_1 \ell \alpha_1 \frac{\partial^2 T_1(X,t)}{\partial x^2} \Big/ \frac{\partial T_1(X,t)}{\partial x} \tag{3.86}$$

$$-k_1 \frac{\partial T_1(X,t)}{\partial x} + k_2 \frac{\partial T_2(X,t)}{\partial x} =$$

$$-\rho_2 \ell_2 \frac{\partial^2 T_2(X,t)}{\partial x^2} \Big/ \frac{\partial T_2(X,t)}{\partial x} \quad . \tag{3.87}$$

The following temperature approximations can be used:

$$T_1 = T_f + a_1(x - X) + a_2\,(x - X)^2 \tag{3.88}$$

$$T_2 = T_f - 2\,\frac{(T_f - T_o)}{\delta}\,(x - X) + \frac{(T_f - T_o)}{\delta^2}\,(x - X)^2 \quad . \tag{3.89}$$

Equation 3.88, representing the temperature in the region which has changed phase, contains two unknown coefficients. One of these can be found from the specified boundary condition at x = 0. Combining eq 3.86 and 3.87 yields

$$\frac{a_2}{a_1} = -\frac{\alpha_{21}}{2\delta} . \tag{3.90}$$

The surface boundary condition is

$$-k_1 \frac{\partial T_1(0,t)}{\partial x} = h[T_a - T_1 (0,t)]. \tag{3.91}$$

Thus

$$a_1 = \frac{-\ell}{c_1} \frac{S_{Te}}{X\left(1+\frac{\alpha_{21}}{2\delta} X\right) + \frac{k_1}{h} \left(1+\frac{\alpha_{21}}{\delta} X\right)}$$

Using eq 3.87

$$\phi = \frac{b(\sigma+1)}{2} + \sqrt{\frac{b^2(\sigma+1)^2}{4} + \alpha_{21}\ \sigma(\frac{\sigma}{2} + 1)b} \tag{3.92}$$

where

$$b = \frac{2k_{21}S_{Tm} + \alpha_{21}}{S_{Te}}.$$

$$\phi = \frac{h}{k_1}\ \delta$$

$$\sigma = \frac{h}{k_1}\ X.$$

The energy balance equation, eq 3.85, can now be written as

$$\frac{dF}{d\tau} = \frac{2(\phi+\alpha_{21}\ \sigma)}{2\phi(\sigma+1) + \alpha_{21}\ (\sigma+2)} \tag{3.93}$$

$$F = \frac{S_{Te}\ \sigma^2\ (\phi + \frac{1}{3}\ \alpha_{21}\sigma)}{2\phi\ (\sigma+1) + \alpha_{21}\ \sigma(\sigma+2)} + \sigma(1+c_{21}S_{Tm}) + \frac{1}{3}\ c_{21}S_{Tm}\phi. \tag{3.94}$$

Equation 3.93 can be written as

$$2\tau = \int_0^\sigma Q\ d\sigma' \tag{3.95}$$

and

$$PQ = (2\phi + \alpha_{2l}\ \sigma)\ \sigma\ S_{Te} + (1 + C_{2l}S_{Tm})g$$

$$+ 2\ [\sigma(1 + C_{2l}S_{Tm}) + \frac{1}{3}\ C_{2l}S_{Tm}\phi](P + \alpha_{2l})$$

$$+ S_{Te}\ \sigma^2 + \frac{1}{3}\ C_{2l}S_{Tm}g$$

$$+2\ [\sigma(1 + C_{2l}S_{Tm}) + \frac{1}{3}\ C_{2l}S_{Tm}\phi]\ (\sigma+1)$$

$$-2\ (\sigma + 1)F\ \frac{b(P + \alpha_{2l})}{2\phi - b(\sigma + 1)} - 2(P + \alpha_{2l})F$$

where $P = \phi + \alpha_{2l}\sigma$ and $g = 2[P(1 - \sigma) - \sigma\phi]$.

It can be shown that when $S_{Tm} = 0$ and $S_{Te} = 0$, eq 3.93 can be solved to yield the quasi-steady solution, eq 3.24.

The numerical solution to eq 3.95, when $S_{Tm} = 0$, is identical to the heat balance integral solution of Section 3.1.4. The surface temperature is

$$\frac{T_l(0,t) - T_f}{T_a - T_f} = \frac{\sigma(2\phi + \alpha_{2l}\sigma)}{\sigma(2\phi + \alpha_{2l}\sigma) + 2(\phi + \alpha_{2l}\sigma)} \tag{3.96}$$

The nondimensional surface heat transfer rate is

$$q^* = \frac{(\phi + \alpha_{2l}\sigma)}{\sigma(\phi + \frac{1}{2}\ \alpha_{2l}\sigma) + (\phi + \alpha_{2l}\sigma)}\ . \tag{3.97}$$

Equation 3.95 was solved by numerical quadrature (Lunardini 1983). Figures 3.14-23 are plots of the solution for some values of Stefan number and S_{Tm}, with property ratios given as functions of the volumetric water content for soils systems. As has been noted, the heat balance integral method yields solutions that compare quite well with the few exact solutions. Thus the graphs presented here should be accurate for normal engineering design, especially since the soil thermal properties will normally be known only to within 10-20%. Figures 3.24-33 give the phase change depth vs time for some possible phase change materials with the properties listed in Table 3.3.

Table 3.3. Thermal properties of some phase change materials.*

Phase change material	Fusion temperature (°F)	Latent heat of fusion (Btu/lbm)	Specific heat at T_f (Btu/lbm °F)		Thermal conductivity at T_f (Btu/hr-°F-ft)		Density at 25°C (lbm/ft^3)
			Solid	Liquid	Solid	Liquid	
B_2O_3	842	142	0.41	0.44	0.9	0.58	115.5
33 LiF-67 KF	918	266	0.32	0.39	2.4-4.8	2.30	157.9
67 NaF-33 MgF_2	1530	265	0.34	0.33	2.4-4.8	2.69	133.6
12 NaF-59 KF-29 LiF	849	257	0.32	0.38	2.4-4.8	2.60	157.9

*Energy Research and Development Agency (1976).

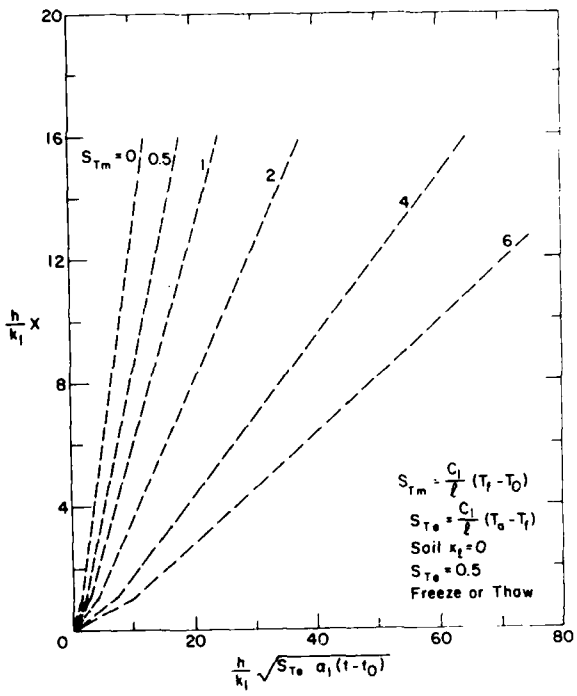


Figure 3.14. Surface convection for soil, x_ℓ (volumetric water content) = 0, S_{Te} = 0.5

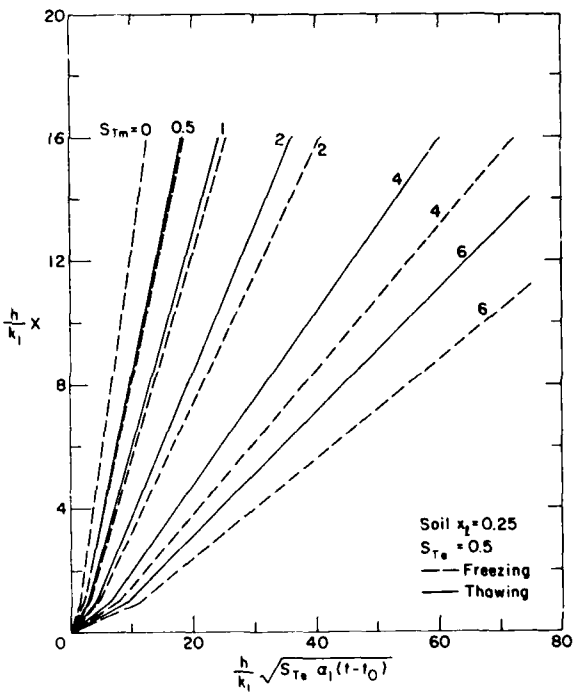


Figure 3.15. Surface convection for soil, x_ℓ = 0.25, S_{Te} = 0.5.

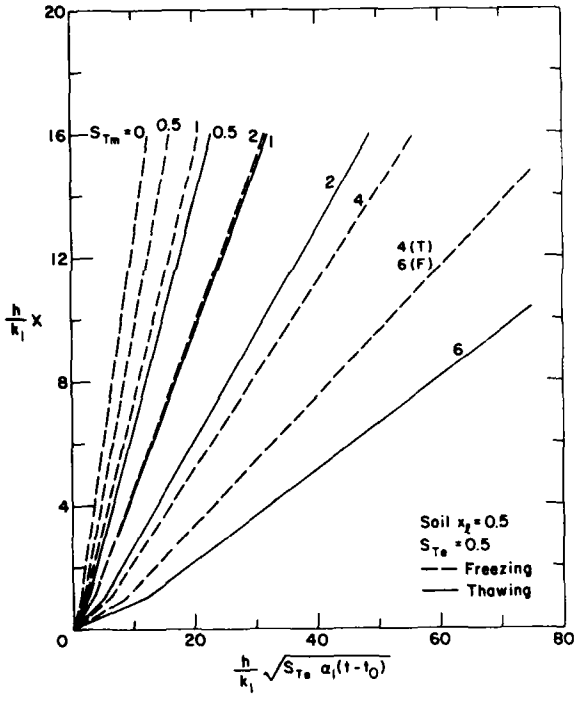


Figure 3.16. Surface convection for soil, $x_\ell = 0.50$, $S_{Te} = 0.5$.

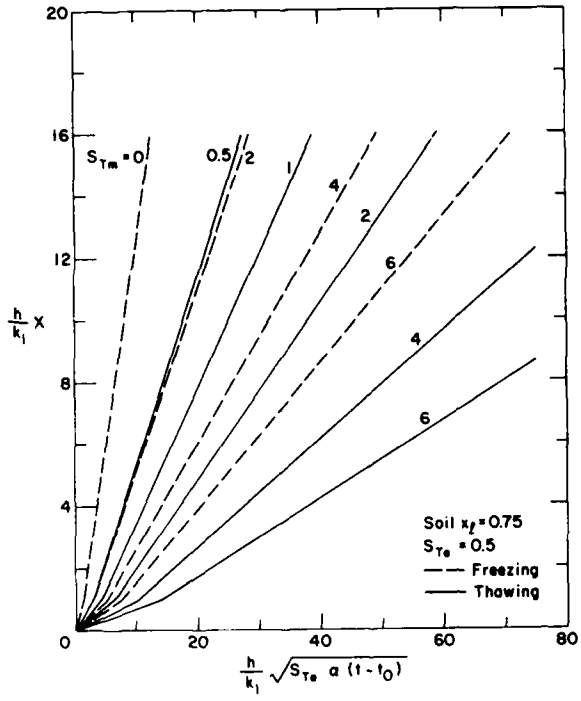


Figure 3.17. Surface convection for soil, $x_\ell = 0.75$, $S_{Te} = 0.5$

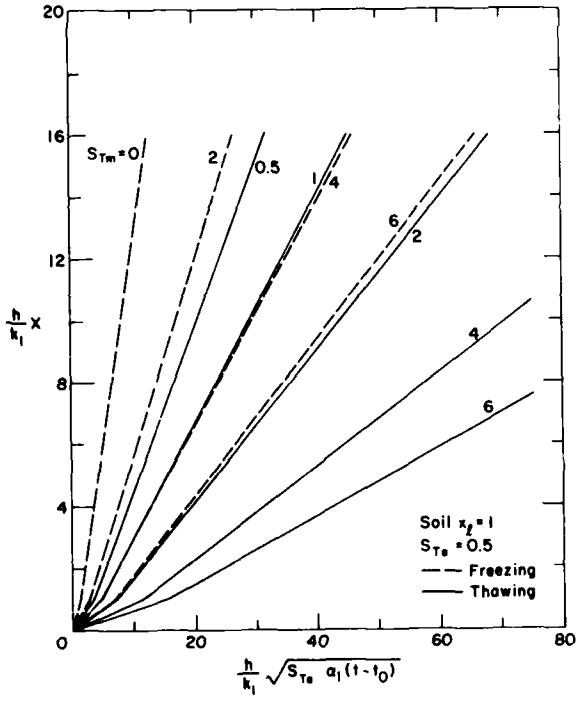


Figure 3.18. Surface convection for soil, $x_\ell = 1.0$, $S_{Te} = 0.5$

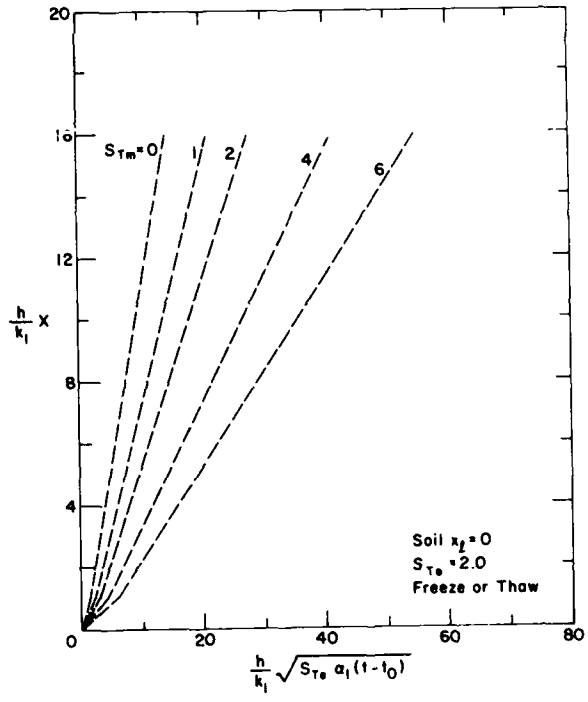


Figure 3.19. Surface convection for soil, $x_\ell = 0$, $S_{Te} = 2$.

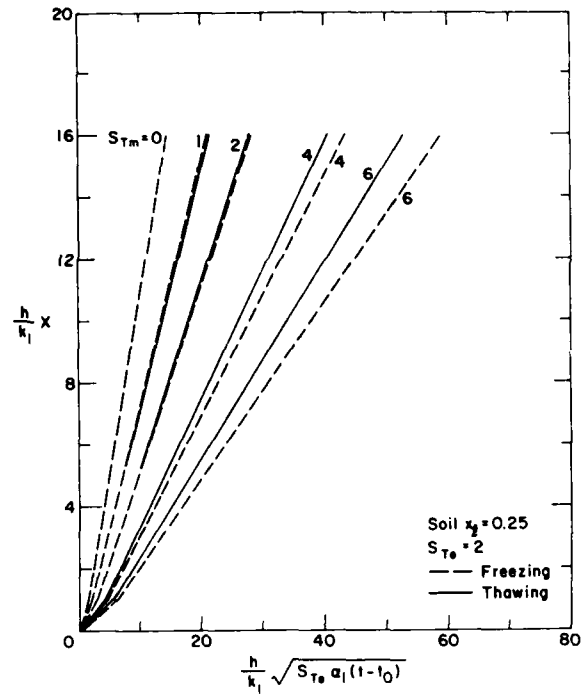


Figure 3.20. Surface convection for soil, $x_\ell = 0.25$, $S_{Te} = 2$.

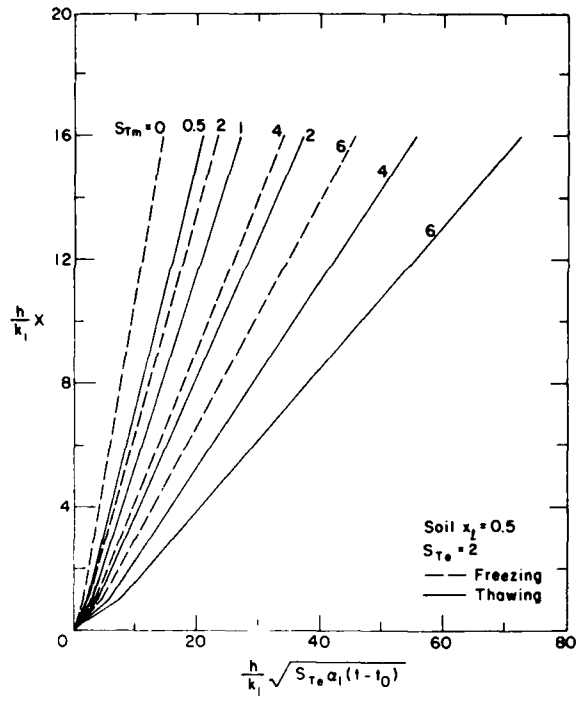


Figure 3.21. Surface convection for soil, $x_\ell = 0.50$, $S_{Te} = 2$.

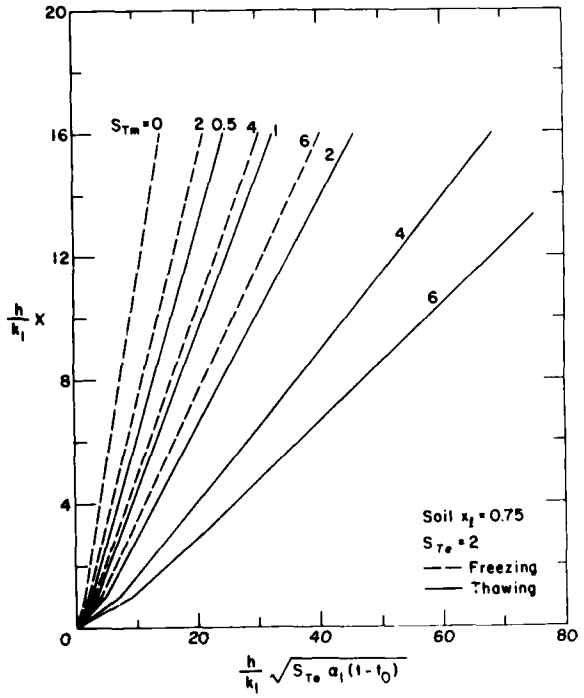


Figure 3.22. Surface convection for soil, $x_\ell = 0.75$, $S_{Te} = 2$.

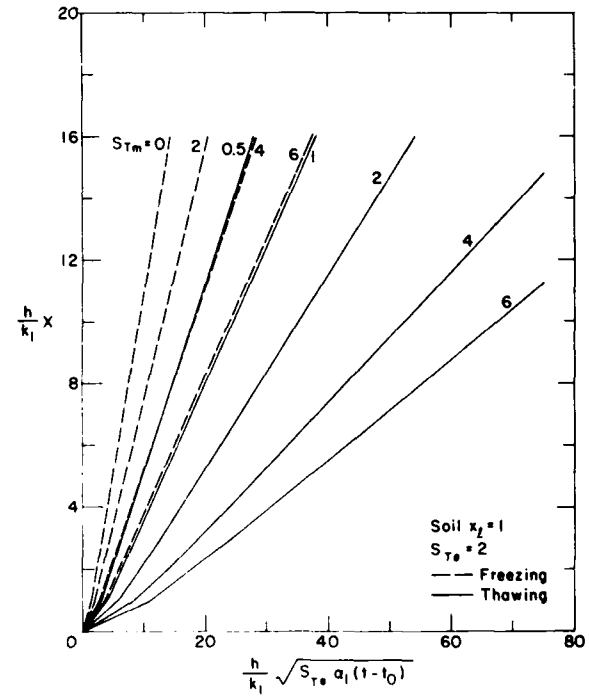


Figure 3.23. Surface convection for soil, $x_\ell = 1.0$, $S_{Te} = 2$.

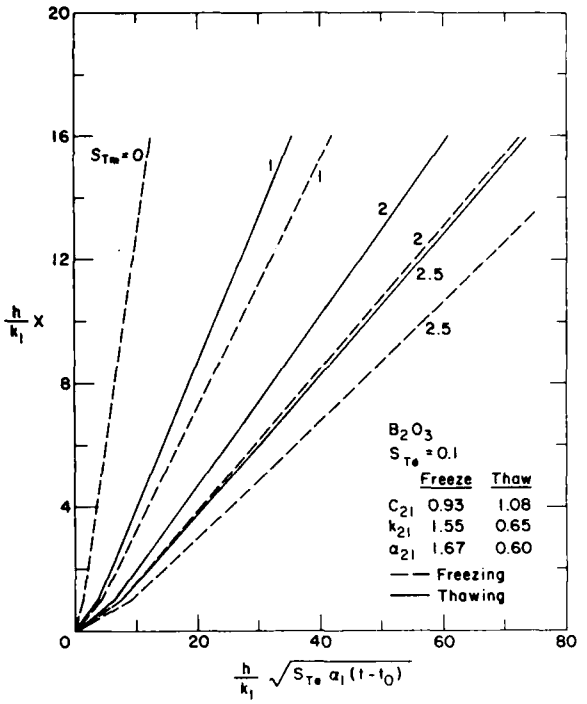


Figure 3.24. Surface convection, B_2O_3 S_{Te} = 0.1.

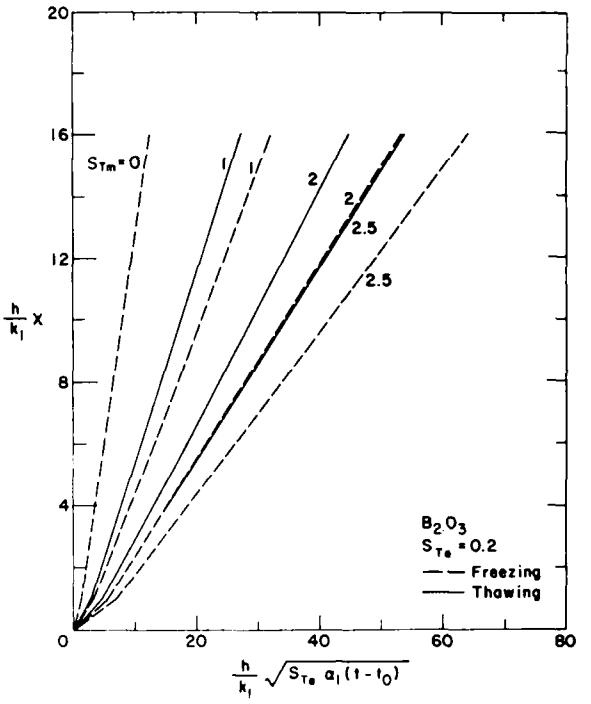


Figure 3.25. Surface convection B_2O_3 S_{Te} = 0.2.

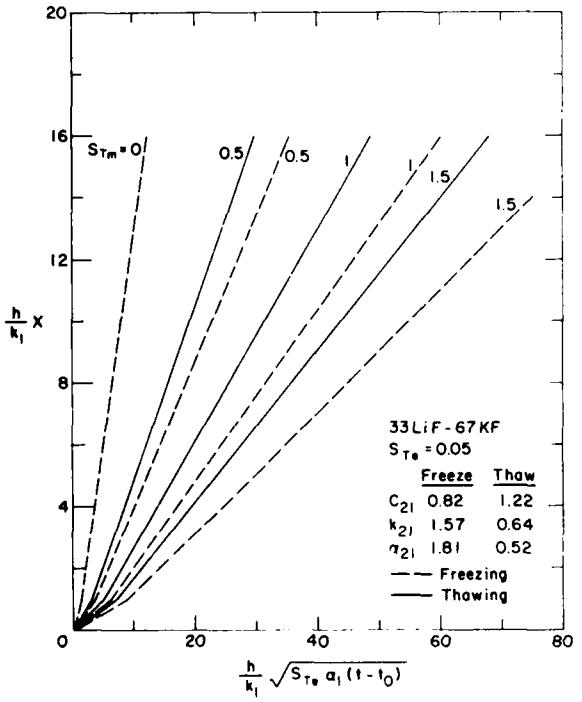


Figure 3.26. Surface convection, 33LiF - 67KF, S_{Te} = 0.05.

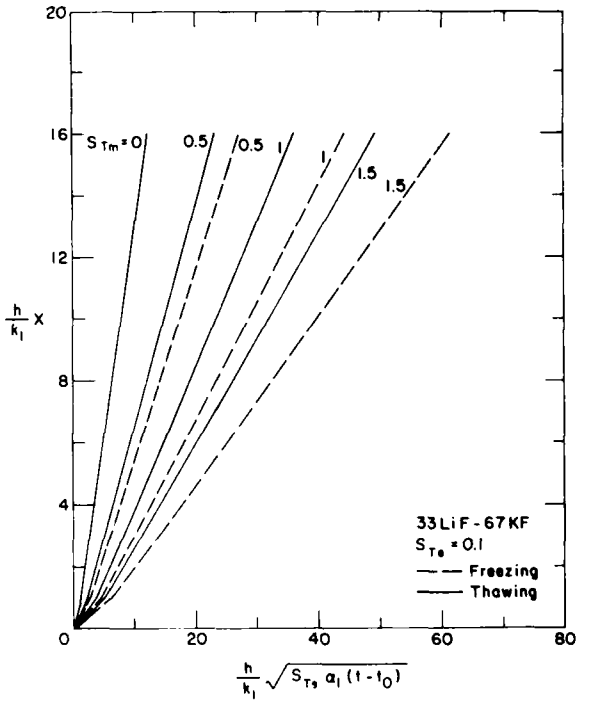


Figure 3.27. Surface convection, 33LiF - 67KF, S_{Te} = 0.1.

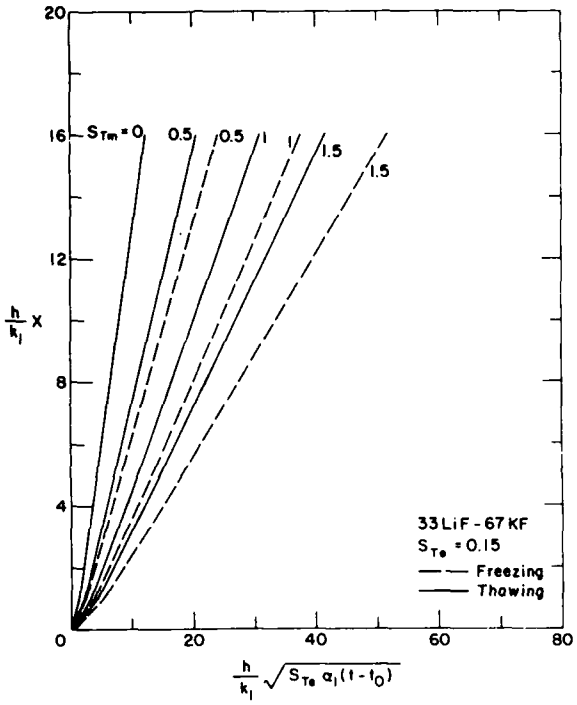


Figure 3.28. Surface convection, 33LiF - 67KF, S_{Te} = 0.15.

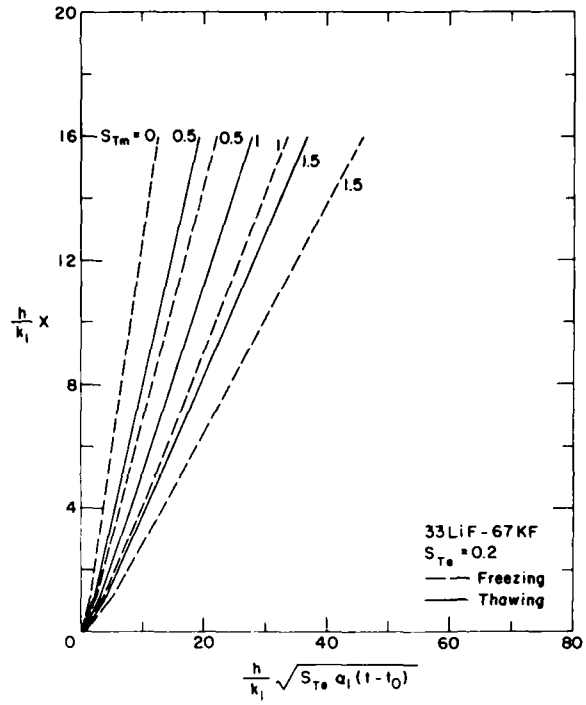


Figure 3.29. Surface convection, 33LiF - 67KF, S_{Te} = 0.20.

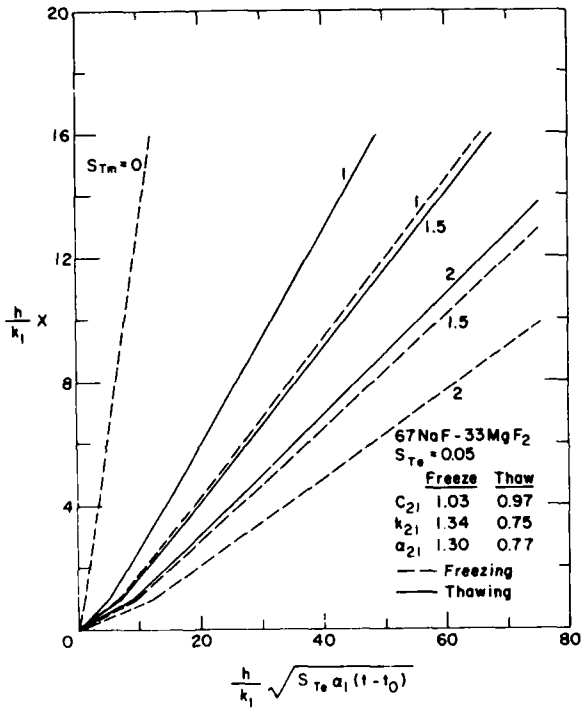


Figure 3.30. Surface convection, 67NaF - 33MgF$_2$, S_{Te} = 0.05.

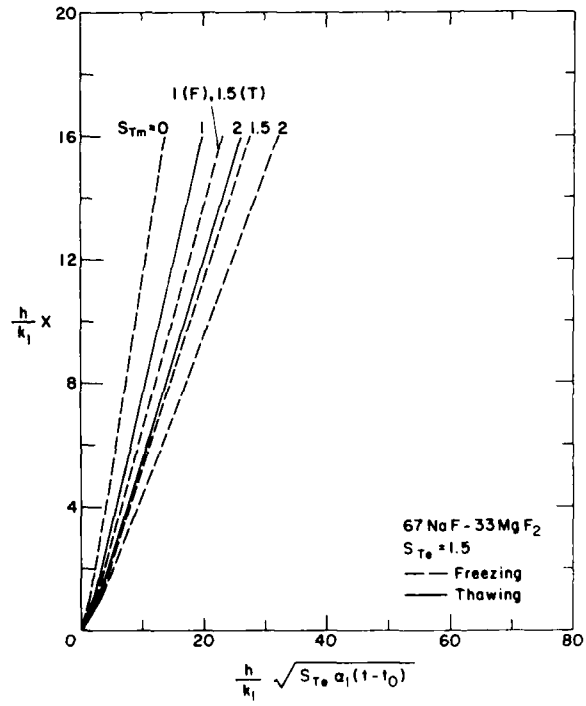


Figure 3.31. Surface convection, 67NaF - 33MgF$_2$, S_{Te} = 1.5.

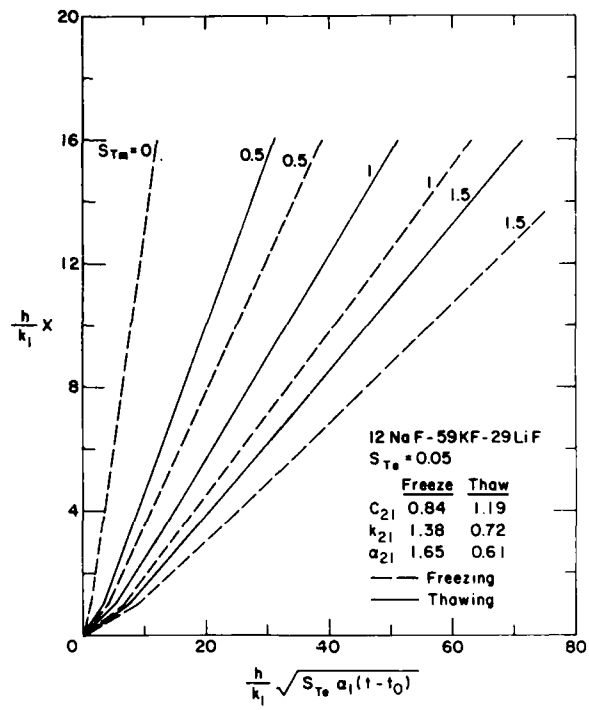


Figure 3.32. Surface convection, 12NaF - 59KF-29LiF, S_{Te} = 0.05.

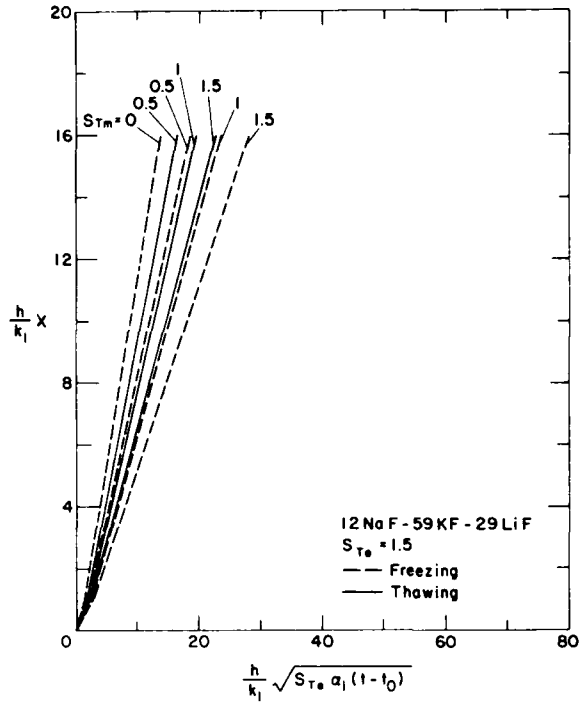


Figure 3.33. Surface convection, 12NaF - 59KF - 29LiF, S_{Te} = 1.5.

3.2.2 Insulated semi-infinite region

Figures 3.14-33 can also be used for the case of a slab insulated with a layer of material when the insulation temperature is T_∞, as shown in Figure 3.34. The conductive resistance of the insulation must equal the convective resistance of the air layer. Then

$$\frac{d}{k_i} = \frac{1}{h} \quad . \tag{3.98}$$

The dimensionless phase change depth is then given by

$$\sigma_c = \frac{k_i}{dk_1} X_c . \tag{3.99}$$

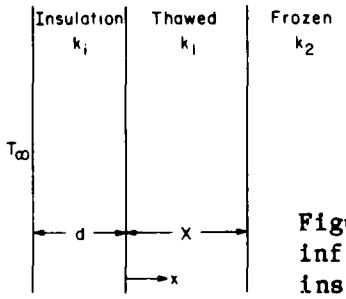


Figure 3.34. Semi-infinite body with insulation layer.

The graphs can then be used by assuming that the insulation layer has no latent heat and phase change starts at $t = t_o$ when the temperature of the insulation/slab interface reaches T_f.

The single-phase solution, with $S_{Te} = 0$, can be rewritten as

$$X_c = \sqrt{k_{1i}^2 d^2 + \frac{2k_i(T_a - T_f)(t - t_o)}{\rho_i \ell}} - k_{1i} d. \qquad (3.100)$$

NOMENCLATURE - CHAPTER 3

a	wall or plate thickness
A	$B_o + \nu + 1$
B_i	$h_o X_s/k$, Biot number
B_o	$h\ a/k$
c	specific heat
C	ρc - volumetric specific heat
d	k_i/h
D	amplitude of ambient temperature above T_m.
F	radiation shape factor
g	$dS/d\tau$
h	surface coefficient of heat transfer
h_c	surface coefficient of convection
h_o	defined by eq 3.46
k	thermal conductivity
k_{ij}	k_i/k_j
k_w	thermal conductivity of wall
ℓ	latent heat of fusion
q	surface heat transfer rate per unit area
q^*	$q/h(T_f - T_a)$
q_w	constant heat flux rate from liquid

q_w^*	$q_w c_1/(h\ \ell)$
S	X/X_s nondimensional phase change depth
S_T	$\dfrac{c(T_f - T_m)}{\ell}$
S_{Te}	$\dfrac{c_1}{\ell}(T_a - T_f)$, thaw or $\dfrac{c_1}{\ell}(T_f - T_a)$ freeze
S_{Tm}	$\dfrac{c_1}{\ell}(T_f - T_o)$
S_{Ta}	$\dfrac{c_1}{\ell}(T_a - T_o)$
t	time
t_o	time when phase change starts
T	temperature
T_a, T_f, T_o	ambient, phase change, initial temperatures
T_m	mean ambient temperature
u	$(T - T_a)/(T_f - T_a)$
v	$(T_f - T)/(T_f - T_m)$
x	Cartesian coordinate
X	phase change depth
X_c	phase change depth for insulated body
X_s	steady state phase change depth
y	dummy variable
α	thermal diffusivity
α_{12}	α_1/α_2
β	$1 + 2\ S_{Te}$
γ	$X/(2\sqrt{\alpha_1 t})$
δ	thermal disturbance
ΔT	$T_f - T_a$

ε surface heat flux parameter

ε' emissivity

ζ $\frac{xh}{k}$

η $\frac{T_f - T_m'}{T_f - T_m}$

θ $(T - T_f)/(T_a - T_f)$ dimensionless temperature

θ_i integrated temperature

μ $2\sqrt{\alpha t}$ time parameter

ν h/h'

ξ x/X_s

ρ density

σ hX/k_1 dimensionless phase change depth

σ_c $k_{i1}X_c/d$ dimensionless phase change depth

σ' Stefan-Boltzmann constant

τ $\alpha_1(t - t_o)/d^2$ dimensionless time

τ_1 $\frac{\alpha S_T h^2 t}{k^2}$

τ^* $k \frac{(T_f - T_a)t}{\rho \ell X_s^2}$

ϕ $h\,\delta/k_1$ dimensionless thermal disturbance

ω $\rho \ell\, k\, \Omega/h^2(T_f - T_m)]$

Ω frequency of ambient temperature

Subscripts

1,2 different phases of medium

i insulation

CHAPTER 4. PLANE PROBLEMS WITH SPECIFIED SURFACE HEAT FLUX

Boundary conditions at the free surface, in terms of heat flow, can be specified as convective heat flow (Chap. 3) or as a prescribed heat flux. This chapter covers some important cases for prescribed heat flux.

4.1 EXACT SOLUTION FOR THE SEMI-INFINITE REGION

Tao (1979) has found an exact solution to the freeze problem shown in Figure 4.1. A liquid, initially at an arbitrary temperature distribution V(x), has a heat flux Q(t) imposed upon its free surface. Solidification starts when the surface temperature of the liquid reaches the melting point such that $Q(t) > 0$ and $V(0) = T_f$.

The basic equations (see Nomenclature at end of chapter) are

$$\alpha_1 \frac{\partial^2 T_1}{\partial x^2} = \frac{\partial T_1}{\partial t} \qquad 0 < x < X(t) \tag{4.1}$$

$$\alpha_2 \frac{\partial^2 T_2}{\partial x^2} = \frac{\partial T_2}{\partial t} \qquad x > X(t) \tag{4.2}$$

$$k_1 \frac{\partial T_1(0,t)}{\partial x} = Q(t) \tag{4.3}$$

$$T_2(x,0) = V(x) \tag{4.4}$$

$$T_1(X,t) = T_2(X,t) = T_f \tag{4.5}$$

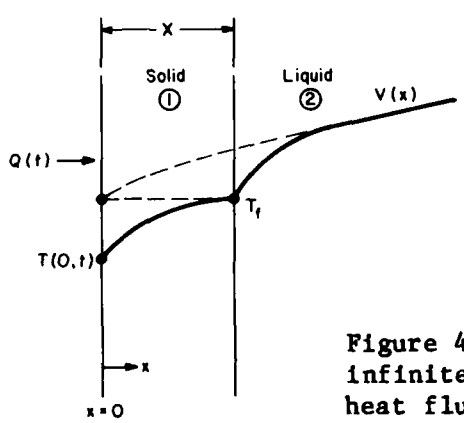


Figure 4.1. Melt of a semi-infinite medium with surface heat flux.

$$k_1 \frac{\partial T_1(X,t)}{\partial x} - k_2 \frac{\partial T_2(X,t)}{\partial x} = \rho \ell \frac{dX}{dt} \quad . \tag{4.6}$$

The heat flux and initial temperature can be represented as

$$Q(t) = \frac{k_1}{\sqrt{\alpha_1}} \sum_0^\infty q_n \, t^n/n! \tag{4.7}$$

$$V(x) = T_f + \sum_1^\infty \frac{v_n}{n!} \left(\frac{x}{\sqrt{\alpha_2}}\right)^n \quad . \tag{4.8}$$

A formal solution to the problem is

$$T_1 = \sum_{n=0} q_n (4t)^{n+\frac{1}{2}} G_{2n+1}(\eta_1) + \sum_{n=0} a_n (4t)^{n/2} E_n(\eta_1) \tag{4.9}$$

$$T_2 = T_f + \sum_{n=1} v_n (4t)^{n/2} G_n(\eta_2) + \sum_{n=0} b_n (4t)^{n/2} i^n \mathrm{erfc}(\eta_2) \tag{4.10}$$

where

$$\eta_i = \frac{x}{2\sqrt{\alpha_i t}}$$

$$X(t) = \sqrt{\alpha_1} \sum_{n=0} \lambda_{n+1} (t)^{\frac{n+2}{2}} \quad . \tag{4.11}$$

The error integral functions G, E, and i^nerfc are discussed in Appendix D. Formulas for the coefficients a_n, b_n, λ_n are given by Tao (1979). The solution method is obviously the same as the solution given in Section 2.3.1, and the numerical work to evaluate a_n, b_n, λ_n is just as laborious.

Tao (1979) gives a numerical example for the special case when $V(x) = T_f$ (thus only one phase is present) and the heat flux is a constant, Q_o. The first four terms of the phase change interface, given by eq 4.11, can be written as

$$\xi = \tau - \frac{1}{2}\tau^2 + \frac{5}{6}\tau^3 - \frac{17}{8}\tau^4 + \ldots \tag{4.12}$$

where

$$\xi = \frac{Q\,X}{\rho \alpha_1 \ell} \qquad \tau = \frac{Q^2 t}{\rho^2 \alpha_1 \ell^2} \quad .$$

Evans et al. (1950) also presented an exact solution for this problem by assuming Taylor series expansions for X(t) about t = 0. The solution is a series

$$\xi = \tau - \frac{1}{2}\tau^2 + \frac{5}{6}\tau^3 - \frac{17}{8}\tau^4 + \frac{827}{120}\tau^5 \ldots \tag{4.13}$$

The equations are only valid for values of $\tau < 0.4$, unless many more terms are included. This will not limit the use of eq 4.13 too significantly if the latent heat is large relative to the heat flux, i.e. $S_{Te} = (Q_o c/h\ell) << 1$.

The exact solution of Lozano and Reemsten (1981), eq 3.8a, has precisely the same form as eq 4.13. All three solutions are exact, although they all appear distinctly different, and they reduce to the identical series for the phase change interface. Since it has been shown that the exact solution to this problem is unique, the above result has to follow.

4.2 APPROXIMATE SOLUTIONS, SINGLE PHASE, SEMI-INFINITE REGION

4.2.1 Analog Solution

Kreith and Romie (1955) present an analog solution for this case with the initial temperature at the fusion temperature. The problem is the same as Section 3.1.2 except that the surface boundary condition is

$$\frac{\partial T_1}{\partial x} = G = Q_o/k_1, \text{ constant} . \tag{4.14}$$

The solutions, in graphical form, are given for the freeze depth and the surface temperature.

4.2.2 Heat Balance Integral Method

The single-phase problem is formulated as

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} \tag{4.15}$$

$$T_1(X,t) = T_f \tag{4.16}$$

$$T_1(x,0) = T_f \tag{4.17}$$

$$X(0) = 0 \tag{4.18}$$

$$k_1 \frac{\partial T_1}{\partial x}(X,t) = \rho_1 \ell \frac{dX}{dt} \tag{4.19}$$

$$\frac{\partial T_1(0,t)}{\partial x} = G(t) = Q(t)/k_1 \ . \tag{4.20}$$

A heat-balance integral solution for the problem has been obtained by Goodman (1958). The heat balance integral is

$$\frac{d}{dt}\left[\theta_1 - \left(T_f + \frac{\alpha_1 \rho_1 \ell}{k_1}\right)X\right] = -\alpha_1 G \tag{4.21}$$

where

$$\theta_1 = \int_0^X T_1(x,t)dx \ .$$

Equation 4.21 can be integrated immediately to give

$$\theta_1(X) - \theta_1(0) - T_f X - \frac{\ell}{c_1} X = -\alpha_1 \int_0^t G(t)\, dt \ . \tag{4.22}$$

The temperature is assumed to be quadratic and with the boundary conditions is

$$T = T_f + a(x-X) + \frac{a - G}{2X}(x - X)^2 \ . \tag{4.23}$$

where a is given by

$$a = \frac{\ell}{2c_1 X}\left(-1 + \sqrt{1 + 4\frac{Gc_1 X}{\ell}}\right) \ . \tag{4.24}$$

The solutions for X and $T_1(0,t)$ are

$$\tau = \frac{\xi}{6}\left(5 + \xi + \sqrt{1 + 4\xi}\right) \tag{4.25}$$

$$\frac{4c_1}{\ell}\left(T_f - T(0,t)\right) = 2\xi - 1 + \sqrt{1 + 4\xi} \tag{4.26}$$

where

$$\tau = \frac{Q}{\rho^2 \alpha_1 \ell^2} \int_0^t Q(t)dt \qquad \xi = \frac{Q\,X}{\rho \alpha_1 \ell} \ .$$

These solutions are plotted on Figures 4.2 and 4.3. When the heat flux is constant, the results agree very well with the analog solution of Kreith and Romie (1955), and also with the exact solution of Evans et al. (1950) for small values of time. Solutions of this type are not valid for a

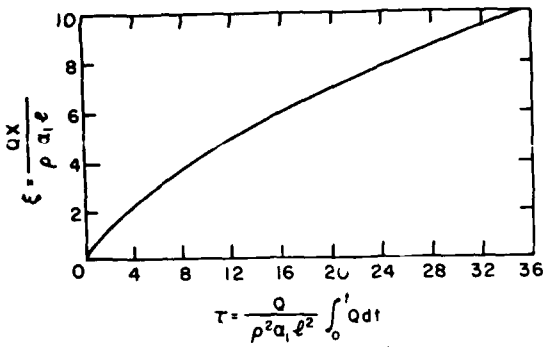


Figure 4.2. Thickness of melt vs time, for a given heat flux at boundary, eq 4.25.

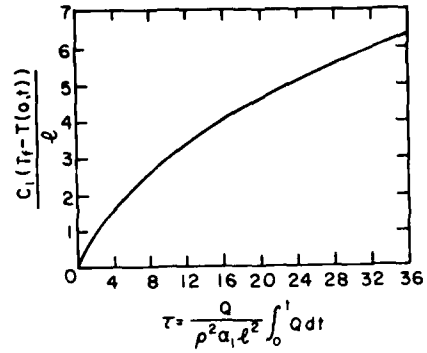


Figure 4.3. Temperature-time history on boundary for a given heat flux, eq 4.26.

pulse-type function for G which vanishes after some finite time. Two-parameter integral heat balance solutions can be used in these cases (see Goodman 1958).

The quasi-steady solution to this problem is extremely simple but valid only for short times:

$$\tau = \xi \tag{4.27}$$

Gutman (1986) used a perturbation method to obtain the following solution (with corrections made to his paper) for a constant heat flux:

$$\xi = \tau - \frac{1}{2}\tau^2 + \frac{5}{6}\tau^3 \tag{4.27a}$$

$$\frac{c}{\ell}\,[T_f - T(\zeta,\tau)] = \tau - \zeta + \frac{1}{2}\zeta^2 - \tau^2 - \tau\,(\zeta^2 - \frac{7}{3}\tau^2)\ . \tag{4.27b}$$

The solution clearly corresponds to the first three terms of eq 4.13.

Cho and Sunderland (1981) present an approximate heat flux solution using an analogy to the nonfreezing solution. The results agree with eq 4.25, but the generality of the method is not shown.

4.3 TWO-PHASE PROBLEM

The two-phase problem has been solved by Lunardini (1982) using the heat balance integral, with collocation. The geometry for a melt problem is shown in Figure 4.4. The overall energy balance for the volume of interest is

$$\frac{d}{dt}\,[\rho_1 c_1 \theta_1 + \rho_2 c_2 \theta_2 + \rho_1 \ell X + (\rho_2 c_2 - \rho_1 c_1)\,T_f X - \rho_2 c_2 T_o (X + \delta)]$$

$$= -k_1 \frac{\partial T_1(0,t)}{dx} \tag{4.28}$$

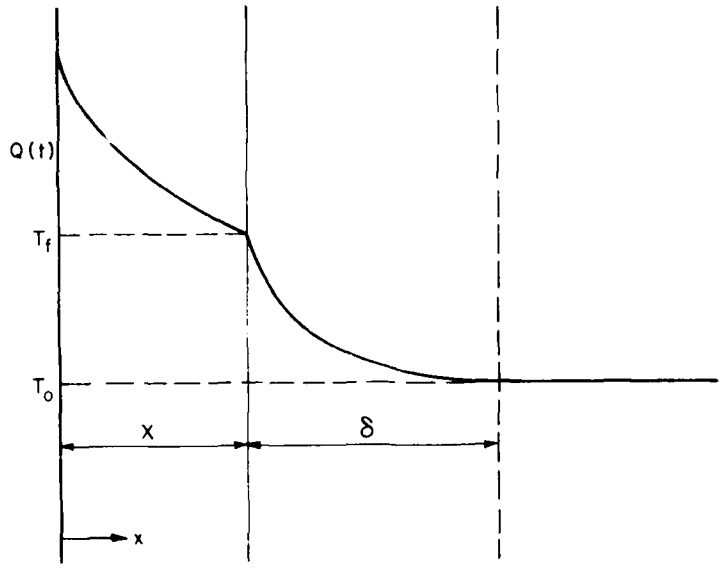


Figure 4.4. Specified surface heat flux for a semi-infinite medium.

where

$$\theta_1 = \int_0^X T_1(x,t)dx$$

$$\theta_2 = \int_X^{X+\delta} T_2(x,t)dx \ .$$

For semi-infinite solids the following temperature approximations can be used:

$$T_1 = T_f + a_1(x - X) + a_2(x - X)^2 \tag{4.29}$$

$$T_2 = T_f - 2\,\frac{(T_f - T_o)}{\delta}(x - X) + \frac{(T_f - T_o)}{\delta^2}(x - X)^2 \ . \tag{4.30}$$

The surface temperature will increase from T_o to the fusion value T_f when melting begins (see Fig. 4.4) and the phase change solution can then be obtained.

The surface boundary condition is

$$-k_1 \frac{\partial T_1(0,t)}{\partial x} = Q(t) \tag{4.31}$$

and

$$a_1 = \frac{Q}{k_1}\left[\frac{\alpha_{21}X}{\delta + \alpha_{21}X} - 1\right]$$

$$a_2 = \frac{Q\,\alpha_{21}}{2k_1\,(\delta + \alpha_{21}X)} \quad .$$

The collocation method allows a simple relation to be derived between δ and X, which is

$$\Delta = \frac{B}{2} + \sqrt{\frac{B^2}{4} + \alpha_{21}\,B\,\xi} \tag{4.32}$$

where

$$B = 2k_{21}\,S_T + \alpha_{21}.$$

$$\Delta = \frac{Q\,\delta}{\rho\alpha_1\ell} \quad .$$

Equation 4.28 can now be solved for the phase change depth ξ. The result is

$$\frac{\alpha_{21}}{6}\xi^3 + \xi^2\left[\frac{\Delta}{2} + \alpha_{21}\,(1 + c_{21}\,S_T)\right] + \xi\left[(1 + c_{21}\,S_T)\,\Delta + \frac{1}{3}k_{21}\,S_T(\Delta - B)\right]$$
$$+ \frac{1}{3}c_{21}\,S_T\,\Delta(\Delta - B) = \tau\,(\Delta + \alpha_{21}\,\xi). \tag{4.33}$$

For the single phase case, when $S_T = 0$, eq 4.33 reduces to eq 4.25, as expected.

The surface temperature (for $t > t_0$) is given by

$$\frac{T_1(0,t) - T_o}{T_f - T_o} = 1 + \left[\frac{\alpha_{21}\,\xi^2 + 2\xi\,\Delta}{2(\Delta + \alpha_{21}\,\xi)}\right]\frac{1}{S_T} \quad . \tag{4.34}$$

4.4 ABLATION WITH COMPLETE REMOVAL OF MELT

Phase change problems for which the melting (or vaporizing) material is removed from the system might be useful for ice melting from vertical surfaces where the water can run off the surface due to gravity.

4.4.1 Constant Surface Heat Flux.

For this problem, there are two time domains to consider: the time before the surface temperature reaches the fusion value, during which no phase change occurs, and the phase change with removal of melt during which the surface temperature remains at T_f. These are shown in Figure 4.5.

4.4.1.1 Pre-Melt Solution (see App. B and C)

Initially, the solid is at T_o, and a heat flux Q is applied at t = 0. The thermal penetration depth is $\delta(t)$; when the surface temperature is T_f, $\delta = \delta_m$ and melting begins. This problem has been solved exactly (Carslaw and Jaeger 1959), but the integral method approximation will be

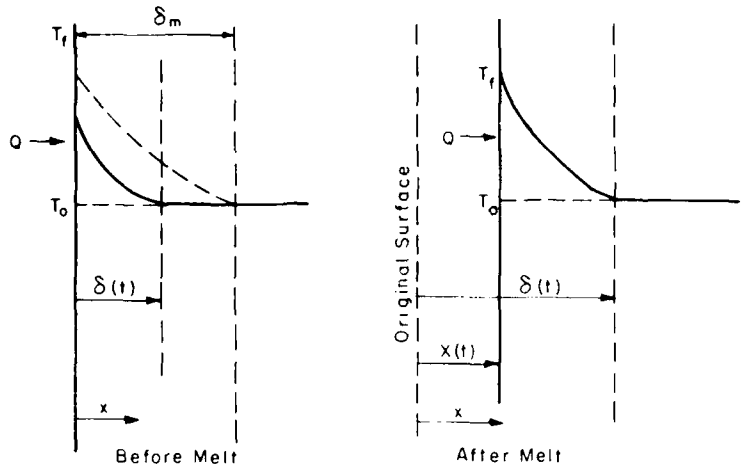


Figure 4.5. Complete removal of melt from surface.

used since this method is used for the melting problem. The problem can be formulated as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.35}$$

$$T(x,0) = T_o \tag{4.35a}$$

$$k\,\frac{\partial T(0,t)}{\partial x} = -Q\ . \tag{4.35b}$$

At the thermal penetration depth the temperature is T_o and the heat flux is zero, then

$$T(\delta,t) = 0 \tag{4.35c}$$

$$\frac{\partial T(\delta,t)}{\partial x} = 0\ . \tag{4.35d}$$

The heat balance integral is given by

$$\frac{d\theta}{dt} + \alpha\,\frac{\partial T(0,t)}{\partial x} - T_o\,\frac{d\delta}{dt} = 0 \tag{4.36}$$

with

$$\theta = \int_0^{\delta} T\,dx\ . \tag{4.37}$$

The temperature is assumed to have a quadratic profile and, with eq 4.35 a,b,c,

$$T = T_o + \frac{Q}{2\delta k}\,(x - \delta)^2 \tag{4.38}$$

From eq 4.36 and 4.37, the equation for δ is

$$\frac{d\delta^2}{dt} = 6\alpha$$

with the solution

$$\delta = \sqrt{6\alpha t} \quad . \tag{4.39}$$

The surface temperature is

$$T(0,t) = T_o + \frac{Q\delta}{2k} \quad .$$

The values of δ and t, when the surface temperature reaches the melt value, are given by

$$\delta_m = \frac{2k(T_f - T_o)}{Q} \tag{4.40}$$

$$t_m = \frac{2}{3} \frac{k^2(T_f - T_o)^2}{\alpha Q^2} \quad . \tag{4.41}$$

The exact value for time t_m is

$$t_m = \frac{\pi}{4} \frac{k_2(T_f - T_o)^2}{\alpha Q^2} \quad .$$

If a quartic temperature profile is used, the additional smoothness relations at δ can be used:

$$\frac{\partial^2 T(\delta,t)}{\partial x^2} = \frac{\partial^3 T(\delta,t)}{\partial x^3} = 0 \quad . \tag{4.42}$$

Then

$$T = T_o + \frac{Q}{4k\delta^3}(x - \delta)^4$$

$$\delta_m = \frac{4k(T_f - T_o)}{Q}$$

$$T_m = 0.800 \frac{k^2(T_f - T_o)^2}{\alpha Q^2} \quad .$$

4.4.1.2 Melt Solution

Once the surface temperature reaches T_f, at $t = t_m$, melting begins and the problem is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.43}$$

$$T(X,t) = T_f \tag{4.43a}$$

$$T(\delta,t) = T_o \tag{4.43b}$$

$$\frac{\partial T(\delta,t)}{\partial x} = 0 \tag{4.43c}$$

$$Q + k\,\frac{\partial T(X,t)}{\partial x} = \rho\ell\,\frac{dX}{dt} \tag{4.43d}$$

$$\delta(0) = \delta_m \tag{4.43e}$$

$$X(0) = 0 \quad . \tag{4.43f}$$

The heat balance integral equation is now

$$\frac{d\theta_2}{dt} - T_o\,\frac{d\delta}{dt} + T_f\,\frac{dX}{dt} + \frac{\alpha\rho\ell}{k}\,\frac{dX}{dt} = \frac{\alpha q}{k} \tag{4.44}$$

with

$$\theta_2 = \int_X^{\delta} T(x,t)dx \quad . \tag{4.45}$$

The quadratic temperature assumption satisfying eq 4.43 a,b,c, yields

$$T = T_f - 2\,\frac{(T_f - T_o)}{(\delta - X)}\,(x - X) + \frac{(T_f - T_o)}{(\delta - X)^2}\,(x - X)^2 \quad . \tag{4.46}$$

Equations 4.44-46 yield

$$\frac{1}{3}\,\frac{d\phi}{d\tau} + \left(\frac{S_T+1}{S_T}\right)\frac{d\sigma}{d\tau} = 1 \quad . \tag{4.47}$$

Equations 4.43d and 4.46 lead to a second differential equation:

$$\frac{1}{S_T}\,\frac{d\sigma}{d\tau} = 1 - \frac{2}{\phi} \tag{4.48}$$

where

$$\phi = \frac{Q(\delta - X)}{(T_f - T_o)k_1} \qquad S_T = \frac{c(T_f - T_o)}{\ell}$$

$$\tau = \frac{Q^2\,t}{\rho^2\alpha_1\ell^2} \qquad \sigma = \frac{XQ}{k_1(T_f - T_o)} \quad .$$

Eliminating $\frac{d\sigma}{d\tau}$ from eq 4.47 and 4.48 gives

$$\frac{d\phi}{d\tau} - \frac{6(1 + S_T)}{\phi} = -3\ S_T\ .$$

The solution to this equation is

$$\tau = -\frac{1}{3S_T}\left\{\phi - 2 + \frac{2(1 + S_T)}{S_T}\ \ln[S_T\ (1 - \frac{\phi}{2}) + 1]\right\}\ . \qquad (4.49)$$

From eq 4.49

$$d\tau = \frac{\phi\ d\phi}{3[2(1 + 1/S_T) - \phi]S_T}$$

and then

$$d\sigma = (1 - \frac{2}{\phi})\ \frac{\phi\ d\phi}{3[2(1 + 1/S_T) - \phi]}\ .$$

The solution to this equation is

$$\sigma = -\frac{1}{3}\ [\phi - 2 + \frac{2}{S_T}\ \ell n\ \left(S_T(1 - \frac{\phi}{2}) + 1\right)]\ . \qquad (4.50)$$

Equations 4.49 and 4.50 are the parametric equations for σ as a function of τ. Goodman (1958) plotted these equations for some values of S_T, as shown on Figure 4.6. Landau (1950) solved this problem numerically and the results agree quite well with eq 4.49 and 4.50, but as S_T become large the values tend to diverge for small times. This can be seen on Figure 4.7. A steady-state solution for σ_∞ can be easily found. If $d\sigma/d\tau = S_T\sigma_\infty$ then, from eq 4.47, ϕ = constant and

$$\sigma_\infty = \frac{1}{1 + S_T}\ .$$

For large values of time

$$\sigma = \frac{S_T}{1 + S_T}\ \tau\ .$$

This equation is identical to the exact solution of Landau (1950).

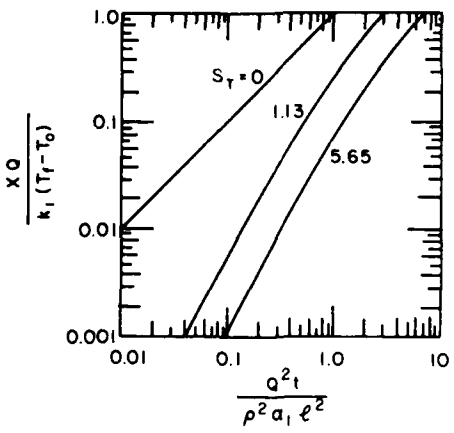


Figure 4.6. Melt-line location vs time, with complete removal of melt, eq 4.50 (adapted from Goodman 1958).

4.4.2 Variable Surface Heat Flux

The procedure detailed in the previous section can be used for the case of variable

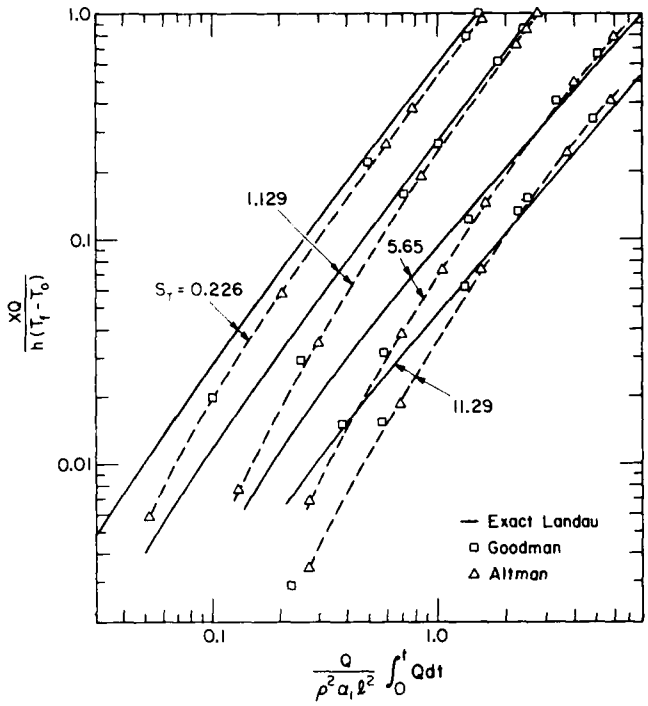


Figure 4.7. Melt-line location vs time, constant surface heat flux.

surface heat flux. Following Altman (1961) a fourth-order polynomial for the temperature will be used, with eq 4.42. Then

$$T = T_o + \frac{Q(t)}{4k\delta^3}(x - \delta)^4 \quad .$$

The surface temperature is

$$T(0,t) = T_o + \frac{Q\delta}{4k} \quad .$$

The equation for δ and the time when the surface temperature is T_f are

$$\delta(t) = \sqrt{20\alpha} \; [\frac{1}{Q} \int_0^t Q \, dt]^{\frac{1}{2}} \tag{4.51}$$

$$\delta_m = \frac{4k(T_f - T_o)}{Q(t_m)} \tag{4.52}$$

$$\int_0^{t_m} Q \, dt = \frac{4k^2(T_f - T_o)^2}{5\alpha \, Q(t_m)} \tag{4.53}$$

Equations 4.52 and 4.53 can be used to evaluate δ_m and t_m if the transient function q(t) is specified.

The temperature profile after melting begins is

$$\frac{T - T_f}{T_f - T_o} = -4\left(\frac{x - X}{\delta - X}\right) + 6\left(\frac{x - X}{\delta - X}\right)^2 - 4\left(\frac{x - X}{\delta - X}\right)^3 + \left(\frac{x - X}{\delta - X}\right)^4 . \qquad (4.54)$$

Using eq 4.43d, 4.44, 4.45, and 4.54 again leads to two differential equations:

$$\frac{d}{dt}(\delta - X) + 5\left(\frac{1 + S_T}{S_T}\right)\frac{dX}{dt} = \frac{5\alpha Q}{k(T_f - T_o)} \qquad (4.55)$$

$$\frac{dX}{dt} = \frac{1}{\rho\ell}\left[Q - \frac{4k(T_f - T_o)}{\delta - X}\right] . \qquad (4.56)$$

Using eq 4.55 in 4.56 produces an equation in $(\delta - X)$:

$$\frac{d}{dt}(\delta - X) - \frac{20k(T_f - T_o)}{\rho\ell}\left(1 + \frac{1}{S_T}\right)\frac{1}{(\delta - X)} = \frac{-5Q(t)}{\rho\ell} . \qquad (4.57)$$

The initial value is again $\delta(0) - X(0) = \delta_m$.

This equation can be solved numerically, given $Q(t)$,

$$X = \frac{1}{\rho\ell}\int_0^t \left[Q(t) - \frac{4k(T_f - T_o)}{(\delta - X)}\right] dt \qquad (4.58)$$

For $q(t)$ constant, the solution is

$$\tau = -\frac{1}{5S_T}\left[\phi - 4 + 4\left(1 + \frac{1}{S_T}\right)\ln\left(1 + S_T\left(1 - \frac{\phi}{4}\right)\right)\right] \qquad (4.59)$$

$$\sigma = -\frac{1}{5}\left[\phi - 4 + \frac{4}{S_T}\ln\left(1 + S_T\left(1 - \frac{\phi}{4}\right)\right)\right] . \qquad (4.60)$$

As often occurs in integral solutions, this fourth-power approximation is not superior to Goodman's (1958) second-power solution except that t_m is much closer to the exact value.

4.4.3 Ablation of a Slab with Finite Thickness

An approximate solution method has been given by Citron (1960), for ablation of a slab with finite thickness. The slab is originally at T_o and is insulated at $x = D$, while a constant heat flux Q_o impinges on the free surface as shown in Figure 4.8. Melting begins when the free surface temperature reaches the fusion value T_f and continues until the entire slab has melted.

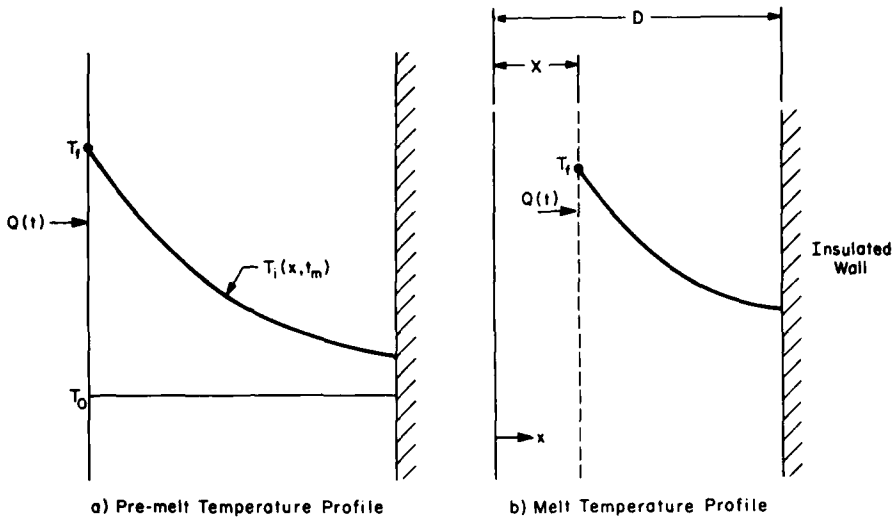


Figure 4.8. Ablation of slab with finite thickness.

The temperature of the slab before melt is reported by Carslaw and Jaeger (1959) as

$$T_i(x,t) = \frac{Q_o D}{k}\left[\frac{\alpha t}{D^2} + \frac{3x^2 - 6xD + 2D^2}{6D^2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2\alpha t}{D^2}} \cos\frac{n\pi x}{D}\right] . \quad (4.61a)$$

The quasi-steady form of this equation will be used:

$$T_i(x,t) = \frac{Q_o D}{k}\left[\frac{\alpha t}{D^2} + \frac{3x^2 - 6xD + 2D^2}{6D^2}\right] . \quad (4.61b)$$

The time for the surface temperature to reach T_f is

$$\frac{\alpha t_m}{D^2} = \frac{k(T_f - T_o)}{Q_o D} - \frac{1}{3} . \quad (4.62)$$

The temperature distribution at t_m is

$$T_i(x,t_m) = \frac{Q_o \ell}{k}\left\{\frac{\alpha t_m}{D^2} + \frac{3x^2 - 6xD + 2D^2}{6D^2}\right\} . \quad (4.63)$$

The mathematical description of the melt problem, $t \geq t_m$, is as follows:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (4.64)$$

$$T(X,t) = T_f \quad (4.64a)$$

$$\frac{\partial T}{\partial x}(D,t) = 0 \quad (4.64b)$$

$$T(x,t_m) + T_i(x,t_m) \tag{4.64c}$$

$$X(t_m) = 0 \tag{4.64d}$$

$$Q_o + k\,\frac{\partial T(X,t)}{\partial x} = \rho\,\ell\,\frac{dX}{dt} \quad . \tag{4.64e}$$

An approximate solution method is facilitated by immobilizing the phase change interface using a Landau Transformation. The following variables can be used:

$$z = \frac{x - X}{D}$$

$$\tau_1 = \frac{\alpha(t - t_m)}{D^2}$$

$$S_T = \frac{c(T_f - T_o)}{\ell}$$

$$r = \frac{k(T_f - T_o)}{Q_o D}$$

$$S = \frac{X}{D} \quad .$$

A new temperature variable is defined as

$$T = T_o + T_i(z,t_m) - \frac{\partial T_i(1-S,t_m)}{\partial z}\,\psi(z,\tau_1) \quad . \tag{4.65}$$

The nondimensional equations are

$$\frac{\partial^2\psi}{\partial z^2} + \dot{S}\,\frac{\partial\psi}{\partial z} - \frac{\dot{S}}{S}\,\psi = \frac{\partial\psi}{\partial\tau_1} - \frac{1}{S} - \frac{\dot{S}(z-1)}{S} \tag{4.66}$$

$$\psi(0,\tau_1) = 0 \tag{4.66a}$$

$$\frac{\partial\,\psi(1 - S,\tau_1)}{\partial z} = 1 \tag{4.66b}$$

$$\lim_{\tau_1 \to o} \left\{\frac{\partial T_i(1 - S,t_m)}{\partial t}\,\psi(z,\tau_1)\right\} = 0 \tag{4.66c}$$

$$S(0) = 0 \tag{4.66d}$$

$$\frac{\partial\psi(0,\tau_1)}{\partial z} = \frac{r}{S_T}\,\frac{\dot{S}}{S} \quad . \tag{4.66e}$$

The solution follows iteratively by finding a first-order solution with the the assumption that the term $\partial\psi/\partial\tau_1$ will be zero in the energy equation. Successive approximations can then be calculated by using the i-th order $\partial\psi_i/\partial\tau$ correction in each solution.

The first order solution is

$$\psi_1 \ (z,\tau_1) = A + z + C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z} \tag{4.67}$$

$$\lambda_{1,2} = -\frac{\dot{S}}{2} \pm \sqrt{\frac{\dot{S}^2}{4} + \frac{\dot{S}}{S}}$$

$$A = S - 1 + \frac{1}{\dot{S}}$$

$$C_1 = \frac{A\ p}{1 - p}$$

$$C_2 = \frac{-A}{1 - p}$$

$$p = \lambda_2/\lambda_1 \ e^{(\lambda_2 - \lambda_1)(1 - S)} \ .$$

The equation for the phase change interface is

$$\frac{r}{S_T}\dot{S} = S\left\{1 - \frac{\lambda_2 A(1 - \frac{\lambda_1}{\lambda_2} p)}{1 - p}\right\} \ . \tag{4.68}$$

The phase change interface equation can be solved easily by numerical techniques.

A starting solution must be used at $\tau_1 = 0$ since the equation is singular there. The phase change interface near $\tau_1 = 0$ is approximately

$$\dot{S} = \frac{S_T}{r} \sqrt{\frac{S}{\dot{S}}} \ . \tag{4.69}$$

The starting solution is

$$S = \frac{S_T}{r} \left(\frac{2\tau_1}{3}\right)^{\frac{3}{2}} \ . \tag{4.70}$$

Citron (1960) showed that the first-order solution, for the complete melt time, is about 9% high for $S_T/r = 5.52$ and is essentially exact for $S_T/r = 56.4$. Citron (1962) also solved the same problem using an expansion of the variables about the phase change interface.

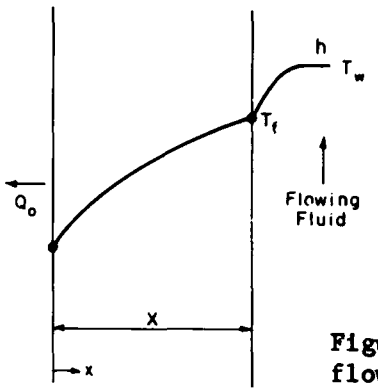


Figure 4.9. Freeze of a flowing fluid.

4.5 FREEZING OF A FLOWING FLUID

The freezing of a flowing fluid with constant temperature and convective coefficient is shown in Figure 4.9. A constant heat flux is extracted from the surface to induce freezing. This problem has no steady-state solution, but a constant heat flux during cooling is difficult to maintain in any case. The equations are

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{4.71}$$

$$\rho \ell \frac{dX}{dt} = k \frac{\partial T(X,t)}{\partial x} - h\,(T_w - T_f) \tag{4.71a}$$

$$k \frac{\partial T(0,t)}{\partial x} = Q_o \tag{4.71b}$$

$$T(X,t) = T_f \ . \tag{4.71c}$$

A perturbation solution will be used, as outlined in Appendix A. Nondimensional equations are

$$\frac{\partial^2 U}{\partial \zeta^2} = S_{Te} \frac{\partial U}{\partial \tau^*} \tag{4.72}$$

$$\frac{d\varepsilon}{d\tau} = \frac{\partial U(\varepsilon,\tau^*)}{\partial \zeta} - \beta \tag{4.72a}$$

$$\frac{\partial U(0,\tau^*)}{\partial \zeta} = 1 \tag{4.72b}$$

$$U(\varepsilon,\tau^*) = 0 \ . \tag{4.72c}$$

Equation 4.72 can be written with the time variable τ^* replaced by ε:

$$\frac{\partial^2 U}{\partial \zeta^2} = g\, S_{Te} \frac{\partial U(\zeta,\varepsilon)}{\partial \varepsilon} \tag{4.72d}$$

$$g = \frac{d\varepsilon}{d\tau} \tag{4.73}$$

where

$$U = (T - T_f)\frac{h}{Q_o} \qquad \varepsilon = X\frac{h}{k} \qquad \zeta = \frac{xh}{k}$$

$$\tau^* = \frac{Q_o ht}{\rho\ell\ k} \qquad S_{Te} = \frac{Q_o c}{h\ell} \qquad \beta = \frac{h(T_w - T_f)}{Q_o} \qquad \gamma = 1-\beta\ .$$

The temperature and the rate of change of the phase change depth are expanded as

$$U = \sum_{i=0}^{N} S_{Te}{}^{i}\ U_i \tag{4.74}$$

$$g = \sum_{i=0}^{N} S_{Te}{}^{i}\ g_i\ . \tag{4.75}$$

Also

$$\tau^* = \sum_{i=0}^{N} \tau_i\ S_{Te}{}^{i} \tag{4.76}$$

where the relations between τ_i and g_i are given in Appendix A.

Substituting into eq 4.72a-d leads to the following systems of asymptotic equations:

$$\frac{\partial^2 U_i}{\partial\ \zeta^2} = \begin{cases} 0 & i = 0 \\ \sum_{j=1}^{i} g_{j-1}\ \dfrac{\partial U_{i-j}}{\partial\ \varepsilon} & i \neq 0 \end{cases} \tag{4.77}$$

$$g_i = \begin{cases} \dfrac{\partial U_i(\varepsilon,\varepsilon)}{\partial\ \zeta} - \beta & i = 0 \\ \dfrac{\partial U_i(\varepsilon,\varepsilon)}{\partial\ \zeta} & i > 0 \end{cases} \tag{4.78}$$

$$\frac{\partial U_i(0,\varepsilon)}{\partial\ \zeta} = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases} \tag{4.79}$$

$$U_i(\varepsilon,\varepsilon) = 0\ . \tag{4.80}$$

The solutions are

$$U_o = \zeta - \varepsilon \tag{4.81a}$$

$$g_o = \gamma \tag{4.81b}$$

$$U_1 = -\frac{\gamma}{2}\ (\zeta^2 - \varepsilon^2) \tag{4.82a}$$

$$g_1 = -\gamma\ \varepsilon \tag{4.82b}$$

$$U_2 = \frac{\gamma(\gamma+1)\varepsilon}{2}\ [\zeta^2 - \varepsilon^2] \tag{4.83a}$$

$$g_2 = \gamma(\gamma+1)\varepsilon^2 \tag{4.83b}$$

$$U_3 = \frac{\gamma^2(\gamma+1)\zeta^4}{24} - \gamma^2\left(\frac{7}{2} + \frac{3}{2}\gamma + \frac{1}{\gamma}\right)\frac{\varepsilon^2\zeta^2}{2} + \left(17\gamma+41+\frac{12}{\gamma}\right)\frac{\gamma^2\varepsilon^4}{24} \tag{4.84a}$$

$$g_3 = -\frac{\varepsilon^2\gamma^2}{3}\left(4\gamma + 10 + \frac{3}{\gamma}\right) \tag{4.84b}$$

$$\tau^* = \frac{1}{\gamma}\left[\varepsilon + \frac{\varepsilon^2}{2} S_{Te} - \frac{\gamma}{3}\varepsilon^3\ S_{Te}{}^2 + \frac{\varepsilon^4}{3}\left(\gamma^2 + \gamma - \frac{3}{4}\right)S_{Te}{}^3\right]\ . \tag{4.85}$$

Seeniraj and Bose (1982) presented this solution and showed that the values compared very well with a numerical solution of Goodling and Khader (1975) for $S_{Te} \leq 2.0$ and $\beta = 0.6$. The result also agrees well with eq 4.13 and 4.25 for values of $\tau < 0.4$.

NOMENCLATURE - CHAPTER 4

B	$2k_{21}S_T + \alpha_{21}$
c	specific heat
D	thickness of slab
g	$d\varepsilon/d\tau$
G	Q/k_1
h	surface coefficient of convection
k	thermal conductivity
ℓ	latent heat of solidification
L	$\rho\ell$ volumetric latent heat
Q	heat flux imposed at surface
Q_o	constant surface heat flux

r	$\frac{k(T_f - T_o)}{Q_o D}$
S	$\frac{X}{D}$
S_{Te}	$Q_o c_1/(h\ell)$ modified Stefan number
S_T	$c_1(T_f - T_o)/\ell$ Stefan number
t	time
t_m	time for T_s to reach fusion temperature
T	temperature
T_f, T_o, T_s	fusion, initial, surface temperature
T_i	temperature distribution at beginning of melt
U	$(T-T_f)\ h/Q_o$
$V(x)$	initial temperature distribution
x	Cartesian coordinate
X	phase-change depth
z	$\frac{x - X}{D}$, Landau transformation
α	thermal diffusivity
β	$\frac{h(T_w - T_f)}{Q_o}$
γ	$1-\beta$
δ	thermal penetration depth
δ_m	value of δ when T_s reaches fusion temperature
Δ	$Q\delta/(\rho\alpha_1\ell)$
ε	Xh/k, dimensionless phase-change depth
ζ	xh/k
η_i	$x/(2\sqrt{\alpha_i t})$

θ_i integrated temperature

ξ $XQ/(\rho\alpha_1\ell)$ dimensionless phase-change depth

ρ density

σ $XQ/[k_1(T_f - T_o)]$

σ_∞ steady-state value of phase-change depth

τ $\dfrac{Q}{\rho^2\alpha_1\ell^2} \displaystyle\int_0^t Q(t)\,dt$

τ_1 $\dfrac{\alpha(t - t_m)}{D^2}$

τ^* $\dfrac{Q_o\ h\ t}{\rho\ell\ k}$

ϕ $Q(\delta-X)/[k_1(T_f - T_o)]$

ψ $[T_o + T_i(z,t_m) - T(x,t)]/\dfrac{\partial T_i(1-S,t_m)}{\partial t}$, dimensionless temperature

Subscripts

1,2 different phases of medium

CHAPTER 5. THAW BENEATH INSULATED STRUCTURES: QUASI-STEADY SOLUTIONS

The effect of heated structures on the underlying medium involves the study of conduction heat transfer in media that can undergo freezing and thawing. This aspect of heat transfer is important for the melting of permanently frozen ground (permafrost) or the freezing of thawed soils which can lead to frost heave complications. Lachenbruch (1957a, b, 1959) and Jumikis (1978) applied linear conduction theory to the effects of heating on permafrost. Linear theory was possible since no phase change was considered despite the direct reference to permafrost problems. However, if phase change is introduced, the conduction problem becomes nonlinear and only a few exact solutions exist (for the simplest geometries and boundary conditions) as has been noted in Chapters 2-4 and Lunardini (1981a).

Geometries of practical interest do not allow exact solutions of the thermal problem to be found. Thus closed-form solutions, as opposed to numerical evaluations, have relied on approximate methods.

The quasi-steady method is not as rigorous as the heat balance integral method, but it can be applied to a wide variety of geometries (see App. A-C). Applications have included uninsulated buried pipes (Hwang 1977, Thornton 1976, Porkhaev 1963), insulated buried pipes (Lunardini 1981b, Seshadri and Krishnayya 1980), and three-dimensional structures (Porkhaev 1970). Widely used calculated results are those of Porkhaev (1970). Lunardini (1982, 1983) derived new quasi-steady relations for insulated geometries including semi-infinite strips (roads, sewers), rectangular buildings, circular storage tanks, and buried pipes. The quantitative results for insulated systems are superior in accuracy to those of Porkhaev (1970). Graphical solutions for practical geometries will be given in this chapter.

The quasi-steady approximation assumes that the temperature field varies successively from one steady state to another. Let us examine the limitations of this approximation. Consider an infinite strip as shown in Figure 5.1. Initially the temperature of the semi-infinite space is uniform at T_o and the insulated surface of the strip jumps to T_p at time zero. The temperature beneath the strip then starts to vary with

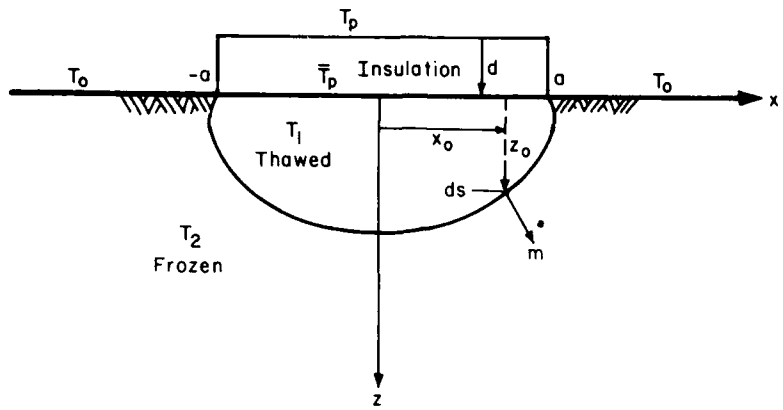


Figure 5.1. Insulated semi-infinite strip.

time. The properties of the material differ for regions below and above the fusion temperature T_f.

The equations (see Nomenclature at end of chapter) for the conduction heat flow problem are

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial z^2} = \frac{1}{\kappa_1}\frac{\partial T_1}{\partial t} \tag{5.1}$$

$$T_1(x,0,t) = \bar{T}_p; \quad -a \leq x \leq a \quad ; \; t > 0 \tag{5.1a}$$

$$T_1(x_o, z_o, t) = T_f \tag{5.1b}$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial z^2} = \frac{1}{\kappa_2}\frac{\partial T_2}{\partial t} \tag{5.2}$$

$$T_2(x, 0, t) = T_o \qquad x < -a \text{ or } x > +a \tag{5.2a}$$

$$T_2(x,z,0) = T_o \tag{5.2b}$$

$$T_2(x_o, z_o, t) = T_f \tag{5.2c}$$

$$\lim_{x,z\to\infty} T_2(x, z, t) \to T_o \; . \tag{5.2d}$$

If thawing is considered, the energy balance over the surface S separating the frozen and thawed zones is

$$\int_S \left(k_1 \frac{\partial T_1}{\partial m} - k_2 \frac{\partial T_2}{\partial m}\right)_{x_o,z_o} ds = -L\frac{dV}{dt} \tag{5.3}$$

where

L = volumetric latent heat of fusion

m = outward normal to the interface

V = volume of the material changing phase.

Equations 5.1-3 can be nondimensionalized using the following quantities:

$$\zeta = \frac{x}{a} \qquad \eta = \frac{y}{a} \qquad \xi = \frac{z}{a} \qquad m' = \frac{m}{a} \qquad s' = \frac{s}{a^2} \qquad V' = \frac{V}{a^3}$$

$$S' = \frac{S}{a^2} \qquad \phi_1 = \frac{T_1 - T_f}{\overline{T}_p - T_f} \qquad \phi_2 = \frac{T_2 - T_o}{T_f - T_o} \qquad \beta = k_{21}\frac{T_f - T_o}{T_f - T_f}$$

$$S_T = \frac{C_1(T_p - T_f)}{L}$$

$$\tau = \frac{2\kappa_1 S_T t}{\pi a^2}$$

$$\phi_o = \frac{T_o - T_f}{\overline{T}_p - T_f}$$

$$\frac{\partial^2\phi_1}{\partial\zeta^2} + \frac{\partial^2\phi_1}{\partial\xi^2} = \frac{2}{\pi} S_T \frac{\partial\phi_1}{\partial\tau} \tag{5.4}$$

$$\phi_1(\zeta, 0, \tau) = 1 \qquad -1 \leq \zeta \leq +1 \tag{5.4a}$$

$$\phi_1(\zeta_o, \xi_o, \tau) = 0 \tag{5.4b}$$

$$\frac{\partial^2\phi_2}{\partial\zeta^2} + \frac{\partial^2\phi_2}{\partial\xi^2} = \frac{2S_T}{\pi\kappa_{21}} \frac{\partial\phi_2}{\partial\tau} \tag{5.5}$$

$$\phi_2(\zeta, 0, \tau) = 0 \qquad \zeta < -1 \text{ or } \zeta > +1 \tag{5.5a}$$

$$\phi_2(\zeta, \xi, 0) = 0 \tag{5.5b}$$

$$\phi_2(\zeta_o, \xi_o, \tau) = 1 \tag{5.5c}$$

$$\lim_{\zeta,\xi\to\infty} \phi_2(\zeta, \xi, \tau) = 0 \tag{5.5d}$$

$$\int_{S'} \left(\frac{\overline{T}_p - T_f}{T_p - T_f} \frac{\partial\phi_1}{\partial m'} - \beta \frac{\partial\phi_2}{\partial m'}\right)_{\zeta_o,\xi_o} ds' = -\frac{2}{\pi}\frac{dV'}{d\tau} . \tag{5.6}$$

Equations 5.4-6 cannot be solved exactly but the diffusion equations, eq 5.4 and 5.5, reduce to the steady-state case if the Stefan number S_T, is small. In this case it is not necessary to solve the transient conduction equation but only the much simpler steady-state conduction equation, the Laplace equation. For simplicity, assume that the properties of the frozen and thawed media are identical. Then the quasi-steady system to be solved is

$$\frac{\partial^2\phi}{\partial\zeta^2} + \frac{\partial^2\phi}{\partial\xi^2} = 0 \tag{5.7}$$

$$\phi\ (\zeta,\ o) = \begin{cases} 1 & -1 \leq \zeta \leq 1 \\ 0 & \zeta > 1 \text{ or } \zeta < -1 \end{cases} \tag{5.7a}$$

$$\lim_{\zeta,\xi\to\infty} \phi(\zeta,\ \xi) = 0\ . \tag{5.7b}$$

Notice that the initial condition cannot be satisfied and that the energy boundary equation is solved by merely substituting in the solution for ϕ. It has been shown (Lunardini 1981a) that eq 5.7-5.7b is the zeroth system of a perturbation solution of eq 5.4-5.5. The quasi-steady method then consists of solving the steady-state form of the problem and locating the phase change boundary with the use of an equation similar to eq 5.6. Clearly the accuracy of the quasi-steady method depends upon the magnitude of the Stefan number, which is the ratio of the sensible to the latent heat for the material. For systems with a large latent heat relative to the sensible heat, it can be expected that the quasi-steady approximation will be reasonably good. This covers soil systems and many phase-change materials used for latent heat storage.

The utility of the quasi-steady methods lies in the fact that for many phase change problems the equivalent steady-state solution can be written down immediately or easily found.

The solution to eq 5.7-5.7b, which is well known and will be useful (Lunardini 1981a), is

$$\phi = \frac{1}{\pi}\left\{\tan^{-1}\left(\frac{1+\zeta}{\xi}\right) + \tan^{-1}\left(\frac{1-\zeta}{\xi}\right)\right\} = \frac{1}{\pi}\tan^{-1}\left(\frac{2\xi}{\zeta^2+\xi^2-1}\right) = f(\zeta,\xi) \tag{5.8}$$

5.1 GENERAL QUASI-STEADY RELATIONS

General equations can be derived which will be valid for a class of important phase-change problems. Assume that the solution to the quasi-steady form of eq 5.4 is

$$\phi_1 = A_1(\tau) + B_1(\tau)\ f(\zeta,\xi)\ . \tag{5.9}$$

This equation, when combined with eq 5.4, gives

$$\phi_1 = \frac{f - f_o}{1 - f_o} \tag{5.10}$$

The function f is the solution to the equivalent steady-state equations and f_o is the value of the function on the phase-change surface where the temperature is at the fusion value.

Also let

$$\phi_2 = A_2(\tau) + B_2(\tau)\ f(\zeta,\xi)\ . \tag{5.11}$$

Then using eq 5.5a,c,

$$\phi_2 = \frac{f}{f_o}\ . \tag{5.12}$$

Once f_o has been determined, the total solution will be known. Equation 5.6 will be evaluated at only one location on the phase-change boundary:

$$\frac{\overline{T}_p - T_f}{T_p - T_f}\ \frac{\partial\phi_1}{\partial m'} - \beta\ \frac{\partial\phi_2}{\partial m'} = -\frac{2}{\pi}\frac{dp}{d\tau}\ . \tag{5.13}$$

The relation between dp', the small amount of material thawed during $d\tau$, and $d\xi_o$ is shown in Figure 5.2:

$$dp' = \left(\frac{\sqrt{(\frac{\partial f}{\partial\zeta})^2 + (\frac{\partial f}{\partial\xi})^2}}{\frac{\partial f}{\partial\xi}}\right)_{\xi_o} d\xi_o\ . \tag{5.14}$$

Also

$$\frac{\partial\phi}{\partial m'} = \sqrt{(\frac{\partial\phi}{\partial\zeta})^2 + (\frac{\partial\phi}{\partial\xi})^2}\ . \tag{5.15}$$

With these relations eq 5.13 is

$$\left(\frac{\overline{T}_p - T_f}{T_p - T_f}\ \frac{1}{f_o - 1} + \frac{\beta}{f_o}\right)\left(\frac{\partial f}{\partial\xi}\right)_{\zeta_o,\xi_o} = \frac{2}{\pi}\frac{d\xi_o}{d\tau}\ . \tag{5.16}$$

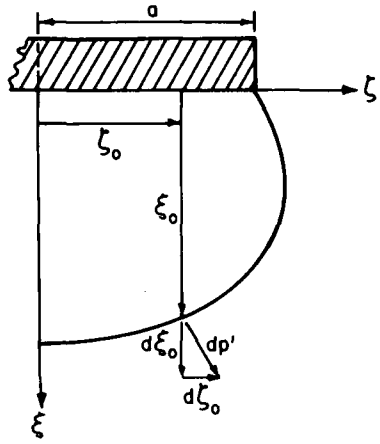


Figure 5.2. Geometry for thaw at (ζ_o, ξ_o).

It is necessary to consider the effect of the insulation since the temperature of the ground insulation interface $\overline{T}_p$ is an unknown function of time. There are several ways to handle the insulation.

5.1.1 Method 1

The heat flow through the insulation will be equated to the heat flow through the thawed soil at $\xi = 0$. This concept has been used by Seshradi and Krishnayya (1980) and by Lunardini (1981b):

$$-\frac{k_1}{a}(\overline{T}_p - T_f)\frac{\partial\phi_1(0,0)}{\partial\xi} = k_i\frac{(T_p - \overline{T}_p)}{d} \quad . \tag{5.17}$$

By using eq 5.10, this yields

$$\frac{\overline{T}_p - T_f}{T_p - T_f} = \frac{1}{1 - \frac{2\alpha}{1-f_o}\frac{\partial f(0,0)}{\partial\xi}} \tag{5.18}$$

where α is the insulation parameter given in the nomenclature.

The general interface equation, eq 5.16, is now

$$\left(\frac{1}{f_o - 1 + 2\alpha\frac{\partial f(0,0)}{\partial\xi}} + \frac{\beta}{f_o}\right)\left(\frac{\partial f}{\partial\xi}\right)_{\zeta_o,\xi_o} = \frac{2}{\pi}\frac{d\xi_o}{d\tau} \quad . \tag{5.19}$$

The final, steady-state, or limiting solution occurs when $d\xi_o/d\tau = 0$ in eq 5.19. This is

$$\frac{1}{f_{o\infty} - 1 + 2\alpha\frac{\partial f(0,0)}{\partial\xi}} + \frac{\beta}{f_{o\infty}} = 0 \tag{5.20}$$

or

$$f_{o\infty} = \frac{\beta}{1+\beta}\left[1 - 2\alpha\frac{\partial f(0,0)}{\partial\xi}\right] \quad . \tag{5.21}$$

The heat flux, into the thawed soil, at the center of the heated surface is

$$q = -k_1 \left(\frac{\partial T_1}{\partial z}\right)_{x,y,z=0} \quad . \tag{5.22}$$

This can be written as

$$q = \frac{-k_1(T_p - T_f)}{a[1 - f_o - 2\alpha \frac{\partial f(0,0)}{\partial \xi}]} \frac{\partial f(0,0)}{\partial \xi} \quad . \tag{5.23}$$

To evaluate eq 5.19 it is only necessary to find the appropriate, steady-state, geometric function f.

5.1.2 Method 2

This approach assumes that the effect of the insulation is accounted for by considering an excess layer of soil with a thermal resistance equal to that of the actual insulation. The excess soil layer is only applied to the thawed zone temperature relations. The concept was introduced by Porkhaev (1963, 1970).

The temperature equations are

$$\phi_1' = \frac{T_1 - T_f}{T_p - T_f} = \frac{f' - f_o'}{1 - f_o'} \tag{5.24}$$

$$\phi_2' = \frac{T_2 - T_o}{T_f - T_o} = \frac{g}{g_o} \tag{5.25}$$

where f' is the steady-state solution, with the excess thawed soil for insulation, and g is the usual steady-state solution. Equation 5.3 then becomes

$$\left\{\frac{1}{1-f_o'} \sqrt{\left(\frac{\partial f'}{\partial \zeta}\right)^2 + \left(\frac{\partial f'}{\partial \xi}\right)^2} - \frac{\beta}{g_o} \sqrt{\left(\frac{\partial g}{\partial \zeta}\right)^2 + \left(\frac{\partial g}{\partial \xi}\right)^2}\right\} \left(\frac{\partial f'}{\partial \xi}\right)_{\zeta_o, \xi_o} =$$

$$- \frac{2}{\pi} \sqrt{\left(\frac{\partial f'}{\partial \zeta}\right)^2 + \left(\frac{\partial f'}{\partial \xi}\right)^2} \frac{d\xi_o}{d\tau} \quad . \tag{5.26}$$

The center heat flux to the thawed soil is

$$q = - \frac{k_1(T_p - T_f)}{a(1-f_o')} \frac{\partial f'(0,0)}{\partial \xi} \quad . \tag{5.27}$$

While both methods approximate the steady state solution, when an insulation layer is present, the first method gives more accurate results (Lunardini 1982). A still more accurate approach would solve exactly the

steady-state problem, probably by using complex variables with a Schwarz-Christoffel transformation.

The same relations apply to the freezing case if $\tau = 2k_1 (T_f - T_p)\, t/(\pi a^2 L)$ and region 1 is the frozen layer.

It is now possible to examine, quantitatively, several practical geometries.

5.2 SEMI-INFINITE STRIP

The semi-infinite strip can represent roads, shallow rivers, or very long rectangular buildings. The geometry of the system is shown in Figure 5.1.

The steady-state solution is given by Lunardini (1981a) - see eq 5.8 - and the geometric function f is

$$f = \frac{1}{\pi} \tan^{-1} \left(\frac{2\xi}{\zeta^2 + \xi^2 - 1}\right) = \frac{1}{\pi} \left\{\tan^{-1} \left(\frac{1+\zeta}{\xi}\right) + \tan^{-1} \left(\frac{1-\zeta}{\xi}\right)\right\} \quad . \tag{5.28}$$

Through the use of eq 5.28 and 5.19 the phase change interface equation is

$$d\tau = \frac{[(\zeta_o^2 + \xi_o^2 - 1)^2 + 4\ \xi_o^2]\ d\xi_o}{(\zeta_o^2 - \xi_o^2 - 1)\left[\dfrac{1}{f_o - 1 - \dfrac{4\alpha}{\pi}} + \dfrac{\beta}{f_o}\right]} \quad . \tag{5.29}$$

Thus

$$\tau = \int_0^{\gamma} \frac{[(\zeta^2 + u^2 - 1)^2 + 4u]\ du}{(\zeta^2 - u^2 - 1)\left[\dfrac{1}{f_o - 1 - \dfrac{4\alpha}{\pi}} + \dfrac{\beta}{f_o}\right]} \tag{5.30}$$

where $f_o = \frac{1}{\pi} \tan^{-1} \left(\frac{2u}{\zeta^2 + u^2 - 1}\right)$.

The limiting interface, at $\tau = \infty$, is

$$\xi_{o\infty} = \cot p_1 + \sqrt{\cot^2 p_1 - (\zeta^2 - 1)} \tag{5.31}$$

where

$$p_1 = \frac{\beta}{1+\beta} (\pi + 4\alpha) \quad .$$

Since eq 5.30 is valid for any location ζ, it will be convenient to evaluate the equation at $\zeta = 0$. This is along the axis of symmetry for the geometry. Then

$$\tau = - \int_0^{\gamma} \frac{(u^2 + 1)du}{\dfrac{1}{f(u) - (1 + \frac{4\alpha}{\pi})} + \dfrac{\beta}{f(u)}} \tag{5.32}$$

where

$$f = \frac{2}{\pi} \operatorname{ctn}^{-1} u$$

γ = the value of ξ_o along the axis of symmetry.

Equation 5.31 reduces to

$$\gamma_\infty = \cot \left[\frac{\beta}{\beta+1} \left(\frac{\pi}{2} + 2\alpha\right)\right] . \tag{5.33}$$

Since the value of f_o is constant, the value of the thaw depth at any ζ, γx, can be found as a function of the center line value γ:

$$\gamma_x = \frac{\gamma^2 - 1}{2\gamma} + \sqrt{\frac{(\gamma^2 - 1)^2}{4\gamma^2} - (\zeta^2 - 1)} . \tag{5.34}$$

When $\gamma < 1$, the phase change at the edge of the strip ($\zeta = 1.0$) is zero. Equation 5.23 for the surface heat flux is

$$q = \frac{2k_1(T_p - T_f)}{\pi a[1 + \frac{4\alpha}{\pi} - f_o]} . \tag{5.35}$$

Equation 5.32 can be evaluated exactly if $\beta = 0$. For this case

$$\tau = (1 + \frac{4\alpha}{\pi})(\frac{\gamma^3}{3} + \gamma) = \frac{2}{\pi}\{(\frac{\gamma^3}{3} - \gamma) \operatorname{ctn}^{-1}\gamma + \frac{1}{3} \ln(1 + \gamma^2) + \frac{1}{6} \gamma^2\} . \tag{5.36}$$

Equation 5.32 was evaluated numerically and plotted as Figures 5.3-14. Equation 5.31 is also plotted as Figures 5.15-17 for various values of ζ.

The temperatures, in the thawed and frozen zones, can be found with eq 5.10 and 5.12.

5.3 RECTANGULAR BUILDING

While the semi-infinite strip can represent a building with a very large length to width ratio, it is useful to have solutions for any aspect ratio. The geometry is as shown in Figure 5.18.

The steady-state solution is given by Lunardini (1981a) and the geometric factor is

$$f = \frac{1}{2\pi} \left\{ \tan^{-1} \frac{(\zeta+1)(\eta+n)}{\xi \sqrt{\xi^2 + (\zeta+1)^2 + (\eta+n)^2}} - \tan^{-1} \frac{(\zeta-1)(\eta+n)}{\xi\sqrt{\xi^2 + (\zeta-1) + (\eta+n)^2}} \right.$$

$$\left. - \tan^{-1} \frac{(\zeta+1)(\eta-n)}{\xi\sqrt{\xi^2 + (\zeta+1)^2 + (\eta-n)^2}} + \tan^{-1} \frac{(\zeta-1)(\eta-n)}{\xi\sqrt{\xi^2 + (\zeta-1)^2 + (\eta-n)^2}} \right\} \tag{5.37}$$

where $n = b/a$ is the aspect ratio.

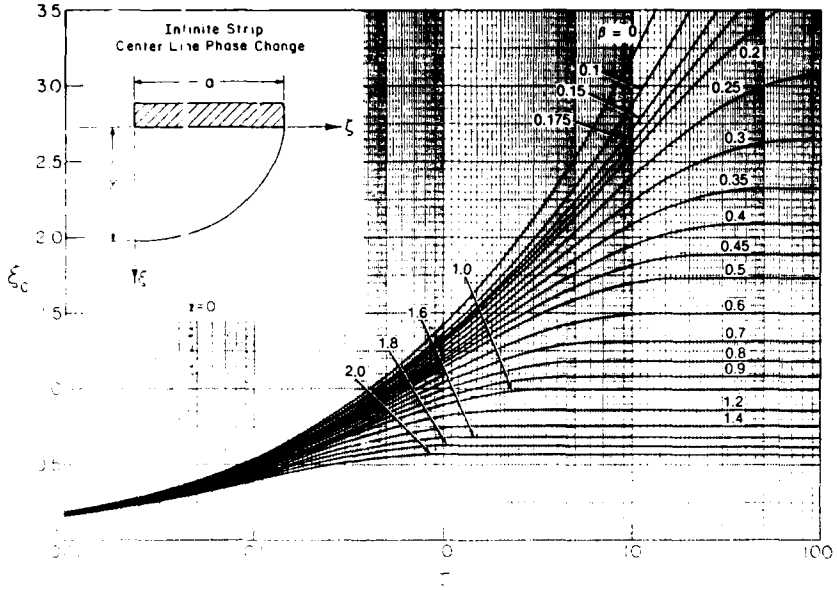


Figure 5.3.

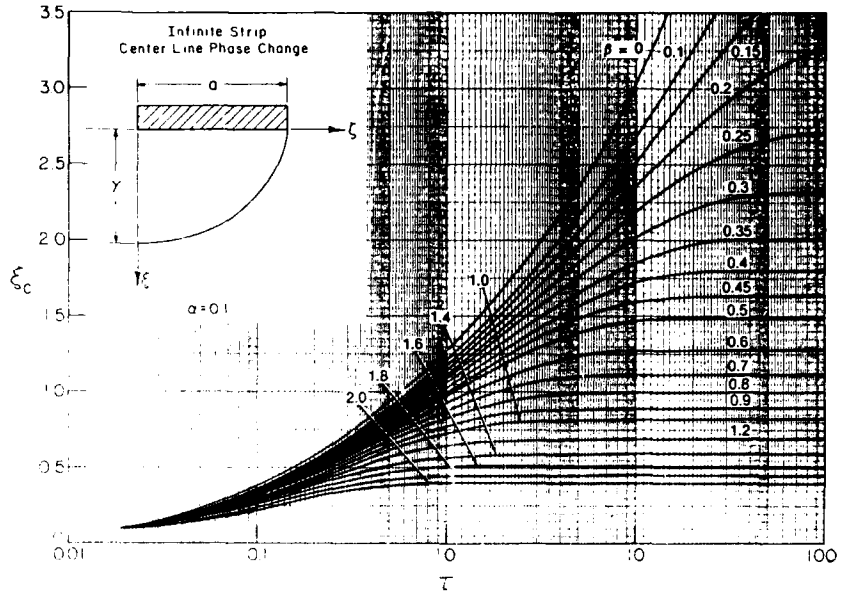


Figure 5.4.

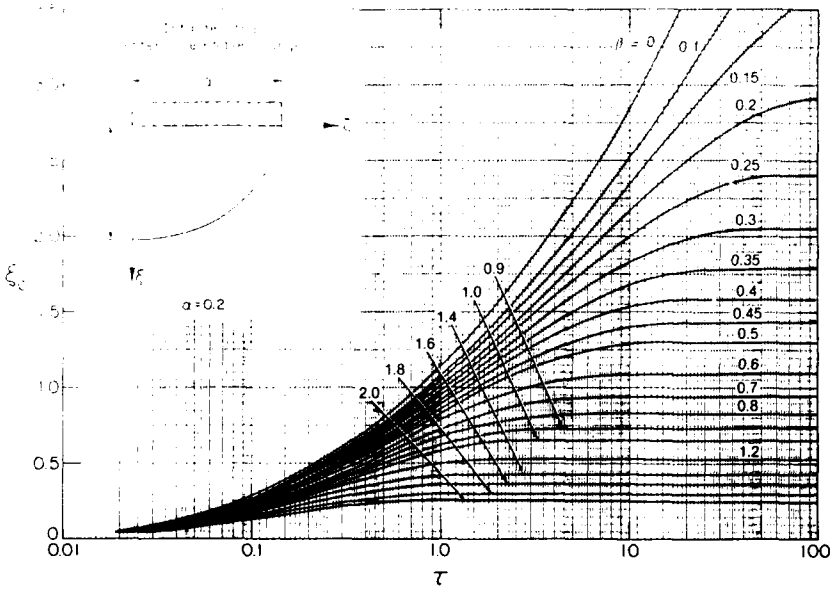


Figure 5.5.

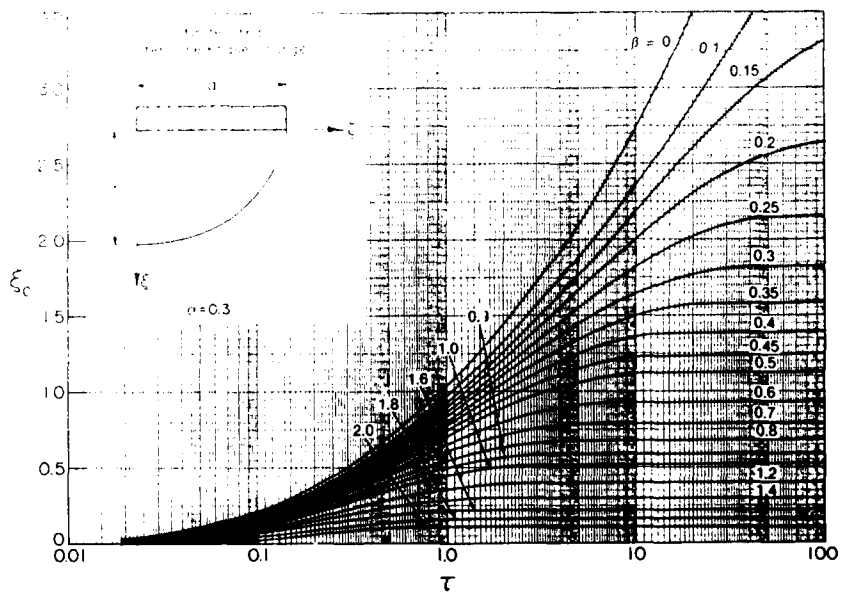


Figure 5.6.

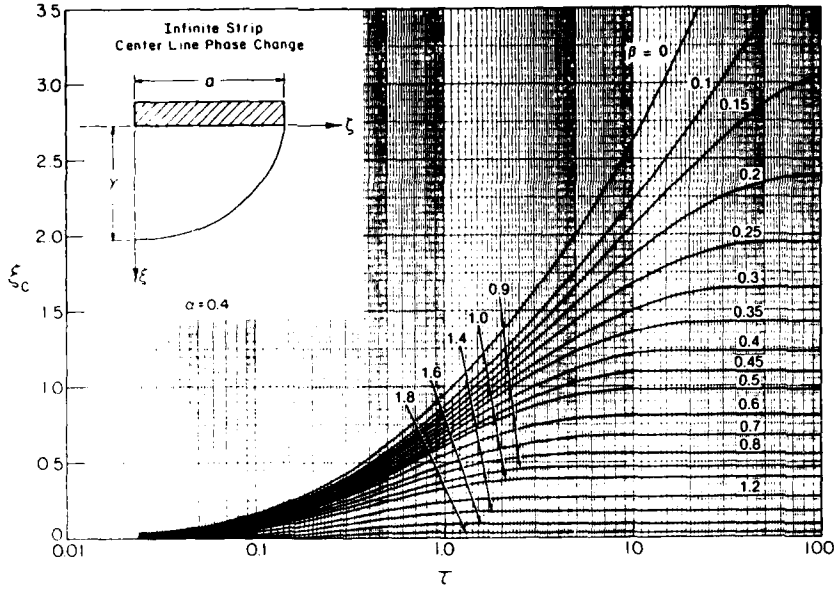


Figure 5.7.

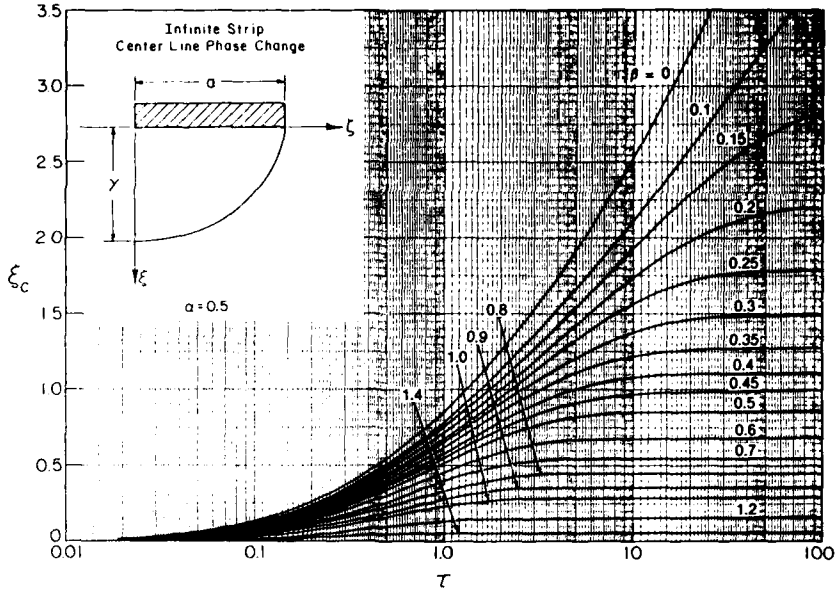


Figure 5.8.

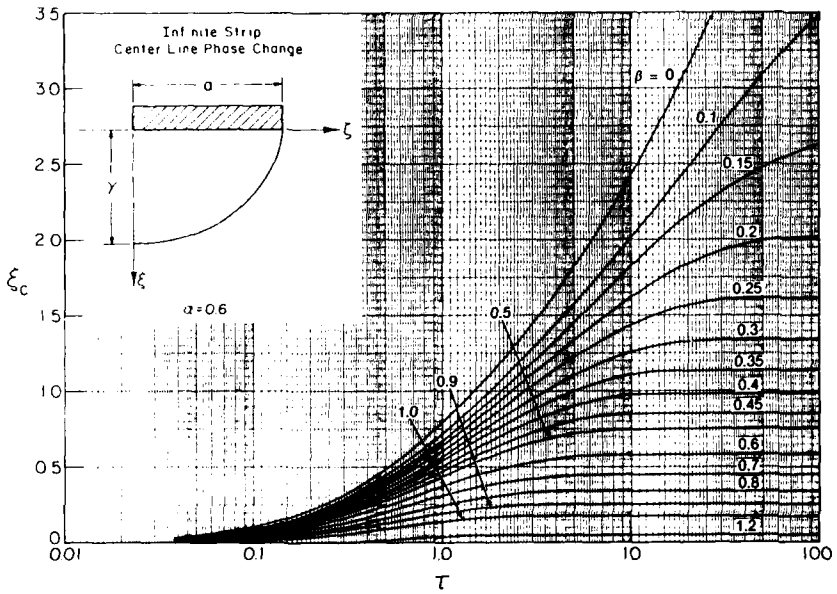


Figure 5.9.

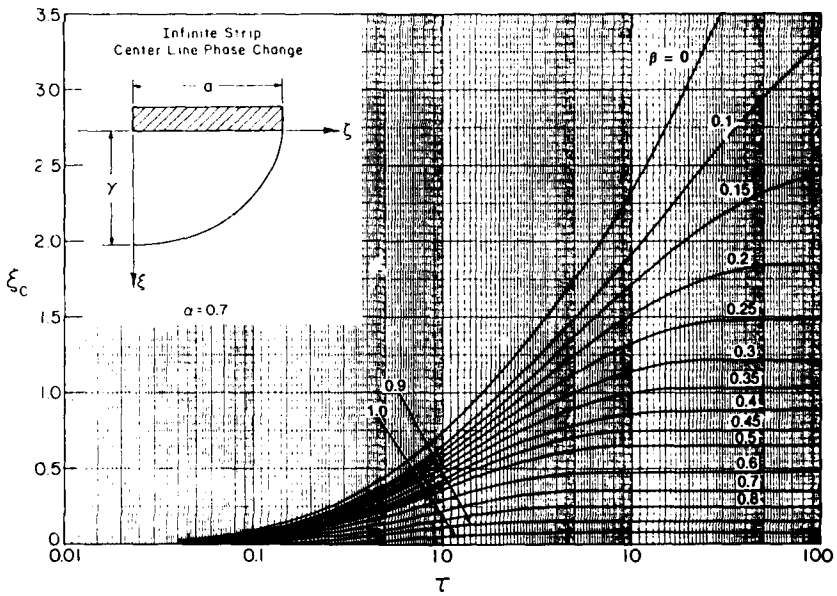


Figure 5.10.

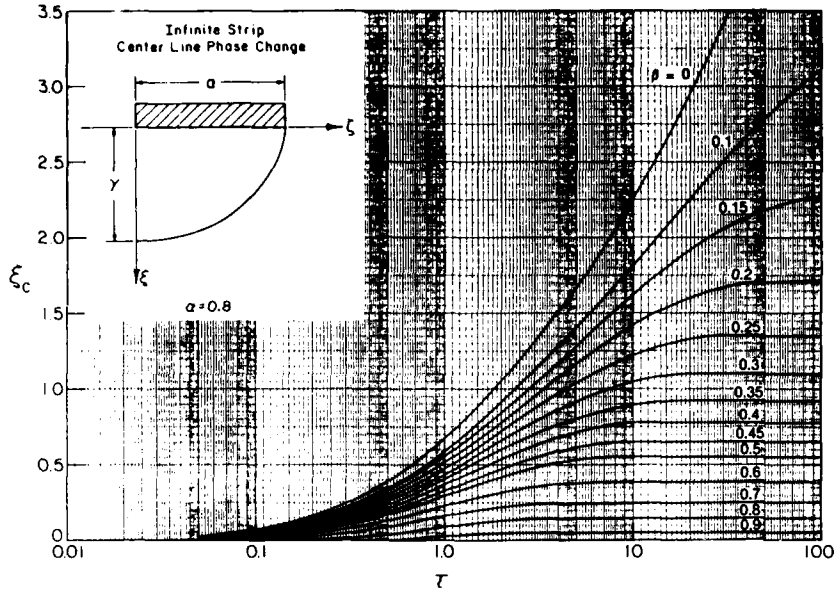


Figure 5.11.

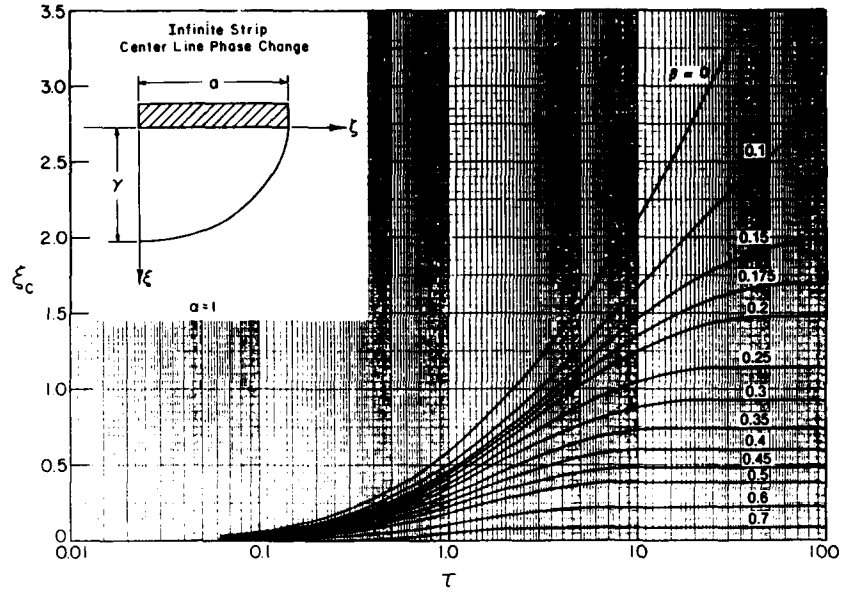


Figure 5.12.

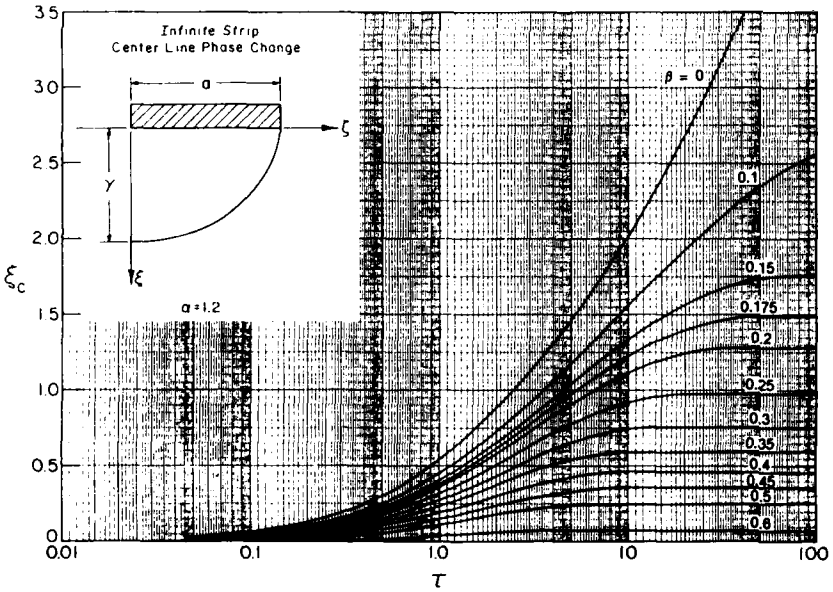


Figure 5.13.

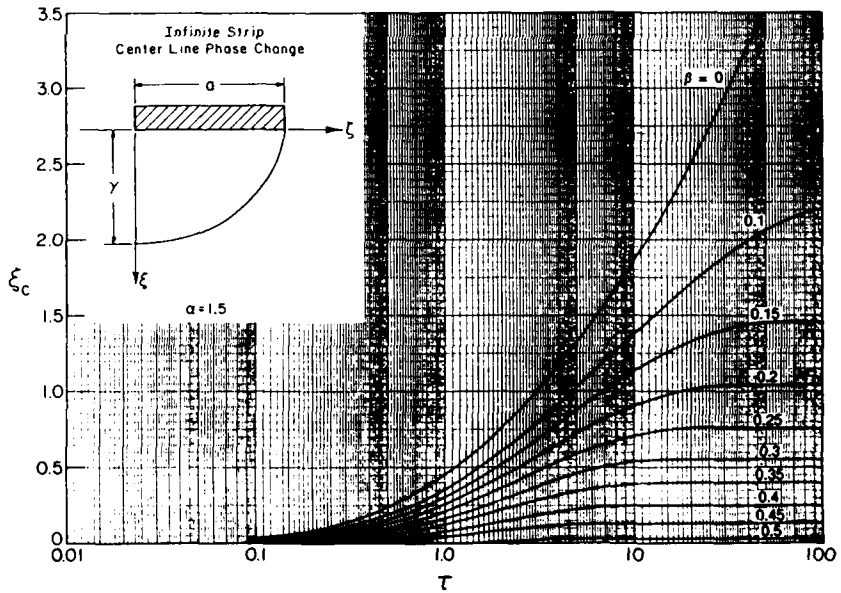


Figure 5.14.

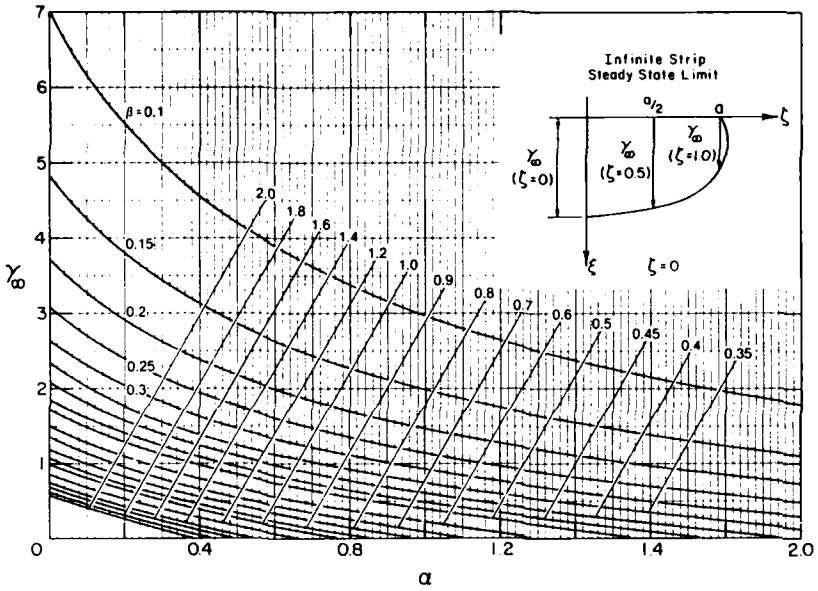


Figure 5.15.

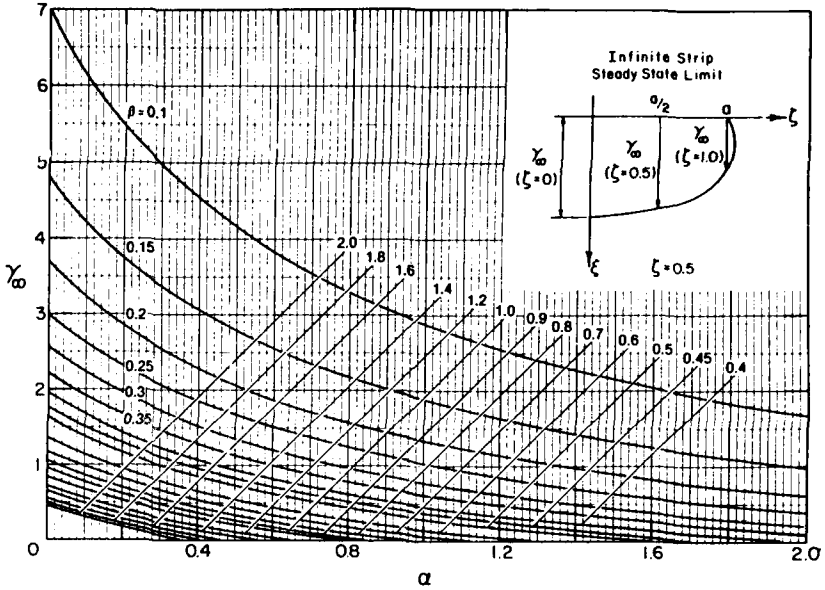


Figure 5.16.

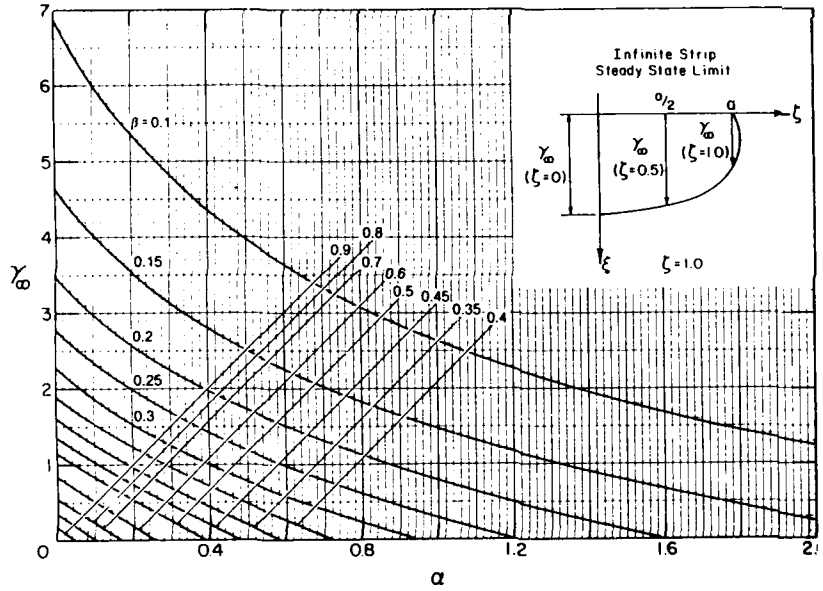


Figure 5.17.

The calculations will be carried out along the axis of symmetry, $\zeta = \eta = 0$, and the geometric function can be written

$$f(\xi) = \frac{2}{\pi} \tan^{-1} \left(\frac{n}{\xi\sqrt{\xi^2+1+n^2}}\right) \quad . \tag{5.38}$$

Equation 5.19, for the phase change interface, is now

$$\tau = -\frac{1}{n} \int_0^{\xi_B} \frac{[u^2(u^2+1+n^2)+n^2]\ \sqrt{u^2+1+n^2}\ du}{(2u^2+1+n^2)\ \dfrac{1}{f_o - 1 - \dfrac{4\alpha\sqrt{1+n^2}}{\pi n}} + \dfrac{\beta}{f_o}} \tag{5.39}$$

where

$$f_o = \frac{2}{\pi} \tan^{-1} \left(\frac{n}{u\sqrt{u^2+1+n^2}}\right) \quad .$$

As the aspect ratio n becomes large, eq 5.39 will reduce to eq 5.32 for the infinite strip. Thus the infinite strip is represented by eq 5.39 when the aspect ratio is large.

The limiting or steady-state solution is

$$\xi_{B\infty} = \left\{ \sqrt{\left(\frac{1+n^2}{2}\right)^2 + n^2 \cot^2 \left[\frac{\beta}{1+\beta} \left(\frac{\pi}{2} + \frac{2\alpha\sqrt{1+n^2}}{n}\right)\right]} - \frac{(1+n^2)}{2} \right\}^{\frac{1}{2}} \quad . \tag{5.40}$$

Numerical quadrature of eq 5.39 leads to the plots given by Figures 5.19-54 for n = 1, 2, 3. The limiting, steady-state values are also given in Figures 5.55-57.

5.4 CIRCULAR TANK

Storage tanks are often used in cold climates. The solution for heat transfer from a circular tank, shown in Figure 5.58, follows in the same way as for the other geometries.

The transient solution for the temperature in a semi-infinite medium initially at T_o, after a surface area S_1 is disturbed with a temperature T_p, is given by Lachenbruch (1957) as

$$\frac{T - T_o}{T_p - T_o} = \frac{z}{2\pi} \iint_{S_1} \left[\frac{e^{\frac{-r_1^2}{4\kappa t}}}{\sqrt{\pi\kappa t}} + \operatorname{erfc}\left(\frac{r_1}{2\sqrt{\kappa t}}\right)\right] \frac{dA}{r_1^3} \tag{5.41}$$

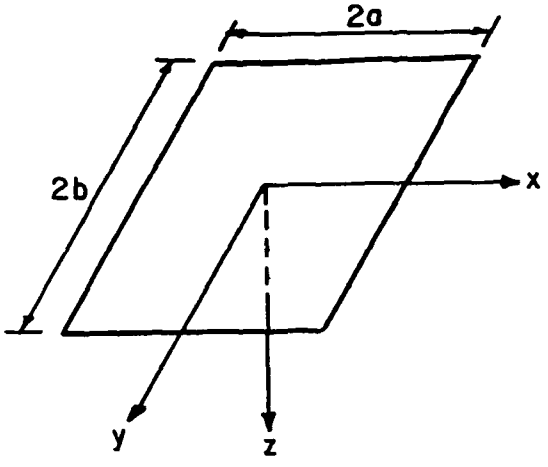


Figure 5.18. Geometry for a rectangular building on the ground surface.

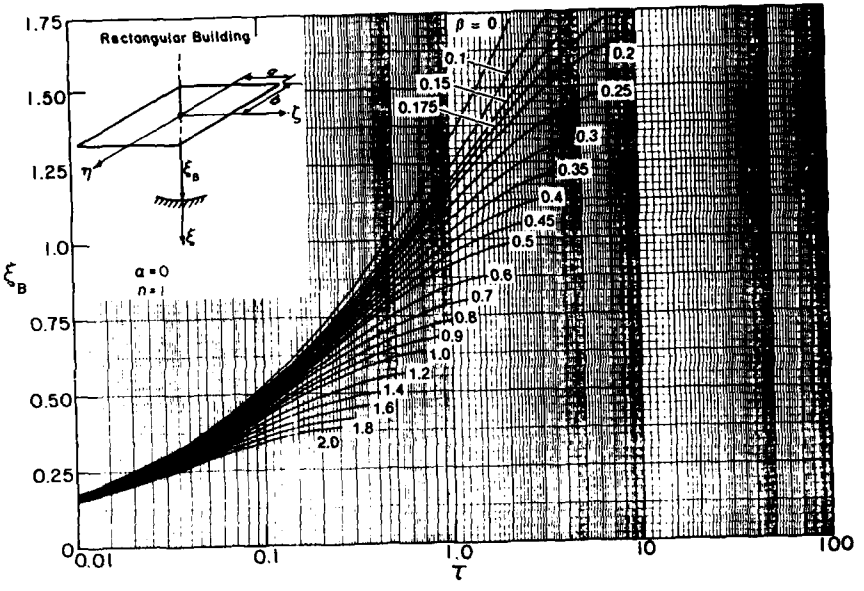


Figure 5.19.

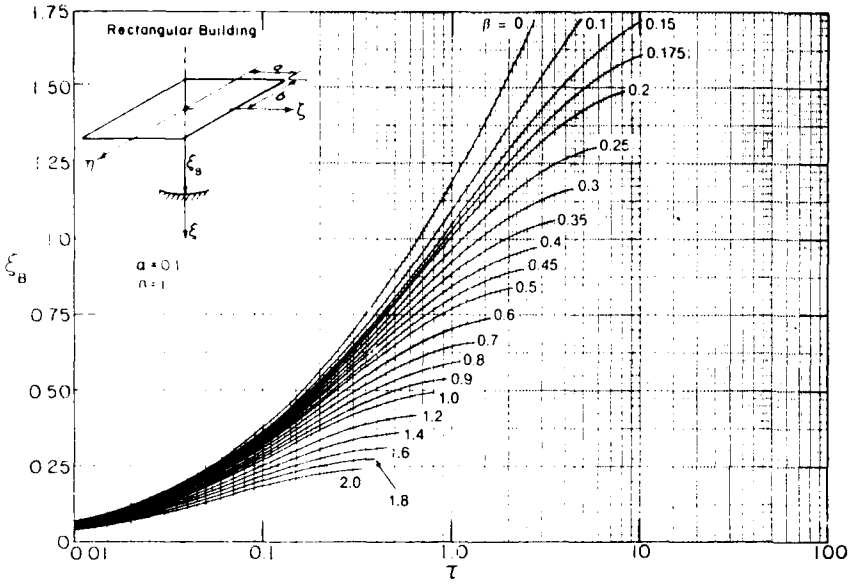


Figure 5.20.

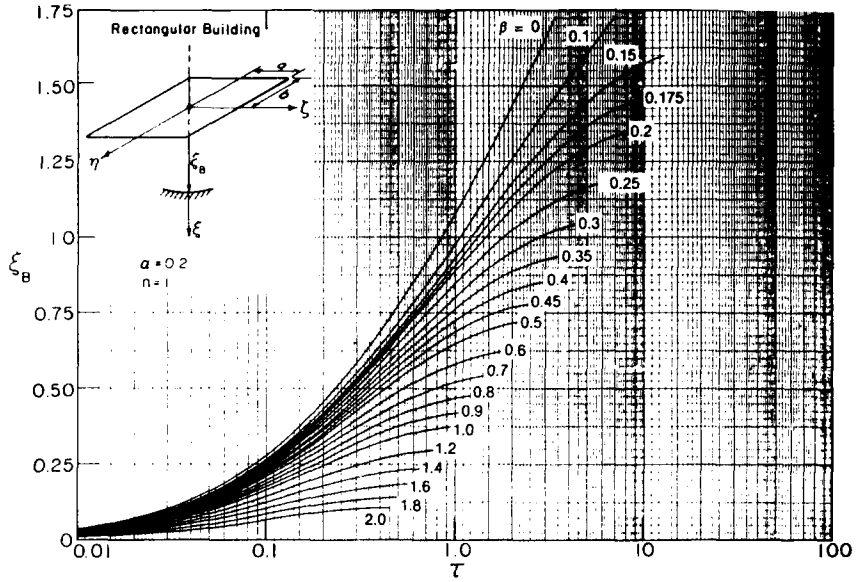


Figure 5.21.

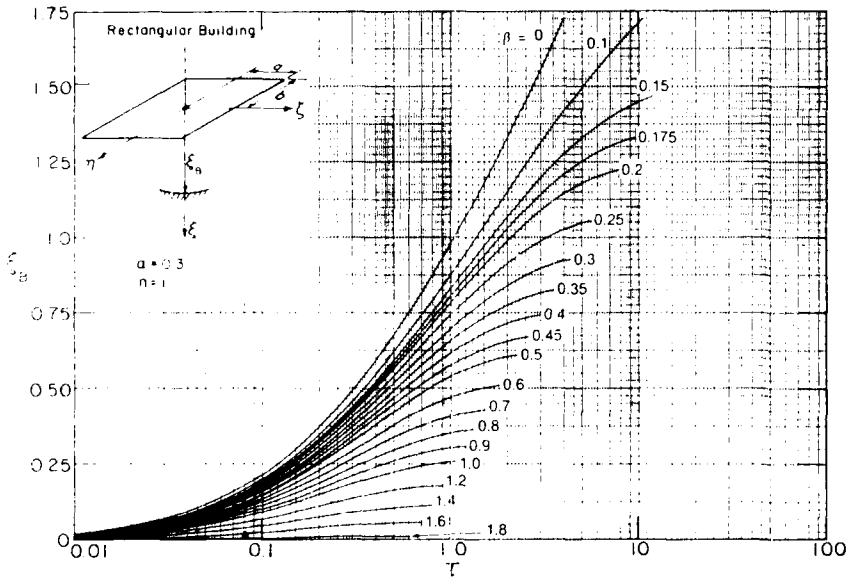


Figure 5.22.

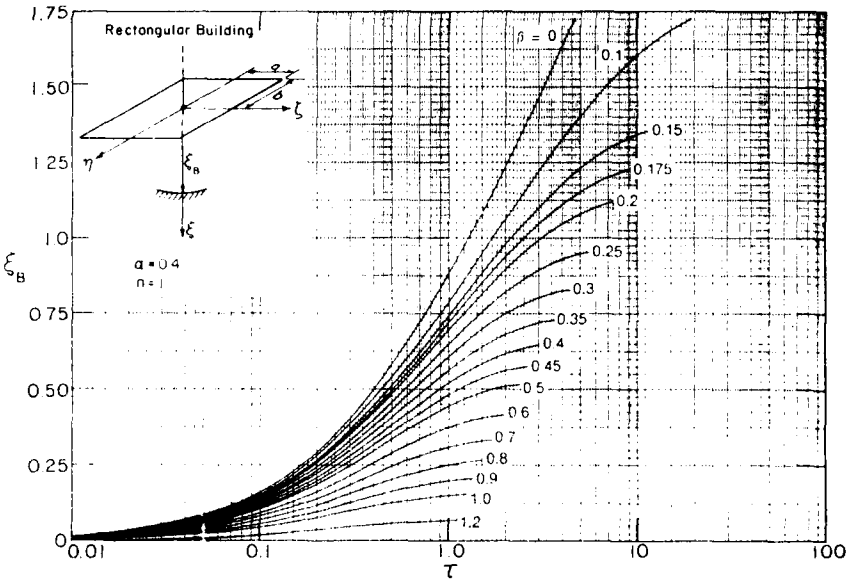


Figure 5.23.

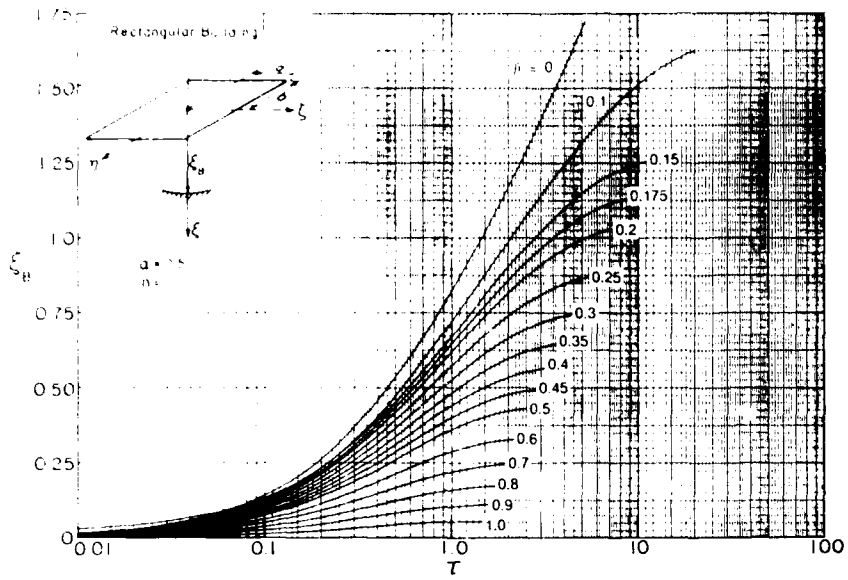


Figure 5.24.

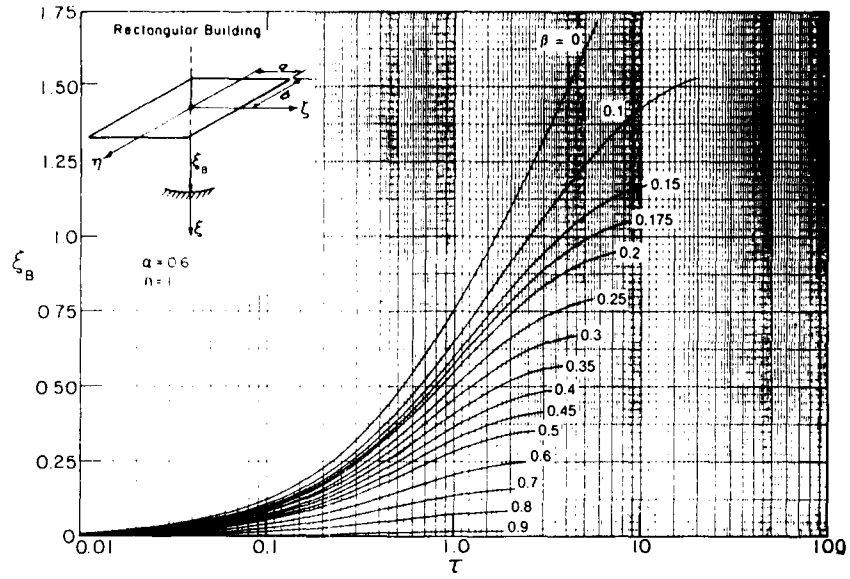


Figure 5.25.

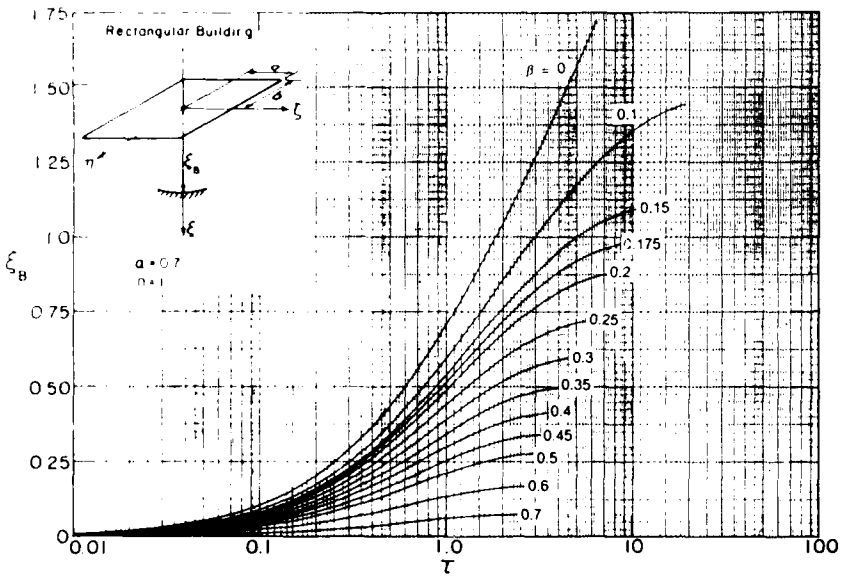


Figure 5.26.

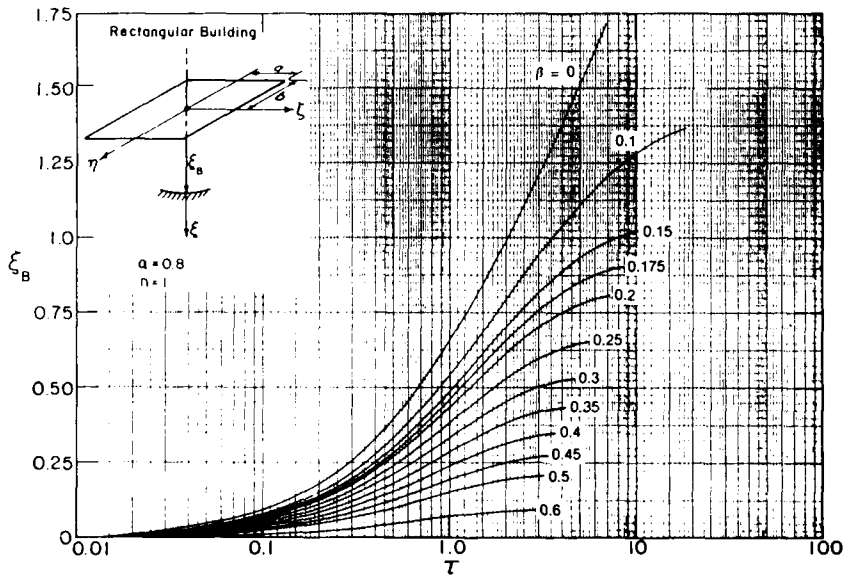


Figure 5.27.

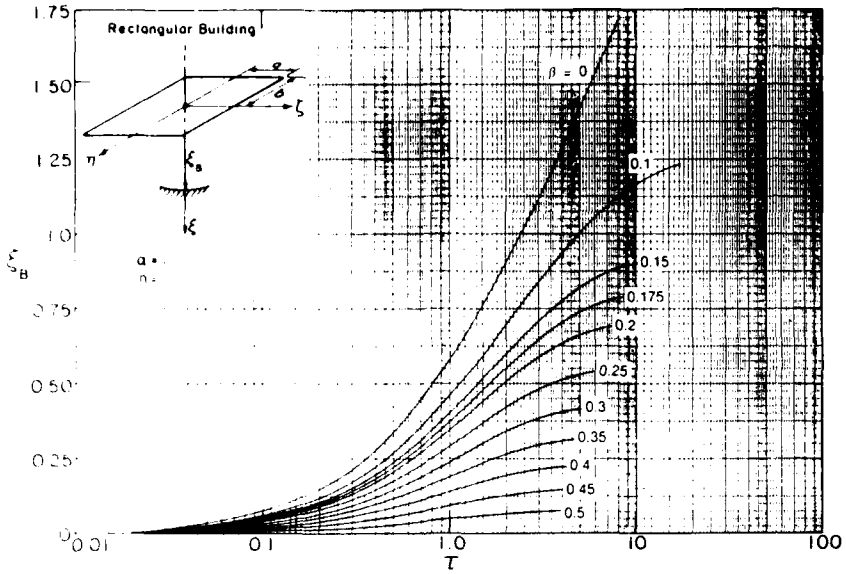


Figure 5.28.

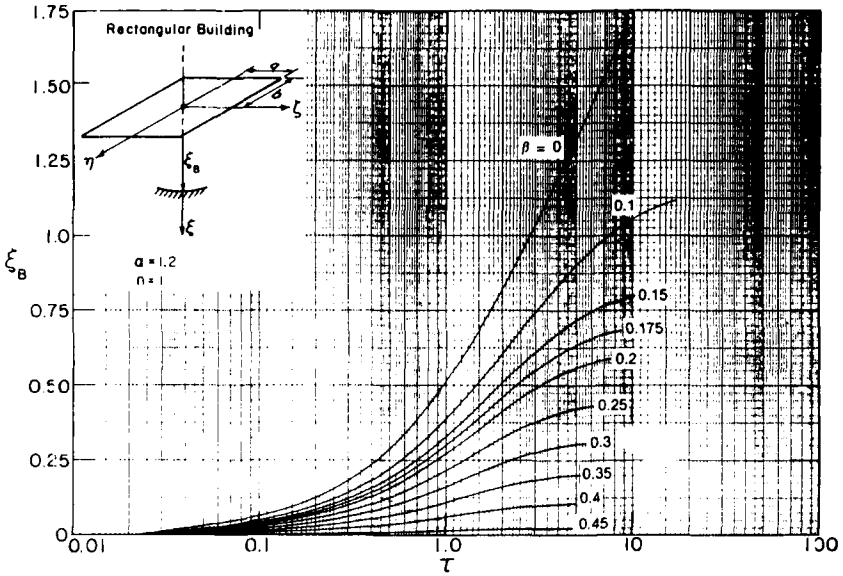


Figure 5.29.

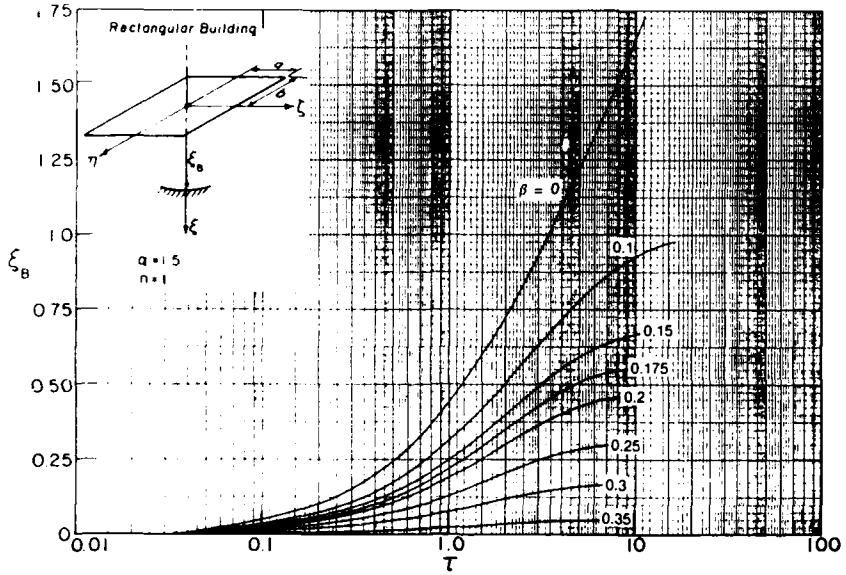


Figure 5.30.

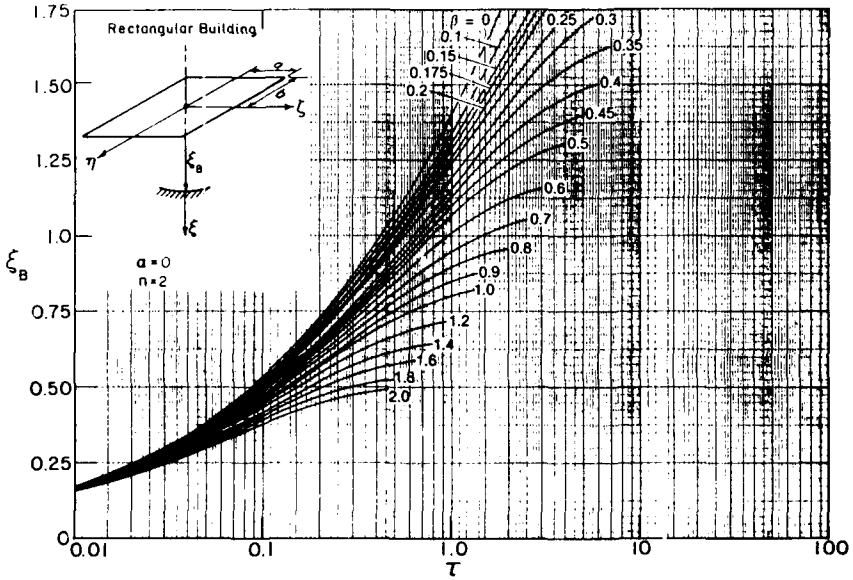


Figure 5.31.

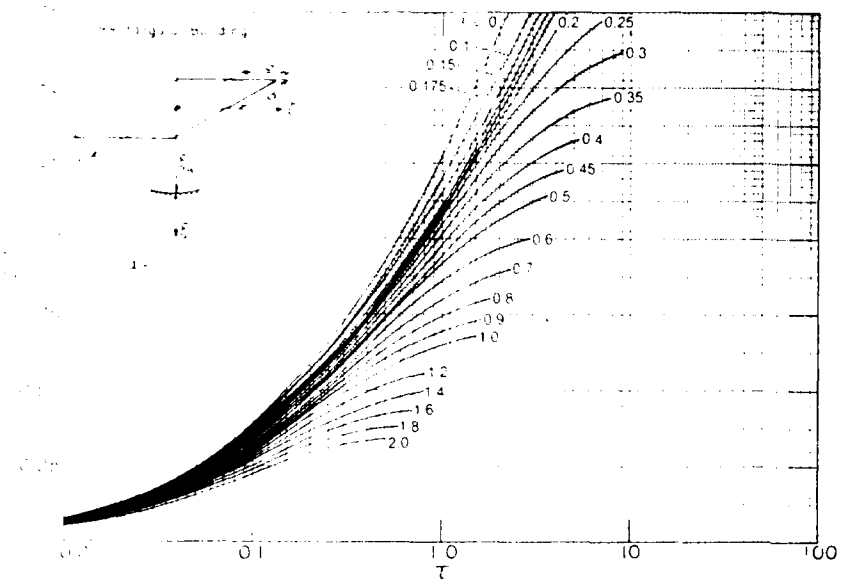


Figure 5.32.

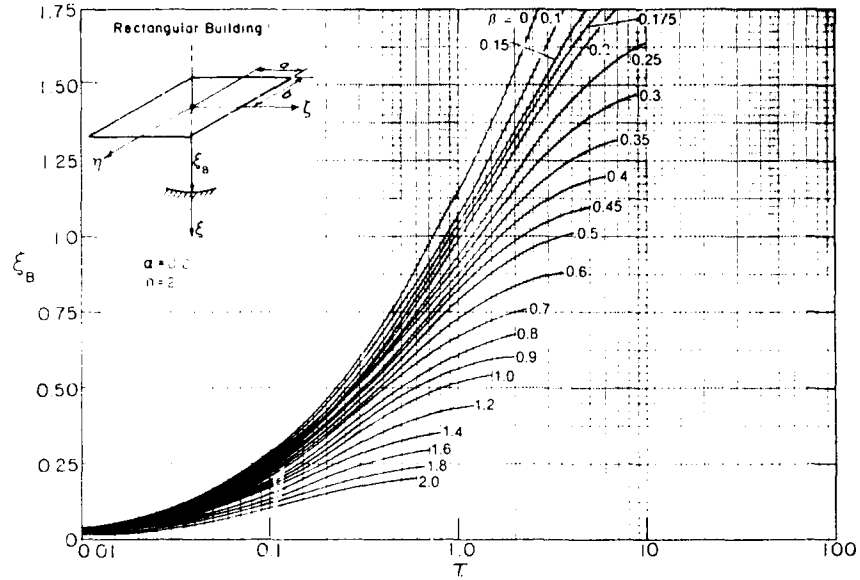


Figure 5.33.

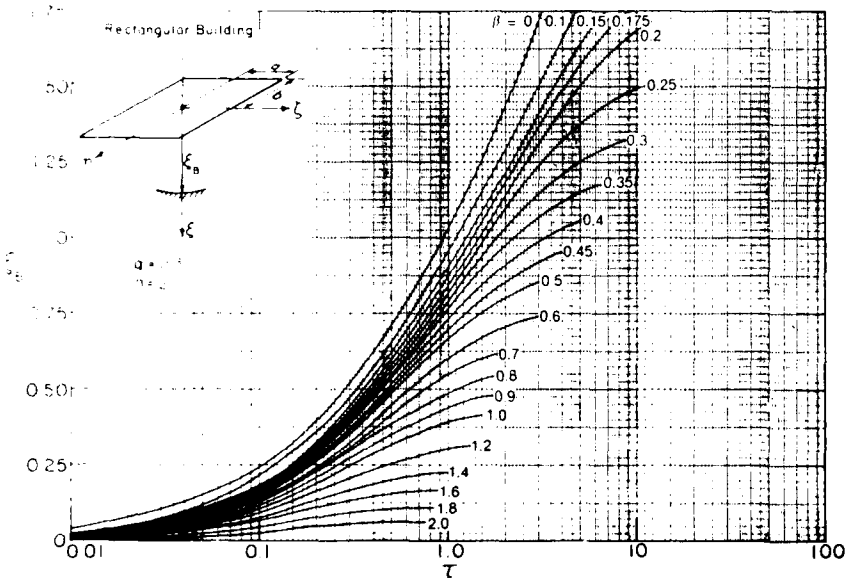


Figure 5.34.

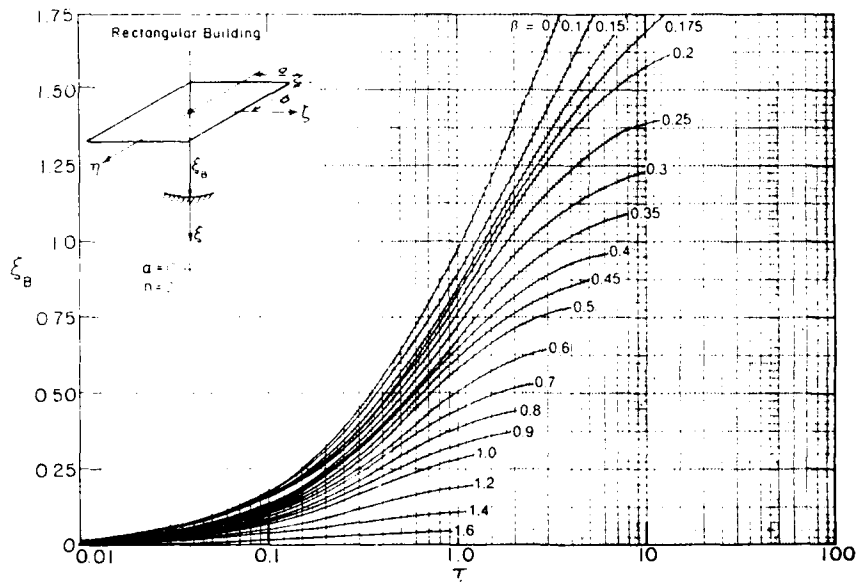
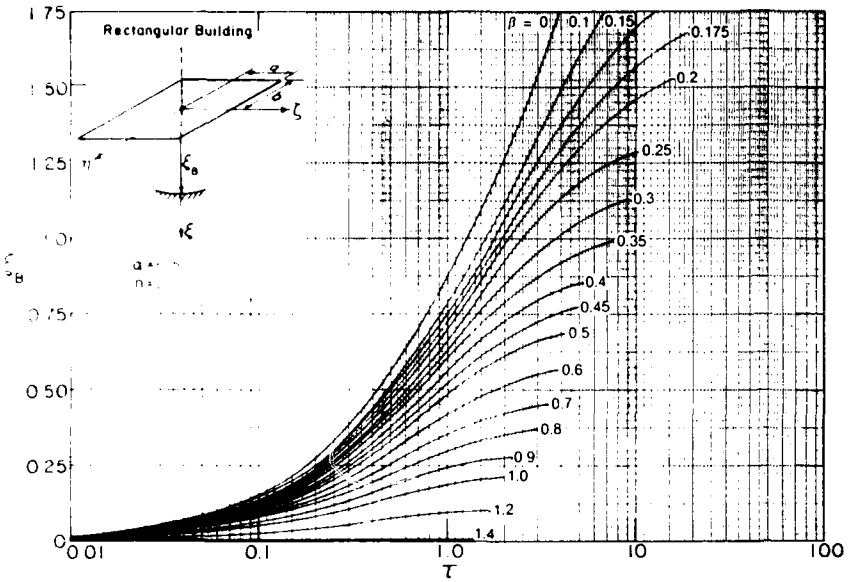


Figure 5.35.



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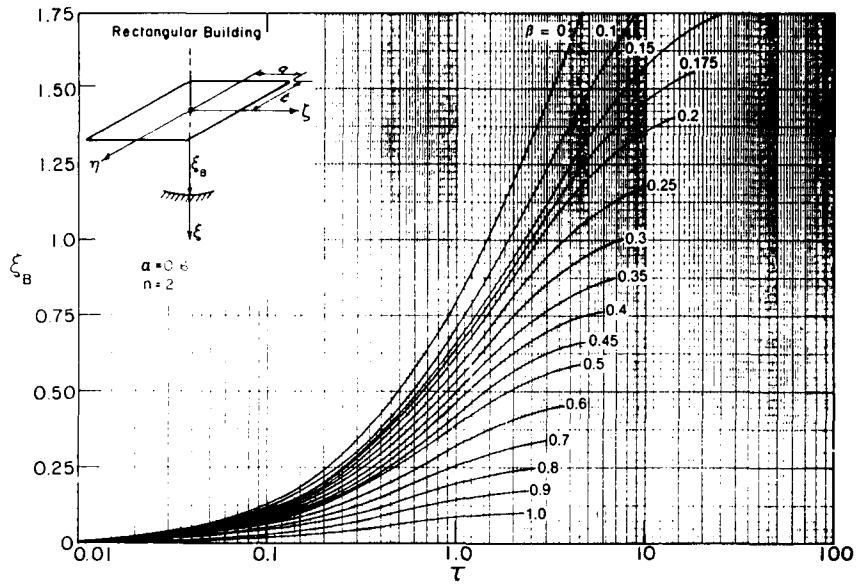

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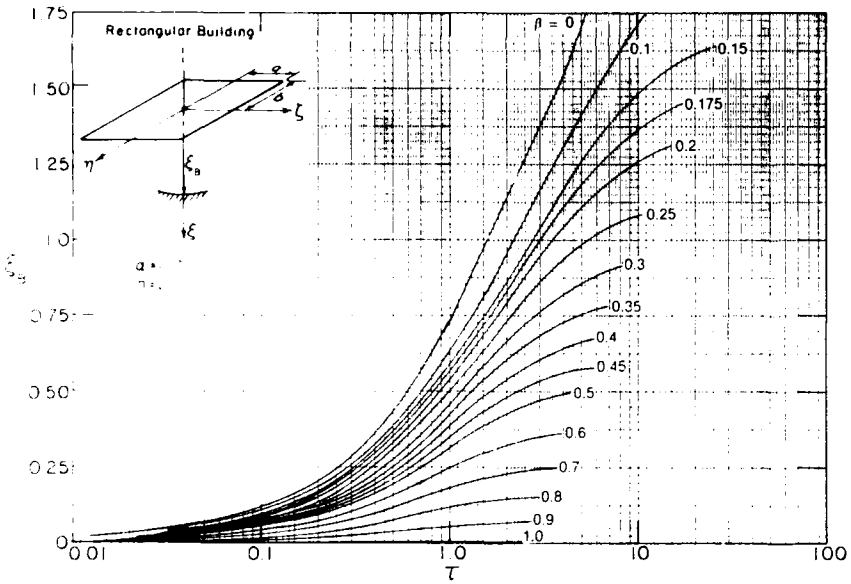

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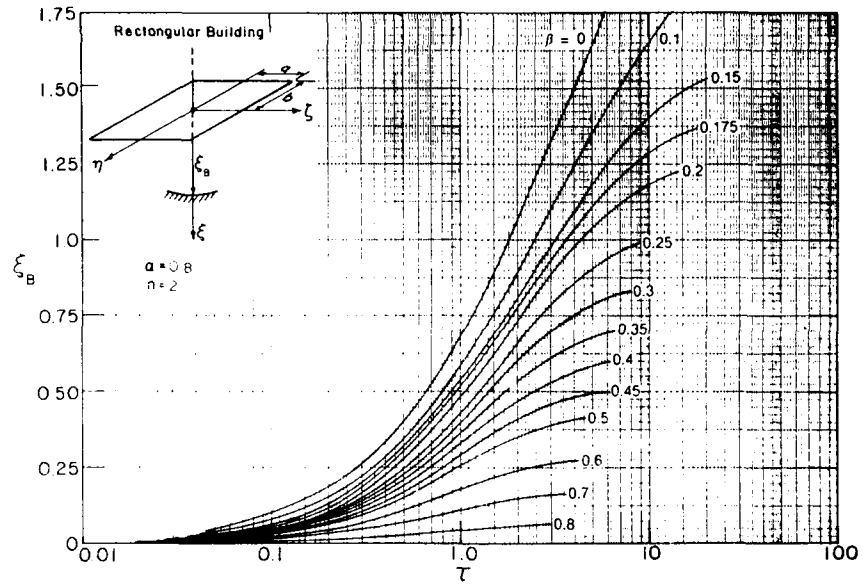

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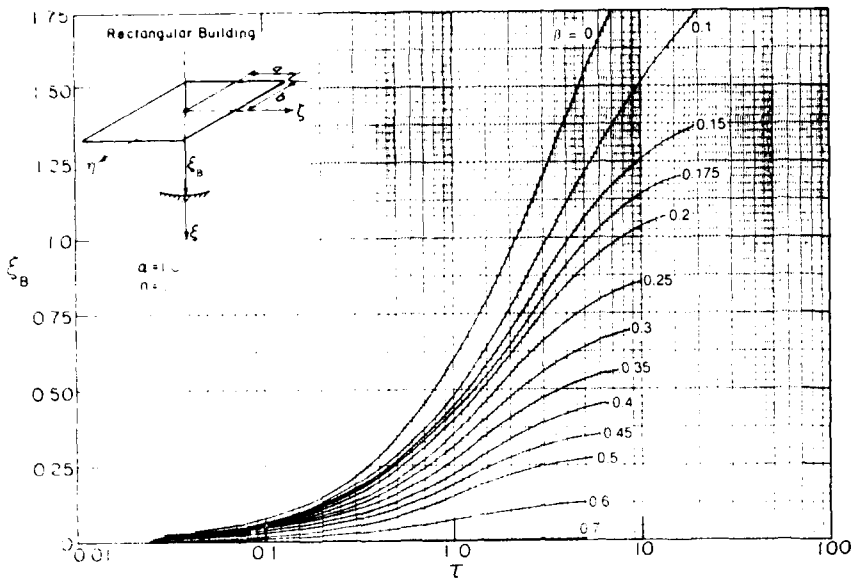

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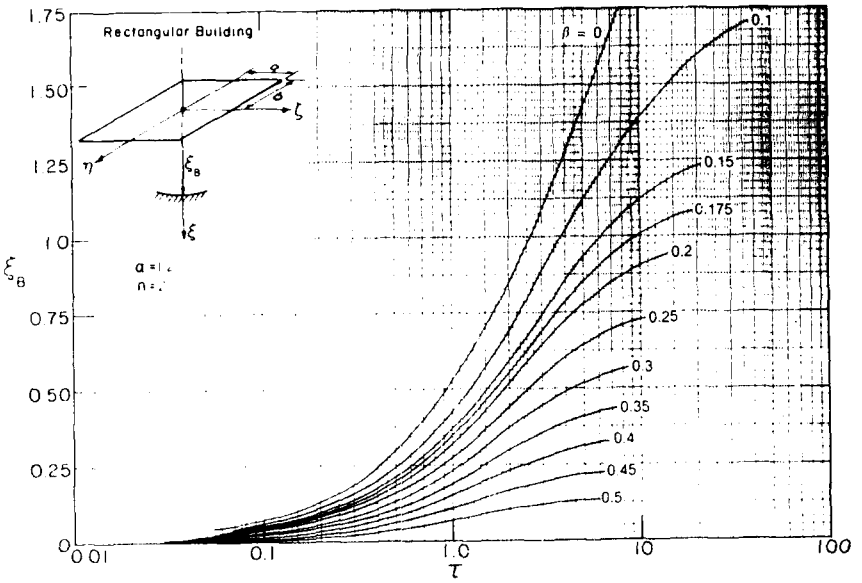

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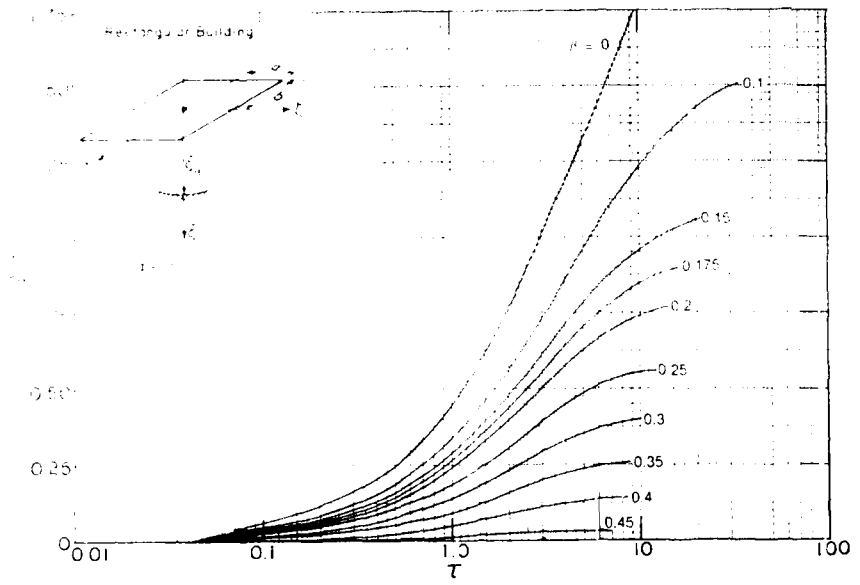

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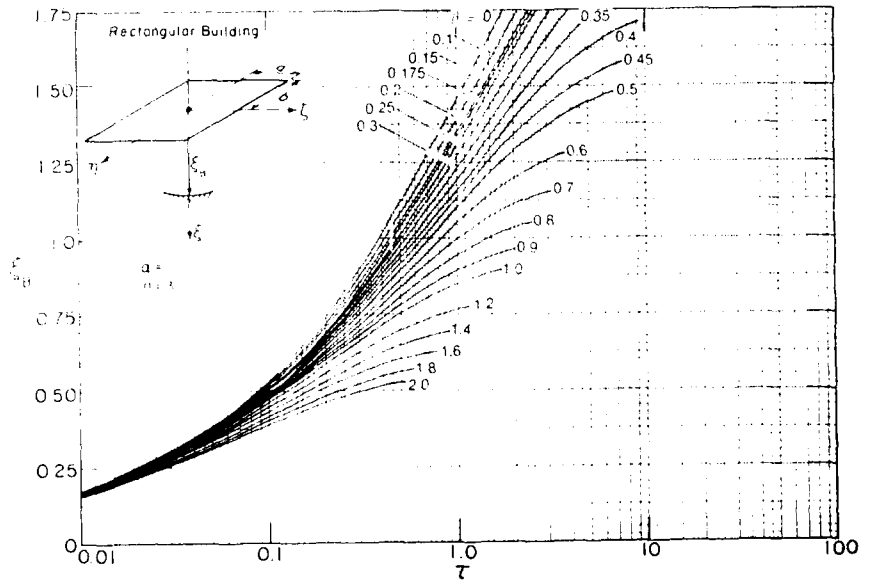

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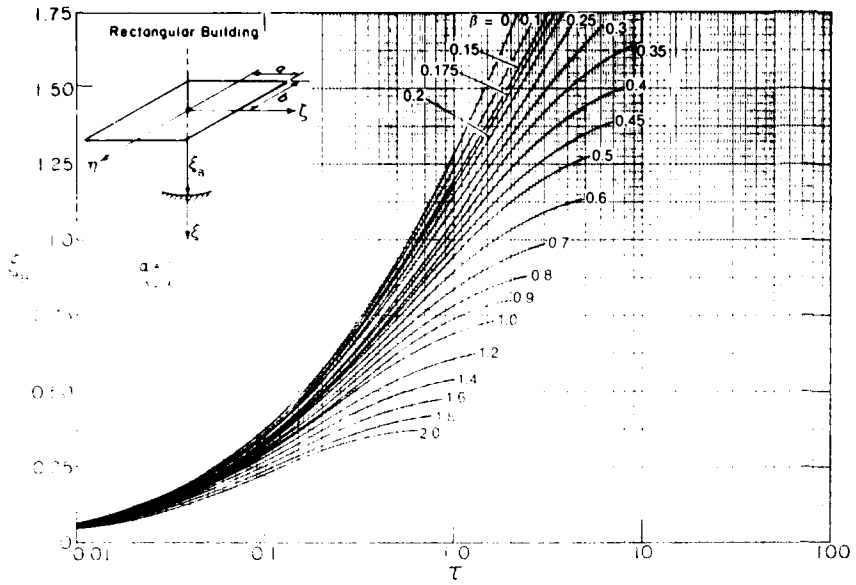

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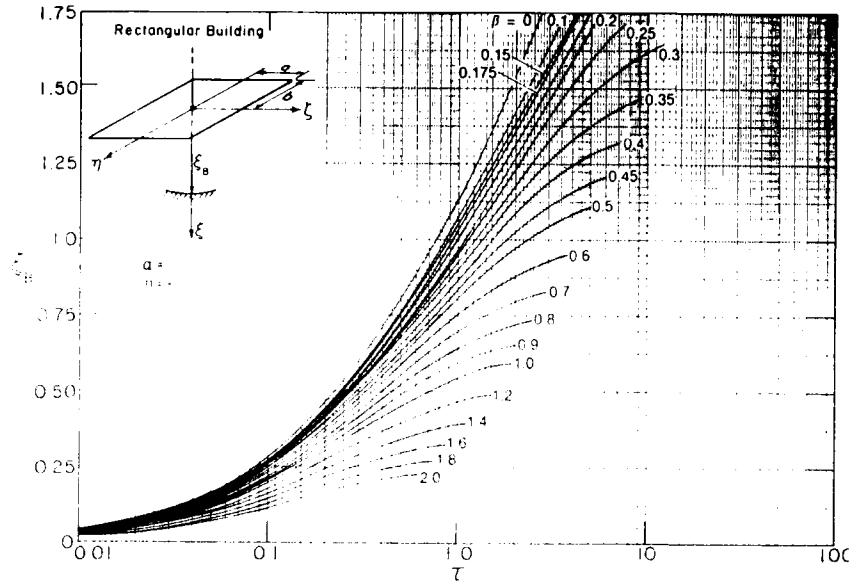

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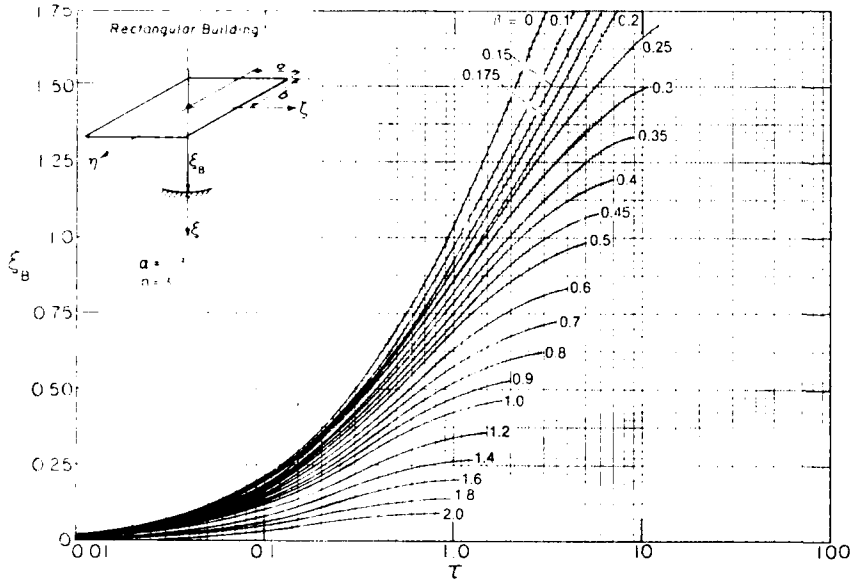

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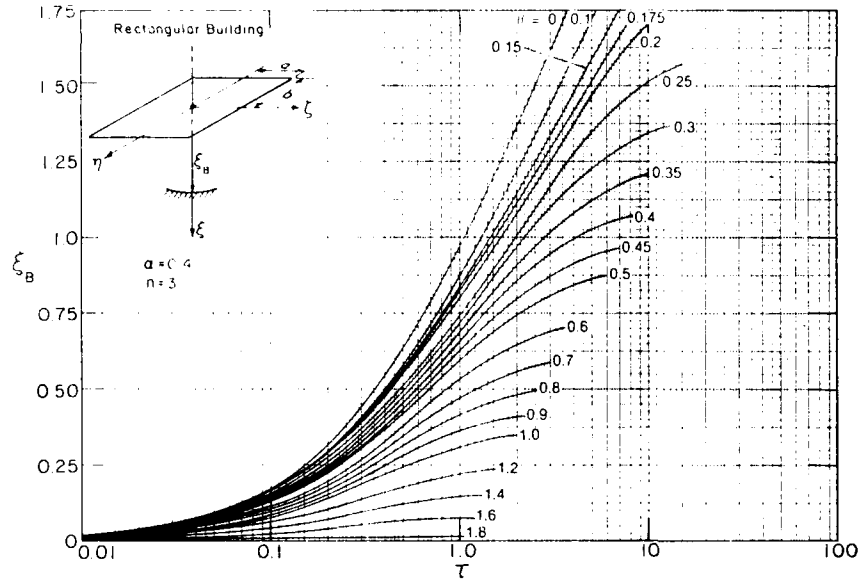

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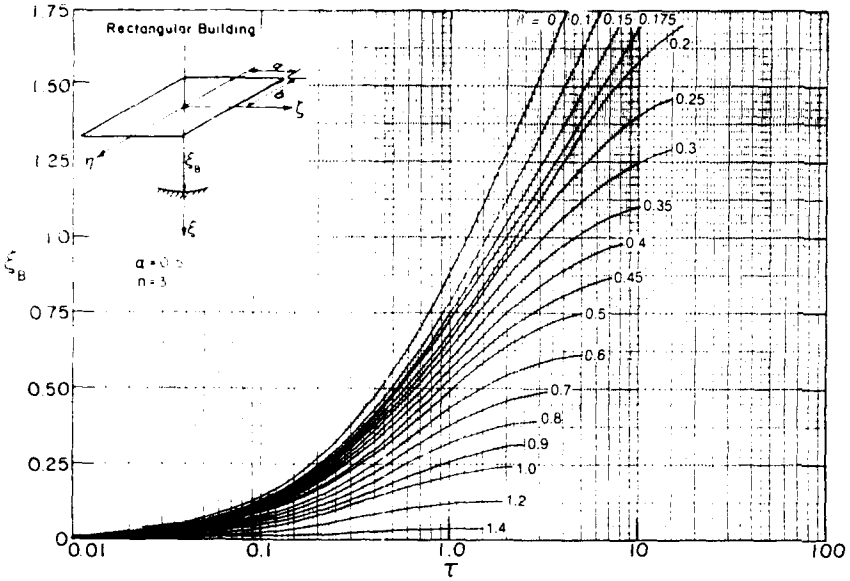

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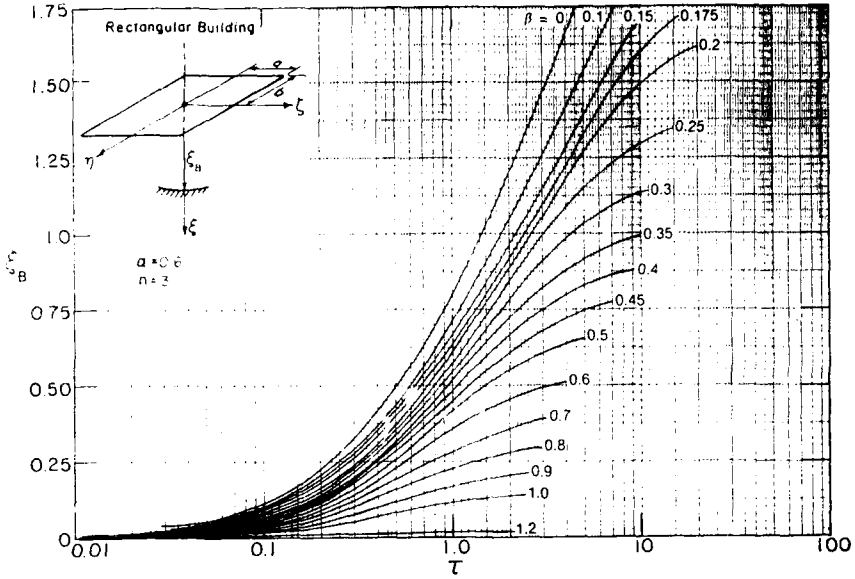

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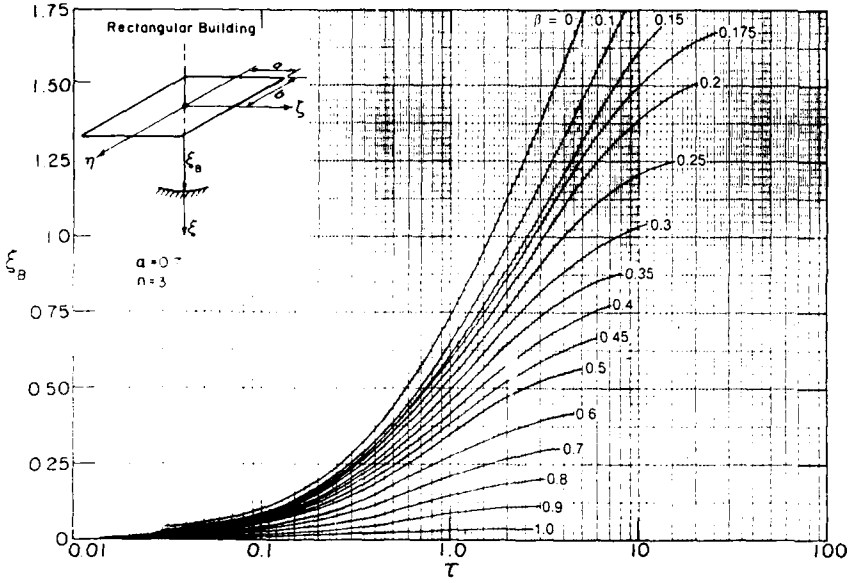

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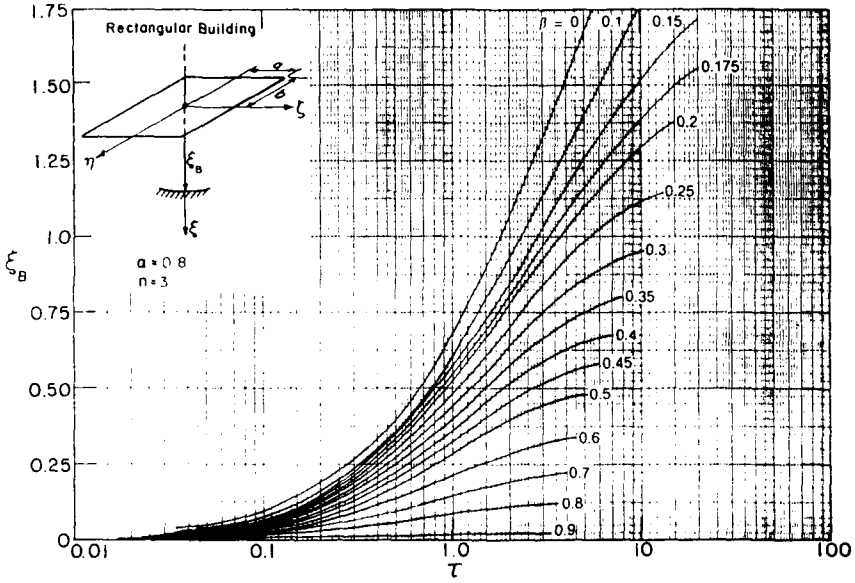

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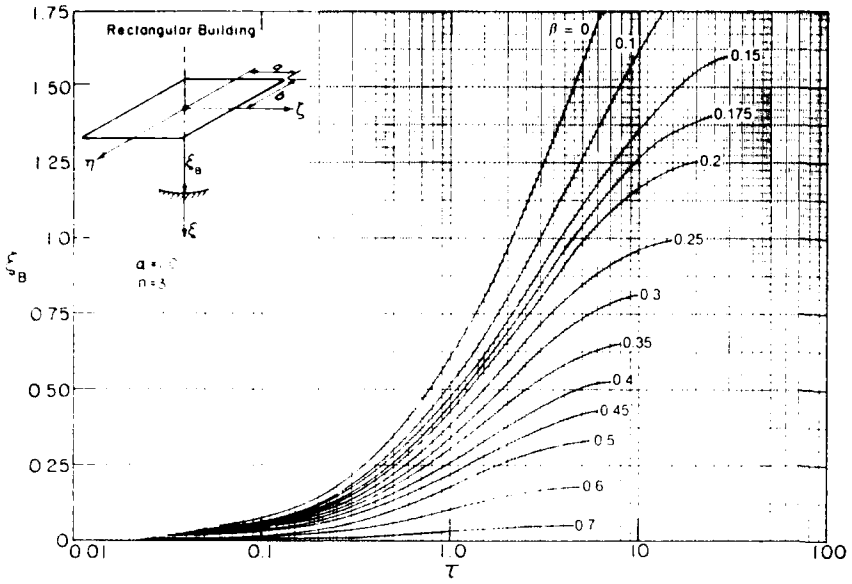

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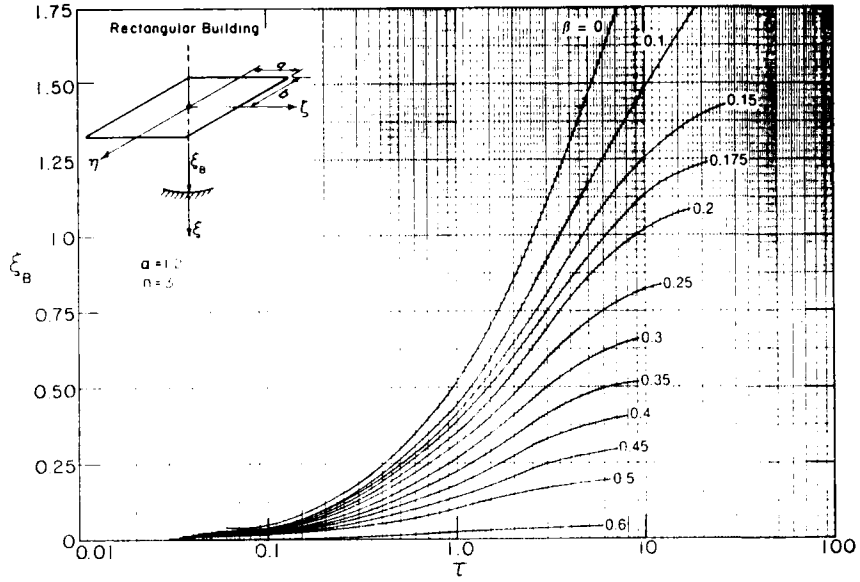

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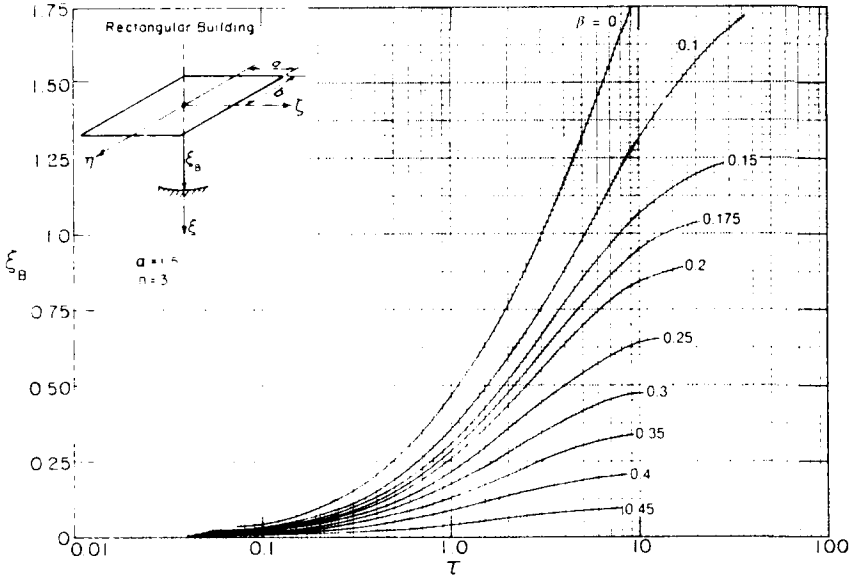

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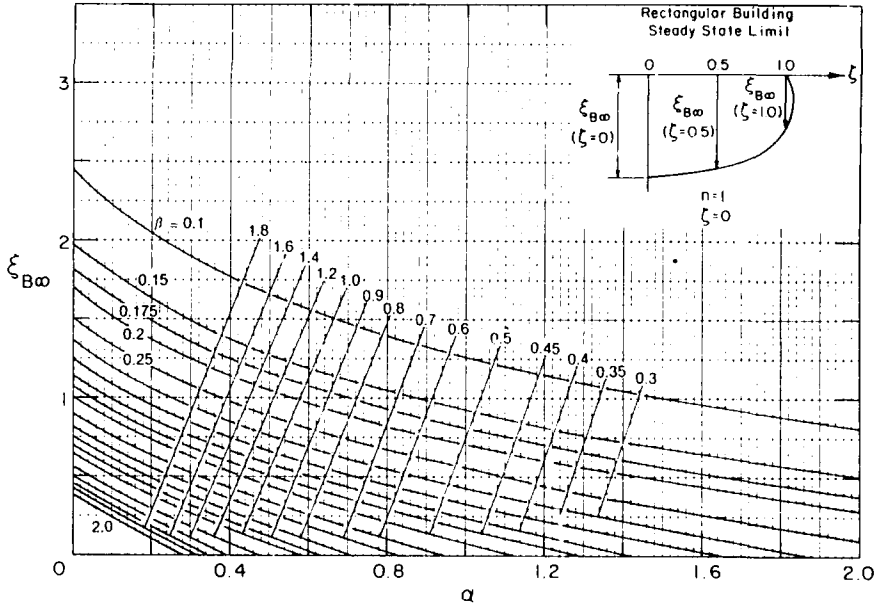

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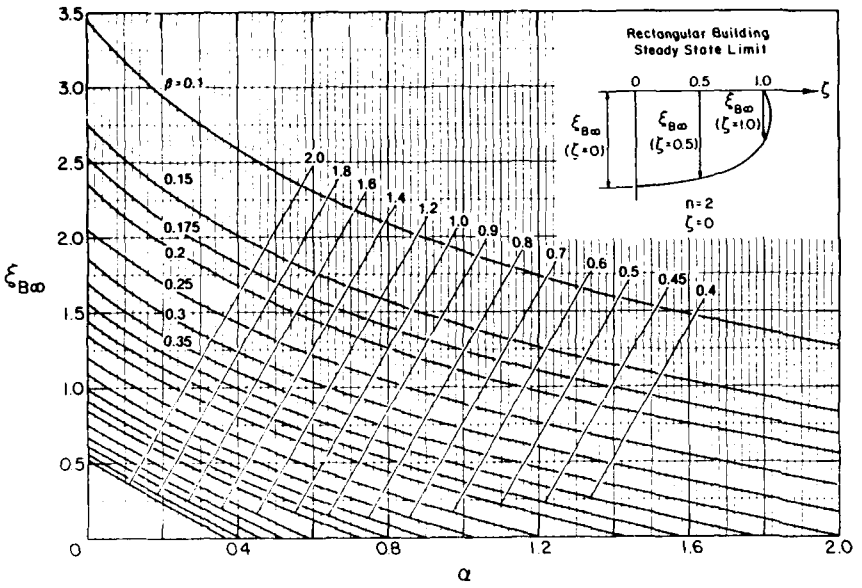

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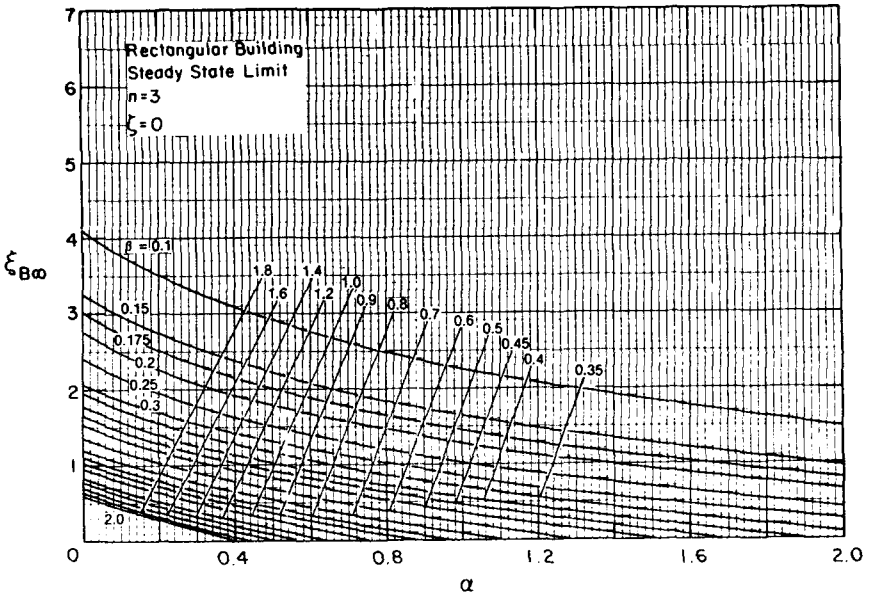

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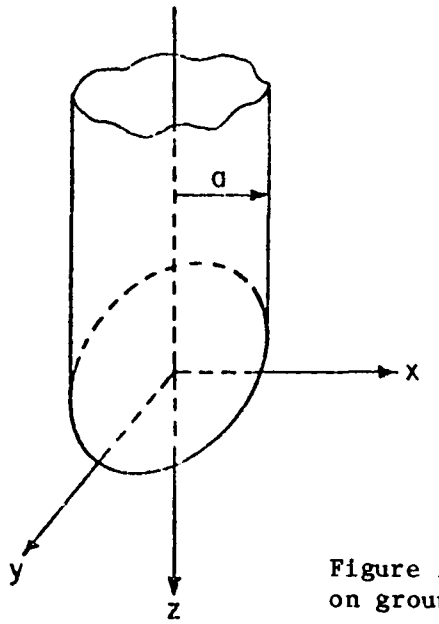


Figure 5.58. Circular tank on ground surface.

where $r_1^2 = (x-x')^2 + (y-y')^2 + z^2$

x', y' = coordinates of dA in S_1.

The steady-state solution, $t \to \infty$, reduces to

$$\frac{T - T_o}{T_p - T_o} = \frac{z}{2\pi} \iint_{S_1} \frac{dA}{r_1^3} \quad . \tag{5.42}$$

Equation 5.42 can be integrated for certain simple geometries such as the infinite strip or the rectangular area already discussed. A general solution for the circle is not available but the temperature along the z-axis, x = y = 0, can be written immediately:

$$\frac{T - T_o}{T_p - T_o} = 1 - \frac{z}{\sqrt{z^2 + a^2}} \quad . \tag{5.43}$$

The geometric function for this case is then

$$f(\xi) = 1 - \frac{\xi}{\sqrt{\xi^2 + a^2}} \quad . \tag{5.44}$$

Equation 5.19 for the phase-change interface depth can then be written along the axis of symmetry ξ:

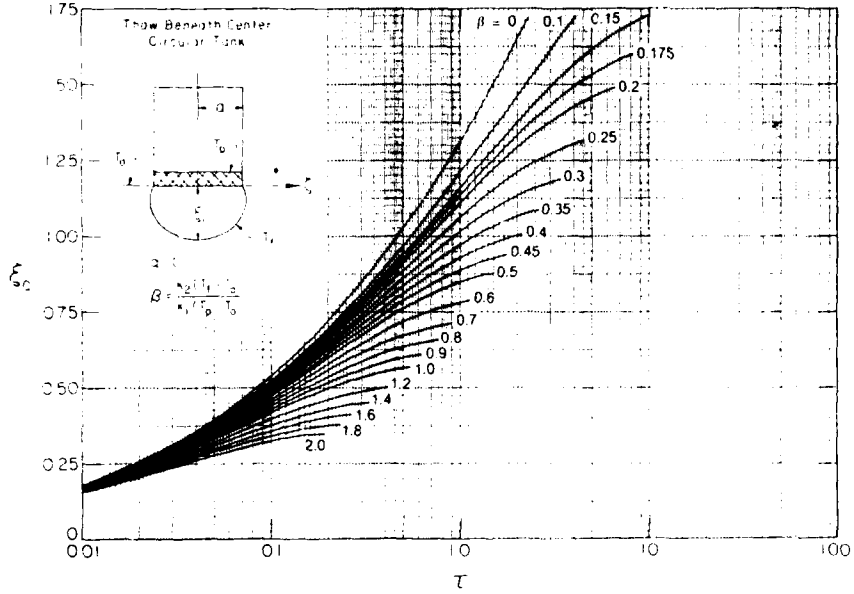


Figure 5.59.

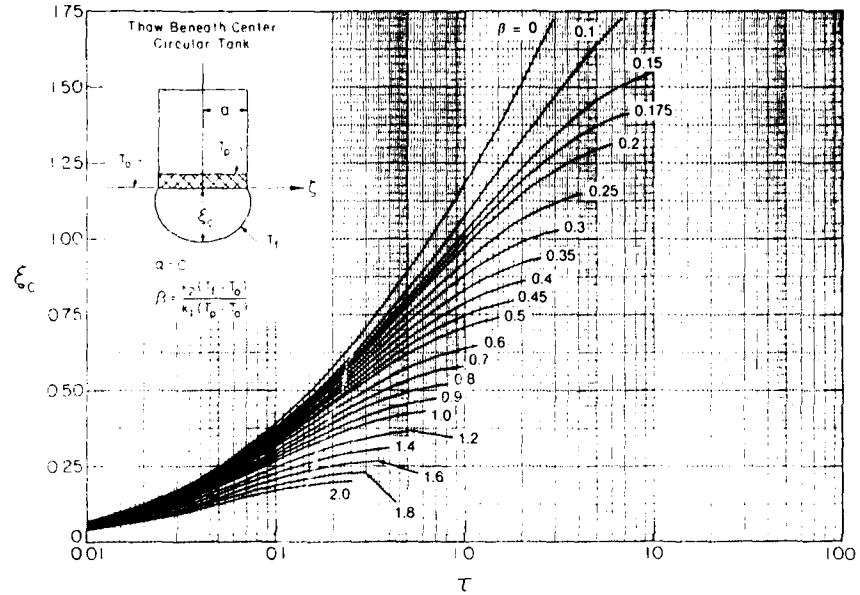


Figure 5.60.

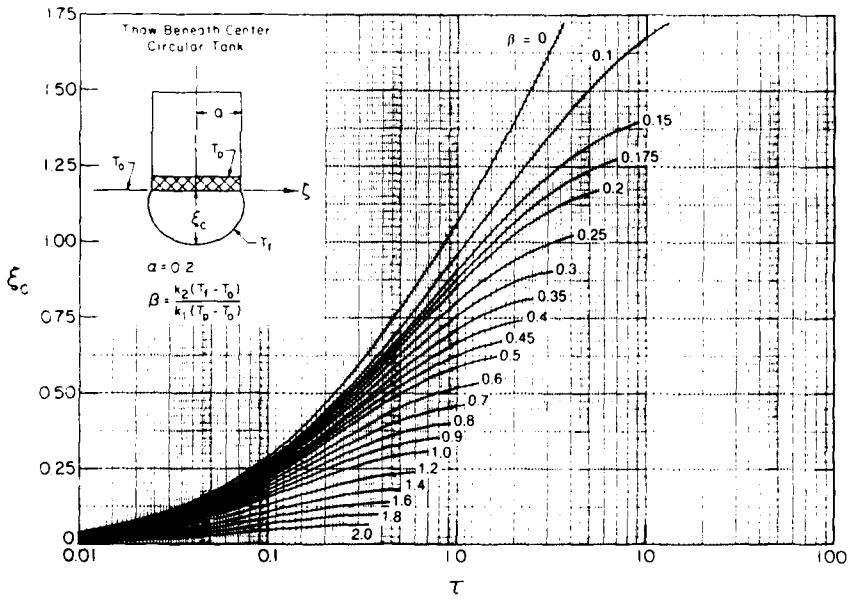


Figure 5.61.

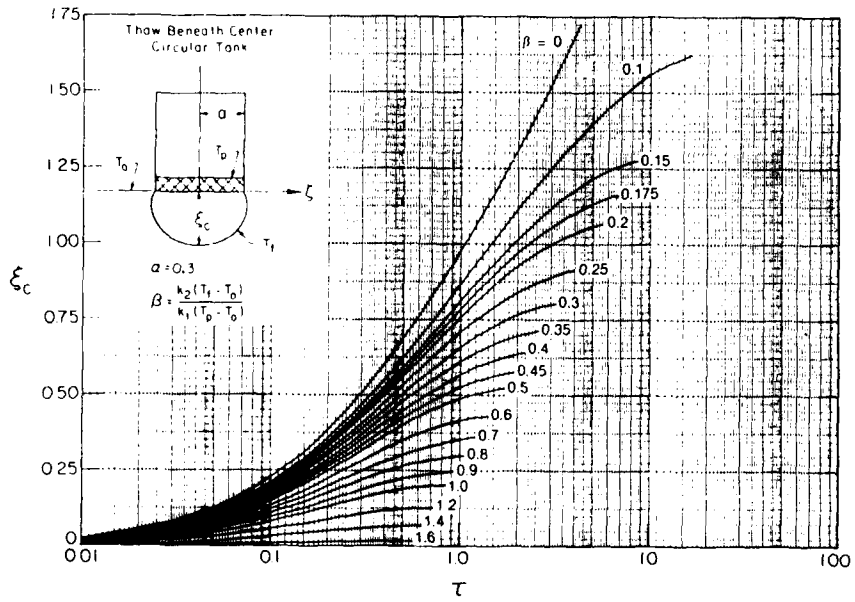


Figure 5.62.

$$\tau = -\frac{2}{\pi} \int_{o}^{\xi_c} \frac{(u^2 + 1)^{3/2} \, du}{\frac{1}{f(u) - 1 - 2\alpha} + \frac{\beta}{f(u)}} \tag{5.45}$$

where $f(u) = 1 - \dfrac{u}{\sqrt{u^2 + 1}}$.

Equation 5.45 is plotted on Figures 5.59-70 for various values of α and β. The limiting solution is

$$\xi = \frac{1}{\sqrt{\left(\frac{1 + \beta}{1 - 2\alpha\beta}\right)^2 - 1}} \quad . \tag{5.47}$$

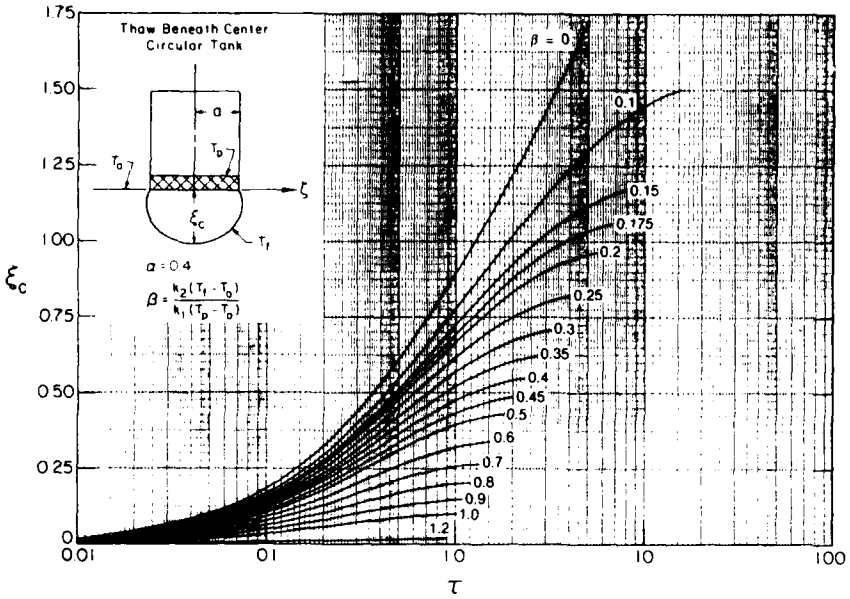
The limiting solution is plotted as Figure 5.71.

The single phase solution ($\beta = 0$) can be written explicitly as

$$\tau = \frac{1}{\pi} \left\{ \frac{\xi_c^4}{2} + \xi_c^2 + \frac{\alpha}{2} \left[\xi_c \, (2\xi_c^2+5) \sqrt{\xi_c^2+1} + 3\ln \left(\xi_c + \sqrt{\xi_c^2+1}\right) \right] \right\} \quad . \tag{5.48}$$

NOMENCLATURE - CHAPTER 5

a	half width of structure or radius of tank
b	half length of structure
C	volumetric specific heat
d	insulation thickness
f	steady-state solution geometric function
f'	modified steady state solution geometric function
f_o	value of f on the phase change interface
g	steady-state solution geometric function
g_o	value of g on the phase change interface
k	thermal conductivity
k_{12}	k_1/k_2
L	volumetric latent heat
L_e	effective latent heat $= L(1 + k_{12}\,\beta S_T + S_T/2)$



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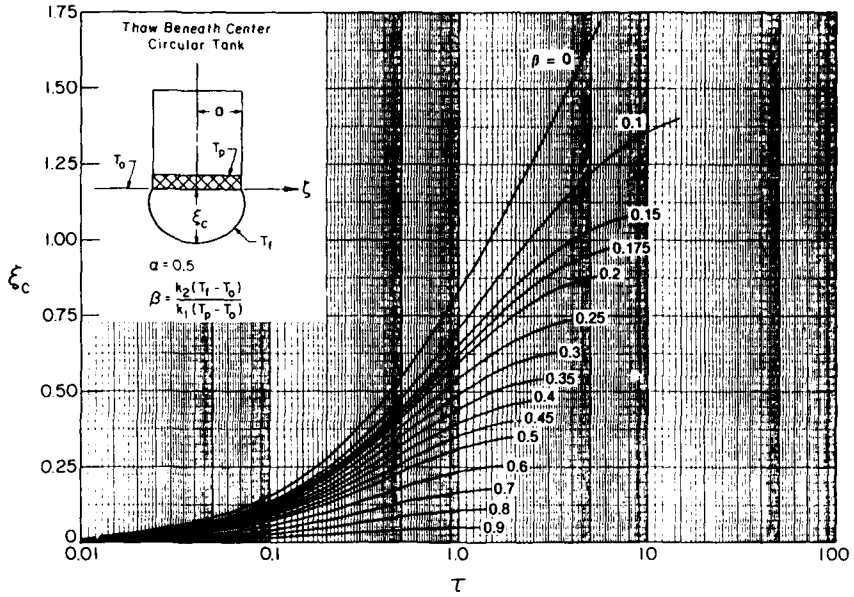

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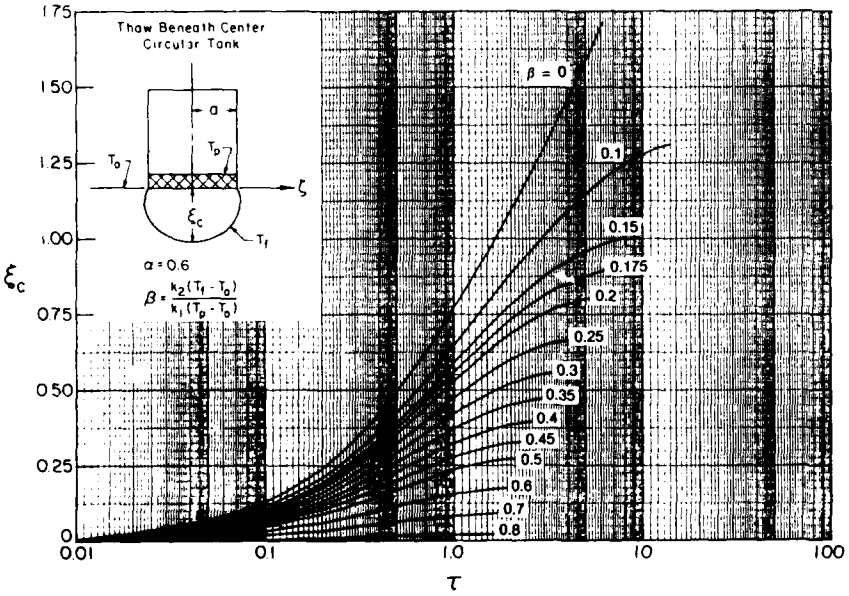

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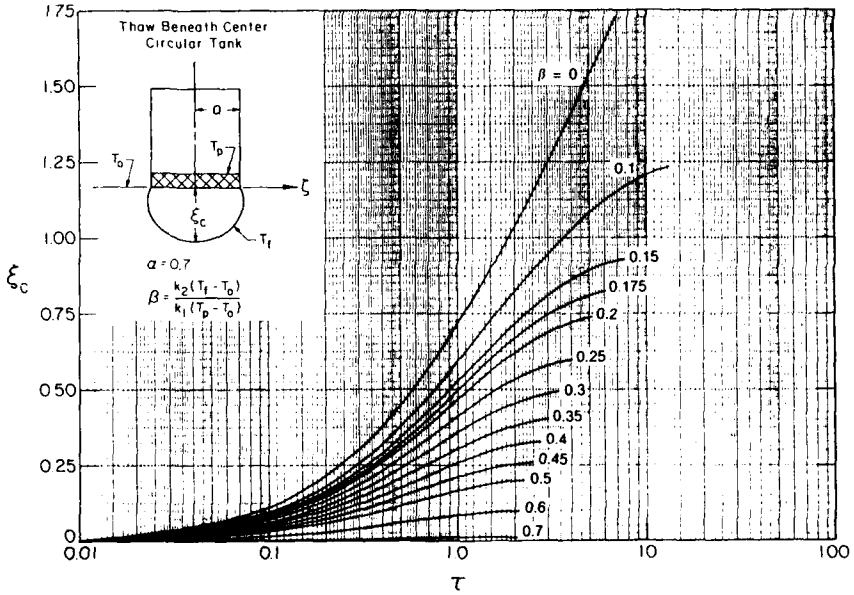
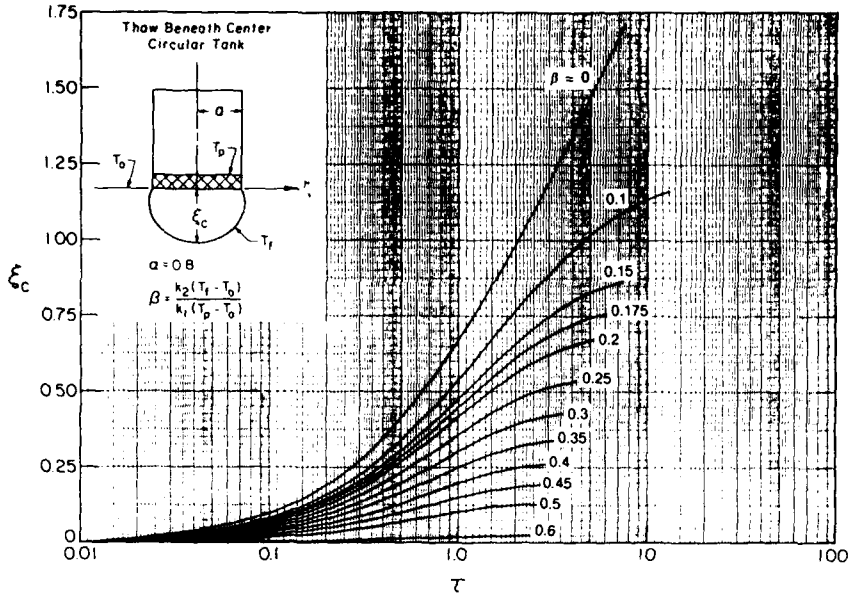


Figure 5.66.

m	outward normal from phase-change interface
m'	m/a
n	b/a
p	distance along outward normal to phase-change interface
p'	p/a
p_1	$\beta(\pi + 4\alpha)/(1 + \beta)$
q	heat flux from bottom of insulated structure
r_1^2	$(x-x')^2 + (y-y')^2 + z^2$
s	distance along phase-change interface
s'	s/a
S	area of phase-change interface
S'	S/a^2
S_1	surface area with disturbed temperature
S_T	$\frac{C_1(T_p - T_f)}{L}$, Stefan number
t	time
T	temperature
T_f	freezing temperature
T_o	ground initial and surface temperature
T_p	temperature of bottom of structure
$\overline{T}_p$	temperature of insulation/ground interface
V	volume of region changing phase
V'	V/a^3
x,y,z	Cartesian coordinates
x_o,y_o,z_o	Cartesian coordinates of phase-change interface



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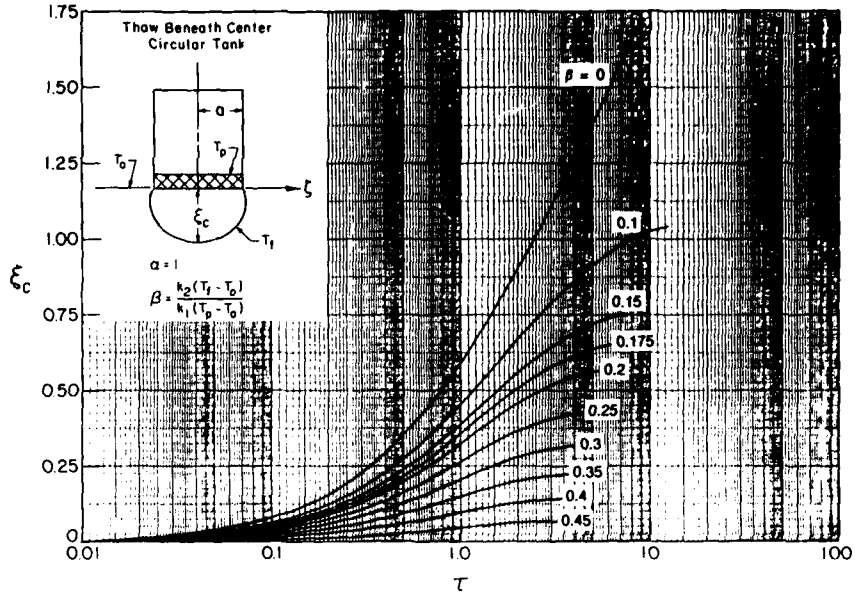

Figure 5.68.

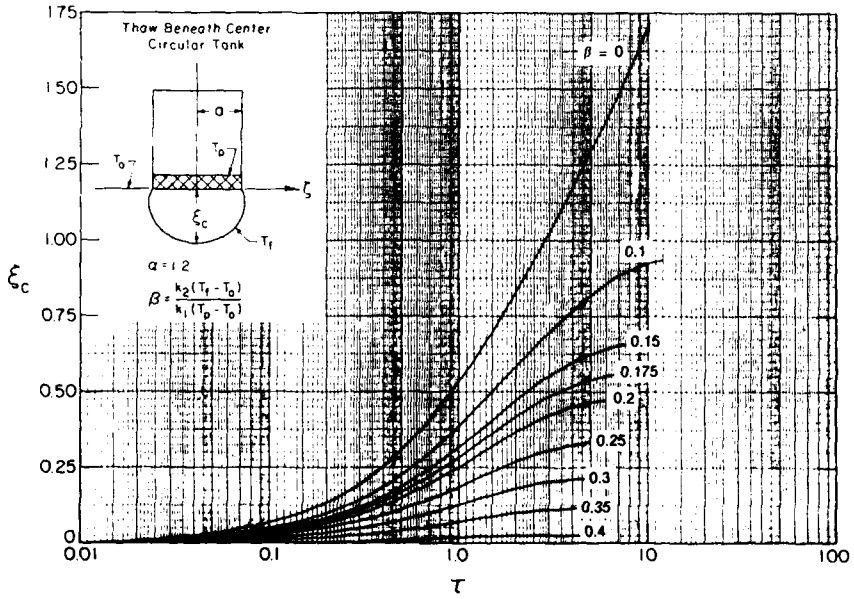

Figure 5.69.

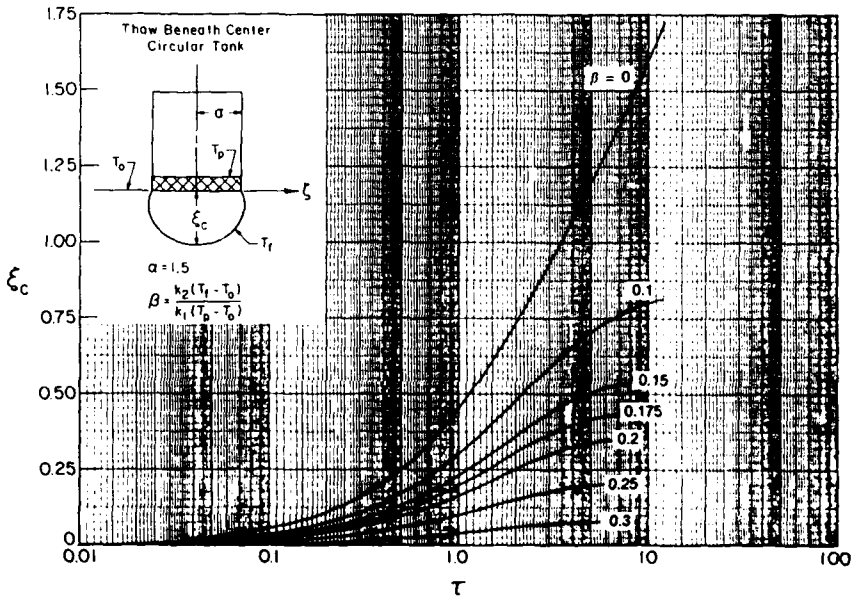

Figure 5.70.

α	$k_{1i}\frac{d}{2a}$	insulation parameter
β	$k_{2i}\frac{(T_f - T_o)}{(T_p - T_f)}$	
γ	value of ξ_c at center of semi-infinite strip	
γ_x	value of γ at any location ζ	
γ_∞	maximum or limiting value of γ	
ζ	x/a	
ζ_o	x_o/a	
η	y/a	
η_o	y_o/a	
κ	thermal diffusivity	
ξ	z/a	
ξ_o	z_o/a	
ξ_c	value of ξ_o beneath center of circular tank	

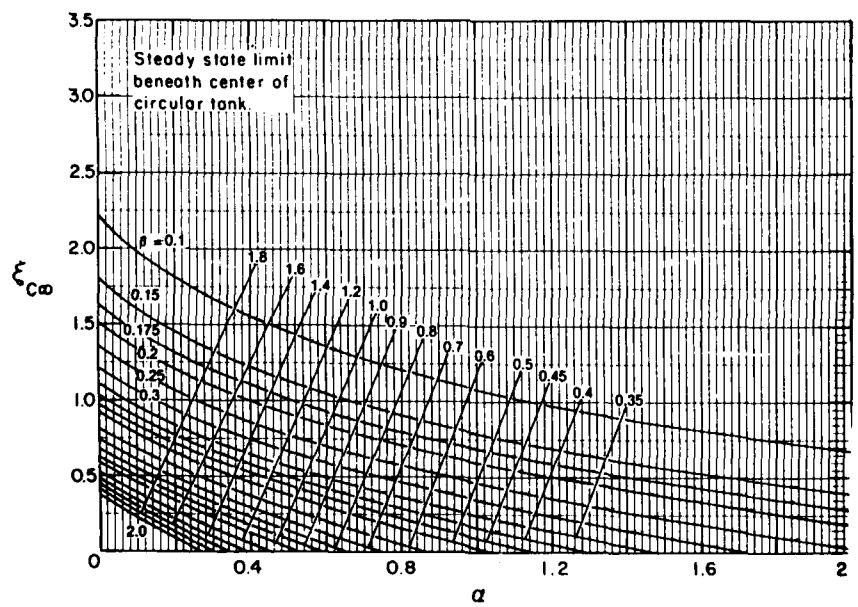

Figure 5.71. Steady state limit beneath center of circular tank.

$\xi_{c\infty}$	limiting value of ξ_c
ξ_B	value of ξ_o at center of rectangle
$\xi_{B\infty}$	limiting value of ξ_B
τ	$\dfrac{2\kappa_1 S_T t}{\pi a^2}$, dimensionless time
τ^*	$2\sqrt{\tau}$
ϕ	$\dfrac{T - T_o}{T_p - T_o}$
ϕ_o	$\dfrac{T_o - T_f}{T_p - T_f}$
ϕ_1	$\dfrac{T_1 - T_f}{T_p - T_f} = \dfrac{f - f_o}{1 - f_o}$
ϕ_2	$\dfrac{T_2 - T_o}{T_f - T_o} = \dfrac{f}{f_o}$

Subscripts

1	thawed region
2	frozen region
f	fusion or frozen
i	insulation

CHAPTER 6. CYLINDRICAL PROBLEMS WITH OUTWARD GROWTH OF INTERFACE

The previous chapters examined problems in the Cartesian or plane coordinate system. Of course, many practical engineering shapes are better adapted to curvilinear coordinates, especially the cylindrical and spherical systems. This chapter will discuss problems expressed in the cylindrical coordinate system.

For plane problems we saw that the similarity solution technique yielded an exact solution to an important practical problem - the Neumann solution. The similarity technique was also adaptable to other useful problems, although they are all essentially extensions of the Neumann problem.

In contrast to the Neumann problem, there are no general, exact, solutions for practical phase change problems in cylindrical coordinates; no exact solution has the same applicability as does the Neumann solution. Since this is an important geometry for engineering applications, a significant effort has been expended upon analytical solutions--limited to certain domains--or approximate solutions.

A formal solution, in cylindrical coordinates, can be found as follows.

The energy equation (see Chapter 1, and Nomenclature at the end of this chapter) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{6.1}$$

or

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{6.2}$$

Let us consider the transformation

$$\eta = \frac{r^2}{4\alpha t} \quad . \tag{6.3}$$

Then eq 6.2, is transformed to

$$\eta\frac{d^2T}{d\eta^2} + (1+\eta)\frac{dT}{d\eta} = 0 \ . \tag{6.4}$$

The transformation

$$\eta = \frac{r}{2\sqrt{\alpha t}}$$

also works, leading to

$$\eta \frac{d^2T}{d\eta^2} + (1 + 2\eta^2)\frac{dT}{d\eta} = 0 \quad .$$

A formal solution to eq 6.4 follows from integration:

$$T - T_i = - C \int_{\eta}^{\infty} \frac{e^{-u}du}{u} \quad ; \eta \neq 0 \ . \tag{6.5}$$

The exponential integral is defined as

$$E_i\ (z) = \int_{z}^{\infty} \frac{e^{-t}}{t}\ dt \tag{6.6a}$$

for the complex argument z.

Also, for the real argument,

$$E_i(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t}\ dt \ . \tag{6.6b}$$

These relations are often written as

$$-E_i\ (-x) = E_i\ (x) = \int_{x}^{\infty} \frac{e^{-t}}{t}\ dt \ . \tag{6.6c}$$

Series expansions can be written (Abramowitz and Stegun 1964) as

$$E_i(-x) = \gamma_e + \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{nn!}$$

$$E_i\ (x) = \gamma_e + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{nn!}$$

where $\gamma_e = 0.5772157\ .\ .\ .$ is the Euler constant.

The exponential integral functions are extensively tabulated (see App. E). Thus a formal solution to the energy equation in cylindrical coordinates is

$$T = -C\ Ei(-x) = C\ E_i(x) \ .$$

While the transformation η will reduce the energy equation to an ordinary differential equation, it cannot transform all boundary and initial conditions appropriately. Consider relations at the phase-change interface. If the phase-change interface radius R has the following form

$$R = 2\lambda \sqrt{\alpha t}$$

then

$$\lim_{\eta \to R} \eta \to \lambda^2$$

and the boundary condition transforms acceptably. Also

$$\lim_{t=0} \eta \to \infty$$

and

$$\lim_{\eta \to \infty} \eta \to \infty,$$

which are appropriate. However

$$\lim_{r \to r_o} \eta \to \frac{r_o^2}{4\alpha t} ;$$

thus boundary conditions at a finite radius will not work. The only similarity solutions possible will be for $r_o = 0$ or for problems with no conditions to satisfy at a finite cylinder radius r_o. These will be discussed shortly.

The energy condition at the phase-change interface is given by the same relation as in the Cartesian case:

$$-k_1 \frac{\partial T_1}{\partial r} + k_2 \frac{\partial T_2}{\partial r} = \pm \rho \ell \frac{dR}{dt} \quad . \tag{6.7}$$

The upper sign is for melting and the lower is for freezing. This assumes that the interface motion is in the positive direction of the coordinate r.

The medium of interest can be the region exterior to a cylinder (outward phase change) or it can be contained within the cylinder (inward phase change).

6.1 OUTWARD PHASE CHANGE, INFINITE DOMAIN

6.1.1 Freezing of Subcooled Liquid

Carslaw and Jaeger (1959) give an exact solution for the case of a subcooled liquid which freezes while the solidified region remains at the

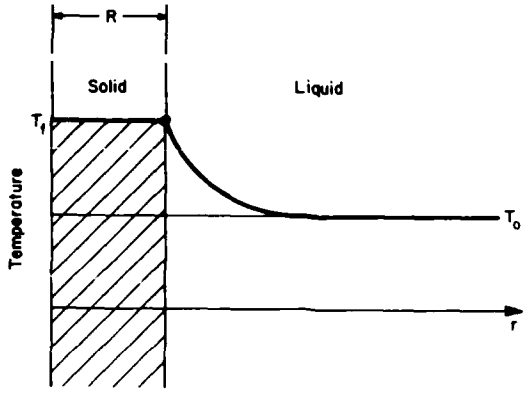

Figure 6.1. Freeze of subcooled liquid.

fusion temperature. The latent given up by the liquid is used to increase the subcooled liquid to its freezing point. For $r < R$, the solid--region (1)--is at the fusion temperature.

The basic equations, referring to Figure 6.1, are

$$T_1 = T_f \; ; \quad r < R$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad ; \; r > R \tag{6.8}$$

$$T(R,t) = T_f \tag{6.9a}$$

$$\lim_{r\to\infty} T(r,t) = T_o \tag{6.9b}$$

$$T(r,o) = T_o \tag{6.9c}$$

$$k\frac{\partial T}{\partial r}(R,t) = -\rho\ell\frac{dR}{dt} \quad . \tag{6.9d}$$

The solution follows directly from eq 6.5:

$$T = T_o + (T_f - T_o)\frac{Ei(-\eta)}{Ei(-\lambda^2)} \tag{6.10}$$

$$R = 2\lambda\sqrt{\alpha t} \tag{6.11}$$

$$\lambda^2 \, Ei\,(-\lambda^2)\, e^{\lambda^2} + S_{To} = 0 \tag{6.12}$$

where

$$S_{To} = \frac{c}{\ell}(T_f - T_o)$$

$$\eta = \frac{r^2}{4\alpha t} \quad .$$

6.1.2. Continuous Line Source Along the Origin of an Infinite Medium.

Consider a line source which extracts energy at the rate Q_s, per unit length, along the origin. An exact solution is possible as discussed in the introduction. Carslaw and Jaeger (1959) give the freezing solution as

$$T_1 = T_f + \frac{Q_s}{4\pi k_1} \left\{E_i\ (-\eta_1) - E_i\ (-\lambda^2)\right\} \qquad 0 < r < R \tag{6.13}$$

$$T_2 = T_o - \frac{(T_o - T_f)}{E_i(-\alpha_{12}\lambda^2)}\ E_i\ (-\eta_2) \qquad r > R \tag{6.14}$$

$$R = 2\lambda\ \sqrt{\alpha_1 t} \tag{6.15}$$

$$Q^*\ e^{-\lambda^2} + \frac{k_{21}\ e^{-\alpha_{12}\lambda^2}}{E_i\ (-\alpha_{12}\lambda^2)} = \frac{\lambda^2}{S_T} \tag{6.16}$$

where

$$\eta_i = \frac{r^2}{4\alpha_i t}$$

$$Q^* = \frac{Q_s}{4\pi k_1\ (T_o - T_f)}$$

$$S_{To} = \frac{c_1\ (T_o - T_f)}{\ell}\ .$$

For the thaw case, one should replace Q_s by $-Q_s$ (heat input) and use S_{To} for thaw.

6.1.2.1 Line Sink at Origin with Extended Fusion Temperature Range

An extension of the method used above can be used for the case of a material with an extended fusion temperature range. Ozisik and Uzzell (1979) considered two relations for the solid fraction-temperature function:

$$f_s = f_{su}\left[1 - \frac{T_2 - T_s}{T_\ell - T_s}\right] \tag{6.17}$$

$$f_s = f_{su}\left(\frac{R_2 - r}{R_2 - R_1}\right) \tag{6.18}$$

where

f_s = fraction of solid material in mixed phase zone (zone 2)

f_{su} = solid fraction in mushy zone at solidus front.

T_ℓ, T_s = liquidus and solidus temperatures of material

R_1 = radius of solid

R_2 = radius of mushy zone (zone 2).

The latent heat released is handled like an energy generation term:

$$q''' = \rho \ell \frac{df_s}{dt} \tag{6.19}$$

In reference to Figure 6.2, the equations are

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} \qquad 0 < r < R_1 \tag{6.20}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) + \frac{\rho \ell}{k_2}\frac{df_s}{dt} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} \qquad R_1 < r < R_2 \tag{6.21}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_3}{\partial r}\right) = \frac{1}{\alpha_3}\frac{\partial T_3}{\partial t} \qquad r > R_2 \ . \tag{6.22}$$

The heat sink, at the origin, satisfies

$$\lim_{r \to o}\left(2\pi k_1 \frac{\partial T_1}{\partial r}\right) = Q_s \tag{6.22a}$$

$$T_1(R_1) = T_2(R_1) = T_s \tag{6.22b}$$

$$k_1 \frac{\partial T_1(R_1,t)}{\partial r} - k_2 \frac{\partial T_2(R_1,t)}{\partial r} = \rho \ell \ (1-f_{su}) \frac{dR_1}{dt} \tag{6.22c}$$

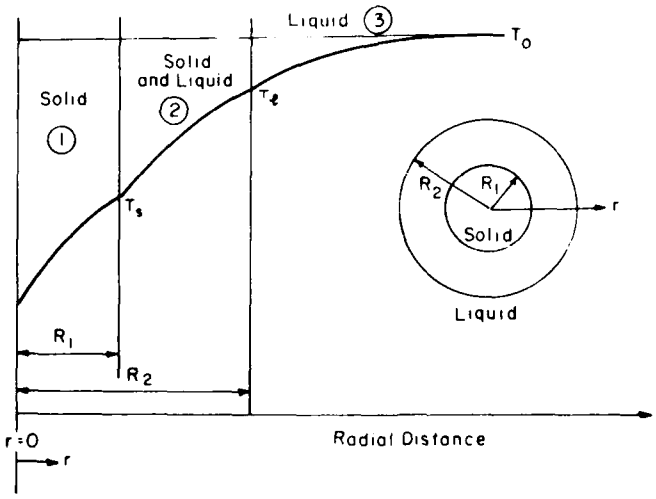

Figure 6.2. Freezing with line sink at origin.

$$T_2(R_2) = T_3(R_2) = T_\ell \tag{6.22d}$$

$$k_2 \frac{\partial T_2(R_2,t)}{\partial r} = k_3 \frac{\partial T_3(R_2,t)}{\partial r} \tag{6.22e}$$

$$\lim_{r\to\infty} T_3(r,t) = T_o \tag{6.22f}$$

$$T_3(r,o) = T_o \ . \tag{6.22g}$$

Equation 6.21 can be written in the form of eq 6.20 if an equivalent thermal diffusivity is defined. By use of eq 6.17 this is

$$\alpha_4 = \frac{\rho\ell\ f_{su}}{k_2(T_\ell - T_s)} + \frac{1}{\alpha_2} \ . \tag{6.23}$$

The solution is as follows:

$$T_i = T_s + \frac{Q_s}{4\pi k_i}\,[Ei(-\eta_i) - Ei\ (-\lambda^2)] \qquad 0 < r < R_i \tag{6.24}$$

$$T_2 = \frac{1}{p}\,[\Delta T E_i(-\eta_4) + T_s\ Ei\ (-\gamma^2) - T_\ell\ Ei(-\alpha_{i4}\lambda^2)] \tag{6.25}$$

$$R_i < r < R_2$$

$$T_3 = T_o - \frac{T_o - T_\ell}{Ei\ (-\alpha_{43}\gamma^2)}\ Ei\ (-\eta_3) \qquad r > R_2 \tag{6.26}$$

$$R_i = 2\lambda_i \sqrt{\alpha_i t} \tag{6.27}$$

$$R_2 = 2\gamma_i \sqrt{\alpha_4 t} \ . \tag{6.28}$$

The parameters γ_1, λ_1 are found from the following two coupled equations:

$$\frac{Q_s}{4\pi}\, e^{-\lambda_1^2} - \frac{k_2 \Delta T e^{-\alpha_{i4}\lambda_1^2}}{p} = \lambda_1^2 \alpha_1 (1 - f_{su})\rho\ell \tag{6.29}$$

$$\frac{Ei\ (-\alpha_{43}\gamma_1^2)}{p} = k_{32}\phi_3 e^{-\gamma_1^2(\alpha_{43}-1)} \tag{6.30}$$

where

$$p = E_i\ (-\gamma^2) - Ei\ (-\alpha_{14} + \lambda_1^2)$$

$$\phi_3 = (T_\ell - T_o)/\Delta T$$

$$\Delta T = T_\ell - T_s \ .$$

6.1.2.2 Line Sink at Origin with Moisture Flow

Boles and Ozisik (1983) obtained an exact solution for freezing about a line sink with moisture flow in the surrounding medium. A moist, porous medium is initially at uniform temperature and moisture content as shown in Figure 6.2. At time zero, a line sink removes energy at the origin, causing freezing and moisture flow toward the freezing interface. No moisture occurs in the frozen zone and the moisture flux follows the relation (see eq 1.70) given by Luikov (1975):

$$\dot{m} = -\alpha_m \rho_o \left[\frac{\partial w}{\partial r} + \delta_G \frac{\partial T}{\partial r}\right] \tag{6.30a}$$

where

$\alpha_m \rho_o$ = hydraulic diffusivity

δ_G = constant factor for the mass flux induced by thermal gradient.

$\dot{m}$ = mass flux rate

w = mass of moisture per unit mass of solid component

ρ_o = density of solid soil skeleton.

The governing equations for energy and mass flow are

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} \qquad 0 < r < R \tag{6.30b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} \qquad R < r < \infty \tag{6.30c}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\delta_G}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_m}\frac{\partial w}{\partial t} \qquad R < r < \infty . \tag{6.30d}$$

The boundary conditions are

$$\lim_{r\to o}\left(2\pi\, k_1\, r\frac{\partial T_1}{\partial r}\right) = Q_s$$

$$T_1(R,t) = T_2(R,t) = T_f$$

$$T_2(r,o) = T_o$$

$$w(r,o) = w_o$$

$$\lim_{r\to\infty} T_2(r,t) = T_o$$

$$\lim_{r\to\infty} w(r,t) = w_o$$

$$\frac{\partial w(R,t)}{\partial r} + \delta_G \frac{\partial T_2(R,t)}{\partial r} = 0$$

$$k_1 \frac{\partial T_1}{\partial r}(R,t) - k_2 \frac{\partial T_2(R,t)}{\partial r} = \rho \ell \, w(R,t) \frac{dR}{dt}$$

Equation 6.30d can be uncoupled by using

$$Z(r,t) = T_2(r,t) + \frac{(\alpha_2 - \alpha_m)(w_o - w(r,t))}{\alpha_m \delta_G} \tag{6.30e}$$

The system of three homogeneous diffusion equations is then solved using the exponential integral:

$$T_1 = T_f + \frac{Q_s}{4\pi k_1} [Ei\ (-\eta_1) - Ei\ (-\lambda^2)] \tag{6.30f}$$

$$T_2 = T_o + (T_f - T_o) \frac{Ei(-\eta_2)}{Ei(-\alpha_{12}\lambda^2)} \tag{6.30g}$$

$$w = w_o - \frac{\alpha_{m2}(T_f - T_o)\,\delta_G}{(1 - \alpha_{m2})Ei(-\alpha_{12}\lambda^2)} \left[\frac{1}{\alpha_{m2}} e^{(\alpha_{1m}-\alpha_{12})\lambda^2} Ei(-\eta_m) - Ei(-\eta_2)\right] \tag{6.30h}$$

where

$$\eta_i = \frac{r^2}{4\alpha_i t}$$

$$R = 2\lambda \sqrt{\alpha_1 t} \tag{6.30i}$$

$$\frac{Q_s}{4\pi k_2} e^{-\lambda^2} - \frac{(T_f - T_o)e^{-\alpha_{12}\lambda^2}}{Ei(-\alpha_{12}\lambda^2)} =$$

$$\frac{\alpha_{12}\lambda^2 \ell w_o}{c_2} \left\{1 - \frac{\delta_G (T_f - T_o)}{w_o(1 - \alpha_{m2})} \left[e^{(\alpha_{1m} - \alpha_{12})\lambda^2} \frac{Ei(-\alpha_{1m}\lambda^2)}{Ei(-\alpha_{12}\lambda^2)} - \alpha_{m2}\right]\right\} . \tag{6.30j}$$

6.1.3 Zero Superheat and Subcooling

A number of solutions are available for the case when the temperature of the medium is initially uniform at the fusion temperature.

6.1.3.1 Constant Phase Change Rate

A simple system, which has an exact solution, requires that the

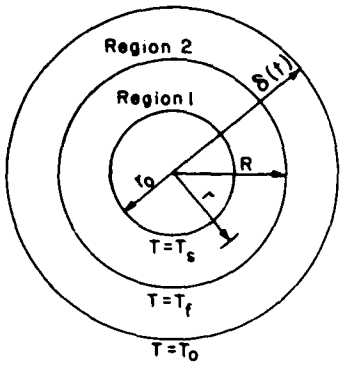

Figure 6.3. Cylindrical phase change notation.

temperature gradient at the phase-change interface be a constant. The problem (see Fig. 6.3) can be written as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{6.31}$$

$$T(R,t) = T_f \tag{6.31a}$$

$$k\frac{\partial T(R,t)}{\partial r} = \rho\ell\frac{dR}{dt} \tag{6.31b}$$

$$T(r_o,0) = T_f\ . \tag{6.31c}$$

The temperature gradient at the phase-change interface is specified as

$$\frac{\partial T(R,t)}{\partial r} = G\ . \tag{6.31d}$$

It is not necessary to solve eq 6.31 in order to find the location of the freezing front. From eq 6.31b and 6.31d,

$$\frac{dR}{dt} = \frac{k}{\rho\ell}\ G\ .$$

Thus the rate of movement of the phase-change interface is a constant. The location is

$$R = \frac{kG}{\rho\ell}\ t + r_o$$

or

$$\beta = 1 + \frac{G\ r_o}{(T_f - T_s)}\tau \tag{6.32}$$

where

$$\tau = \frac{\alpha_1 S_T t}{r_o^2}$$

$$S_T = \frac{c_1}{\ell}(T_f - T_p)$$

$$\beta = \frac{R}{r_o}\ .$$

Kreith and Romie (1955) used an iterative series solution for the temperature of the solid. This will not be given here since the problem is

rather impractical because of the necessity of imposing a complicated transient temperature at the cylinder surface.

The previous solutions comprise the known exact solutions for cylindrical phase change. The following details relate to approximate results.

6.1.3.2 Zero Sensible Heat, $S_T = 0$

If the sensible heat of the material is also neglected, then particularly simple solutions are available. Since the Stefan number is a measure of the sensible heat, this situation is equivalent to assuming that the Stefan number is zero. Carslaw and Jaeger (1959) presented a thaw solution when the surrounding medium is at the phase-change temperature T_f, using the quasi-steady approximation. The problem is as follows:

$$\frac{d}{dr}\left(r\,\frac{dT}{dr}\right) = 0 \tag{6.33}$$

$$T(R) = T_f \tag{6.33a}$$

$$T(r_o) = T_p \ . \tag{6.33b}$$

From eq 6.7, the interface energy balance is now

$$-\left(k\,\frac{\partial T}{\partial r}\right)_t + \left(k\,\frac{\partial T}{\partial r}\right)_f = \rho\ell\,\frac{dR}{dt} \ . \tag{6.33c}$$

The solution for the phase change location β is straightforward and is

$$\frac{1}{2}\,\beta^2 \ln\beta - \frac{\beta^2}{4} + \frac{1}{4} = \tau \ . \tag{6.34}$$

Equation 6.34 will overestimate the thaw around the pipe since all of the energy transfer from the pipe will go into thawing. A crude correction can be made by using an effective latent heat in place of ℓ (see App. A). The phase change interface is then given by

$$\tau = \left(1 + \frac{S_T}{2}\right)\left(\frac{1}{2}\,\beta^2 \ln\beta - \frac{\beta^2}{4} + \frac{1}{4}\right) \ . \tag{6.34a}$$

The heat transfer at the cylinder surface is

$$q = \frac{2\pi(T_p - T_f)}{\ln\beta}$$

$$q_c^* = \frac{q\,c_1}{2\pi k_1 \ell} = \frac{S_T}{\ln\beta} \ .$$

6.1.3.3 Heat Balance Integral Approximation

The heat balance integral can also be conveniently used, for the zero superheating problem. The equations are

$$\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) = \frac{r}{\alpha}\,\frac{\partial T}{\partial t} \tag{6.35}$$

$$T(R,t) = T_f \tag{6.35a}$$

$$T(r_o,t) = T_s \tag{6.35b}$$

$$T(r,o) = T_f \tag{6.35c}$$

$$-k\,\frac{\partial T(R,t)}{\partial r} = \rho\ell\,\frac{dR}{dt}\ . \tag{6.35d}$$

Integration of eq 6.35 once, over the space coordinate, yields the heat balance integral equation:

$$\alpha\left[R\,\frac{\partial T(R,t)}{\partial r} - r_o\,\frac{\partial T(r_o,t)}{\partial r}\right] = \frac{d\theta}{dt} - RT_f\,\frac{dR}{dt} \tag{6.36}$$

where the integrated temperature is

$$\theta = \int_{r_o}^{R} rT\,dr\ . \tag{6.37}$$

Lardner and Pohle (1961) noted that the logarithmic approximation is more appropriate than a polynomial in r since the area is varying with r. They suggest that $T = f(r)\ \ln r$ be used as an approximation. The temperature is thus assumed to be a logarithmic relation satisfying eq 6.35a, b:

$$T = T_p - (T_p - T_f)\,\frac{\ln\frac{r}{r_o}}{\ln\frac{R}{r_o}}\ . \tag{6.38}$$

Equations 6.36-6.38 then yield a differential equation for the phase-change location:

$$\left[\left(\frac{\beta}{2} - \frac{\beta^2 - 1}{4\beta\ln\beta}\right) S_T + \beta\ \ln\beta\right]d\beta = d\tau\ .$$

This can be integrated to give

$$\frac{S_T}{4}\left[\beta^2 - 1 - \sum_{n=1}^{\infty}\frac{2^n(\ln\beta)^n}{nn!}\right] + \frac{\beta^2}{2}\ \ln\beta - \frac{\beta^2}{4} + \frac{1}{4} = \tau\ . \tag{6.39a}$$

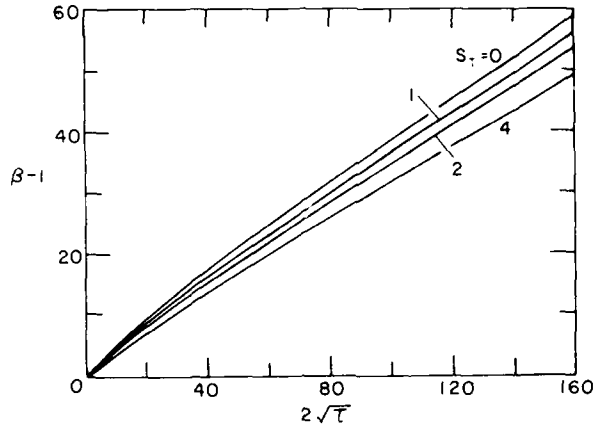

Figure 6.4. Thaw beneath a pipe, infinite system.

The solution can also be written in terms of the exponential integral as

$$\frac{S_T}{4}\left[\beta^2 - 1 - Ei(2\ln\beta) + \ln(2\ln\beta) + \gamma_e\right] + \frac{\beta^2}{2}\ln\beta - \frac{\beta^2}{4} + \frac{1}{4} = \tau \ . \quad (6.39b)$$

Equation 6.39 reduces to the zero sensible heat solution, eq 6.34, when the Stefan number is zero. The phase change vs time is shown in Figure 6.4.

Bell (1979) used up to eight subdivisions with linear temperatures in each subregion. Even with eight subdivisions the results differed from eq 6.39a by less than 1%. The use of multiple subregions eliminates the problem of the temperature guess but at the price of a large increase in computational effort (see App. B for details).

6.1.3.4 Outward Freeze of Cylinder with Convection/Radiation

We can use the heat balance integral to obtain an approximate solution. The equations are

$$\alpha\left[R\frac{\partial T(R,t)}{\partial r} - r_o\frac{\partial T(r_o,t)}{\partial r}\right] = \frac{d}{dt}\int_{r_o}^{R} rT(r,t)dr - T_f R\frac{dR}{dt} \quad (6.39c)$$

$$T(R,t) = T_f \quad (6.39d)$$

$$\frac{\partial T(r_o,t)}{\partial r} = \frac{B_i}{r_o}\left[T(r_o,t) - T_\infty\right] \quad (6.39e)$$

$$k\frac{\partial T(R,t)}{\partial r} = \rho\ell\frac{dR}{dt} \ . \quad (6.39f)$$

A logarithmic temperature profile satisfying eq 6.39d,e is

$$T = T_f + \frac{B_i\ (T_f - T_\infty)}{1 + B_i \ln \frac{R}{r_o}} \ln \frac{r}{R} \tag{6.39g}$$

where

$$S_{T\infty} = \frac{c\ (T_f - T_\infty)}{\ell}$$

$$\tau_4 = \frac{\alpha\ S_{T\infty} t}{r_o^2}$$

$$\beta = \frac{R}{r_o}$$

$$B_i = \frac{h\ r_o}{k} \ .$$

The energy equation, eq 6.39c, then yields a relation for the thaw interface radius R:

$$d\tau_4 = \left\{\left(\frac{1 + Bi\ \ln\beta}{Bi}\right)\beta + \frac{S_{T\infty}}{4}\left[2\left(\beta - \frac{1}{\beta}\right) + \frac{Bi(1 + 2\ \ln\beta - \beta^2)}{\beta(1 + B_i\ \ln\beta)}\right]\right\}d\beta . \tag{6.39h}$$

The solution to eq 6.39h is

$$\tau_4 = \frac{S_{T\infty}}{4}\left\{\left(1\frac{2}{Bi}\right)\ln\ (1+B_i\ \ln\beta)+\beta^2-1-e^{\frac{-2}{Bi}}\ \left[E_i\left(\frac{2}{Bi} + 2\ln\beta\right)-E_i\left(\frac{2}{Bi}\right)\right]\right\}$$

$$+ \frac{1}{2}\left[\left(\frac{1}{Bi} - \frac{1}{2}\right)(\beta^2 - 1) + \beta^2\ln\beta\right] . \tag{6.39i}$$

It can be shown that eq 6.39i reduces to the case of a cylinder with a constant inner temperature when $Bi \to \infty$. This solution has been given in Section (6.1.3.3), eq 6.39b.

Vives and Perry (1986) measured the solidification of tin in an annular region with and without superheating. Equation 6.39i predicted the solidification time very well for the small superheat case.

6.1.4 Finite Superheat

The more practical problems, for which the initial temperature is not at the fusion value, present significantly more difficult analyses. Several methods have been utilized. All of the problems assume that the pipe is buried at an infinite depth. A finite burial depth will present very severe restraints on the solution methods.

6.1.4.1 Quasi-Steady Solution

A simple solution can again be obtained with the quasi-steady state approximation. Khakimov (1957) investigated the problem and introduced the concept of a thermal layer of influence around the buried pipe. Let us consider the case of thawing of a medium initially frozen at $T = T_o$. A hot cylinder with a surface temperature T_p is inserted in the medium at time zero. The thaw effect is asumed to extend, at any time t, to a finite distance δ as shown in Figure 6.3. That is, the temperature of the medium will be T_o at some location sufficiently far from the hot cylinder. This concept of a temperature penetration is also used in the heat balance integral method and in boundary layer theory.

Based on experimental evidence, Khakimov (1957) assumed that $\Omega = \delta/R$ was a constant equal to 4.5. This assumption of constant Ω for a given S_T and $\phi = (T_f - T_o)/(T_s - T_f)$ is correct for the Neumann problem (see Lunardini and Varotta 1979), but it is invalid for a cylindrical system. Nevertheless it does simplify the equations and allows a reasonable solution.

The temperature will be the solution of

$$\frac{d}{dr}\left(r\,\frac{du}{dr}\right) = 0 \tag{6.40}$$

for each region with the boundary conditions

$$u_1 = u_2 = u_f \qquad r = R \tag{6.40a}$$

$$u_1 = u_s \qquad r = r_o \tag{6.40b}$$

$$u_2 = 0 \qquad r = \delta \tag{6.40c}$$

where

$$u = T - T_o \ .$$

The solutions for the temperatures are

$$u_1 = \frac{(T_f - T_p)}{\ln\left(\frac{R}{r_o}\right)} \ln\left(\frac{r}{r_o}\right) + u_s$$

$$u_2 = \frac{u_f}{\ln\left(\frac{R}{\delta}\right)} \ln\left(\frac{r}{\delta}\right) \ .$$

The total energy added to the thawing medium will be the latent heat to

thaw the layer between r_o and R and the sensible heat to increase the temperature of all the layers up to $r = \delta$. Thus,

$$Q_m = \pi(R^2 - r_o^2)[L+C_2(T_f-T_o)] + 2\pi\, C_1 \int_{r_o}^{R} r\,(T_1-T_f)\,dr + 2\pi\, C_2 \int_{R}^{\delta} r\, u_2\, dr.$$

Carrying out the integration yields

$$Q_m = \pi\,(R^2-r_o^2)[L+C_2(T_f-T_o)] + \pi C_1\{(R^2-r_o^2)(T_p-T_f)$$

$$+ \frac{(T_f-T_p)}{\ln\frac{R}{r_o}}\left[R^2(\ln\frac{R}{r_o} - \frac{1}{2}) + \frac{r_o^2}{2}\right]\} + \pi\, C_2(T_f-T_o)R^2 p \qquad (6.41)$$

where L = volumetric latent heat

C_1, C_2 = volumetric specific heat of thawed and frozen layers

$$p = (\frac{\Omega^2 - 1}{2\ln\Omega} - 1)\ .$$

During a small time increment the change in the energy absorbed by the system must equal the energy added to the system at the cylinder surface:

$$q_p = -k_1\, 2\pi r_o\, (\frac{du_1}{dr})_{r=r_o} = \frac{dQ_m}{dt}\ . \qquad (6.42)$$

Using eq 6.41 and 6.42 yields

$$d\tau = \{\beta\, \ln\beta[1 + S_T C_{21}\phi(1 + p)] + \frac{S_T}{4}\left[2\beta - \frac{\beta^2 - 1}{\beta\, \ln\beta}\right]\}d\beta\ .$$

The solution to this equation is

$$4\tau = (2\beta^2\ln\beta-\beta^2+1)[1+S_T C_{21}\phi(1+p)]+S_T[\beta^2-1-Ei(2\ln\beta)+\gamma_e+\ln(\ln\beta)]\ . \qquad (6.43)$$

The quasi-steady method gives a solution which reduces to that of the heat balance integral approximation, eq 6.39, when $\phi = 0$.

The method is simple but limited in application to first estimates of thaw depth.

6.1.4.2 Heat Balance Integral Solution

The heat balance integral method may also be applied to the problem of finite superheat but the labor is considerably increased. This problem is the cylindrical analog to the Neumann problem. Note that the labor required to arrive at a solution far exceeds that for the plane case. The problem may be stated as follows, for a melting system (see Fig. 6.3 again):

$$\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) = \frac{r}{\alpha_1}\frac{\partial T_1}{\partial t} \tag{6.44}$$

$$T_1(R,t) = T_f \tag{6.44a}$$

$$T_1(r_o,t) = T_p \tag{6.44b}$$

$$T_2(r,o) = T_o \tag{6.44c}$$

$$-k_1\frac{\partial T_1(R,t)}{\partial r} + k_2\frac{\partial T_2(R,t)}{\partial r} = \rho_1 \ell \frac{dR}{dt} \tag{6.44d}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T_2}{\partial r}\right) = \frac{r}{\alpha_2}\frac{\partial T_2}{\partial r} \tag{6.45}$$

$$T_2(R,t) = T_f \tag{6.45a}$$

$$T_2(\delta,t) = T_o \tag{6.45b}$$

$$\frac{\partial T_2(\delta,t)}{\partial r} = 0 \ . \tag{6.45c}$$

The integrated temperatures are defined as usual:

$$\theta_1 = \int_{r_o}^{R} r\ T_1 dr$$

$$\theta_2 = \int_{R}^{\delta} r\ T_2 dr \ .$$

The heat balance integral equations are

$$\alpha_1\ \left[R\frac{\partial T_1(R,t)}{\partial r} - r_o\frac{\partial T_1(r_o,t)}{\partial r}\right] = \frac{d\theta_1}{dt} - RT_f\frac{dR}{dt}$$

$$-\alpha_2\ R\frac{\partial T_2(R,t)}{\partial r} = \frac{d\theta_2}{dt} - T_o\delta\frac{d\delta}{dt} + T_f R\frac{dR}{dt} \ .$$

An approximation for T_1 that satisfies eq 6.44 a,c is

$$T_1 = T_p - \frac{(T_p - T_f)\ln\left(\frac{r}{r_o}\right)}{\ln\frac{R}{r_o}} \ .$$

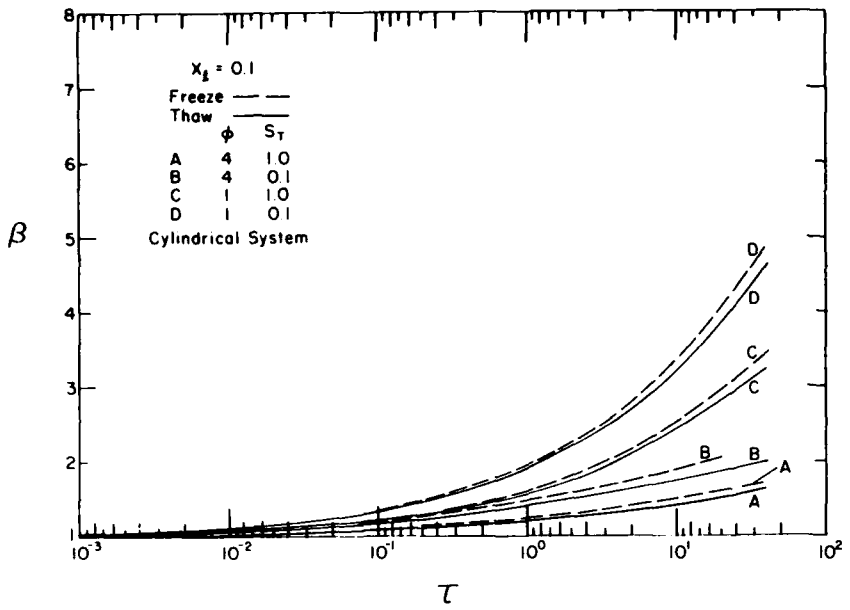

Figure 6.5. Radius vs time (k_{12} = 0.8730, α_{12} = 0.7920, thaw).

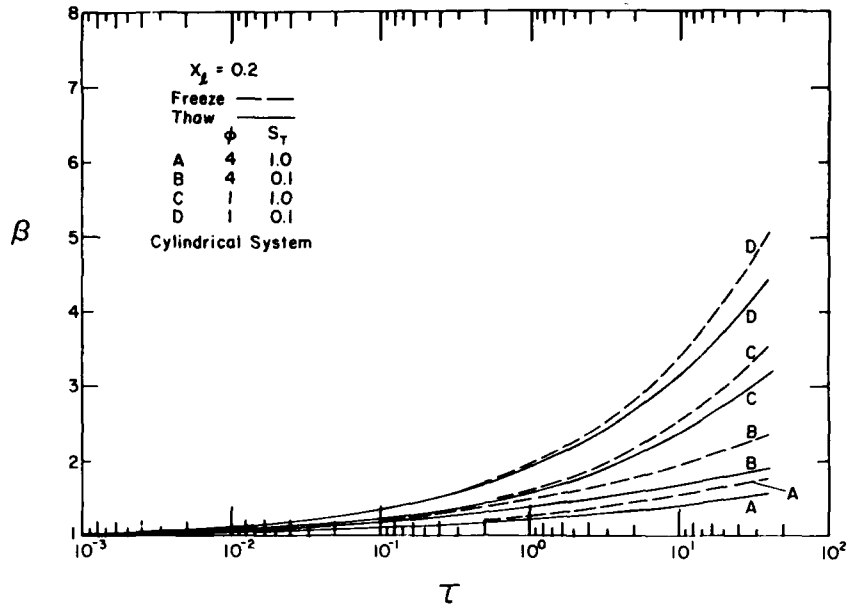

Figure 6.6. Radius vs time (k_{12} = 0.7621, α_{12} = 0.6326, thaw).

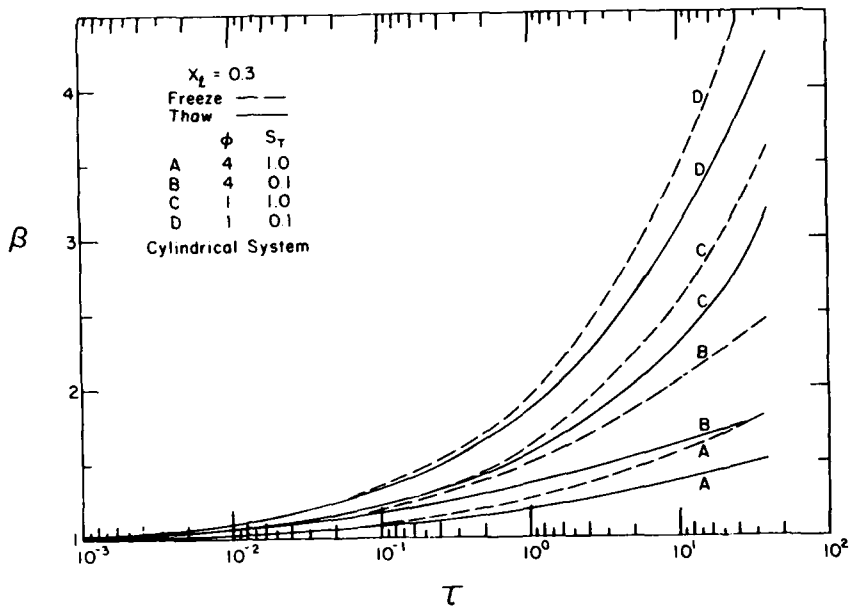

Figure 6.7. Radius vs time (k_{12} = 0.6653, α_{12} = 0.5091, thaw).

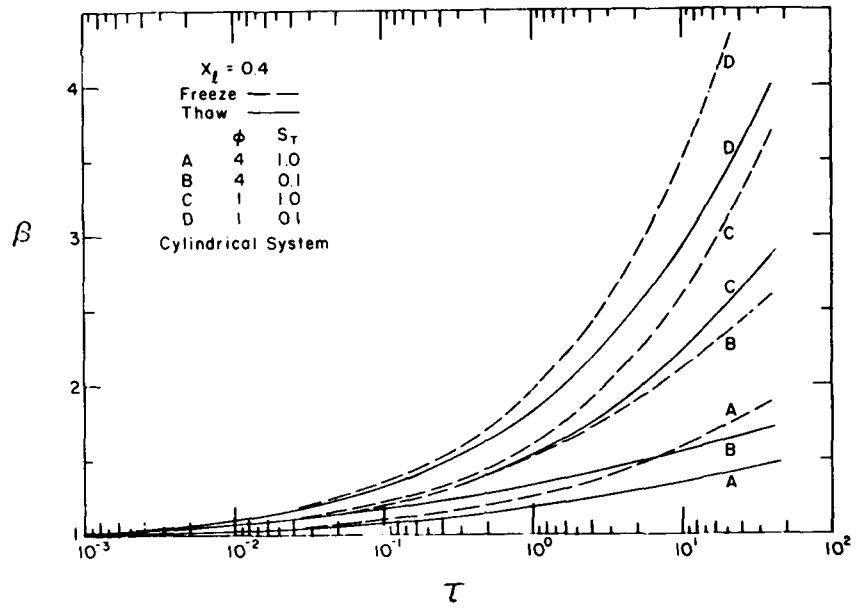

Figure 6.8. Radius vs time (k_{12} = 0.5808, α_{12} = 0.4121, thaw).

A similar approximation for T_2 that satisfies eq 6.45a-c is

$$T_2 = T_o + (T_f - T_o)\,\frac{\delta - r}{\delta - R}\,\frac{\ln\left(\frac{r}{\delta}\right)}{\ln\frac{R}{\delta}} \quad .$$

The previous relations now lead to the following system of equations to solve:

$$\alpha_{12}\, S_T \frac{dF}{d\tau} = \frac{1}{\Omega-1} + \frac{1}{\ln\Omega} \tag{6.46}$$

$$F(\beta,\Omega) = \frac{\beta^2}{2} + \frac{\beta^2}{(1-\Omega)\ln\Omega}\left[-\frac{\Omega}{4}(\Omega^2-1)+\left(\frac{\Omega}{2}-\frac{1}{3}\right)\ln\Omega + \frac{1}{9}(\Omega^3-1)\right] \tag{6.47}$$

$$\left[S_T\left(\frac{\beta}{2\ln\beta} - \frac{\beta^2-1}{4\beta(\ln\beta)^2}\right) + \beta\right]\frac{d\beta}{d\tau} = \frac{1}{\ln\beta} - k_{21}\,\phi\left(\frac{1}{\Omega-1} + \frac{1}{\ln\Omega}\right)$$

where $\Omega = \delta/R$.

Equations 6.46-48 were solved numerically with a fourth-order Runge-Kutta, predictor-corrector technique (Lunardini 1980, 1981). Since the problem, as specified, is initially singular at the origin, the Neumann solution was used to start the solution. Sparrow et al. (1978) solved this problem numerically, with $\alpha_{12} = k_{12} = 1$, for a range of S_T and ϕ, also using the Neumann solution for starting. The much simpler method presented here is within 5% of Sparrow's results. The calculations have been generalized for a range of α_{12}, k_{12}, pertinent for soil systems, and are presented in Figures 6.5-6.21.

Tien and Churchill (1965) also numerically evaluated freezing outside a circular cylinder using a numerical method that was totally different than that of Sparrow (1978). Their calculations are more extensive than those of Sparrow et al. (1978) and also agree very well with the solution presented here.

The numerical solutions, while very good, are not as convenient as analytical solutions. Thus it is valuable to further examine methods with which to obtain approximate solutions that yield acceptable accuracy.

6.1.4.3 Coordinate Transformation

A method suggested by Lin (1971) uses a coordinate transformation to reduce a problem with a variable phase-change area, such as a cylinder, to one of constant phase-change area, the semi-infinite solid. Since the relations for the semi-infinite solid are well known, this is a useful procedure but it is limited in accuracy.

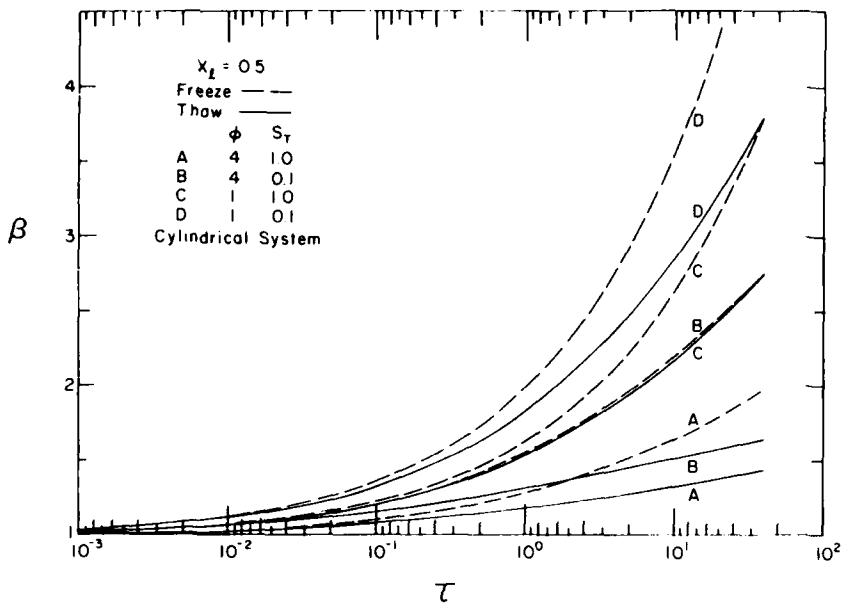

Figure 6.9. Radius vs time (k_{12} = 0.5070, α_{12} = 0.3354, thaw).

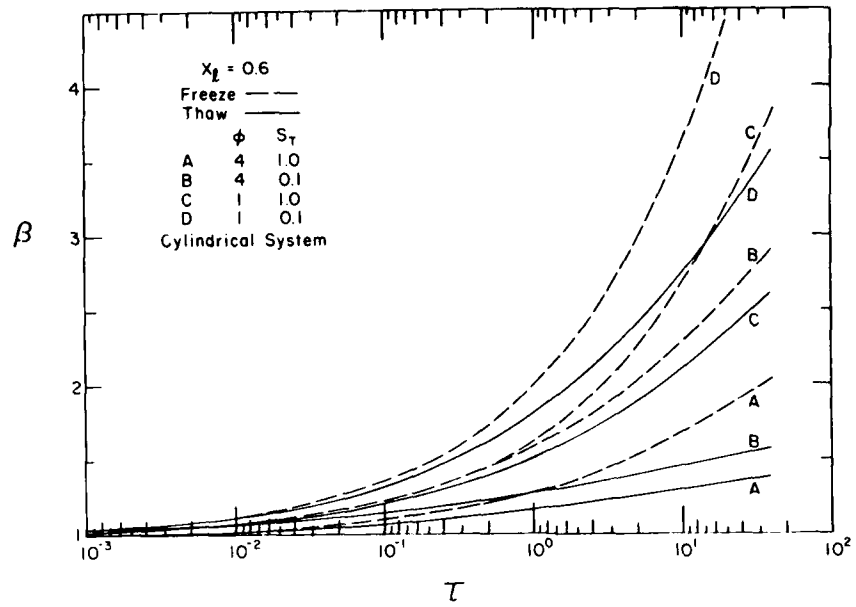

Figure 6.10. Radius vs time (k_{12} = 0.4426, α_{12} = 0.2742, thaw).

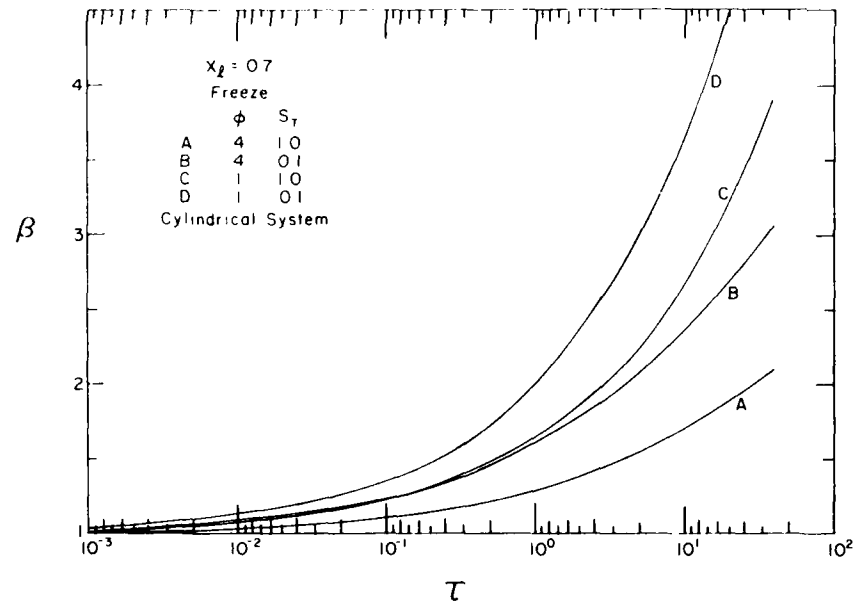

Figure 6.11. Radius vs time (k_{12} = 2.588, α_{12} = 4.441).

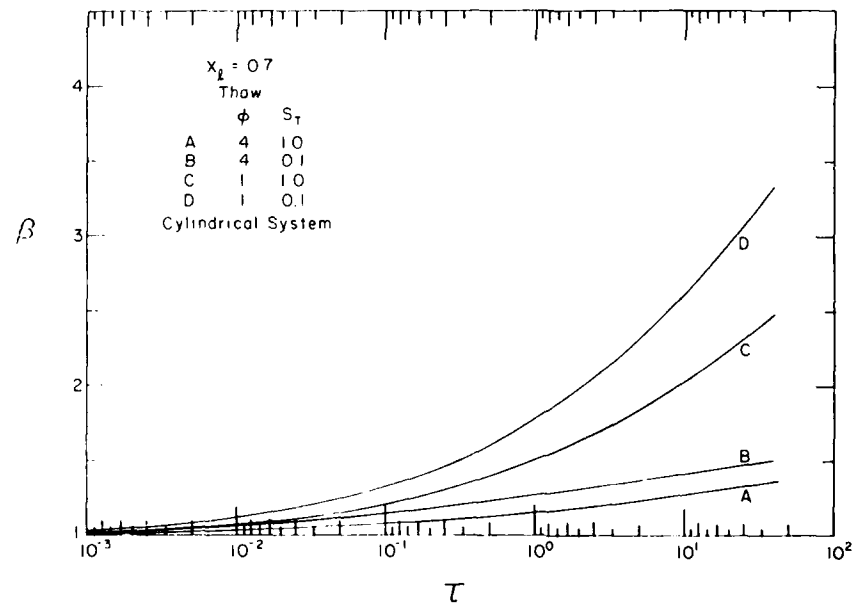

Figure 6.12. Radius vs time (k_{12} = 2.588, α_{12} = 4.4405).

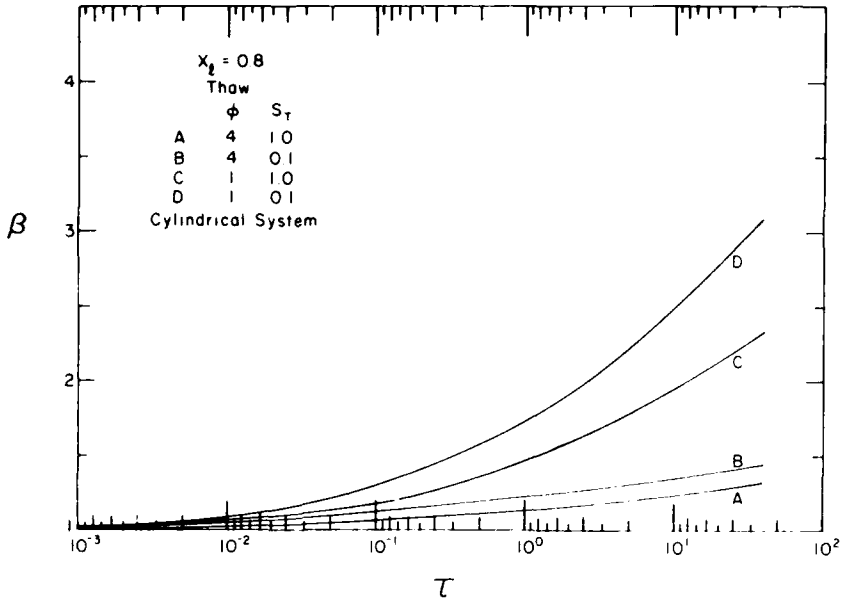

Figure 6.13. Radius vs time, k_{12} = 0.3373, α_{12} = 0.1855.

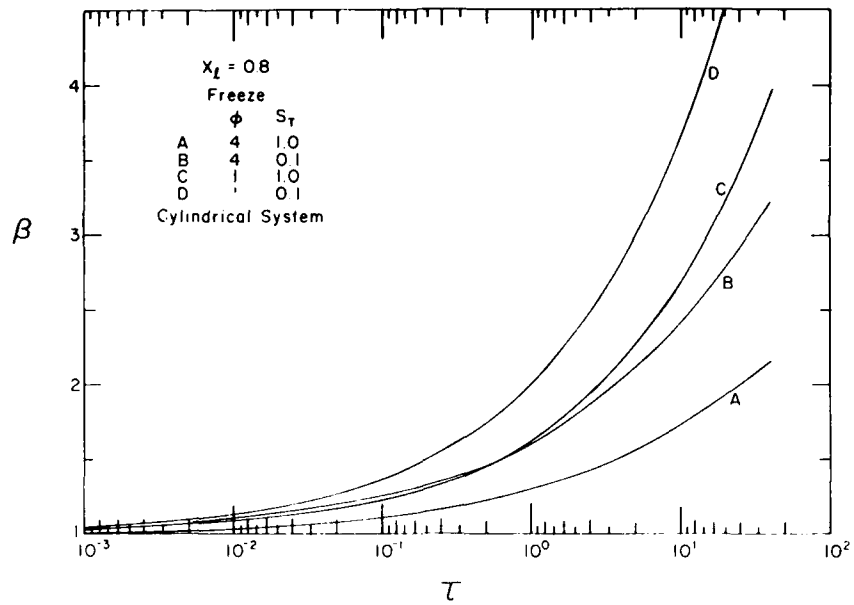

Figure 6.14. Radius vs time, k_{12} = 2.965, α_{12} = 5.391.

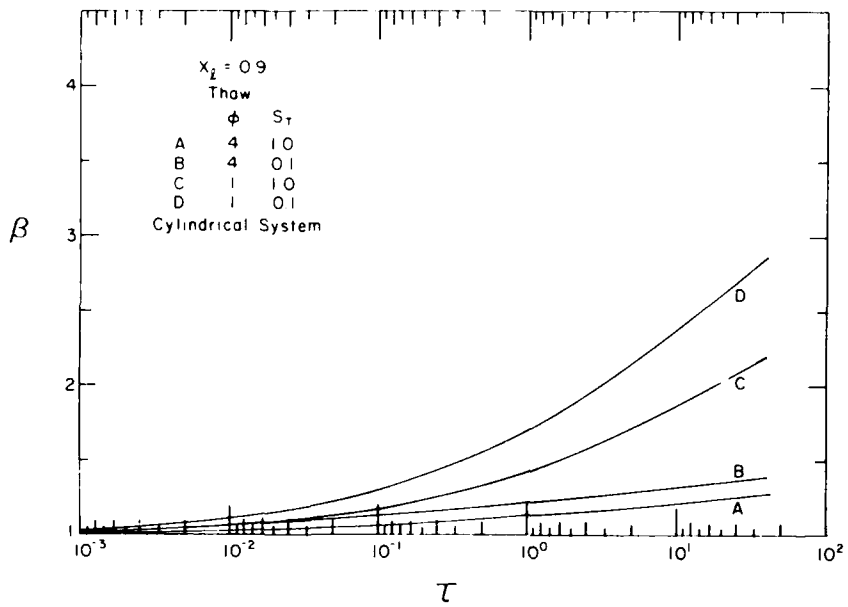

Figure 6.15. Radius vs time, k_{12} = 0.2945, α_{12} = 0.1533.

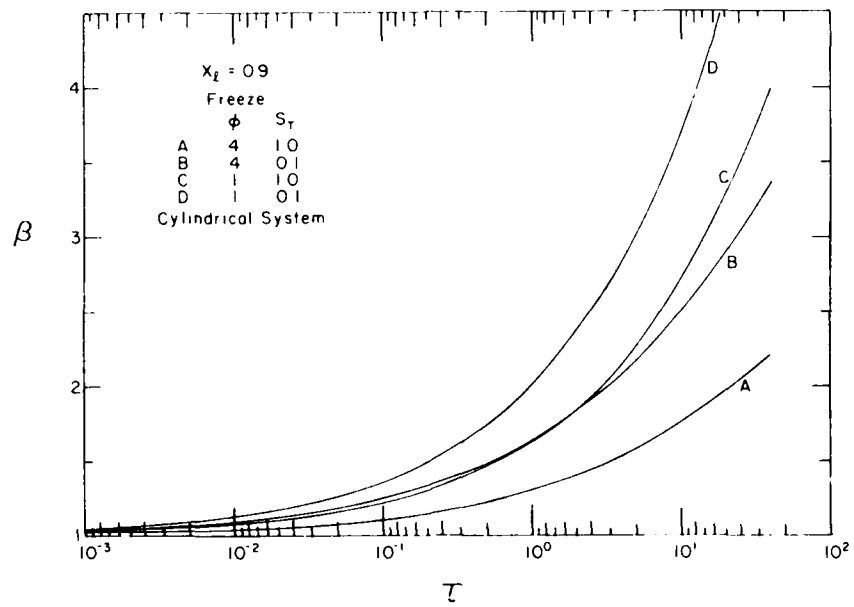

Figure 6.16. Radius vs time, k_{12} = 3.396, α_{12} = 6.5232.

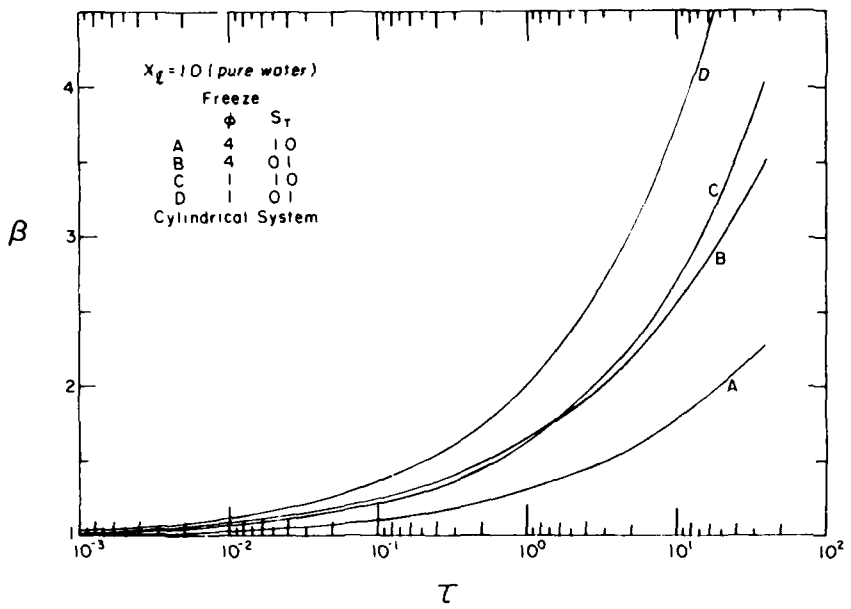

Figure 6.17. Radius vs time, k_{12} = 3.890, α_{12} = 7.869.

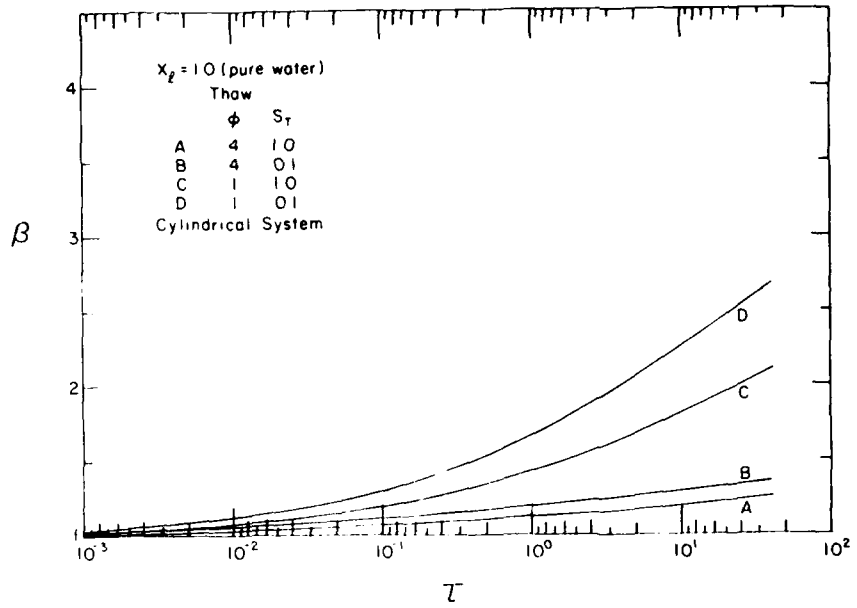

Figure 6.18. Radius vs time, k_{12} = 0.2571, α_{12} = 0.1271.

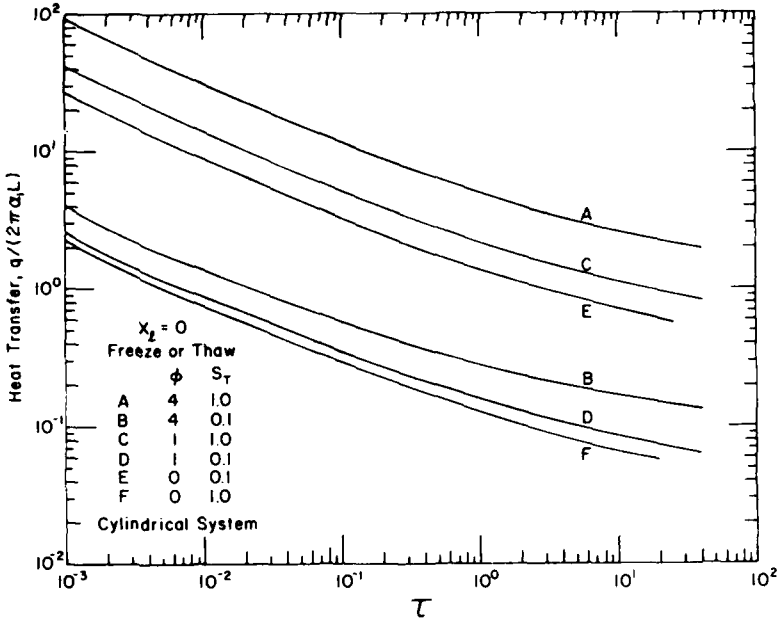

Figure 6.19. Surface heat transfer rate vs time, x_ℓ = 0.

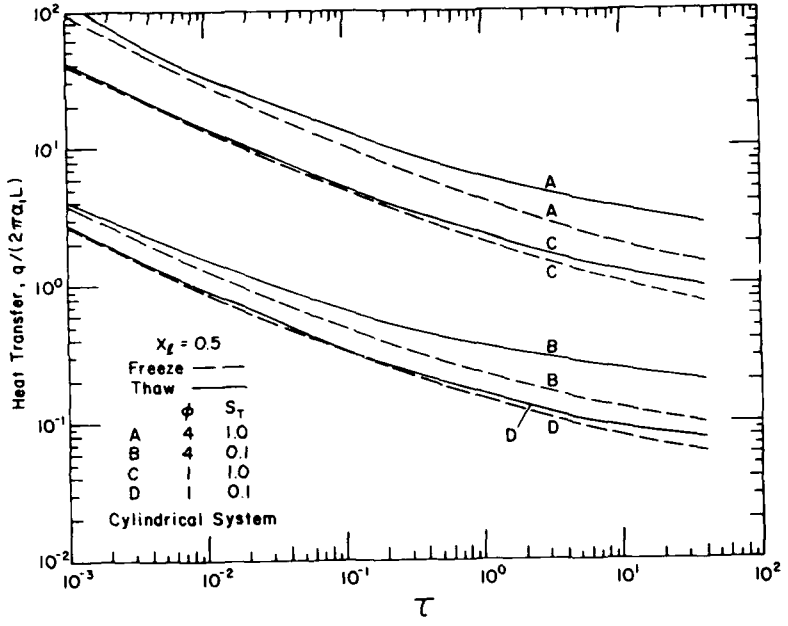

Figure 6.20. Surface heat transfer rate vs time, x_ℓ = 5.0.

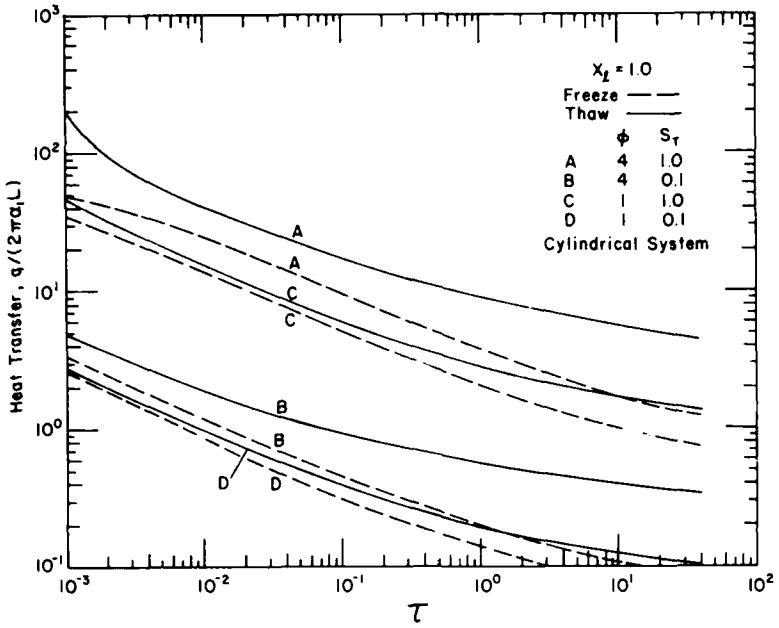

Figure 6.21. Surface heat transfer rate vs time, $x_\ell = 1.0$.

The following transformation will reduce the cylindrical system to the constant area case:

$$y = r_o \ln \left(\frac{r}{r_o}\right) . \tag{6.49}$$

The phase change interface, which is the value of y when r = R, is related by

$$= r_o \ln \left(\frac{R}{r_o}\right) . \tag{6.50}$$

The governing equations, for the cylindrical system, then transform into the following system, valid near the phase change interface where r ≈ R:

$$\frac{\partial^2 T}{\partial y^2} = \frac{k}{\rho \ell \alpha} \frac{\partial T(\eta,\eta)}{\partial y} \frac{\partial T}{\partial \eta}$$

$$T(y,0) = T_f$$

$$T(\eta,\eta) = T_f$$

$$T(0,\eta) = T_p .$$

Solutions of this system of equations are universal functions for all cross-sectional areas. However, the solutions are only valid near the phase-change interface. The system of equations need not be solved to use

the method. The phase-change interface rate of movement is given by

$$\frac{d\eta}{dt} = \frac{k}{\rho\ell} \left[\frac{A(r_o)}{A\ (R)}\right]^2 \frac{\partial T(\eta,\eta)}{\partial y} \quad .$$

For the constant area case $A(r_o) = A(R)$ and

$$\frac{d\eta}{dt} = \frac{k}{\rho\ell} \frac{\partial T(\eta,\eta)}{\partial y} = g(\eta) \ .$$

Then the generalized case is

$$\frac{dR}{dt} = \frac{r_o}{R} g(\eta) \ .$$

Thus if the velocity of the phase-change interface for the constant area case (Neumann) is given by

$$\frac{dX}{dt} = g(X) \ ,$$

then the phase-change interface velocity for the cylindrical system is

$$\frac{dR}{dt} = \frac{r_o}{R} g(r_o \ \ln \frac{R}{r_o})$$

The plane solution is given in Section 2.1.1. From this

$$g(X) = \frac{2\gamma^2\alpha_l}{X}$$

where $\gamma = X/(2\sqrt{\alpha_l t})$.

Finally, the velocity of the cylindrical interface is

$$\frac{dR}{dt} = \frac{2\gamma^2\alpha_l}{R \ \ln \frac{R}{r_o}} \quad . \tag{6.51}$$

Equation 6.51 may be integrated to give

$$2\beta^2 \ \ln\beta - \beta^2 + 1 = \frac{8\gamma^2}{S_T} \tau \ . \tag{6.52}$$

This solution may be compared to the quasi-steady solution, eq 6.34, for the case of no superheat, i.e. $S_T = 0$, $\phi = 0$. In the limit as $\phi \to 0$, the parameter γ is given by $\gamma^2 = S_T/(2 + S_T)$ and in this case, eq 6.52 is identical to eq 6.34.

While eq 6.52 is an extremely convenient form for cylindrical systems, its accuracy is limited to certain ranges of S_T, ϕ, and time.

In comparison to numerical solutions, the result is accurate if $\tau/S_T < 1.0$, when $\phi = 4$. For lesser values of ϕ the time limit when eq 6.52 is accurate will increase. The relation is so simple that it may be of value for quick, more or less crude, estimates, especially when $\phi \approx 0.0$.

6.1.4.4 Effective Thermal Diffusivity

Churchill and Gupta (1977) have introduced another method which allows the Neumann solution to be used for more complex geometries.

The effective diffusivity procedure involves replacing the nonlinear phase change problem with its linear analog which does not include phase change. The thermal diffusivity of the latter problem is then replaced by an effective diffusivity which includes the latent heat. Since many solutions are available for no-phase-change problems, the method has potential for application to numerous freeze/thaw problems.

The basis of the derivation of the desired effective diffusivity is that the solution to the zero latent heat analog of the Neumann problem (simply transient conduction in the semi-infinite medium) can be forced to agree with the Neumann solution if an effective diffusivity replaces the actual diffusivity. The location of the isotherm with the phase change value and the heat flux rate for the zero latent heat plane problem can be found from the standard solution of Carslaw and Jaeger (1959):

$$x_f = 2\sqrt{\alpha t}\ \mathrm{erf}^{-1}\left(\frac{T_f - T_p}{T_o - T_p}\right) \tag{6.53}$$

$$\frac{q_p}{A} = \frac{k(T_o - T_p)}{\sqrt{\pi \alpha t}} \quad . \tag{6.54}$$

These equations assume that the surrounding medium has constant properties, k and α. Equating eq 6.53 and 6.54 with the appropriate relations from the Neumann solution yields the effective diffusivities for heat flux and phase change position:

$$\alpha_e = \alpha_1\ [(\phi + 1)\ \mathrm{erf}\gamma]^2 \tag{6.55}$$

$$\alpha_e = \alpha_1\ \left[\frac{\gamma}{\mathrm{erf}^{-1}\left(\frac{1}{\phi + 1}\right)}\right]^2 \tag{6.56}$$

where α_1 is the actual thermal diffusivity.

Churchill and Gupta (1977) applied the method to cylinders and to

freezing in a corner with good results. They used the exact solution for the cylindrical, no-phase-change problem given by Carslaw and Jaeger (1959). This required the use of tabulated numerical values.

Lunardini (1981b) corrected the equivalent diffusivities to account for the variable properties of frozen and thawed regions. The corrected effective diffusivities are then

$$\alpha_e/\alpha_1 = \{[1 + (\frac{1 + k_{12}}{2})\phi]\ \mathrm{erf}\gamma\}^2 \tag{6.57}$$

$$\alpha_e/\alpha_1 = \{\frac{\gamma}{\mathrm{erf}^{-1}\ (\frac{1}{1 + k_{12}\phi})}\}^2 \ . \tag{6.58}$$

The first step in the solution of the phase-change problem is to solve for the temperature of the conduction problem with zero latent heat. For the case of a cylinder surrounded by an infinite medium, an exact solution for the surface step-change situation is given by Carslaw and Jaeger (1959). However, the solution involves an infinite series of Bessel functions which were approximated by error functions for small values of time. The final results are in graphical form. A closed form approximate solution to this problem will be found by using the heat balance integral method. This will introduce a further approximation in the final results and the procedure need be used only when convenient, exact solutions cannot be found. The freezing case will be examined but the results apply to melting also. In reference to Figure 6.3, the heat balance integral equations are

$$-\alpha r_o \frac{\partial v\ (r_o,t)}{\partial r} = \frac{d}{dt} \int_{r_o}^{\delta} rv\ dr \tag{6.59}$$

$$T(\delta,t) = T_o \tag{6.59a}$$

$$T(r_o,t) = T_p \tag{6.59b}$$

$$\frac{\partial T(\delta,t)}{\partial r} = 0 \tag{6.59c}$$

where $v = (T - T_o)/(T_p - T_o)$. For simplicity, a polynomial in r is assumed for the temperature,

$$v = (\frac{\delta - r}{\delta - r_o})^n \qquad r_o \leq r \leq \delta \ . \tag{6.60}$$

The solution for the penetration distance δ is

$$\frac{4}{3(n+2)}\Delta^3 + \Delta^2 = 2n(n+1)\frac{\tau}{S_T} \tag{6.61}$$

where $\Delta = (\delta - r_o)/r_o$.

Volkov and Li-Orlov (1970) noted that the accuracy of the integral method could be improved by integrating the energy equation twice over the space coordinate. El-Genk and Cronenberg (1979) applied this idea to the Neumann problem with apparent success. However, for the cylindrical system, this resulted in considerably poorer results than those for the simple integral heat balance for any given value of $n > 2$.

The solution for the actual phase change interface location is then obtained by solving eq 6.61 with the value of the freezing isotherm and the equivalent diffusivity of eq 6.57:

$$\beta = \left[1 - \left(\frac{\phi}{\phi+1}\right)^{0.05}\right]\left\{(a+d)^{1/3} + (a-d)^{1/3} - 5.5\right\} + 1 \tag{6.62}$$

$$a = 6930\,\frac{\alpha_e}{\alpha_i}\,\frac{\tau}{S_T} - 166.375$$

$$d = \left[a^2 - 27680.6406\right]^{1/2} .$$

The root of the cubic equation used in eq 6.62 is not correct if $a < d$, which occurs if τ is very small. The value of δ is found from the Neumann solution.

The instanteous heat flow from the cylinder and the total heat loss or gain during a given time are also quantities of interest. The surface heat flux rate is given by

$$q^* = \frac{qc_i}{2\pi k_i \ell} = \frac{20\,S_T(1+\phi)}{e} \tag{6.63}$$

where

$$e = (a+d)^{1/3} + (a-d)^{1/3} - 5.5$$

as before, and eq 6.58 is used for the effective diffusivity.

The integrated heat transfer can be written in nondimensional form as

$$Q_t^* = \frac{\int_0^t q\,dt}{2\pi r_o^2 \rho_1 \ell} = (1+\phi)\,\frac{S_T}{21}\left(\frac{e^2}{22} + e\right) . \tag{6.64}$$

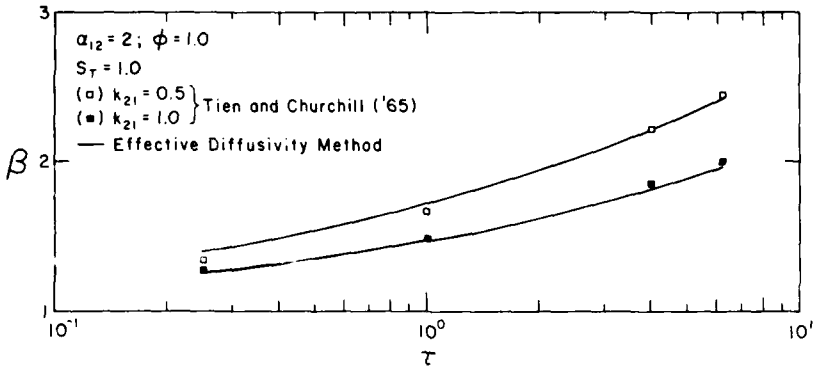

Figure 6.22. Phase change location vs time. $\phi = 1.0$, $S_T = 1.0$, $\alpha = 2.0$.

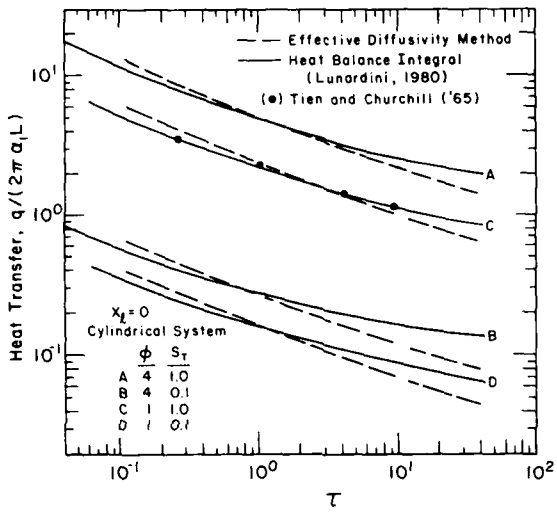

Figure 6.24. Surface heat transfer vs time, $x_\ell = 0.00$.

The approximate solution given here compared favorably with numerical solutions of Lunardini (1980), Sparrow et al. (1978), and Tien and Churchill (1965) as can be seen in Figures 6.22 and 6.23.

The comparisons confirm that the effective diffusivity method is valid, for the cylindrical problem, over a wide range of parameters, but the method cannot be applied when the superheat parameter ϕ is zero.

6.1.4.5 Constant Surface Heat Flux

The problems discussed so far have used a constant surface temperature for the cylinder. Goodling and Khader (1975) discuss numerical solutions for a constant heat flux, Q, at the cylinder surface. The heat flux from the liquid region is accounted for by a surface coefficient of heat transfer as shown in Figure 6.24. Thus the conduction equation need be considered only in the solid region. Nondimensional equations are

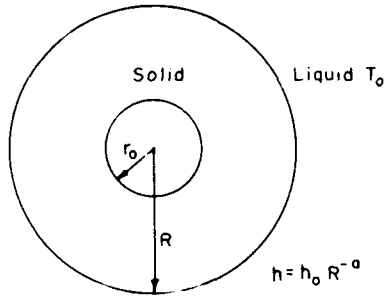

Figure 6.24. Cylindrical phase change with convection.

$$\frac{\partial^2\theta}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial\theta}{\partial\nu} = S\frac{\partial\theta}{\partial\tau_2} \qquad (6.65)$$

$$\theta(\beta,\tau) = 0$$

$$\frac{\partial\theta}{\partial\nu}(1,\tau_2) = 1$$

$$\frac{d\beta}{d\tau_2} = \frac{\partial\theta(\beta,\tau_2)}{\partial\nu} - p\beta^{-a}$$

$$\beta(0) = 1$$

where

$$\theta = \frac{(T - T_f)k}{Q\, r_o}$$

$$\nu = r/r_o$$

$$\tau_2 = Qt/\rho\ell r_o$$

$$S = Q\frac{c r_o}{\ell k}$$

$$p = \frac{h_o(T_o - T_f)}{Q\, r_o^a}$$

and the energy extracted is a positive quantity. Numerical solutions are given for a = 0.25, p = 0.6, and S = 0.2-10.

An integral heat balance solution is given where the approximate perature is defined as a logarithmic function. The integral solution is

$$\tau_2 = \int_1^{\beta} \frac{2d\beta}{\left[\left(\frac{1}{S\,\beta\ln\beta} - \frac{p}{\beta^a}\right) + \frac{4}{S\,\beta^2\ln\beta} - \left(\frac{1}{S\,\beta\ln\beta} + \frac{p}{\beta^a}\right)\right]^{1/2}} \qquad (6.65a)$$

6.1.4.6 Convection Boundary Condition

The quasi-steady method of Section 6.1.4.1 can yield an approximate solution for freezing (thawing) around a circular pipe with convection at the cylinder surface and an initial temperature of T_o.

The quasi-steady solutions for the temperatures of the frozen and thawed zones are

$$T_1 = T_f + \frac{B_i(T_f - T_\infty)}{1 + B_i \ln\frac{R}{r_o}} \ln\frac{r}{R} \qquad r_o < r < R \qquad (6.65b)$$

$$T_2 = T_o + \frac{(T_f - T_o)}{\ln \frac{R}{\delta}} \ln \frac{r}{\delta} \qquad R < r < \delta \ . \tag{6.65c}$$

The energy extracted at the pipe surface is

$$Q_m = \pi(R^2 - r_o^2)[L + C_2(T_o - T_f)] + 2\pi C_1 \int_{r_o}^{R} r(T_f - T_1)dr$$

$$+ 2\pi C_2 \int_{R}^{\delta} r(T_o - T_2)dr \ . \tag{6.65d}$$

Equating the conduction heat transfer at r_o to the time rate of change of Q_m leads to

$$d\tau = [1 + \phi_\infty S_{T\infty} C_{21} p] \frac{\beta}{B_i} [1 + B_i \ln\beta] + \frac{C_{21} S_{T\infty}}{4} [2(\beta - \frac{1}{\beta}) + \frac{B_i(1+2 \ln\beta - \beta^2)}{\beta(1+B_i \ln\beta)}].$$

The solution is

$$\tau = \frac{S_{T\infty}}{4} \{(1 - \frac{2}{B_i}) \ln(1+B_i \ln\beta) + \beta^2 - 1 - e^{-2/B_i}[E_i(\frac{2}{B_i}+2\ln\beta) - E_i(\frac{2}{B_i})]\}$$

$$+ \frac{1}{2} [(\frac{1}{B_i} - \frac{1}{2})(\beta^2 - 1) + \beta^2 \ln\beta][1 + \phi_\infty S_{T\infty} C_{21} p] \tag{6.65e}$$

where

$$p = \frac{\Omega^2 - 1}{2 \ln \Omega}$$

$$\Omega = \frac{\delta}{R}$$

$$\phi_\infty = \frac{T_o - T_f}{T_f - T_\infty} \ .$$

In general the value of p decreases as the superheat $(T_o - T_f)$ increases. This is due to a slowing down of the thermal penetration as the degree of superheat increases.

The solution given here will reduce to the constant inner surface temperature as the Biot number becomes large.

6.2 OUTWARD PHASE CHANGE, FINITE GEOMETRY

The solutions for an infinite medium, discussed in Section 6.1, give results which indicate a growth rate of the thaw zone around a buried pipe that is too fast when compared to the numerical results of Lachenbruch

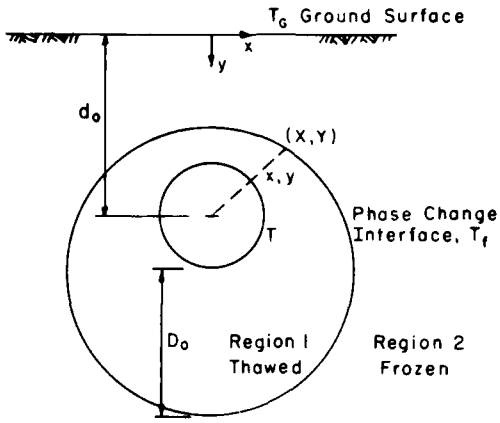

Figure 6.25. Thaw around a buried pipe.

(1970) for the transient thaw depth around a hot pipe buried in permafrost. The solution of Lachenbruch assumes two-dimensional heat transfer with no heat transfer along the axis of the pipe.

6.2.1 Problem Formulation

The problem is formulated as follows (see Fig. 6.25):

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = \frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} \qquad y > 0; \quad T < T_f \tag{6.66}$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = \frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} \qquad T < T_f \tag{6.67}$$

$$T(0,x,t) = T_G + \theta_c \sin \omega t$$

$$T = T_G \qquad y \to \infty$$

$$\frac{\partial T}{\partial x} = 0 \qquad x = 0 \text{ or } x \to \infty$$

$T = T_p$ on the pipe surface

$T_1(X,Y) = T_2(X,Y) = T_f$

where X,Y = the coordinates of the phase change interface

T_G = mean ground air temperature

θ_c = amplitude of yearly air or ground temperature fluctuation

ω = frequency of temperature fluctuation (1 year).

The energy balance at the phase-change interface (X,Y) is

$$\left.\begin{aligned} k_1 \frac{\partial T_1}{\partial x} - k_2 \frac{\partial T_2}{\partial x} &= L \frac{\partial X}{\partial t} \\ k_1 \frac{\partial T_1}{\partial y} - k_2 \frac{\partial T_2}{\partial y} &= L \frac{\partial Y}{\partial t} \end{aligned}\right\} \quad (x,y) = (X,Y) \ . \qquad (6.68)$$

The ground is initially frozen with the following initial temperature distribution:

$$T(x,y,0) = T_G + \theta_c \exp\left(-y\sqrt{\frac{\omega}{2\alpha_2}}\right) \sin\left(t - y\sqrt{\frac{\omega}{2\alpha_2}}\right) \qquad (6.69)$$

An exact solution to this problem is beyond present mathematical techniques. Numerical solutions for the thaw depth beneath the pipe, as a function of soil properties and temperatures, have been given by Lachenbruch (1970), Hwang (1972), and Gold (1972). While numerical techniques allow great flexibility with respect to properties, temperature conditions, etc., they are usually cumbersome and often expensive. In many cases preliminary estimates will suffice, especially where site conditions and data do not warrant the use of more exact analyses.

6.2.2 Quasi-Steady Approximation

The quasi-steady approximation requires steady state solutions as the starting point for a phase change problem.

Solutions are available for the steady state (perhaps after phase change ceases) using source-sink images or conformal transformations (Carslaw and Jaeger 1959, and Lunardini 1981c). The results will be given for a thaw problem, as noted in Figure 6.26, but freezing is essentially the same. The thermal properties of the medium are constant, but different, for temperatures above and below the phase change value.

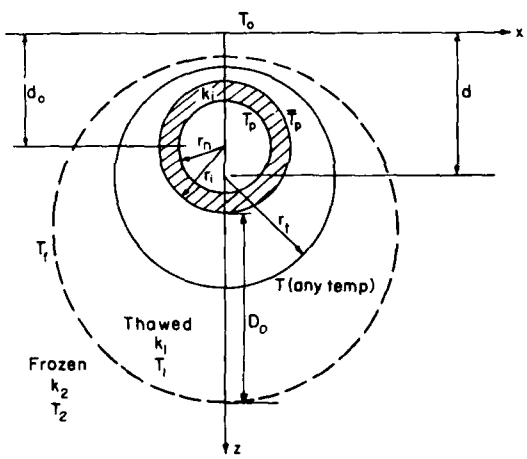

Figure 6.26. Phase change around insulated buried pipe.

6.2.2.1 Steady State Equations

The insulation surface temperature is assumed to be a constant T_p. This is acceptable if the insulation is not extremely thick. The temperature T_p is

evaluated by equating the integrated heat flow from the pipe to the heat flow into the medium at the insulation surface. Then

$$\bar{T}_p - T_f = (T_p - T_f)\left(\frac{1 - k_{21}\varepsilon\phi}{1 + b\varepsilon}\right) \tag{6.70}$$

where

$$b = \frac{1}{\ln\left(\mu + \sqrt{\mu^2 - 1}\right)}$$

$$\varepsilon = k_{1i} \ln (r_i/r_o)$$

$$\mu = d_o/r_i$$

$$\phi = \frac{T_f - T_o}{T_p - T_f} .$$

The temperatures in the thawed and frozen regions are

$$\frac{T_1 - T_f}{T_p - T_f} = \frac{1 + k_{21}\phi}{1 + b\varepsilon} \frac{\cosh^{-1}\left(\frac{d}{r_t}\right)}{\cosh^{-1}\mu} - k_{2i}\,\phi \tag{6.71}$$

$$\frac{T_2 - T_o}{T_f - T_o} = \frac{1 + k_{21}\phi}{k_{21}\phi(1+b\varepsilon)} \frac{\cosh^{-1}\left(\frac{d}{r_t}\right)}{\cosh^{-1}\mu} . \tag{6.72}$$

Equations 6.71 and 6.72 are written in terms of the origin and radius (h, r) of an isotherm. The geometric function f is defined as

$$f(d,r_t) = \frac{\cosh^{-1}\left(\frac{d}{r_t}\right)}{\cosh^{-1}\mu} \tag{6.73a}$$

where $d^2 - r_t^2 = d_o^2 - r_i^2$

This can also be written in Cartesian coordinates as

$$f(x,y) = \frac{\ln\left\{\dfrac{x^2 + [y + \sqrt{d_o^2 - r_o^2}]^2}{x^2 + [y - \sqrt{d_o^2 - r_i^2}]^2}\right\}}{2 \ln\left\{\dfrac{d_o}{r_i} + \sqrt{\left(\dfrac{d_o}{r_i}\right)^2 - 1}\right\}} \tag{6.73b}$$

In nondimensional form

$$f(\zeta,\xi) = \frac{b}{2} \ln\left\{\frac{\zeta^2 + [\xi + a]^2}{\zeta^2 + [\xi - a]^2}\right\} . \tag{6.74}$$

The value of the function f on the phase change interface is

$$f_o = \frac{b}{2} \ln \left\{ \frac{\zeta_o^2 + (\xi_o + a)^2}{\zeta_o^2 + (\xi_o - a)^2} \right\} \quad . \tag{6.75}$$

For the steady state the phase change interface is simply the T_f isotherm. This is given, from eq 6.71, by

$$f_{o\infty} = \frac{k_{2i}\phi}{1+k_{2i}\phi} (1 + b\varepsilon) \tag{6.76}$$

The limiting thaw depth can be evaluated from

$$\xi_{o\infty}/a = \frac{(\mu + a)^{f_{o\infty}} + 1}{(\mu + a)^{f_{o\infty}} - 1} \quad . \tag{6.77}$$

In this equation $\xi_{o\infty}$ denotes the depth to the bottom of the thaw interface on the plane of symmetry where $\zeta_o = 0$.

Also the depth and radius of the final thaw bowl are

$$d_{f\infty}/r_i = a\left(\frac{e^{2f_{o\infty}/b} + 1}{e^{2f_{o\infty}/b} - 1}\right) \tag{6.78}$$

$$(R_\infty/r_i)^2 = (d_{f\infty}/r_i)^2 - a^2 \quad . \tag{6.79}$$

The insulation thickness needed to keep the soil frozen is given by $\xi_{o\infty} \leq \mu + 1$. Thus, from eq 6.75, $b\varepsilon = 1/k_{2i}\phi$ and

$$\left(\frac{\mu_o}{\mu}\right)^{k_{1i}} = (\mu + a)^{\frac{1}{k_{2i}\phi}} \quad . \tag{6.80}$$

The heat transfer, from the pipe to the surface, is

$$q = 2\pi k_2 \frac{(1 + k_{21}\phi)b}{k_{2i}\phi(1 + b\varepsilon)} (T_f - T_o) \quad . \tag{6.81}$$

6.2.2.2 Quasi-Steady Equations

The quasi-steady approximation does not attempt to solve the phase change problem exactly but assumes that the phase-change interface moves slowly so that at any instant the temperature profiles satisfy the Laplace equation. This condition will be approached as the ratio of the sensible to the latent heat - the Stefan number - approaches zero (Lunardini 1981c). The motion of the interface then follows by satisfying the energy boundary condition between the two phases. The energy balance may be

formulated and solved at various locations on the phase-change interface, including the plane of symmetry.

The pipe and ground surface temperatures are always circular isotherms. The approximate isotherms will be circles, as has been noted for the steady-state solution, although the true isotherms are not circular since they are not constant coordinate surfaces of a bicircular transformation. After an infinite time, when the steady state is reached, all isotherms become circular. Thus, the intermediate-time, circular isotherms required by the quasi-steady method should be reasonable approximations of the actual isotherms.

The temperatures, at any instant, satisfy the Laplace equations with the form of the steady-state temperatures (eq 6.71, 6.72):

$$\phi_1 = \frac{T_1 - T_f}{T_p - T_f} = \frac{f}{1+b\varepsilon} + k_{21} \frac{(T_f - T_{of})}{(T_p - T_f)} \left(\frac{f}{1+b\varepsilon} - 1\right) \qquad (6.82)$$

$$\phi_2 = \frac{T_2 - T_o}{T_f - T_o} = \left[\frac{T_{pf} - T_f}{T_f - T_o} + k_{21}\right] \frac{f}{k_{21}(1+b\varepsilon)} \quad . \qquad (6.83)$$

The value of the geometric function on the plane of symmetry, $\zeta = 0$, is

$$f(\xi) = b \ln \left(\frac{\xi + a}{\xi - a}\right) \quad . \qquad (6.84)$$

T_{of} and T_{pf} are fictitious, transient boundary temperatures of the frozen and thawed zones that allow the phase change interface to move from the pipe (or insulation) surface to the final steady-state position. These auxiliary temperatures - a concept developed by Porkhaev (1963, 1970) - are needed since only one location of the interface, that given by eq 6.76, is possible with the actual boundary temperatures.

Since the value of f is f_o on the phase change interface, where the temperature is T_f, then eq 6.82 and 6.83 lead to

$$\frac{T_f - T_{of}}{T_p - T_f} = \frac{f_o}{k_{21}(1+b\varepsilon - f_o)} \qquad (6.85)$$

$$\frac{T_{pf} - T_f}{T_f - T_o} = \frac{k_{21}(1+b\varepsilon - f_o)}{(1 + b\varepsilon)} \quad . \qquad (6.86)$$

Thus the thawed and frozen zone dimensionless temperatures are

$$\phi_1 = \frac{f - f_o}{1+b\varepsilon - f_o} \qquad (6.87)$$

$$\phi_2 = \frac{f}{f_o} \quad . \tag{6.88}$$

Different solutions for the movement of the phase-change interface can be obtained by satisfying the phase change energy balance (eq 6.68) at different locations. Hwang (1977) and Lunardini (1981d) satisfied the energy equation at the lowest point of the symmetry plane while assuming a plane interface. Seshadri and Krishnayya (1980) used the same concept but evaluated the insulation temperature by equating heat flows only at the bottom of the pipe. Thornton (1976) integrated the energy balance over the entire phase change interface, rather than directly beneath the pipe. This automatically restricts the phase change to a circular geometry. We will follow Lunardini (1983), where the heat flux is assumed to be constant around the entire phase change interface, using the value at the bottom of the pipe.

The phase change radius is governed by

$$ab\tau_3 = \int_1^{R_o} \frac{z^2 \; (p+1) \; dz}{\dfrac{1}{1+b\varepsilon - f_o} - \dfrac{k_{21}\phi}{f_o}} \tag{6.89}$$

where

$$f_o = b \ln (p + \sqrt{p^2 - 1})$$

$$p = \sqrt{1 + \frac{a^2}{z^2}} \quad .$$

Equation 6.89 can be written as

$$4a\tau_3 = \int_{\mu+1}^{\xi_o} \frac{[x^2 - (a/x)^2]dx}{\dfrac{1}{\left\{ \dfrac{1}{b} + \varepsilon - \ln \left(\dfrac{x+a}{x-a}\right) \right.} - \dfrac{k_{21}\phi}{\ln \left(\dfrac{x+a}{x-a}\right) \left. \right\}}} \quad . \tag{6.89a}$$

The solution to eq 6.89a, when $\phi = 0$, is

$$4a\tau = \left(\frac{\xi_o^3}{3} + \frac{a^4}{\xi_o}\right) \ln \left[\frac{(\mu+a)(\xi_o - a)}{\xi_o + a}\right] + \varepsilon \left[\frac{1}{3} (\xi_o^3 - g^3) + a^4 \left(\frac{1}{\xi_o} - \frac{1}{g}\right)\right]$$

$$- \frac{a}{3} (\xi_o^2 - g^2) + 2a^3 \ln \left(\frac{\xi_o}{g}\right) - \frac{4}{3} a^3 \ln \left(\frac{\xi_o^2 - a^2}{2g}\right) \tag{6.90}$$

where $g = \mu + 1$.

Equation 6.89 yields the same steady state solutions given by eq 6.78.

The quasi-steady solution was compared (Lunardini 1983) to the numerical cases of Gold et al. (1972), Lachenbruch (1970), and Hwang et al. (1972), and found to be accurate to about 20%. Numerical quadrature of eq 6.89 was carried out and values are plotted in Figures 6.27-6.44 for practical ranges of ϕ, ϵ and μ (Lunardini 1983).

These graphs are felt to be acceptable for engineering estimates if accuracies on the order of 20% are acceptable. For insulated pipes, where the phase change is expected to be more limited, the graphs should be more accurate. For hot oil pipes the sensible heat can be considerably greater than the latent heat. This means that the Stefan number is quite large and in order to account for the sensible heat, an effective latent heat is defined as

$$L_e/L = 1 + C_{21}\phi\ S_T + S_T/2\ . \tag{6.91}$$

The equations and graphs presented for the thawing case are all valid for the freezing problem if

$$\tau_3 = \frac{k_1(T_f - T_p)t}{r_1^2\ L_e}$$

and k_1, etc., refer to the frozen values.

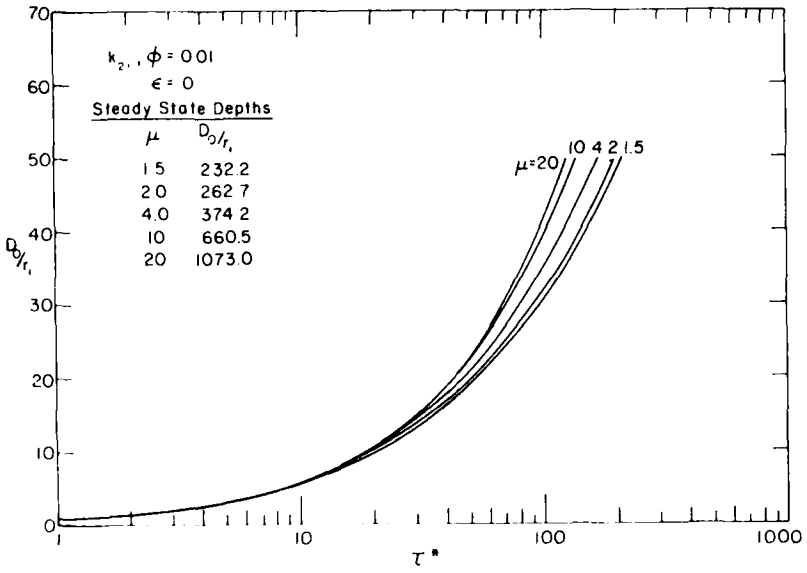

Figure 6.27. Phase change beneath a buried pipe, $k_{21}\phi = 0.01$, $\epsilon = 0$.

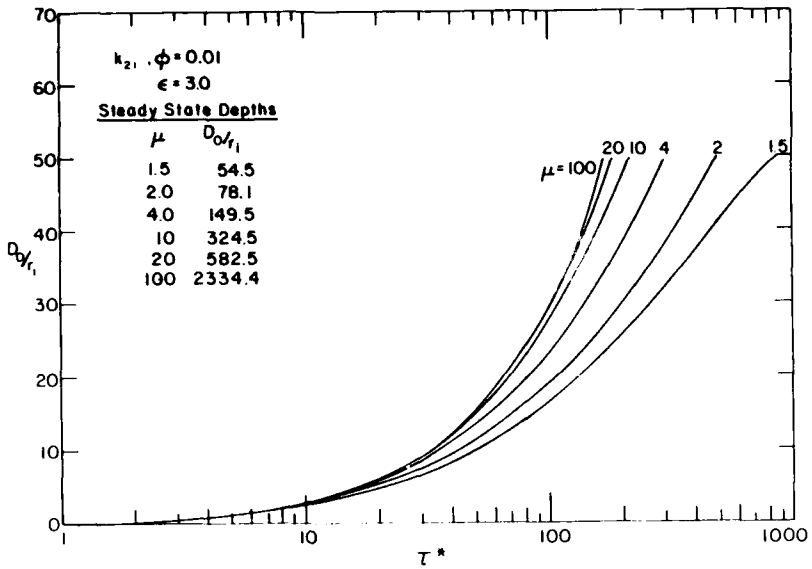

Figure 6.28. Phase change beneath a buried pipe, $k_{21}\phi = 0.01$, $\epsilon = 3.0$.

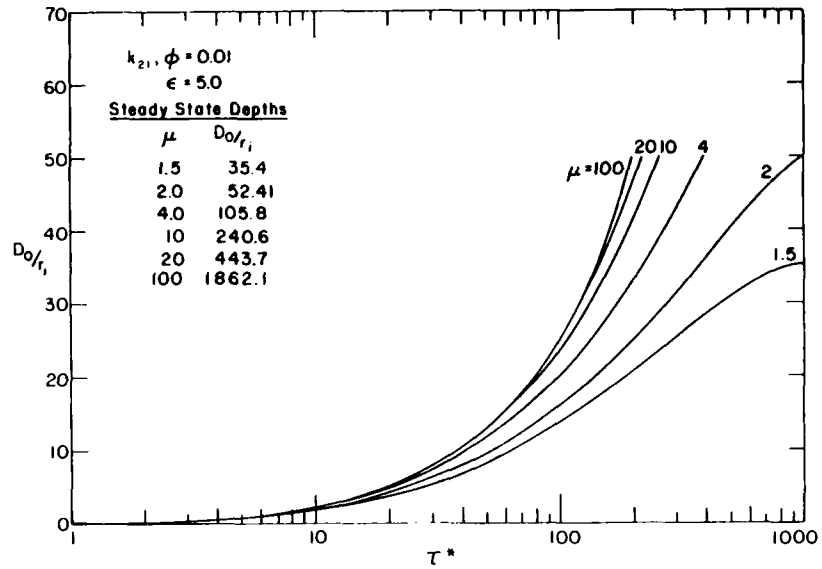

Figure 6.29. Phase change beneath a buried pipe, $k_{21}\phi = 0.01$, $\epsilon = 5.0$.

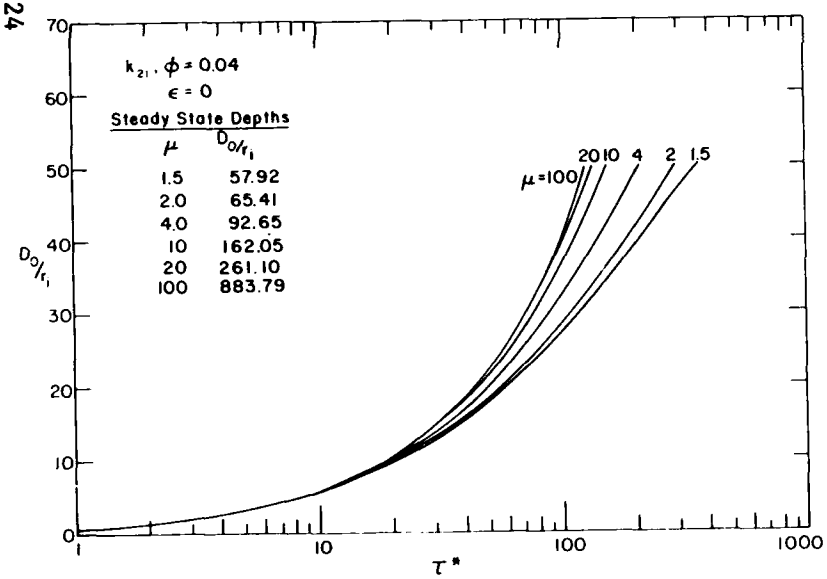

Figure 6.30. Phase change beneath a buried pipe, $k_{21}\phi = 0.04$, $\epsilon = 0$.

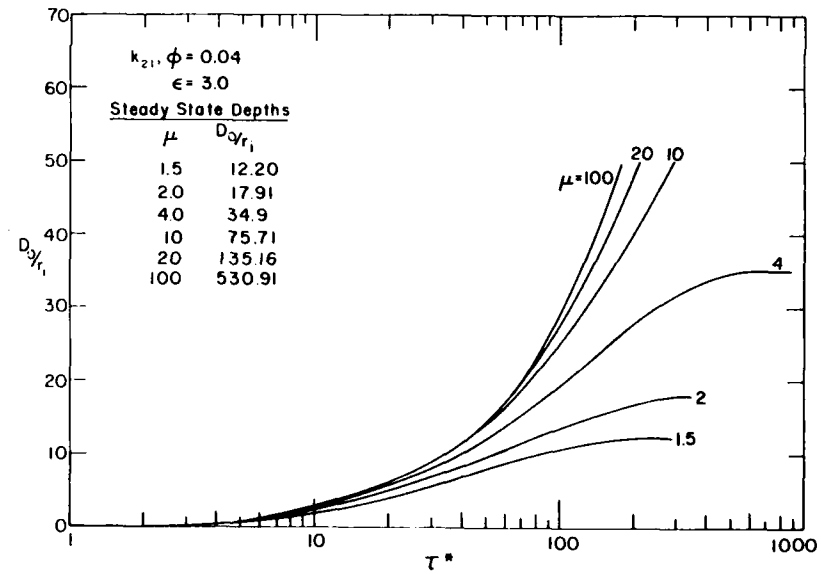

Figure 6.31. Phase change beneath a buried pipe, $k_{21}\phi = 0.04$, $\epsilon = 3.0$.

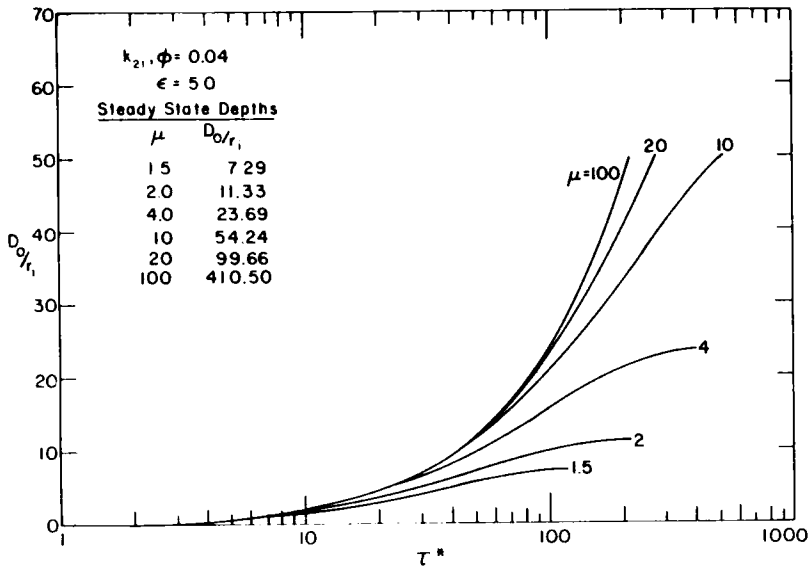

Figure 6.32. Phase change beneath a buried pipe, $k_{21}\phi = 0.04$, $\varepsilon = 5.0$.

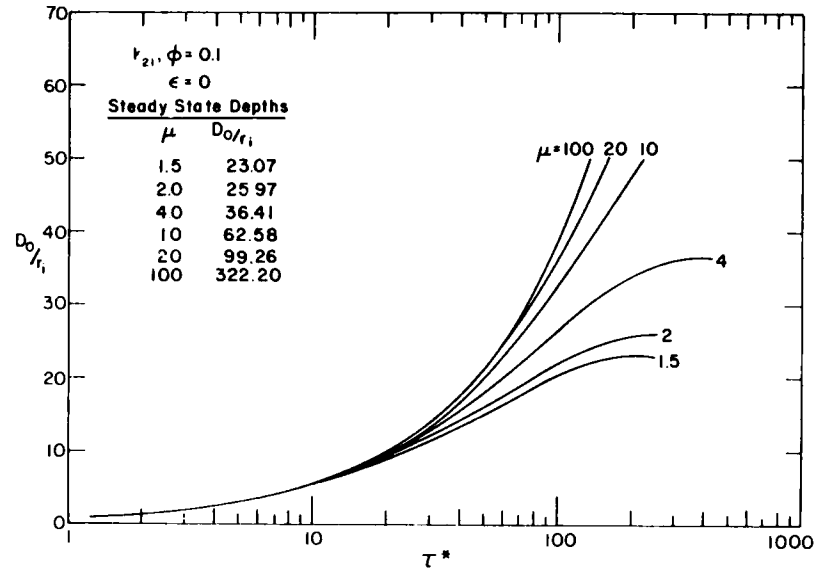

Figure 6.33. Phase change beneath a buried pipe, $k_{21}\phi = 0.1$, $\varepsilon = 0$.

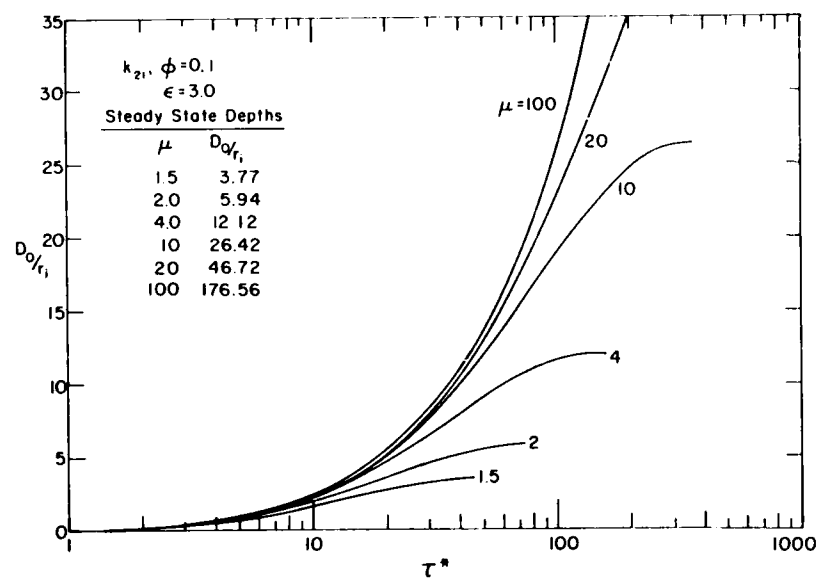

Figure 6.34. Phase change beneath a buried pipe, $k_{21}\phi = 0.1$, $\varepsilon = 3.0$.

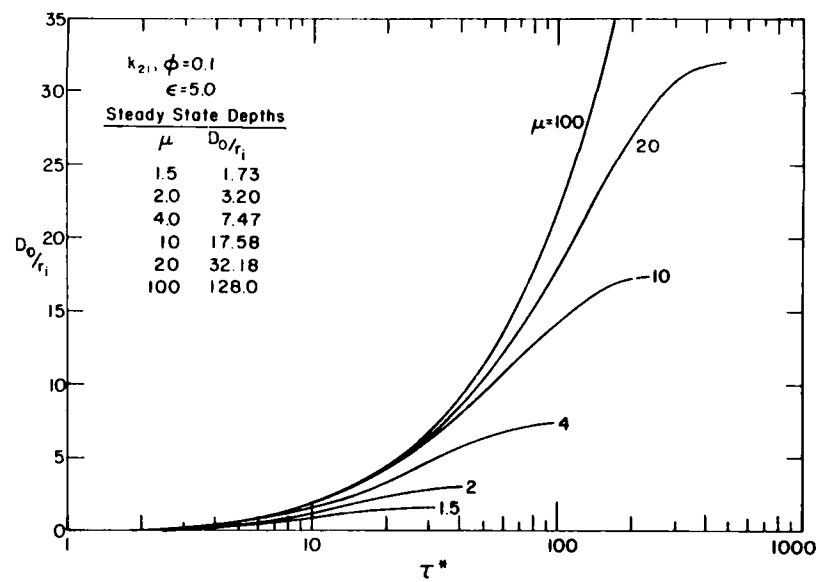

Figure 6.35. Phase change beneath a buried pipe, $k_{21}\phi = 0.1$, $\varepsilon = 5.0$.

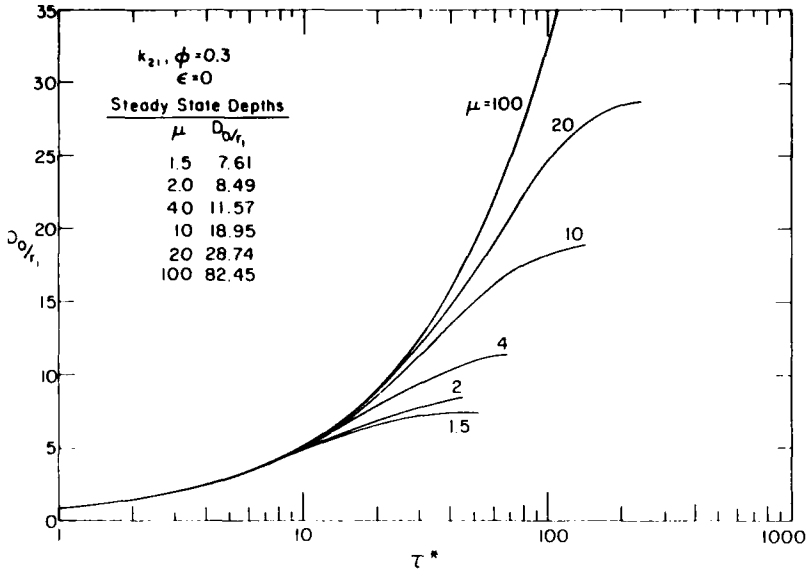

Figure 6.36. Phase change beneath a buried pipe, $k_{21}\phi = 0.3$, $\varepsilon = 0$.

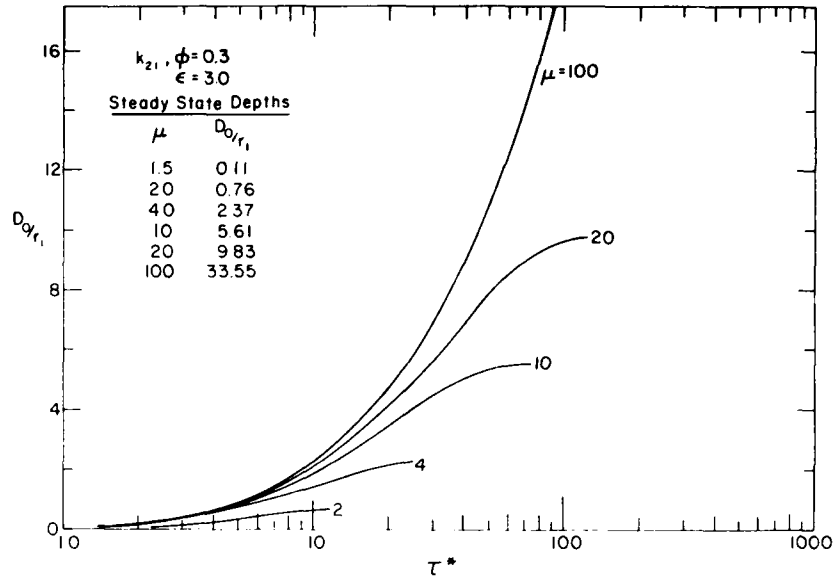

Figure 6.37. Phase change beneath a buried pipe, $k_{21}\phi = 0.3$, $= 3.0$.

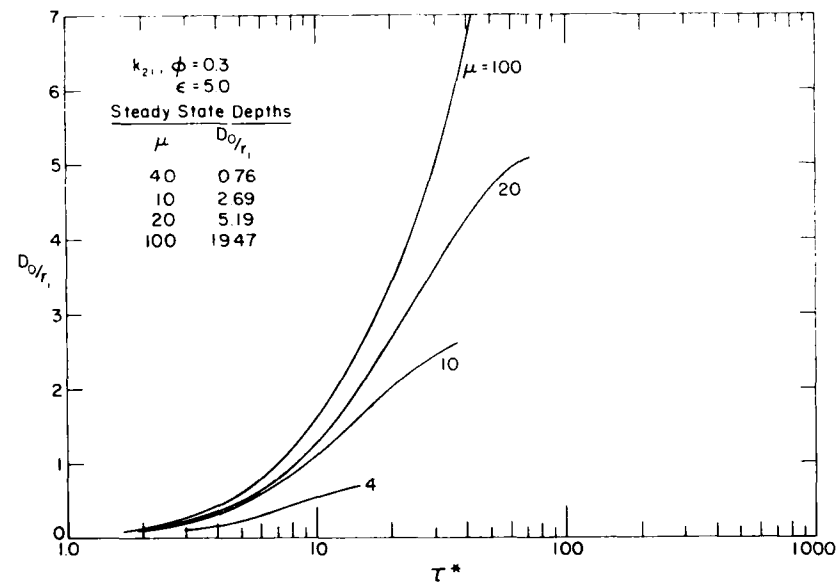

Figure 6.38. Phase change beneath a buried pipe, $k_{21}\phi = 0.3$, $\varepsilon = 5.0$.

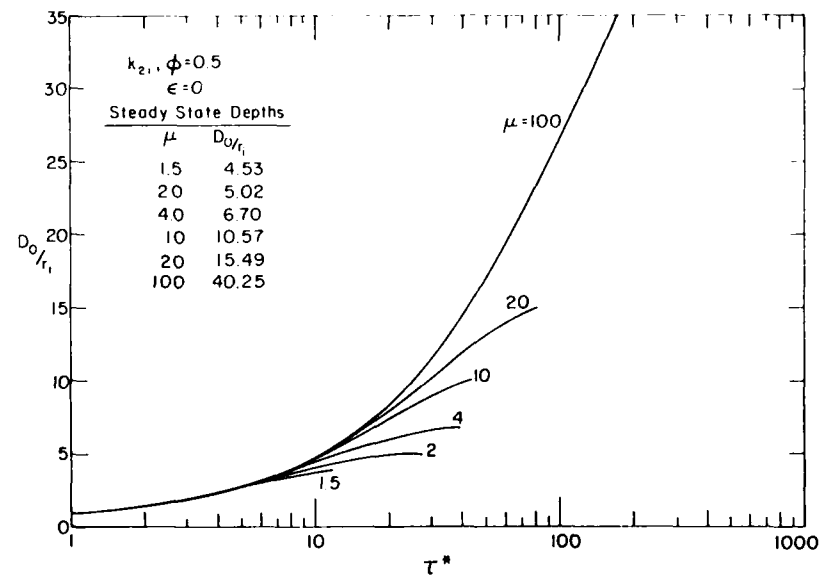

Figure 6.39. Phase change beneath a buried pipe, $k_{21}\phi = 0.5$, $\varepsilon = 0$.

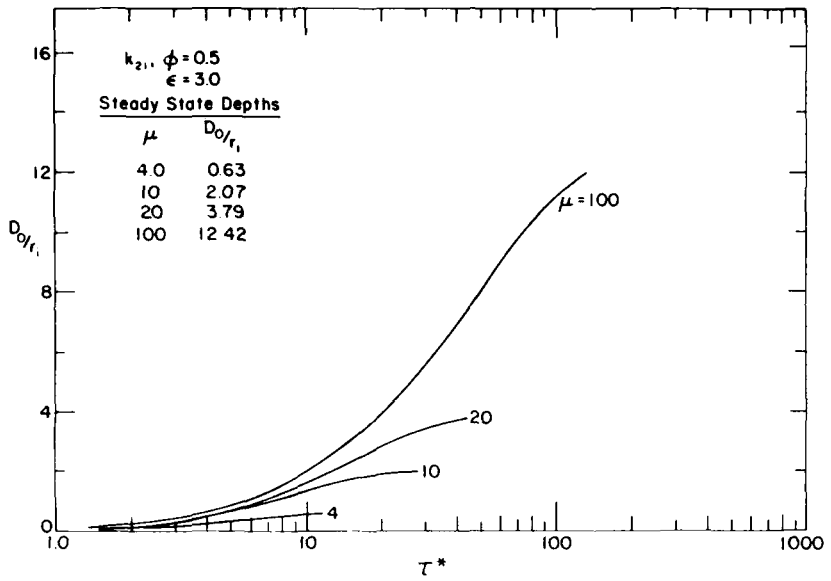

Figure 6.40. Phase change beneath a buried pipe, $k_{21}\phi = 0.5$, $\varepsilon = 3.0$.

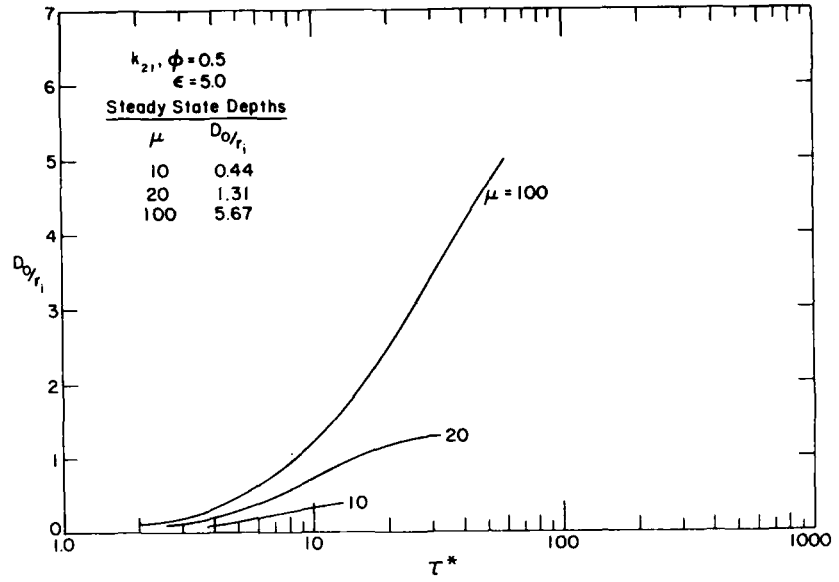

Figure 6.41. Phase change beneath a buried pipe, $k_{21}\phi = 0.5$, $\varepsilon = 5.0$.

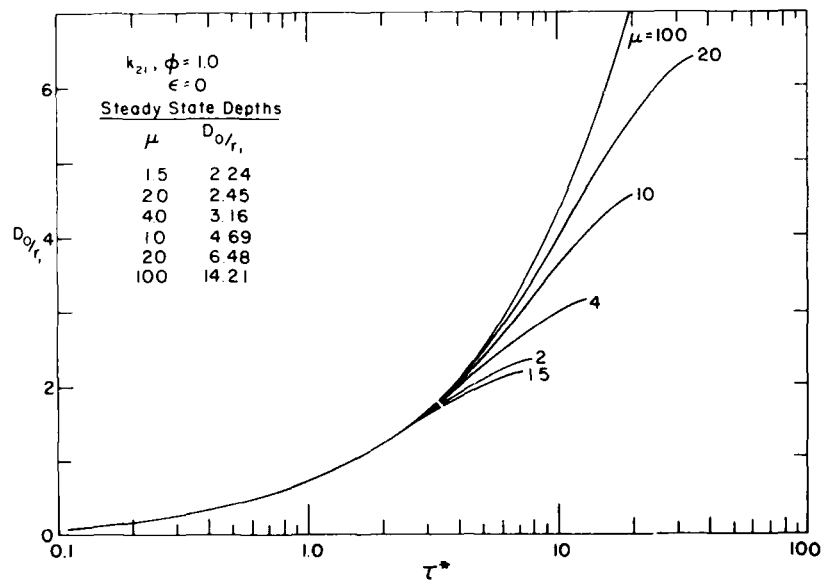

Figure 6.42. Phase change beneath a buried pipe, $k_{21}\phi = 0.1$, $\varepsilon = 0$.

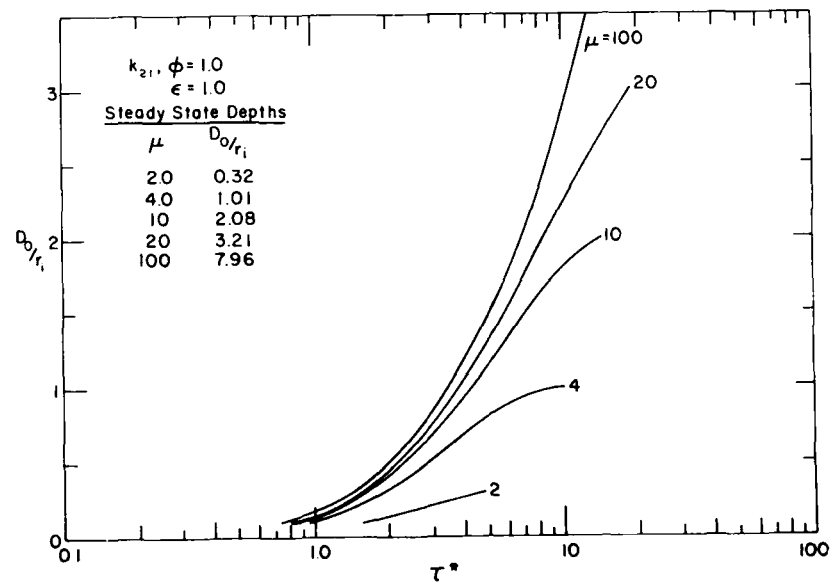

Figure 6.43. Phase change beneath a buried pipe, $k_{21}\phi = 1$, $\varepsilon = 1.0$.

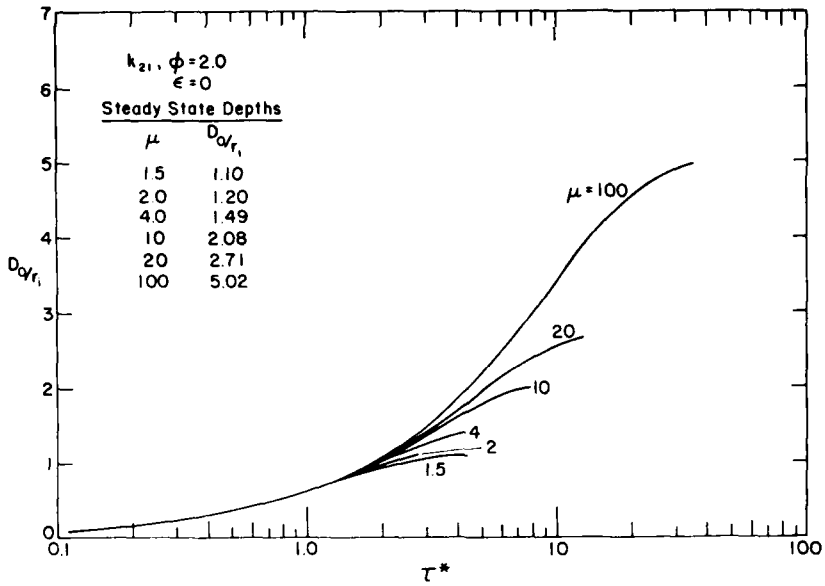

Figure 6.44. Phase change beneath a buried pipe, $k_{21}\phi = 0.2$, $\varepsilon = 0$.

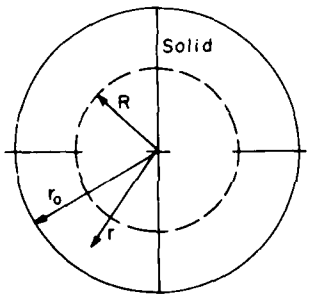

Figure 6.45. Inward solidification in a cylinder.

6.3 INWARD PHASE CHANGE

The problem of inward phase change from a cylindrical surface is important in dealing with the freezing of water in pipes and allied problems. A number of solutions are available for this problem in a finite domain.

6.3.1 Zero Superheat or Subcooling

As expected when the initial temperature of the system is at the fusion value the analysis is considerably simplified.

6.3.1.1 Constant Phase Change Rate

The solution to this problem, as given by Kreith and Romie (1955), is similar to that discussed in Section 6.1.3.1. With reference to Figure 6.45, the problem is expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{6.92}$$

$$T(R,t) = T_f \tag{6.92a}$$

$$\left.\begin{aligned} k\frac{\partial T}{\partial r} &= +\rho\ell\frac{dR}{dt} \qquad (6.92b) \\ -\frac{\partial T}{\partial r} &= G = Q/k \qquad (6.92c) \end{aligned}\right\} \quad r = R$$

$$T(r_o,0) = T_f \ . \tag{6.92d}$$

The interface moves at a constant rate given by

$$R = r_o - \frac{kGt}{\rho\ell} \ . \tag{6.93}$$

The time to attain complete solidification is

$$t_f = \frac{\rho\ell r_o}{kG} \ . \tag{6.94}$$

6.3.1.2 Zero Sensible Heat

Zero sensible heat implies that the Stefan number is small. Our definition of S_T means that the latent heat is very large. Once again the quasi-steady procedure will lead to simple and useful results. The quasi-steady method always neglects the sensible heat. The problem may be described as follows:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \tag{6.95}$$

$$\left.\begin{aligned} T &= T_f \\ k\frac{\partial T}{\partial r} &= +\rho\ell\frac{dR}{dr} \end{aligned}\right\} \quad r = R$$

$$T(r,0) = T_f$$

$$T(r_o,t) = T_p$$

Equation 6.95 is easily solved for the temperature which is given by

$$T = T_s + \frac{(T_f - T_p)}{\ln\left(\frac{R}{r_o}\right)} \ln\left(\frac{r}{r_o}\right) \ .$$

The solidification interface can easily be evaluated:

$$2\beta^2 \ln \beta - \beta^2 + 1 = 4\tau \tag{6.96}$$

where

$$\beta = R/r_o$$

$$\tau = \frac{\alpha t}{r_o^2} S_T$$

$$S_T = \frac{c\,(T_f - T_p)}{\ell} \quad .$$

The solidification time is

$$\frac{\alpha\, t_f}{r_o^2} S_T = \tau_f = \frac{1}{4} \quad . \tag{6.97}$$

6.3.1.3 Finite Sensible Heat

A solution for the constant surface temperature problem was given by Riley et al. (1974), using a perturbation method valid for small values of the Stefan number. The first two terms of the series for the solidification time gave good results. The relation is

$$\tau_f = \frac{1}{4}\,(1 + S_T) \quad . \tag{6.98}$$

It is clear that this result will be close to that of eq 6.97 for small values of the Stefan number.

6.3.1.4 Integral Method

An integral method can be used to examine this problem. The basic equations are those of Section 6.3.1.1 with eq 6.92c replaced by

$$T(r_o,t) = T_p \quad . \tag{6.99}$$

An equivalent form of eq 6.92b has been discussed previously and is

$$\frac{\partial T}{\partial t} = + \frac{k}{\rho \ell} \left(\frac{\partial T}{\partial r}\right)^2 \qquad r = R \quad . \tag{6.100}$$

The integrated energy equation is arrived at by integrating eq 6.92 over the solidified region and using eq 6.92b:

$$\frac{\rho \ell}{k} R \frac{dR}{dt} + r_o \frac{\partial T(r_o,t)}{\partial r} = - \frac{1}{\alpha} \int_{r_o}^{R} r \frac{\partial T}{\partial t}\, dr \quad .$$

A second equation is obtained by multiplying eq 6.92 by $r\partial T/\partial r$, integrating and using eq 6.100 to obtain

$$\frac{\rho\ell}{k} R^2 \frac{\partial T(R,t)}{\partial t} + r_o^2 \left[\frac{\partial T(r_o,t)}{\partial r}\right]^2 = -\frac{2}{\alpha} \int_{r_o}^{R} r^2 \frac{\partial T}{\partial r} \frac{\partial T}{\partial t} \, dr \quad .$$

A single parameter relation for the temperature was used

$$\frac{T - T_p}{T_f - T_p} = \frac{r_o - r}{r_o - R}$$

which then led to a closure time of

$$\tau_f = \frac{1}{4}(2 + S_T) - \frac{1}{9}(3 + S_T) \quad . \tag{6.101}$$

This result was not too accurate and Poots (1962) used a two-parameter relation

$$\frac{T - T_p}{T_f - T_p} = \frac{r_o - r}{r_o - R} + \frac{r_o - r}{r_o - R}\left(1 - \frac{r_o - r}{r_o - R}\right)g \quad .$$

Unfortunately, the resulting equations for R and g had to be numerically integrated but the closure time was much more accurate.

The results of these calculations are summarized in Table 6.1. Equation 6.98 is probably the best relation to use but the simple quasi-steady

Table 6.1 Complete solidification of a circular cylinder.

		$\tau_f/S_T = \alpha t_f/r_o^2$					
S_T	Eq 6.102	Quasi-steady	Riley (1974)	Poots (1962)	Beckett (1971)†	Allen & Severn (1962)†	Asfar et al. (1979) Tao (1967)
0.01	25.125	25.00	25.25	--	--	--	25.200
0.05	5.125	5.00	5.25	3.47	5.30	--	--
0.1	2.625	2.50	2.75	1.81	2.69	--	2.720
0.2	1.375	1.25	1.50	0.97	--	--	--
0.25	1.25	1.00	1.25	0.81	1.19	--	--
0.50	0.625	0.500	0.75	--	--	--	0.642
0.641	0.515	0.39	0.64	0.40) (0.52)*	--	0.47	--
1.0	0.375	0.25	0.50	0.31	--	--	0.379
2.0	0.25	0.125	0.375	--	--	--	0.239
3.0	0.208	0.083	0.333	--	--	--	0.189

*Two parameter method.
†Numerical solutions.

analysis is surprisingly good. The average of eq 6.98 and 6.97 gives

$$\frac{\alpha t_f}{r_o^2} = \frac{1}{4S_T} + \frac{1}{8} \tag{6.102}$$

and this relation is very good.

Asfar et al. (1979) used a perturbation method with strained coordinates. Simple equations were presented for the closure time which agreed quite well with the numerical results of Tao (1967).

6.3.2 Surface Heat Flux Specified

When the heat flux at the surface of the cylinder is specified, approximate solutions have been given by Shamsundar and Sparrow (1974) for two cases.

6.3.2.1 Constant Heat Flux

At the cylinder the heat flux is specified as

$$-k \frac{\partial T}{\partial r} = Q \ . \tag{6.103}$$

The basic equations are as given in 6.3.1.1. An approximate solution is obtained by expanding the equations in a truncated series such that the energy equation and interface boundary condition are satisfied exactly only at the phase change interface. The method is from Megerlin (1968). The phase change interface is given by

$$2 \frac{S\tau}{S_T} = \frac{1-\beta}{2} (1 + \beta) + \frac{1}{2} [1 - \beta^2 \sqrt{1 - 4S\ln\beta}]$$

$$+ \frac{1}{4}\sqrt{\pi S} \ \exp (1/(2 S_T) \left\{\mathrm{erf} \sqrt{\frac{1}{2S} - 2 \ln\beta} - \mathrm{erf} \sqrt{\frac{1}{2S}}\right\} \tag{6.104}$$

where

$S = cQr_o/k\ell$ (a form of Stefan number)

$S_T = \dfrac{c(T_f - T_s)}{\ell}$ (the usual Stefan number)

$\beta = R/r_o$.

The solidification time is given by

$$\frac{S\tau_f}{S_T} = \frac{1}{2} + \frac{\sqrt{2\pi S}}{8} \ \exp \left(\frac{1}{2S_T}\right) \mathrm{erfc} \sqrt{\frac{1}{2S}} \ . \tag{6.105}$$

The quasi-steady solution to this problem is

$$T = T_f - \frac{Qr_o}{k} \ln \left(\frac{r}{R}\right) . \tag{6.106}$$

The surface temperature is

$$T(r_o,t) = T_f - \frac{Qr_o}{k} \ln \beta \quad .$$

The phase change interface moves as

$$\frac{S\tau}{S_T} = \beta - \frac{\beta^2}{2} \quad . \tag{6.107}$$

The closure time is

$$\frac{S\tau_f}{S_T} = \frac{1}{2} \tag{6.108}$$

or

$$t_f = \frac{1}{2S} \frac{r_o^{\,2}}{\alpha} = \frac{\rho \ell r_o^2}{2Q} \quad .$$

6.3.2.2 Convection at Surface

The second case involves convection at the surface into an ambient fluid at T_a

$$-k \frac{\partial T}{\partial r} = h(T - T_a) \qquad r = r_o \; . \tag{6.109}$$

In this case numerical evaluation is generally needed to find the location of β. However, for small specific heats (Stefan number) an approximation for the closure time is given by the quasi-steady method (London and Seban 1944). The solidification interface is

$$2\tau_4 = \beta^2 \left(\ln \beta - \frac{1}{2} - \frac{1}{B_i}\right) + \frac{1}{2} + \frac{1}{B_i} \quad . \tag{6.110}$$

The solidification time is then

$$\tau_f = \frac{1}{2} \left(\frac{1}{B_i} + \frac{1}{2}\right) \quad . \tag{6.111}$$

When the Biot number is very large the ambient temperature will be the same as the surface temperature. Then

$$\tau_f = \frac{1}{4} \quad .$$

This is exactly the result of the simple quasi-steady procedure, eq 6.97,

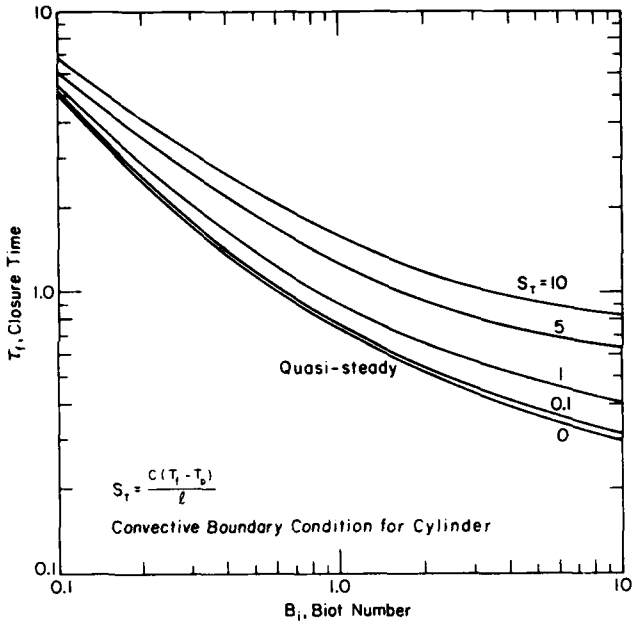

Figure 6.46. Complete freeze of a circular cylinder.

Table 6.2 Closure time for cylinders, numerical solutions, zero superheat, surface convection.

Biot number B_i	Stefan number S_T	Dimensionless time τ Gupta & Kumar (1983)	Tao (1967)	Baxter (1962)
1	1	1.044	1.045	1.00
	2	1.240	1.243	1.18
2	1	0.737	0.736	0.70
	2	0.897	0.897	0.82
10	1	0.467	0.463	0.42
	2	0.583	0.580	0.52

for a constant surface temperature.

The temperature is

$$T(r,t) = T_p + (T_f - T_p)\frac{\ln\left(\frac{r}{r_o}\right)}{\ln} \quad . \tag{6.112}$$

The pipe surface temperature is

$$T_p = T(r_o,t) = T_f - \frac{B_i T_a \ln}{1 - B_i \ln} \quad . \tag{6.113}$$

Figure 6.46 is a plot of the numerical evaluation of the approximate solution for the closure time of Shamsundar and Sparrow (1974). The results are in good agreement (less than 10% difference) with those of Goodling and Khadar (1974) if the Stefan number is less than one. Above S_T = 1.0 the results are less accurate. Numerical solutions for the closure time of cylinders have been evaluated for large Stefan and Biot numbers by several investigators. The results are given in Table 6.2.

6.2.3 Surface Convection and Radiation

Goodling and Khader (1974) numerically solved this problem with the addition of radiation at the cylinder surface. Unfortunately their calculations were such that only one value of the ambient to fusion

temperature ratio was studied, which was not in the right range for water solidification. One can estimate that if $\sigma_s \varepsilon_s T_f^3 \, r_o/k < 1/3 \, B_i^2$, the solidification time with radiation will be at least 90% of the value for convection only. This estimate will vary with the ratio of the ambient to fusion, absolute temperatures.

Seeniraj and Bose (1982) used regular perturbation to obtain solutions up to the first order for the single-phase problem. For zero radiation the zeroth solution was exactly the quasi-steady solution of Section 6.3.2.1. For $(T_f - T_a)$ small, an analytic expression for the cylinder closure time was found.

The convection/radiation boundary condition at r_o reduces to the convection case

$$-\frac{\partial T}{\partial r}(r_o,t) = \frac{B_r}{r_o}\left[T(r_o,t) - T_a\right]$$

$$\tau_f = \left(\frac{1}{4} + \frac{1}{2B_r}\right)\left(S_{T\infty} + 1\right) \tag{6.114}$$

where

$$B_r = B_i + \frac{4\sigma_s \varepsilon_s}{k} F r_o T_f^3$$

$$S_{T\infty} \ll 1.0$$

F = radiation shape factor (one for a small cylinder in a large enclosure).

For zero radiation eq 6.114 agrees reasonably well with the results of Gupta and Kumar (1983), if $S_{T\infty} < 0.5$. It reduces to the constant temperature perturbation solution of Riley et al. (1974), eq 6.98, when B_i becomes infinite.

The quasi-steady approximation, for small $(T_f - T_a)$, is

$$T = T_f - \frac{(T_f - T_a)\ln\left(\frac{R}{r}\right)}{\ln\beta - \frac{1}{B_r}} \quad . \tag{6.115}$$

The surface temperature is

$$T(r_o,t) = T_f + \frac{(T_a - T_f)\ln\beta}{\ln\beta - \frac{1}{B_r}} \quad . \tag{6.116}$$

The phase change interface and closure time are

$$\tau_4 = \frac{\beta^2}{2}\left[\ln \beta - \frac{1}{B_r} - \frac{1}{2}\right] + \frac{1}{4} + \frac{1}{2B_r} \tag{6.117}$$

$$\tau_f = \frac{1}{4} + \frac{1}{2B_r} \quad . \tag{6.118}$$

6.3.3 Finite Superheat (Two-Phase Problems)

Very few analytical results are available for the inward solidification of cylinders if the initial temperature is not at the fusion value. Jiji and Weinbaum (1978) used a perturbation method to arrive at approximate solutions for annular regions. They considered the cases of an insulated inner cylinder and an isothermal inner cylinder. For the insulated case at small Stefan numbers, which was used as the perturbation parameter, the complete solidification time of a tube was the same as that of the quasi-steady solution. They concluded that the initial temperature was not too significant for the insulated system; thus the zero sensible heat solutions presented in Section 6.3.1.2 should be reasonable at small Stefan numbers. For isothermal systems the superheat is quite significant even at small S_T.

For the annular region, with radiation/convection at outer cylinder, there are two cases to consider: a) inner cylinder insulated, b) inner cylinder at constant temperature. The quasi-steady equations, referred to Figure 6.47, are

$$\frac{d}{dr}\left(r\,\frac{dT_1}{dr}\right) = 0 \tag{6.119}$$

$$\frac{d}{dr}\left(r\,\frac{T_2}{dr}\right) = 0 \tag{6.120}$$

$$T_1\,(R,t) = T_2(R,t) = T_f \tag{6.121a}$$

$$T_1(r_n,t) = T_o \quad \text{or} \quad \frac{\partial T_1}{\partial r}(r_n,t) = 0 \tag{6.121b}$$

$$\frac{-\partial T_2(r_o,t)}{\partial r} = \frac{B_r}{r_o}\left[T_2(r_o,t) - T_\infty\right] \tag{6.121c}$$

$$k_1\,\frac{\partial T_1(R,t)}{\partial r} - k_2\,\frac{\partial T_2(R,t)}{\partial r} = -\rho\ell\,\frac{dR}{dt} \tag{6.121d}$$

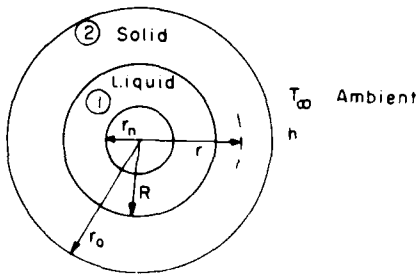

Figure 6.47. Solidification of annular region.

where

$$B_r = B_i + \frac{4\sigma_s \varepsilon_s F r_o}{k_2} \overline{T}^3$$

F = radiation shape factor (probably 1)

$$\overline{T} = \frac{T_f + T_\infty}{2} .$$

1. <u>Constant temperature inner surface</u>

$$T_1 = T_f + (T_o - T_f) \frac{\ln (\frac{R}{r})}{\ln (\frac{R}{r_n})} \tag{6.122}$$

$$T_2 = T_f - \frac{B_r (T_f - T_\infty) \ln \frac{r}{R}}{(1 + B_r \ln \frac{r_o}{R})} . \tag{6.123}$$

The interface radius is governed by

$$\tau_a = - \int_1^\beta \frac{x\, dx}{\frac{\phi_4}{\ln \frac{\Delta}{x}} + \frac{B_r}{1 - B_r \ln x}} \tag{6.124}$$

where

$$\beta = \frac{R}{r_o}$$

$$\phi_4 = k_{12} (\frac{T_o - T_f}{T_f - T_\infty})$$

$$\Delta = \frac{r_n}{r_o}$$

$$\tau_a = \frac{k_2 (T_f - T_\infty) t}{\rho \ell r_o^2} .$$

The solution to the interface energy equation is

$$\tau_a = \frac{e^a}{b} \{ \frac{e^c}{2} [\frac{B_r}{2} (c - 1) + g] + \frac{e^{-a}}{2} [\frac{B_r}{2} (a + 1) - g]$$

$$+ d[\ln (\frac{c}{-a}) + \sum_{n=1}^{\infty} \frac{c^n - (-a)^n}{n\, n!}]\} \tag{6.125}$$

where

$$a = \frac{2(\phi_4 + B_r \ln \Delta)}{(\phi_4 + 1) B_r}$$

$$b = (\phi_4 + 1) B_r$$

$$c = \frac{2[B_r \ln \frac{\beta}{\Delta} + \phi_4 (B_r \ln \beta - 1)]}{(\phi_4 + 1) B_r}$$

$$d = \ln \Delta - \frac{(\phi_4 + B_r \ln \Delta)(1 - B_r \ln \Delta)}{(\phi_4 + 1)^2 B_r}$$

$$g = \frac{(\phi_4 - 1)[1 - B_r \ln \Delta]}{(\phi_4 + 1)} \quad .$$

The cylinder will not freeze completely if $T_o > T_f$. The steady-state location of the interface is

$$\beta_\infty = \exp \left[\frac{\phi_4 + B_r \ln \Delta}{(1 + \phi_4) B_r}\right] \tag{6.126}$$

The solution to the same problem with a constant outer cylinder temperature can be found by letting $B_r \to \infty$, in the above equations. The quasi-steady solution to the constant temperature outer cylinder problem is the zeroth order solution given by Jiji and Wienbaum (1978). The phase-change interface is

$$\tau_a = \frac{1}{2(\phi_4 + 1)} \left\{(1 - \beta^2)\left[\phi_4 + \ln\rho + \frac{1}{2}\right] + \beta^2 \ln\beta\right.$$

$$- 2\rho^2 (\ln \rho)^2 \left[\ln\left(1 - \frac{(\phi_4 + 1) \ln\beta}{\ln\Delta}\right) + \sum_{n=1}^{\infty} \frac{[[2 \ln \beta/\rho]^n - [-2 \ln \rho]^n}{n \; n!}\right] \quad . \tag{6.127}$$

where $\rho = \Delta^{1/(\phi_4+1)}$.

2. Insulated Inner Cylinder

The temperature in the solid region is identical to that for case (a) whi. the liquid temperature is simply

$$T_l = T_f \quad .$$

This is due to the quasi-steady assumption which cannot accommodate the sensible heat of the liquid. The cylinder can freeze completely with

$$\tau_a = \frac{\beta^2}{2} \ln\beta - \frac{\beta^2}{4} + \frac{1}{4} - \frac{1}{2B_r} (\beta^2 - 1) \ . \qquad (6.128)$$

This is easily seen to be the single phase solution (Sec. 6.1.3.2) for $B_r \to \infty$, or constant temperature outer cylinder.

Sinha and Gupta (1982) considered outward freezing from the inner cylinder. Limited experimental and numerical results are given for superheated water with an insulated outer cylinder. The results agree with single phase quasi-steady solutions and heat balance integral approximations for small superheat. The numerical solutions are very sensitive to superheat.

6.3.4 Freezing of Water in Pipes

The question of the freezing of water in pipes is of importance in terms of water supply, sewage removal, etc. Figure 6.48 graphically describes the growth of ice in a tube. The density of water is at a maximum at about 39°F, and as the water near the pipe surface cools to 39°F, the cooler, denser water sinks as shown. As the temperature drops below 39°F, the pattern is reversed, and the cooler, lighter water rises. A certain amount of subcooling is necessary before nucleation of ice will occur. Zarling (1978) and Gilpin (1977a) report values of 8 or 9°F of subcooling, but this will vary from system to system. Nucleation will be followed by dendritic ice growth (see Section 6.1.1) and a rapid return of the water to the fusion temperature, after which the usual annular growth will begin. The effects of dendritic ice are usually ignored, but they may have significance for pipe blockage, as will be seen.

The time required to cool water in a pipe from an arbitrary temperature to the fusion value can be easily estimated.

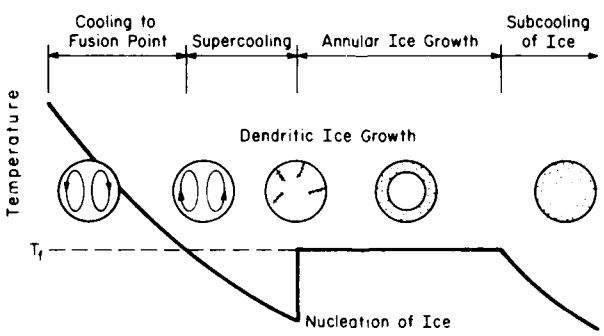

Figure 6.48. Cooling of water in pipes.

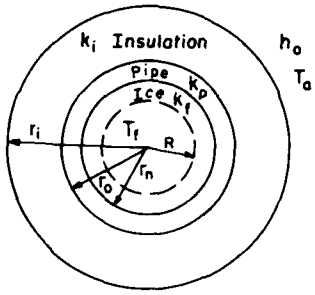

Figure 6.49. Ice growth inside pipe.

6.3.4.1 Quasi-Steady Solutions

Consider an analysis where the water in the tube does not vary spatially and the effects of the density maximum are neglected. For a unit length of insulated pipe, with the notation of Figure 6.49, an energy balance of the water yields

$$\frac{2(T_w - T_a)}{\frac{1}{r_n h_i} + \frac{1}{k_p}\ln\frac{r_o}{r_n} + \frac{1}{k_i}\ln\frac{r_i}{r_o} + \frac{1}{r_i h_o}} = -r_n^2 C_w \frac{dT_w}{dt} \quad . \tag{6.129}$$

The solution to this equation is

$$T_w = (T_o - T_a)e^{-t/S_1} + T_a \tag{6.130}$$

where

$$S_1 = \frac{r_n^2 \rho_w c_w}{2}\left[\frac{1}{k_p}\ln\frac{r_o}{r_n} + \frac{1}{k_i}\ln\frac{r_i}{r_o} + \frac{1}{r_i h_o} + \frac{1}{r_n h_i}\right] \quad .$$

The time for the water to cool to T_f is

$$t_c = S_1 \ln\left(\frac{T_o - T_a}{T_f - T_a}\right) \quad . \tag{6.131}$$

Consider the growth of an annular ice layer as shown in Figure 6.49. If the water temperature is T_f and the air temperature is T_a, then the heat liberated by the ice formation must be transferred outward through the layers of ice, pipe, and insulation. The quasi-steady equation is

$$\frac{(T_f - T_a)}{\frac{\ln\left(\frac{r_n}{R}\right)}{k_f} + \frac{\ln\frac{r_o}{r_i}}{k_p} + \frac{\ln\frac{r_i}{r_o}}{k_i} + \frac{1}{r_i h_o}} + L\,R\frac{dR}{dt} = 0 \quad . \tag{6.132}$$

The solution to this equation for the time required to form an ice layer of thickness R is

$$t = \frac{L}{k_f\ (T_f - T_a)} \left\{ \left[\frac{1}{2} + S\right] \frac{r_n^2 - R^2}{2} + R^2 \ln \frac{R}{r_n} \right\} \qquad (6.133)$$

where

$$S = k_f \left\{ \frac{1}{k_p} \ln \frac{r_o}{r_n} + \frac{1}{k_i} \ln \frac{r_i}{r_o} + \frac{1}{r_n h_o} \right\} \quad .$$

The water will freeze completely when R = 0

$$t_f = \frac{L}{k_f\ (T_f - T_a)} \frac{r_n^2}{2} \left[\frac{1}{2} + S\right] \ . \qquad (6.134)$$

Lock et al. (1970) presented the same solution, but neglected the insulating effect of the ice and the pipe.

Equation 6.134 can be written in nondimensional form as

$$\tau_f = \frac{1}{2} \left(\frac{r_n}{r_i}\right)^2 \left[\frac{1}{2} + \frac{k_f}{k_p} \ln \frac{r_o}{r_n} + \frac{k_f}{k_i} \ln \frac{r_i}{r_o} + \frac{1}{B_f}\right] \qquad (6.135)$$

where

$$\tau_f = \frac{k_f (T_f - T_a) t_f}{\rho \ell\ r_i^2}$$

$$B_f = \frac{h_o r_i}{k_f} \quad .$$

Equation 6.135 is interesting since, for the special case of a thin-walled pipe with no insulation, it reduces to

$$\tau_f = \frac{1}{2} \left[\frac{1}{2} + \frac{1}{B_i}\right] \quad . \qquad (6.136)$$

Note that this is exactly the theoretical result described by eq 6.111.

6.3.4.2 Dendritic Ice in Horizontal Pipes

Dendritic ice is that formed when supercooled water suddenly nucleates and rapidly changes phase (see Section 6.1.1). The latent heat evolved raises the surrounding water temperature back to the normal fusion temperature where the dendritic ice is in the form of thin plate-like crystals that can block the flow of water in a pipe even if only a small part of the water volume has changed to ice. This is due to the interlocking of the plates, later further cemented in place by annular ice growth. The

possibility of flow blockage depends upon the nucleation temperature, cooling rate of water at the center of the pipe, pipe density, and flow velocity.

Typical nucleation temperatures of water are listed below:

Water source	Nucleation temperature (°C)	Reference
Cold tap	-4 to -5	Gilpin (1977a)
Hot tap	-5 to -6	Gilpin (1977a)
Fresh lake or river	-3 to -5	Gilpin (1977b)

The nucleation temperature is independent of container material and cooling rate for rates less than 1°C/min.

A well-insulated pipe will be most susceptible to dendritic blockage, as the water center temperature can cool down to -2°C before the colder surface water causes nucleation. If the center temperature is higher than -2°C, at nucleation, the dendritic ice formed will tend to disappear because of convection effects at the dendritic growth location.

Gilpin (1977b) found that, with dendritic growth, a definite pressure gradient was required to start the water in the pipe flowing. This relation is shown in Figure 6.50. If the cooling of the water is less than a critical value, then dendritic blockage is possible.

The critical cooling rate is given by

$$H_c = C\left(\frac{-2 - T_n}{D}\right)^{5/4} \tag{6.137}$$

where

$C = \begin{cases} 21\text{-}29, \text{ copper tubes (uniform wall temperature)} \\ 6.3\text{-}13.7, \text{ plastic pipes (uniform heat flux)} \end{cases}$

H_c = cooling rate of water temperature, °C/min

D = pipe inside diameter (mm)

T_n = nucleation temperature (°C).

For most cases the plastic pipe relation will be most appropriate. If the cooling rate is less than the critical value, Figure 6.50 can be used to find the ice volume fraction at blockage. Then the time for blockage to occur is estimated as

$$t_{fd} = x_i t_f \tag{6.138}$$

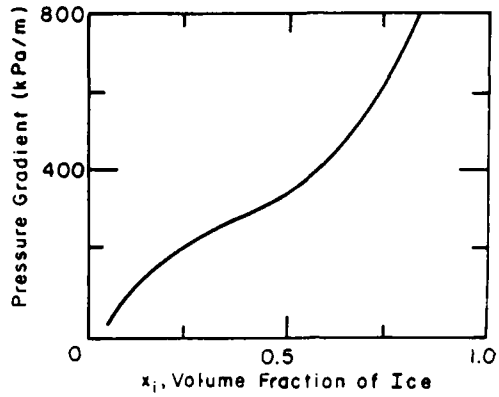

Figure 6.50. Pressure gradient needed to start flow during freezing history of a pipe (adapted from Gilpin 1977b).

where t_f is given by eq 6.134. These relations are only estimates, as the dendritic ice relations are not sufficiently known, but it appears that dendritic blockage may be significant.

The mean cooling rate for a pipe can be estimated from

$$H_c = \frac{(T_n - T_o)}{S_1 \ln\left(\frac{T_o - T_a}{T_n - T_a}\right)} \quad . \tag{6.139}$$

NOMENCLATURE--CHAPTER 6

a	$\sqrt{\mu^2-1}$
A	area
b	$1/\ln(\mu+a)$
B_i	$\frac{h_o r_o}{k}$, Biot Number
B_f	$\frac{h_o r_i}{k_f}$
B_r	$B_i + 4\sigma_s \varepsilon_s Fr_o \overline{T}^3/k_2$
c	specific heat
C	ρc, volumetric specific heat
d	depth to center of arbitrary isotherm

d_f depth to center of thaw bowl

d_o depth to center of buried pipe

D pipe diameter

D_o phase change depth beneath insulated pipe

$E_i(x)$ exponential integral function

f_s solid fraction

f_{su} solid fraction of mushy zone at solidus front

F radiation shape factor (1, for large enclosure)

G Q/k

h surface coefficient of convection

h_i, h_o value of h inside and outside pipe

H_c cooling rate of water

k thermal conductivity

k_{i2} k_i/k_2

k_i thermal conductivity of insulation

K hydraulic conductivity

ℓ latent heat of solidification

L $\rho\ell$, volumetric latent heat

L_e $(1+C_{21}\phi S_T+S_T/2)L$, effective latent heat

$\dot{m}$ mass flux rate

q heat transfer rate at cylinder surface per unit length

q_p plane heat flux, non-melt

q^* $\dfrac{q\ c_i}{2\pi k_i \ell}$ dimensionless heat transfer

q''' volumetric latent energy release

Q	surface heat flux
Q_m	energy added to thawing medium
Q_s	strength of source/sink
Q^*	$\frac{Q_s}{4\pi k_1 (T_o - T_f)}$, dimensionless source/sink
Q_t^*	$\frac{\int_o^t q\, dt}{2\pi r_o^2 \rho_1 \ell}$, dimensionless total heat flow
r	radial coordinate
r_i, r_n, r_o	insulation, inner pipe, outer pipe, radii
r_t	radius of arbitrary isotherm
R	radius of phase change or thaw bowl
R_1	radius of frozen mushy zone
R_2	radius of mushy zone (liquid limit)
R_o	R/r_i, dimensionless radius of thaw bowl
S	$\frac{c_1 Q r_o}{k_1 \ell}$
S_T	$\frac{C_1}{L}(T_p - T_f)$, thaw $\frac{C_1}{L}(T_f - T_p)$, freeze
S_{Ta}	$\frac{C_1}{L}(T_f - T_a)$
S_{To}	$\frac{c_1}{\ell}(T_f - T_o)$, freeze Stefan number $\frac{c_1}{\ell}(T_o - T_f)$, thaw
$S_{T\infty}$	$\frac{c_1}{\ell}(T_f - T_\infty)$
t	time
t_f	closure time for solidification

t_{fd}	closure time for solidification with dendritic ice
T	temperature
T_a	air temperature
T_f	fusion temperature
T_G	mean annual ground surface
T_ℓ, T_s	liquidus and solidus, fusion temperatures
$T_o, T_p, \overline{T}_p$	initial, pipe surface, insulation surface, temperatures
T_n	water nucleation temperature
T_∞	ambient temperature
$\overline{T}$	$\frac{T_f + T_\infty}{2}$
u	$T - T_o$
v	$\frac{T - T_o}{T_p - T_o}$
x,y,z	Cartesian coordinates
x_f	depth of T_f isotherm, non-melt, plane
x_i	ice volume fraction
X,Y	plane, phase-change depths
y_o	phase-change depth beneath center of pipe
w	water concentration in soil system
α	thermal diffusivity
α_{i2}	α_i/α_2
α_e	equivalent thermal diffusivity
β	R/r_o dimensionless phase-change depth
γ	$\frac{x}{2/\alpha_i t}$ planar phase-change parameter

γ_e	Euler's constant
γ_i	$R^2/2\sqrt{\alpha_4 t}$
δ	thermal penetration depth
δ_G	mass flux parameter
Δ	$(\delta - r_o)/r_o$
ΔT	$T_\ell - T_s$
ε	$k_{li} \ln (r_i/r_o)$ insulation parameter
ε_s	surface emmissivity
ζ	x/r_i
η_i	$r^2(4\alpha_i t)$
θ_i	integrated temperature
θ	$\dfrac{(T - T_f)\,k}{Q\,r_o}$
λ	$R/2\sqrt{\alpha t}$ phase-change parameter
λ_i	$R_i/2\sqrt{\alpha_i t}$
μ	d_o/r_i
μ_o	d_o/r_o
ν	r/r_o
ξ	y/r_i
ξ_o	y_o/r_i
ρ	density
σ_s	Stefan-Boltzmann constant
τ	$\dfrac{\alpha\, t\, S_T}{r_o^2}$

τ_1 $\dfrac{\alpha_1 S_{To} t}{r_o^2}$

τ_2 $\dfrac{Qt}{\rho \ell r_o}$

τ_3 $\dfrac{k_1 (T_p - T_f) t}{r_i^2 \; L_e}$

τ_4 $\dfrac{\alpha S_{T\infty} t}{r_o^2}$

τ^* $2/\sqrt{\tau_3}$

τ_a $\dfrac{k_2 \; (T_f - T_\infty) t}{\rho \ell \; r_o^2}$

τ_f dimensionless closure time

ϕ $\dfrac{T_f - T_o}{T_p - T_f}$

ϕ_1 $\dfrac{T_1 - T_o}{T_p - T_f} = \dfrac{f - f_o}{1 + b\varepsilon - f_o}$

ϕ_2 $\dfrac{T_2 - T_o}{T_f - T_o} = \dfrac{f}{f_o}$

ϕ_3 $\dfrac{T_\ell - T_o}{\Delta T}$

ϕ_4 $\dfrac{k_{12} \; (T_o - T_f)}{T_f - T_\infty}$

ω frequency of surface temperature

Ω δ/R

Subscripts

1,2,3 different phases

f,t frozen, thawed

i	insulation
m	soil system
o	location on phase change interface
w	water

CHAPTER 7. PROBLEMS IN SPHERICAL GEOMETRY

The techniques associated with spherical systems follow closely those of cylindrical coordinates. Once again, no complete, practical solutions are available.

The energy equation (see Nomenclature at end of chapter), in spherical coordinates, is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7.1}$$

or

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad . \tag{7.1a}$$

The familiar transformation variable

$$\eta = r/2\sqrt{\alpha t}$$

will transform eq 7.1 to

$$\frac{d^2T}{d\eta^2} + 2\left(\eta + \frac{1}{\eta}\right)\frac{dT}{d\eta} = 0 \ .$$

Direct integration leads to

$$T - T_\infty = A\int_\eta^\infty \eta^{-2}\, e^{-\eta^2}\, d\eta \quad .$$

Further integration by parts yields

$$T - T_\infty = B\left[\frac{1}{2\eta}\, e^{-\eta^2} - \frac{\sqrt{\pi}}{2}\,\mathrm{erfc}\ \eta\right] \ .$$

A formal solution of the energy equation in spherical coordinates is

$$T - T_\infty = A\ F(\eta) \ .$$

where the spherical function is

$$F(\eta) = \frac{1}{2\eta}\, e^{-\eta^2} - \frac{\sqrt{\pi}}{2}\ \mathrm{erfc}\ \eta$$

$$\eta = \frac{r}{2\sqrt{\alpha t}} \quad .$$

The function F(x) can be expanded as

$$F(x) = \sum_{n=o}^{\infty} (-1)^n \frac{x^{2n}}{n!} \left(\frac{1}{2x} + \frac{x}{2n+1}\right) - \frac{\sqrt{\pi}}{2} \quad .$$

For large values of x, an asymptotic expansion is

$$F(x) = -e^{-x^2} \sum_{n=o}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdots (2n+1)}{2^{n+2} x^{2n+3}} \quad .$$

A boundary condition at a finite radius r_o will preclude the use of this similarity solution. Thus exact solutions are restricted to point sources at the origin or problems with no boundary conditions prescribed at $r = r_o$.

7.1 OUTWARD PHASE CHANGE

7.1.1 Continuous Point Source at Origin (Freeze)

We will now consider the continuous extraction of energy at the origin. The equations for the frozen and thawed regions are

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T_1}{\partial r}\right) = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t} \tag{7.3}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T_2}{\partial r}\right) = \frac{1}{\alpha_2} \frac{\partial T_2}{\partial t} \tag{7.4}$$

$$T_2(r,o) = T_o \tag{7.5a}$$

$$\lim_{r \to \infty} T_2(r,t) = T_o \tag{7.5b}$$

$$T_1(R,t) = T_2(R,t) = T_f \tag{7.5c}$$

$$\lim_{r \to o} \left(4\pi k_1 r^2 \frac{\partial T_1}{\partial r}\right) = Q_s(t) \tag{7.5d}$$

$$-k_1 \frac{\partial T_1(R,t)}{\partial r} + k_2 \frac{\partial T_2(R,t)}{\partial r} = -\rho \ell \frac{dR}{dt} \quad . \tag{7.5e}$$

The solutions are

$$T_1 = T_f + \frac{Q_o}{4\pi k_1 \sqrt{\alpha_1}} [F(\eta_1) - F(\lambda)] \tag{7.6}$$

$$T_2 = T_o + \frac{T_f - T_o}{F(\lambda\sqrt{\alpha_{12}})} F(\eta_2) \tag{7.7}$$

$$R = 2\,\lambda\sqrt{\alpha_1 t} \tag{7.8}$$

where

$$F(\eta) = \frac{1}{2\eta} e^{-\eta^2} - \frac{\sqrt{\pi}}{2} \operatorname{erfc} \eta \ .$$

The above equations are valid only if energy is extracted at an increasing rate given by

$$Q_s(t) = Q_o\sqrt{t} \ . \tag{7.9}$$

The equation for λ is

$$\frac{Q_o \ e^{-\lambda^2}}{4\pi k_1\sqrt{\alpha_1}\,(T_o - T_f)} - \frac{k_{21}\sqrt{\alpha_{21}}\ e^{-\lambda^2\alpha_{12}}}{F(\lambda\sqrt{\alpha_{12}})} = \frac{4\lambda^2}{S_{To}} \tag{7.10}$$

where

$$S_{To} = \frac{c_1}{\ell}(T_o - T_f) \ .$$

This problem has limited practical use because of the transient nature of the sink term.

7.1.2 Freeze of a Subcooled Liquid

An exact solution has been found for the problem with a solid region for $r < R$ and a subcooled liquid for $r > R$, as shown in Figure 7.1 (Frank 1950).

The equations are

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7.11}$$

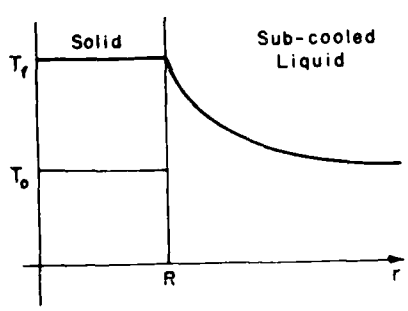

Figure 7.1. Freezing of a sub-cooled liquid in spherical geometry.

$$T(R,t) = T_f \tag{7.11a}$$

$$\lim_{r\to\infty} T(r,t) = T_o \tag{7.11b}$$

$$T(r,o) = T_o \tag{7.11c}$$

$$k \frac{\partial T(R,t)}{\partial r} = -\rho \ell \frac{dR}{dt} \quad . \tag{7.11d}$$

The solution follows directly and is

$$T = T_o + (T_f - T_o) \frac{F(\eta)}{F(\lambda)} \tag{7.12}$$

$$R = 2\lambda\sqrt{\alpha t} \tag{7.13}$$

$$\lambda^2 e^{\lambda^2} \left[e^{-\lambda^2} - \sqrt{\pi}\, \lambda \operatorname{erfc} \lambda\right] = \frac{S_{To}}{2} \quad . \tag{7.14}$$

7.1.3 Zero Superheat

7.1.3.1 Constant Temperature Boundary Condition

The quasi-steady approximation will yield a simple solution. The freeze problem will be given following the notation of Figure 7.2:

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \tag{7.15}$$

$$T(r_o,t) = T_s$$

$$T(R,t) = T_f$$

$$\frac{dR}{dt} = \frac{k}{\rho \ell} \frac{\partial T(R,t)}{dr} \quad .$$

The solution for the temperature is

$$\frac{T - T_s}{T_f - T_s} = \frac{1 - \frac{r_o}{r}}{1 - \frac{r_o}{R}} = \frac{1 - \frac{r_o}{r}}{1 - \frac{1}{\beta}} \quad . \tag{7.16}$$

The phase change interface is given by

$$\tau = \frac{1}{3}\beta^3 - \frac{\beta^2}{2} + \frac{1}{6} \tag{7.17}$$

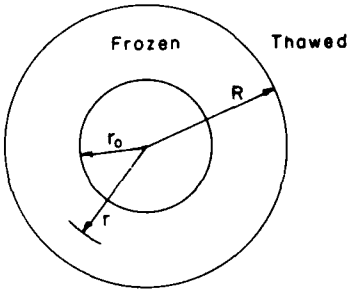

Figure 7.2. Phase change for spherical systems.

where

$$\beta = R/r_o$$

$$\tau = \frac{\alpha\ t}{r_o^2} S_T$$

$$S_T = \frac{c_1}{\ell} (T_f - T_s) \ .$$

The heat transfer into the sphere, at the surface, is

$$q^* = \frac{q}{4\pi(T_f - T_s)kr_o} = \frac{\beta}{\beta-1} \quad . \tag{7.18}$$

These relations are, as usual, for negligible sensible heat, or $S_T \to 0$.

Pedroso and Domoto (1973) used a perturbation method to obtain a solution valid for small Stefan numbers. The problem is

$$\frac{1}{r}\frac{\partial^2(rT)}{\partial r^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7.19}$$

$$T(r_o,t) = T_s$$

$$T(R,t) = T_f$$

$$\frac{dR}{dt} = \frac{k}{\rho\ell}\frac{\partial T(R,t)}{\partial r} \quad .$$

The temperature and the phase interface velocity were expanded in a Taylor series, about $S_T = 0$:

$$u = u_o + u_1 S_T + u_2 S_T^{\ 2} + \ . \ . \ .$$

$$g = g_o + g_1 S_T + g_2 S_T^{\ 2} + \ . \ . \ .$$

where

$$u = \frac{T - T_s}{T_f - T_s}$$

$$g = \frac{d\beta}{d\tau} \quad .$$

The zeroth-order solution is exactly the quasi-steady solution just discussed. The first three terms yield

$$\frac{u}{u_o} = 1 + \frac{1}{6}\left[1-\left(\frac{r}{R}\right)^2 u_o^{\ 2}\right]\frac{S_T}{\beta^2} - \left\{\frac{1}{36}\left[1 - \left(\frac{r}{R}\right)^2 u_o^{\ 2}\right] + \frac{4\beta-1}{120}\left[1-\left(\frac{r}{R}\right)^4 u_o^{\ 4}\right]\right\}\left(\frac{S_T}{\beta^2}\right)^2 \tag{7.20}$$

$$\frac{g}{g_o} = 1 - \frac{1}{3}\frac{S_T}{\beta} + \frac{1-6\beta}{45}\frac{S_T^{\,2}}{\beta^3} \tag{7.21}$$

where

$$u_o = \frac{1 - \frac{r_o}{r}}{1 - \frac{1}{\beta}} \qquad g_o = \frac{1}{\beta(\beta-1)} \quad .$$

Equation 7.21 can be integrated to give the phase change position:

$$\tau = \frac{3(\beta-1)^2 + 2(\beta-1)^3}{6} + \frac{(\beta-1)^2}{6} S_T - \frac{1}{45}\frac{(\beta-1)^2}{\beta} S_T^{\,2} \tag{7.22}$$

The first term of this series is the quasi-steady solution, eq 7.3.

7.1.3.2 Convection at Surface of Sphere

The geometry for the phase change is shown in Figure 7.3. The constant temperature at r_o is replaced by

$$k \frac{\partial T}{\partial r}(r_o,t) = h[T(r_o,t) - T_\infty] \quad .$$

The temperature of the frozen shell is

$$\frac{T_1 - T_\infty}{T_f - T_\infty} = \frac{B_i(1 - \frac{r_o}{r}) + 1}{B_i(1 - \frac{r_o}{R}) + 1}$$

where $B_i = h\, r_o/k$ is the Biot number.

The motion of the interface is given by

$$\tau = \frac{1}{3}\left(\frac{1}{B_i} + 1\right)(\beta^3 - 1) - \frac{1}{2}(\beta^2 - 1) \quad .$$

As the inside surface coefficient increases ($B_i \to \infty$), this case will reduce to the constant temperature problem.

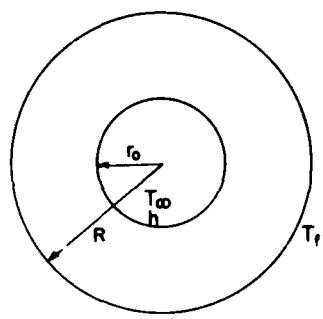

Figure 7.3. Phase change outside sphere with convection.

7.1.4 Finite Superheating

Gupta (1973) used the quasi-steady method of Khakimov (1957)--see Chapter 6--to include the effects of the surrounding medium at temperatures other than the fusion value (see Fig. 7.4).

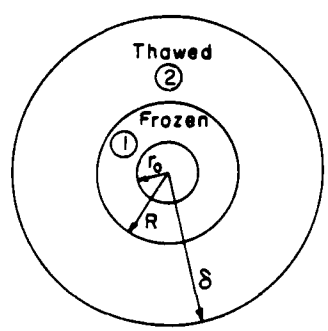

Figure 7.4. Temperature penetration for freeze outside a sphere.

The equations are

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \tag{7.23}$$

for each region:

$$T_1(r_o,t) = T_s$$

$$T_1(R,t) = T_2(R,t) = T_f$$

$$T_2(\delta,t) = T_o \ .$$

The temperatures are

$$\frac{T_1 - T_s}{T_f - T_s} = \frac{1 - \frac{r_o}{r}}{1 - \frac{r_o}{R}}$$

$$\frac{T_2 - T_o}{T_f - T_o} = \frac{1 - \frac{\delta}{r}}{1 - \frac{\delta}{R}} \ .$$

The heat extracted at the surface of the sphere, in a small time, must equal the change in the latent and sensible heats of the surrounding medium. The phase-change interface position is then

$$\tau = \left(\frac{\beta^3}{3} - \frac{\beta^2}{2} + \frac{1}{6}\right)\left[1-\phi S_T \left\{1+k_{21}\alpha_{12}\left(\frac{\Omega^2}{2} + \frac{\Omega}{2} - 1\right)\right\} + \frac{S_T}{6}\left(\beta^2-\beta-\ln\beta\right)\right. \tag{7.24}$$

where $\Omega = \delta/R$

$$\phi = \frac{T_o - T_f}{T_f - T_s} \ .$$

A value of $\Omega = 4.5$ was suggested by Khakimov (1957). The surface heat transfer is again given by eq 7.18.

Equation 7.24 reduces to the zero superheat solution when $S_T = \phi = 0$. Equations 7.24 and 7.22 are quite close for small values of S_T. As the Stefan number increases, eq 7.22 increasingly differs from eq 7.24, at values of β less than 10. Since eq 7.24 is not limited to small values of S_T, it should be more accurate at large Stefan numbers.

The method of Lin (1971, see Section 6.1.4.3), can also be used. This will give a solution for the phase change interface as follows:

$$\frac{\beta^3}{3} - \frac{\beta^2}{2} + \frac{1}{6} = \frac{2\gamma^2}{S_T}\tau \quad . \tag{7.25}$$

The parameter γ is given by the exact solution to the Neumann problem. This will be acceptable for small superheat parameters ϕ, and small values of τ but should be used with caution.

7.1.5 Constant Surface Heat Flux

The case of a constant surface heat flux has been numerically examined by Goodling and Khadar (1975). The equations are as given in Section 6.1.4.5 except that the diffusion equation is

$$\frac{\partial^2\theta}{\partial y^2} + \frac{2}{y}\frac{\partial\theta}{\partial y} = S_1 \frac{\partial\theta}{\partial\tau_1}$$

where

$$\theta = \frac{(T - T_f)k}{Q\ r_o} \qquad y = \frac{r}{r_o}$$

$$\tau_1 = \frac{Q\ t}{\rho\ell\ r_o} \qquad S_1 = Q\frac{c r_o}{\ell\ k}$$

$$p = \frac{h_o(T_f - T_o)}{Q\ r_o^a} \qquad h = h_o R^{-a} \quad .$$

Very limited ranges of the parameters are given, as noted in Section 6.1.4.5.

An integral solution for the phase change interface is

$$\tau_1 = \int_1^\beta \frac{2d\beta}{\left[\left(\frac{1}{S_1\beta(\beta-1)} - \frac{p}{\beta^a}\right)^2 + \frac{4}{S_1\beta^3(\beta-1)}\right]^{1/2} - \left(\frac{1}{S_1(\beta-1)\beta} + \frac{p}{\beta^a}\right)} \quad . \tag{7.26}$$

The time to reach a given solidification location with eq 7.26 is about 10-30% greater than the numerical solution of Goodling and Khadar (1975), with $S_1 \approx 10$, $a = 0.25$, and $p = 0.6$. For lesser values of S_1 the agreement is much better.

7.2 SPHERICAL PROBLEMS, INWARD GROWTH

Several approximate solutions are available for the inward solidification of spheres. No solution exists for problems with finite superheat.

7.2.1 Constant Interface Temperature Gradient

Kreith and Romie (1955) have examined the linear problem of a constant temperature gradient applied at the interface location:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{7.27}$$

$$T(R) = T_f$$

$$\left.\begin{aligned} k\frac{\partial T}{\partial r} &= +\rho\ell\frac{dR}{dt} \\ \frac{\partial T}{\partial r} &= G = \frac{Q}{k} \end{aligned}\right\} \quad r = R \tag{7.27a}$$

$$T(r_o,0) = T_f \quad . \tag{7.27b}$$

The solution for the interface position is given by

$$\beta = 1 - \frac{G\,\tau}{(T_f - T_s)} \quad . \tag{7.28}$$

The closure time occurs when $\beta = 0$; therefore

$$\tau_f = \frac{(T_f - T_s)}{G} \quad . \tag{7.29}$$

This relation for the closure time is not exact, as a solution for the surface temperature showed that, at $\beta \approx 0.1$, the surface temperature will exceed T_f if G is a constant. Thus, realistically, when T_s reaches a value of T_f, it will be held constant and closure will occur with a variable temperature gradient, if surface melting is to be avoided. Solutions for the transient surface temperature are given in the next section.

7.2.2 Prescribed Phase Change Interface Velocity

Rubinsky and Shitzer (1978) solved an inverse Stefan problem to obtain the temperature as a function of the interface velocity and heat flux. The temperature in the changed phase (see Fig. 7.5) is

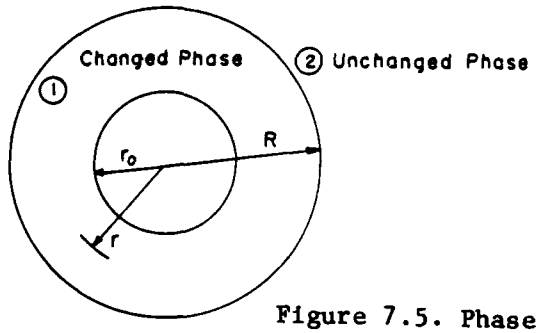

Figure 7.5. Phase change with prescribed interface velocity.

$$T_1\,(r,t) = T_f + \frac{1}{r}\sum_{n=o}^{\infty} a_n(R)\,\frac{(r-R)^n}{n!} \tag{7.29a}$$

$$a_o = 0$$

$$a_1 = R(t)\,\frac{\partial T(R,t)}{\partial r}$$

$$a_n = \frac{dR}{dt}\,\frac{1}{\alpha}\left(\frac{da_{n-2}}{dR} - a_{n-1}\right) \quad .$$

The interface heat flux is given by the usual energy balance

$$k_1\,\frac{\partial T(R,t)}{\partial r} - Q_i = \rho\ell\,\frac{dR}{dt} \tag{7.29b}$$

where

$$Q_i = k_2\,\frac{\partial T_2}{\partial r}\,(R,t) \quad \text{is one possible heat flux.}$$

It is assumed that $dR/dt = V$ and Q_i are prescribed at the interface.

An exact solution can be found if $Q_i = 0$ and $dR/dt = V =$ constant. After manipulation eq 7.29a becomes

$$T(r,t) = T_f + \frac{\rho\ell\alpha}{k\ r}\left[(1 - e^{\mu})R(t) + \frac{\alpha}{V}\left\{(2-\mu)(1+e^{\mu}) - 4\right\}\right] \tag{7.29c}$$

where

$$\mu = \frac{-V}{\alpha}\,[r - R(t)] \quad .$$

This is the solution given by Kreith and Romie (1955) in Section 7.2.1.

7.2.3 Constant Surface Temperature

A quasi-steady solution can be obtained for this case. Referring to Figure 7.6, the equations are

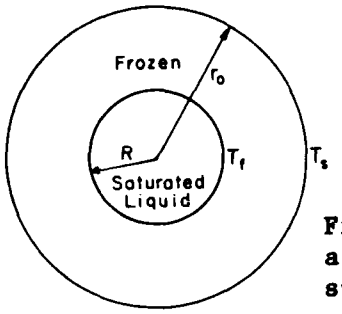

Figure 7.6. Freeze of a sphere with constant surface temperature.

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \quad .$$

Boundary conditions, eq 7.27a,b apply. The temperature is given by

$$\frac{T - T_s}{T_f - T_s} = \frac{-(r_o - R)}{R}\left(1 - \frac{r_o}{r}\right) \quad .$$

The interface position is

$$\tau = \frac{\beta^3}{3} - \frac{\beta^2}{2} + \frac{1}{6} \quad . \tag{7.30}$$

This yields a closure time of

$$\tau_f = \frac{1}{6} \quad . \tag{7.31}$$

Pedroso and Domoto (1973) used a perturbation solution to obtain the following relation for the phase change interface:

$$\tau = \frac{S_T}{9K}\,[3+(3-S_T)K]\left[1-\beta-\left(\frac{3-S_TK}{3K}\right)\ln\frac{3+(3-S_T)K}{3+(3\beta-S_T)K}\right] + \frac{3-S_T}{6}(1-\beta^2) - \frac{1-\beta^3}{3} \tag{7.32}$$

where K is a function of S_T, given below.

K	14.96	9.099	6.699	5.350	4.474	3.854	3.391	3.030	2.791	2.503	1.351	.9287
S_T	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	2	3

Thus the closure time is

$$\tau_f = \frac{S_T}{9K}\,[3+(3-S_T)K]\left[1-\left(\frac{3-S_TK}{3K}\right)\ln\frac{3+(3-S_T)K}{3-S_TK}\right] + \frac{3-S_T}{6} - \frac{1}{3} \quad . \tag{7.33}$$

This relation was shown to be accurate for $S_T \leq 1.0$.

Table 7.1 Closure time for constant surface temperature spheres initially at freezing temperatures.

	Closure time τ_f				
S_T	Numerical solution, (Tao 1967)	Quasi-steady (eq 7.31)	(eq 7.33)	(eq 7.34)	(eq 7.35)
0	--	0.1667	0.1667	0.1667	0.0833
0.1	0.192	0.1667	0.1805	0.1791	0.0924
0.3	--	0.1667	0.2168	0.1948	0.1104
0.5	0.237	0.1667	0.2302	0.2030	0.1285
0.8	--	0.1667	0.2663	0.2048	0.1556
1	0.284	0.1667	0.2901	0.2004	0.1736
2	0.360	0.1667	0.408	0.1239	0.2639

Riley et al. (1974) also used a perturbation method to give a closure solution of

$$\tau_f = \frac{1}{6} + \frac{1}{6} S_T - \frac{S_T^{3/2}}{3\sqrt{2\pi}} + O(S_T^2) \ . \tag{7.34}$$

This relation is also valid at small Stefan numbers but is less accurate than eq 7.33 at $S_T > 0.5$.

The heat balance integral technique was used by Poots (1962). This method, following Goodman (1958), did not give accurate results for the closure time, which was

$$\tau_f = \frac{1}{4} (S_T+2) - \frac{2}{9} (S_T+3) + \frac{1}{16} (S_T + 4.0) \ . \tag{7.35}$$

A two-parameter method led to much better results but required numerical evaluation, which tends to lessen the value of the heat balance integral method.

The results for the closure time of spheres are shown in Table 7.1. Equation 7.33 clearly gives superior results.

7.2.4 Surface Convection, Quasi-Steady

London and Seban (1943) used the quasi-steady method to solve the case of solidification of a spherical mass of fluid initially at the fusion temperature. Convection into the ambient occurs at the outer boundary, as shown by Figure 7.7.

The temperature is

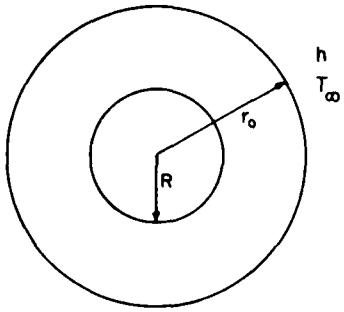

Figure 7.7. Freeze of a spherical mass with surface convection.

$$\frac{T - T_\infty}{T_f - T_\infty} = \frac{B_i \left(\frac{r_o}{r} - 1\right) + 1}{B_i \left(\frac{r_o}{R} - 1\right) + 1} \quad . \tag{7.36}$$

The interface position is

$$\frac{\alpha_1 S_{T\infty} t}{r_o^2} = \tau = \frac{1}{3}\left(\frac{1}{B_i} - 1\right)(1 - \beta^3) + \frac{1}{2}(1 - \beta^2) \quad . \tag{7.37}$$

As the surface coefficient h becomes very large ($B_i \to \infty$), the surface temperature approaches the value of the ambient. The above equations then reduce to those of Section 7.2.3, the constant temperature solution.

The closure time is

$$\tau_f = \frac{1}{3B_i} + \frac{1}{6} \quad .$$

Hill and Kucera (1983) immobilized the moving boundary and used a series solution. The closure time is

$$\tau_f = \frac{1}{6} + \frac{1}{3B_i} + S_T\left[\frac{3}{2} - 2B_i + \frac{4}{3}B_i^2 - \frac{B_i^3}{3}\right]$$

$$- S_T^2 \frac{B_i}{15}\left(32\,B_i^2 - 94B_i + 77\right) - S_T^3 \frac{8}{15} B_i^2 \left(7B_i - 11\right)$$

$$- \frac{29}{15} B_i^3 \; S_T^4 \quad .$$

This solution is only valid for small values of B_i; for $S_T = 0.5$, $B_i < 1.2$. Clearly as $B_i \to \infty$ the result becomes infinite.

The closure time was also shown to be

$$\frac{1}{6}\left(1 + \frac{2}{B_i}\right) < \tau_f < \frac{1}{6}\left(1 + \frac{2}{B_i}\right)(1 + S_T) \quad .$$

This inequality can be used when $B_i > 1.0$. This relation reduces to the constant temperature solution of Seeniraj and Bose (1982), for $B_i \to \infty$ (see next section).

7.2.5 Convection and Radiation Boundary Condition

Goodling and Khadar (1974) numerically evaluated the freezing of a sphere with convection and radiation at the surface of the sphere. The numerical results were too limited to reproduce but, if $\sigma_s \varepsilon T_f^3/kr_o < B_i^2/3$, the solidification time with radiation will exceed 90% of the time with convection only. Thus for soil systems, or atmospheric conditions, it can be anticipated that surface radiation will not be significant.

Seeniraj and Bose (1982) presented a perturbation solution for small $(T_f - T_\infty)$ and zero superheat. The closure time is, to the first order, given as

$$\tau_f = \frac{1}{6} + \frac{1}{3B_r} + \frac{S_{T\infty}}{6(1 - B_r)^2}\left[B_r^2 + \frac{2 - B_r}{B_r}\right] \tag{7.38}$$

where

$$B_r = B_i + \frac{4\,\sigma_s \varepsilon}{k} F\, r_o\, T_f^3 \qquad S_{T\infty} = \frac{c\,(T_f - T_\infty)}{\ell}$$

$$F = \text{radiation shape factor} \qquad \tau = \frac{\alpha\, S_{T\infty} t}{r_o^2} \; .$$

The relation clearly breaks down at $B_r = 1$ which has no physical significance. The result is valid only for $S_{T\infty} < 0.3$. As B_r increases above 0.5, eq 7.38 tends to significantly overestimate the closure time.

If radiation is absent and $h \to \infty$ $(B_i \to \infty)$ then

$$\tau_f = \frac{1}{6}\,(1 + S_{T\infty}) \; .$$

This is identical to the first two terms of the perturbation solution (eq 7.34) of Riley et al. (1974), eq 7.34.

For no radiation, the zeroth-order perturbation solution is exactly the quasi-steady solution of London and Seban (1943). The solution of Hill and Kucera (1983) is also valid here if $B_r = B_i$. The quasi-steady solution for this problem is

$$\frac{T - T_\infty}{T_f - T_\infty} = \frac{1 + B_r \left(\frac{r_o}{r} - 1\right)}{1 + B_r \left(\frac{r_o}{R} - 1\right)} \tag{7.39}$$

$$\tau = \frac{1}{3}\left(\frac{1}{B_r} - 1\right)(1 - \beta^3) + \frac{1}{2}(1 - \beta^2) \tag{7.40}$$

$$\tau_f = \frac{1}{3B_r} + \frac{1}{6} \tag{7.41}$$

The quasi-steady closure time clearly is the zeroth term of the perturbation solution.

NOMENCLATURE - CHAPTER 7

B_i	$\frac{hr_o}{k}$, Biot number
B_r	$B_i + \frac{4\sigma_s \varepsilon}{k} F\, r_o\, T_f^3$
c	specific heat
C	ρc - volumetric specific heat
F	radiation shape factor
$F(\eta)$	spherical function
g	$\frac{d\beta}{dT}$
G	Q/k
h	surface coefficient of convection
k	thermal conductivity
k_{12}	k_1/k_2
ℓ	latent heat of solidification
q	heat transfer rate from sphere surface
q^*	$\frac{q}{4\pi k r_o (T_f - T_s)}$
Q	prescribed surface heat flux

Q_i	prescribed heat flux at phase interface
Q_o	source coefficient
Q_s	source or sink energy rate
r	radial coordinate
r_o	radius of sphere
R	phase change interface radius
S_T	$\frac{c_\perp}{\ell}(T_f - T_s)$
S_{To}	$\frac{c_\perp}{\ell}(T_o - T_f)$
$S_{T\infty}$	$\frac{c_\perp}{\ell}(T_f - T_\infty)$
S_i	$\frac{Qc\ r_o}{k\ \ell}$
t	time
t_f	closure time for sphere
T	temperature
T_f, T_o, T_s	fusion, initial, surface temperatures
T_∞	ambient temperature
u	$\frac{T - T_s}{T_f - T_s}$
X	planar phase change depth
y	r/r_o
α	thermal diffusivity
β	R/r_o
γ	$\frac{X}{2\sqrt{\alpha t}}$ Neumann phase change parameter

δ	thermal penetration radius
ε	surface emmissivity
η_i	$r/(2\sqrt{\alpha_i t})$
θ	$\dfrac{(T - T_f)\,k}{r_o\,Q}$
λ	$R/(2\sqrt{\alpha_l t})$ phase change depth parameter
ρ	density
σ_s	Stefan-Boltzmann constant
μ	$-\dfrac{V}{\alpha}\,[r - R(t)]$
τ	$\dfrac{\alpha\,t\,S_T}{r_o^2}$ dimensionless time
τ_l	$\dfrac{Qt}{\rho \ell r_o}$
τ_f	dimensionless closure time
ϕ	$\dfrac{T_o - T_f}{T_f - T_s}$
Ω	δ/R

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APPENDIX A. QUASI-STATIC APPROXIMATIONS AND PERTURBATION METHODS

Problems of freezing and thawing arise frequently in such diverse applications as thermal design in permafrost regions, thermal storage of latent heat for solar systems, and the heat treatment of metals. One is often interested in the penetration rate of the phase change interface, the temperature field, and the boundary heat transfer rates. From an engineering design viewpoint, exact solutions are sought for geometries and boundary conditions that are simple and yet representative of significant systems. Unfortunately the mathematical difficulties are such that exact solutions to this class of problems are limited to a few very special geometries and boundary conditions (Lunardini 1981). The differential equations or the boundary conditions for these moving value problems are inherently nonlinear and thus solution methods are rare.

The Neumann problem has shown that error functions can lead to an exact solution, suitable for certain boundary conditions. However, the method cannot be applied in general. Such a similarity solution requires that the differential equation and all of the initial and boundary conditions be expressed with a single independent variable. Similarity solutions will not exist for finite domains, two phases present initially, nonuniform initial temperatures, and boundary temperatures which are arbitrary functions of time. Thus, there are very few other exact solutions. This has prompted considerable interest in approximate methods that can yield solutions acceptable for engineering design. Aside from the usual numerical procedures, several analytical methods have been of great value, including the quasi-static approximation, the heat balance integral method, and variational methods.

If the phase change interface moves relatively slowly, then an assumption can be made that the moving interface will not exert a major influence upon the temperature field during short time periods. Two approximations have been used.

The quasi-stationary assumption neglects any convection in the diffusion equation and neglects the moving interface in evaluating the temperature field and consequently the diffusive flux anywhere in the volume of interest. The assumption can handle initial conditions but is not general-

ly valid for all times if the temperature ahead of the interface is not uniform or is changing because of the interface motion. The problem reduces to one of transient conduction with no phase change. The actual phase change is then solved through the interface boundary condition.

The quasi-steady approximation further simplifies the quasi-stationary problem by dropping the transient term in the energy equation. The justification for the method is somewhat tenuous since it cannot satisfy the initial conditions; however, the problem is mathematically so simple that the quasi-steady approximation has been used more than the quasi-stationary method.

QUASI-STATIONARY APPROXIMATION

The melting system will be examined again, for a Neumann problem with variable density, in order to illustrate the quasi-stationary idea (see Fig. A1). The exact solution to this problem is discussed in Section 2.1.1. Neglect the temperature variations in the solid region and examine only the liquid equations. This is done so that a convection term can be maintained since in the solid region there is no convective term. The energy equation for the liquid region has been derived earlier. The equations for the melting system (see Nomenclature at end of appendix) are as follows:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\left[\frac{\partial T}{\partial t} + \left(1 - \frac{\rho_2}{\rho}\right)\frac{dX}{dt}\frac{\partial T}{\partial x}\right] \qquad \text{(A1)}$$

$$T(x,0) = T_o$$

$$T(X,t) = T_f$$

$$T(0,t) = T_s$$

$$-k\,\frac{\partial T(X,t)}{dx} = \rho \ell \frac{dX}{dt}$$

$$X(0) = X_o \ .$$

Here ρ_2/ρ is the ratio of the solid to liquid phase densities. We can introduce the following nondimensionalizing variables:

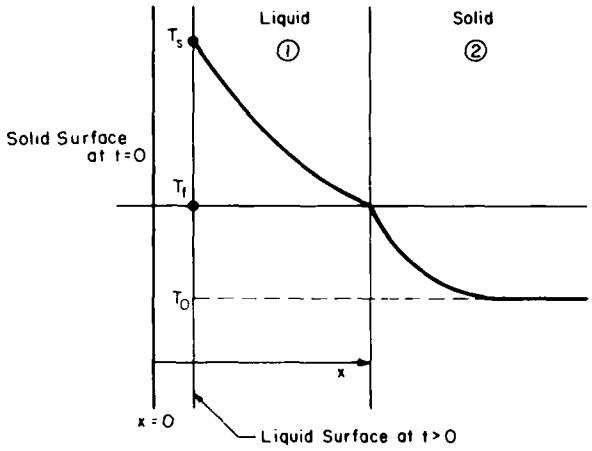

Figure A1. Geometry for melting system with variable density.

$$x_1 = \frac{x}{X_o} \qquad \theta = \frac{T-T_o}{T_s-T_f} \qquad \phi = \frac{T_f-T_o}{T_s-T_f} \qquad \phi_s = \frac{T_s-T_o}{T_s-T_f}$$

$$\xi = \frac{X}{X_o} \qquad \tau = \frac{\alpha t}{X_o^2} \qquad S_T = \frac{c}{\ell}(T_s-T_f)$$

The dimensionless time τ, uses a characteristic time X^2_o/α. This is the diffusion time for the liquid region and is a measure of how long it takes the interface to move a distance X_o. The time domain will be relatively short, in τ, if $X < X_o$.

The equations are

$$\frac{\partial^2\theta}{\partial x_1^2} = \frac{\partial\theta}{\partial\tau} + (1 - \rho_2/\rho)\frac{d\xi}{d\tau}\frac{\partial\theta}{\partial x_1} \tag{A2a}$$

$$\frac{\partial^2\theta}{\partial x_1^2} = \frac{\partial\theta}{\partial\tau} + (\frac{\rho_2}{\rho} - 1)\, S_T \frac{\partial\theta}{\partial x_1}\left(\frac{\partial\theta}{\partial x_1}\right)_\xi \tag{A2b}$$

$$\begin{aligned} &\theta(x_1,0) = 0 \\ &\theta(0,\tau) = \phi_s \\ &\theta(\xi,\tau) = \phi \\ &\frac{d\xi}{d\tau} = -S_T\left(\frac{\partial\theta}{\partial x_1}\right)_\xi \\ &\xi(0) = 1 \end{aligned} \tag{A3}$$

The quasi-stationary approximation, which tends to be valid if $S_T \ll 1$, is

$$\frac{\partial^2\theta}{\partial x_1^2} = \frac{\partial\theta}{\partial\tau} \tag{A4}$$

$$\begin{aligned} &\theta(x_1,0) = 0 \\ &\theta(0,\tau) = \phi_s \\ &\theta(\xi,\tau) = \phi \quad . \end{aligned}$$

After solving eq 4, the interface location is evaluated with eq 3. Because of the limitations already mentioned, the method is best suited to single phase problems.

Duda and Vrentas (1969a) showed that the quasi-stationary solution is the first term of an asymptotic series. If the temperature and interface position are expanded as follows, with the Stefan number as the perturbation parameter,

$$\theta = \theta_o + S_T\theta_1 + S_T^2\,\theta_2 + S_T^3\,\theta_3 + \ldots \tag{A5}$$

$$\xi = 1 + S_T\,\xi_1 + S_T^2\,\xi_2 + S_T^3\,\xi_3 + \ldots \tag{A6}$$

The temperature relations at the interface position can be expanded by Taylor series, about the initial interface location, $\xi_o(0) = 1.0$:

$$\left(\frac{\partial\theta_i}{\partial x_1}\right)_\xi = \left(\frac{\partial\theta_i}{\partial x_1}\right)_1 + \left(\frac{\partial^2\theta_i}{\partial x_1^2}\right)_1 (S_T\xi_1 + S_T^2\,\xi_2 + S_T^3\,\xi_3 + \ldots) \tag{A7}$$

$$\theta_i(\xi,\tau) = \theta_i(1,\tau) + \left(\frac{\partial\theta_i}{\partial x_1}\right)_1 (S_T\xi_1 + S_T^2\,\xi_2 + S_T^3\,\xi_2 + \ldots) \tag{A8}$$

Let $\varepsilon = (\rho_2/\rho - 1)S_T$. The constant ε will be of order S_T^2 if $\rho_2/\rho \approx 1.0$.

The procedure is called a surface-volume perturbation since the differential equation and the boundary conditions both contain nonlinearities. Van Dyke (1964) and Cole (1968) provide further details on regular perturbation methods. Applying eq A5-8 to eq A2 leads to the following system of equations up to order 2:

$$\frac{\partial^2\theta_o}{\partial x_1^2} = \frac{\partial\theta_o}{\partial\tau} \tag{A9}$$

$$\theta_o(x_1,0) = 0$$
$$\theta_o(0,\tau) = \phi_s$$
$$\theta_o(1,\tau) = \phi$$

$$\frac{d\xi_o}{d\tau} = -\,S_T\left(\frac{\partial\theta_o}{\partial x_1}\right)_1 \tag{A10a}$$

$$\xi_o(0) = 1.0 \tag{A10b}$$

$$\frac{\partial^2\theta_1}{\partial x_1^2} = \frac{\partial\theta_1}{\partial\tau} \tag{A11}$$

$$\theta_1(x_1,0) = \theta_1(0,\tau) = 0$$

$$\theta_1(1,\tau) = -\xi_1\left(\frac{\partial\theta_o}{\partial x_1}\right)_1$$

$$\frac{d\xi_1}{d\tau} = -\left(\frac{\partial\theta_o}{x_1}\right)_1$$

$$\xi_1(0) = 0$$

$$\frac{\partial^2\theta_2}{\partial x_1^2} = \frac{\partial\theta_2}{\partial\tau} + \left(\frac{\rho_2}{\rho} - 1\right)/S_T \frac{\partial\theta_o}{\partial x_1}\left(\frac{\partial\theta_o}{\partial x_1}\right)_1 \qquad \text{(A12)}$$

$$\theta_2(x_1,0) = \theta_2(0,\tau) = 0$$

$$\theta_2(1,\tau) = -\xi_2\left(\frac{\partial\theta_o}{\partial x_1}\right)_1 - \xi_1\left(\frac{\partial\theta_1}{\partial x_1}\right)_1$$

$$\frac{d\xi_2}{d\tau} = -\xi_1\left(\frac{\partial^2\theta_o}{\partial x_1^2}\right)_1 - \left(\frac{\partial\theta_1}{\partial x_1}\right)_1$$

$$\xi_2(0) = 0 \quad .$$

It is clear that the zeroth-order solution, eq A9, is essentially the quasi-stationary approximation, as defined earlier. Equations 10a,b are used only if the zeroth solution is calculated alone.

A regular perturbation (volume) can also be used if the nonlinearities are concentrated in the differential equation. This can be done by immobilizing the interface with $\eta = \frac{x_1}{\xi} = \frac{x}{X}$.

Equation A2b is then

$$\frac{\partial^2\theta}{\partial\eta^2} + \left(\frac{\eta}{2}\frac{d\xi^2}{d\tau}\right)\frac{\partial\theta}{\partial\eta} = \xi^2\frac{\partial\theta}{\partial\tau} + \varepsilon\frac{\partial\theta}{\partial\eta}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} \qquad \text{(A13)}$$

$$\theta(\eta,0) = 0$$

$$\theta(0,\tau) = \phi_s$$

$$\theta(1,\tau) = \phi$$

$$\xi(0) = 1$$

$$\frac{d\xi}{d\tau} = -\frac{S_T}{\xi}\left(\frac{\partial\theta}{\partial\eta}\right)_1$$

$$\xi^2 - 1 = -2S_T\int_0^\tau \frac{\partial\theta}{\partial\eta}(1,\tau')d\tau' \quad .$$

A regular perturbation will yield a set of equations equivalent to eq A9-12.

Since the quasi-stationary method involves the solution of a transient conduction problem, the result will often be in the form of an infinite series solution. The method reduces the phase change problem to one of transient heat conduction with no phase change. Analytic solutions of transient conduction problems can be quite difficult in their own right.

QUASI-STEADY APPROXIMATION

The quasi-stationary method can be further simplified if the unsteady terms in the diffusion equation are also neglected. To accomplish this a new characteristic time will be used.

Let

$$\tau_1 = \frac{\alpha t}{X_o^2} S_T \ . \tag{A14}$$

The new characteristic time is now larger than the diffusion time X_o^2/α, if $S_T < 1.0$. The new time domain is ideally suited to long time movement of the interface when the initial conditions have become less significant. Jiji and Weinbaum (1978) used two time domains, the quasi-stationary for initial growth and eq 14 for later growth. The two times were joined at a suitable intermediate time. In this way a two-phase problem could be handled.

The nondimensional equations are

$$\frac{\partial^2\theta}{\partial x_1^2} = S_T \frac{\partial\theta}{\partial\tau_1} - \varepsilon \frac{d\xi}{d\tau_1} \frac{\partial\theta}{\partial x_1} \tag{A15}$$

$$\theta(x_1,0) = 0$$

$$\theta(0,\tau_1) = \phi_s$$

$$\theta(\xi,\tau_1) = \phi$$

$$\frac{d\xi}{d\tau_1} = -\left(\frac{\partial\theta}{\partial x_1}\right)_\xi$$

$$\xi(0) = 1 \ .$$

It is now obvious that for small Stefan numbers the diffusion equation reduces to

$$\frac{\partial^2\theta}{\partial x_1^2} = 0 \ . \tag{A16}$$

Thus, no transient term need be considered and the solution is extremely simple. Solutions are far easier to obtain, compared to the quasi-stationary equations, but the utility of the solution is limited since the initial conditions cannot be met and the sensible heat is not accounted for. Nevertheless, this concept is very widely used for freezing and thawing problems.

The quasi-steady method can also be examined from the viewpoint of perturbation solutions. The following series expansions are used:

$$\theta = \theta_o + S_T\,\theta_1 + S_T^2\,\theta_2 + \ldots \tag{A17}$$

$$\xi = \xi_o + S_T\,\xi_2 + S_T^2\,\xi_2 + \ldots \tag{A18}$$

Conditions at the phase change interface can be evaluated using the following Taylor formulae:

$$\left(\frac{\partial\theta_i}{\partial x_1}\right)_\xi = \left(\frac{\partial\theta_i}{\partial x_1}\right)_{\xi_o} + \left(\frac{\partial^2\theta_i}{\partial x_1^2}\right)_{\xi_o}(S_T\,\xi_1 + S_T^2\,\xi_2 + \ldots)$$
$$+ \left(\frac{\partial^3\theta_i}{\partial x_1^3}\right)_{\xi_o}(S_T\,\xi_1 + S_T^{\,2}\,\xi_2 + \ldots)^2 + \ldots \tag{A19}$$

$$\theta_i(\xi,\tau) = \theta_i(\xi_o,\tau) + \left(\frac{\partial\theta_i}{\partial x_1}\right)_{\xi_o}(S_T\,\xi_1 + S_T^2\,\xi_2 + \ldots)$$
$$+ \left(\frac{\partial^2\theta_i}{\partial x_1}\right)_{\xi_o}(S_T\,\xi_1 + S_T^{\,2}\,\xi_2 + \ldots)^2 + \ldots \tag{A20}$$

The following systems of equations are generated:

$$\frac{\partial^2\theta_o}{\partial x_1^2} = 0 \tag{A21}$$

$$\theta_o(0,\tau_1) = \phi_s$$

$$\theta_o(\xi_o,\tau_1) = \phi$$

$$\frac{d\xi_o}{d\tau_1} = -\left(\frac{\partial\theta_o}{\partial x_1}\right)_{\xi_o}$$

$$\xi_o(0) = 1$$

$$\frac{\partial^2\theta_1}{\partial x_1^2} = \frac{\partial\theta_o}{\partial\tau_1} \tag{A22}$$

$$\theta_1(0,\tau_1) = 0$$

$$\theta_1(\xi_o,\tau_1) = -\xi_1 \left(\frac{\partial\theta_o}{\partial x_1}\right)_{\xi_o}$$

$$\frac{d\xi_1}{d\tau_1} = -\left[\xi_1 \left(\frac{\partial^2\theta_o}{\partial x_1^2}\right)_{\xi_o} + \left(\frac{\partial\theta_1}{\partial x_1}\right)_{\xi_o}\right]$$

$$\xi_1(0) = 0$$

$$\frac{\partial^2\theta_2}{\partial x_1^2} = \frac{\partial\theta_1}{\partial\tau} + \frac{\varepsilon_1}{S_T^2}\left(\frac{\partial\theta_o}{\partial x_1}\right)\left(\frac{\partial\theta_o}{\partial x_1}\right)_{\xi_o} \tag{A23}$$

$$\theta_2(0,\tau_1) = 0$$

$$\theta_2(\xi_o,\tau_1) = -\xi_1\left(\frac{\partial\theta_1}{\partial x_1}\right)_{\xi_o} - \xi_2\left(\frac{\partial\theta_o}{\partial x_1}\right)_{\xi_o}$$

$$\frac{d\xi_2}{d\tau_1} = -\xi_2\left(\frac{\partial^2\theta_o}{\partial x_1^2}\right)_{\xi_o} - \xi_1\left(\frac{\partial^2\theta_1}{\partial x_1^2}\right)_{\xi_o} - \left(\frac{\partial\theta_2}{\partial x_1}\right)_{\xi_o}$$

$$\xi_2(0) = 0 \ .$$

The zeroth solution is the quasi-steady approximation. Pedroso and Domoto (1973a) demonstrated this for a spherical system. Lock (1969) derived the zeroth and first order systems shown here for the solidification of the semi-infinite medium. Duda and Vrentas (1969a,b) discussed the usefulness of perturbation methods for phase change problems. Pedroso and Domoto (1973a,b,c) noted the difficulty of using perturbation methods for inward, spherical, solidifcation. Jiji (1970) used perturbation for curvilinear solidification.

The quasi-steady method is so simple that solutions for the above system can be written down:

$$\theta_o = \frac{-x_1}{\xi_o} + \theta_s$$

$$\xi_o = \sqrt{1 + 2\tau_1}$$

$$\theta_1 = \left(\frac{\xi_1}{\xi_o}\right) - \frac{1}{6}\,\frac{x_1}{\xi_o} + \frac{1}{6\xi_o^3}\,x_1^3$$

$$\xi_1 = \frac{1}{6}\left(\frac{1}{\xi_o} - \xi_o\right)$$

$$\theta_2 = dx_1 + cx_1^2 + ax_1^3 + bx_1^5$$

$$\xi_o d = -\frac{1}{36}\left(\frac{1}{\xi_o^4} - 1\right) + \frac{\xi_2}{\xi_o} - \frac{1}{6}\left(-\frac{1}{2\xi_o^2} + \frac{1}{3}\right) - \left(\frac{1}{40} + \frac{\varepsilon}{2S_T^2}\right)$$

$$a\xi_o^3 = \frac{1}{6}\left(-\frac{1}{2\xi_o^2} + \frac{1}{3}\right)$$

$$\xi_o^5\, b = \frac{1}{40}$$

$$c\xi_o^2 = \frac{\varepsilon}{2S_T^2}$$

$$\xi_2 = \frac{1}{4}\left(\frac{1}{5} + \frac{\varepsilon}{S_T^2}\right)\frac{1}{\xi_o} + \frac{1}{4}\left(\frac{1}{18} - \frac{1}{5} - \frac{\varepsilon}{S_T^2}\right)\xi_o - \frac{1}{72}\,\xi_o^{-3} \quad .$$

PERTURBATION WITH TIME VARIABLE REPLACEMENT

Huang and Shih (1975) introduced a useful concept for perturbation methods. Consider the usual nondimensional equations for a planar system:

$$\frac{\partial^2\theta}{\partial x_1^2}(x_1,\tau^*) = S_T\,\frac{\partial\theta}{\partial\tau_1}(x_1,\tau^*) \qquad \text{(A24)}$$

$$\theta(x_1,0) = 0 \qquad \text{(A24a)}$$

$$\theta(0,\tau^*) = \phi_s \qquad \text{(A24b)}$$

$$\theta(\xi,\tau^*) = \phi \qquad \text{(A24c)}$$

$$\frac{d\xi}{d\tau^*} = -\frac{\partial\theta}{\partial x_1}(\xi,\tau^*) \quad . \qquad \text{(A24d)}$$

Replace the time variable τ^* by $\xi(\tau^*)$, the phase change interface position. This transformation is acceptable if ξ is a monotonic function of τ^*, a common relation for many practical problems. Thus, the energy equation is

$$\frac{\partial^2\theta(x_1,\xi)}{\partial x_1^2} = S_T\, g\, \frac{\partial\theta(x_1,\xi)}{\partial\xi} \tag{A25}$$

and

$$g(\tau^*) \equiv \frac{d\xi}{d\tau^*} \quad . \tag{A26}$$

Now expand the temperature and the rate of change of the phase change interface as asymptotic series with the Stefan number as the parameter:

$$U(x_1,\xi) = U_o + U_1 S_T + U_2 S_T^2 + U_3 S_T^3 + \ldots \tag{A27}$$

$$g(\tau^*) = g_o + g_1 S_T + g_2 S_T^2 + g_3 S_T^3 + \ldots \tag{A28}$$

These expansions allow many problems to be solved in a particularly simple fashion (Seeniraj and Bose 1982, Huang and Shih 1975).

Once the coefficients g_i have been found, the interface relation can be expressed as

$$\tau_1^* = \tau_o + \tau_1 S_T + \tau_2 S_T^2 + \tau_3 S_T^3 + \ldots \tag{A29}$$

where

$$\tau_o = \int_0^\xi \frac{ds}{g_o(s)}$$

$$\tau_1 = -\int_0^\xi \frac{g_1}{g_o^2}\, ds$$

$$\tau_2 = \int_0^\xi \frac{1}{g_o^2}\left(\frac{g_1^2}{g_o} - g_2\right) ds$$

$$\tau_3 = -\int_0^\xi \frac{1}{g_o^3}\left(g_o g_3 - 2g_1 g_2 + \frac{g_1^3}{g_o}\right) dS \ .$$

If the rate and range of convergence of eq A24 is slow or if eq A29 diverges, its performance can be improved using the non-linear transformations of Shanks (1955).

An advantage of this procedure is that it is no longer necessary to expand the unknown functions about ξ_0 when considering the conditions at the phase change interface. This greatly simplifies the formulation of the equations but the solutions are not necessarily simpler.

The Stefan Problem

It is of interest to return to the solution of Section 2.1.3 for the case of a constant surface temperature. The solution was based on the assumption that, $S_T = c_1(T_f - T_s/\ell$, is small which is equivalent to the quasi-steady approximation. The quasi-steady problem reduces to

$$\frac{\partial^2 T_1}{\partial x^2} = 0 \tag{A30}$$

$$T_1\ (0,t) = T_s$$

$$T_1\ (X,t) = T_f$$

$$k_1 \frac{\partial T_1\ (X,t)}{\partial x} = \rho_1 \ell \frac{dX}{dt} \quad .$$

The solution to this system is

$$T_1 = (T_f - T_s)\frac{x}{X} + T_s \tag{A31}$$

$$X = \sqrt{\frac{2k_1}{\rho_1 \ell}(T_f - T_s)t} \ . \tag{A32}$$

The phase change depth X given by eq 32 is identical to that of Section 2.1.3. Thus, Stefan (1891), in effect, seems to have been the first to use the quasi-steady method.

The results are also valid if the surface temperature is a function of time. Then

$$T_1 = [T_f - T_s(t)]\frac{x}{X} + T_s\ (t) \tag{A33}$$

$$X = \sqrt{\frac{2k_1}{\rho_1 \ell}\int_0^t [T_f - T_s(t')]dt'} \ . \tag{A34}$$

A major limitation of the quasi-steady approximation is the failure to account for the sensible heat during the phase change. The heat flow from the surface of the semi-infinite medium is given by

$$q = - k_1 A \left(\frac{\partial T}{\partial x}\right)_{x=0} .$$

Using eq 31 this is

$$q = - k_1 A \frac{(T_f - T_s)}{X} .$$

The total heat flow, at the surface, during a given time is

$$Q_p = \int_0^t q \, dt .$$

This can be evaluated using eq 32 as

$$Q_p = - k_1 A (T_f - T_s) \int_0^t \frac{dt'}{\left[\frac{2k_1 (T_f - T_s)t'}{\rho_1 \Gamma}\right]^{1/2}}$$

or

$$Q_p = - A X L .$$

The total surface energy flow, during the time that a layer of thickness X freezes, is simply the latent heat. Thus, the method does not take into account any sensible heat although the temperature of the frozen layer does decrease with time. This is in contrast to the exact solution of Section 2.1, with $T_o = T_f$, where it can be shown that the heat removed equals the latent heat plus the sensible heat involved in lowering the temperature of the frozen layer. This limitation is directly associated with the assumption of a Stefan number of zero.

A correction for the sensible heat can be made by using an effective latent heat (see Lunardini 1981):

$$\ell_e = \ell(1 + c_{21}\phi S_T + S_T/2) . \qquad \text{(A35)}$$

NOMENCLATURE - APPENDIX A

c	specific heat
g	$d\xi/d\tau^*$
k	thermal conductivity
ℓ	latent heat

ℓ_e	effective latent heat
q	rate of energy flow from surface
Q_p	total heat flow from surface
S_T	$\frac{c_1}{\ell}(T_s - T_f)$ thaw $\quad \frac{c_1}{\ell}(T_f - T_s)$ freeze
t	time
T	temperature
T_f, T_o, T_s	fusion, initial, surface temperatures
x	Cartesian coordinate
x_1	x/X_o
X	phase change depth
α	thermal diffusivity
ε	$(\rho_2/\rho - 1)S_T$
η	x/X
θ	$\frac{T - T_o}{T_s - T_f}$
ξ	X/X_o
ρ	density

Subscripts

1,2	different phases of material

APPENDIX B. THE HEAT BALANCE INTEGRAL METHOD

An approximate method that has been used with good results for solidification phase change problems involves the concept of the temperature penetration depth. The integral method introduced by Goodman (1958) is based on the same concepts as the momentum integral method of the boundary layer in fluid mechanics (Pohlhausen 1921, von Karman 1921). Consider the semi-infinite solid shown in Figure B1. At a time t, after the surface temperature has dropped to T_s, the temperature in the solid will be disturbed to a depth $\delta(t)$. Beyond this depth, the temperature of the solid remains at the initial temperature T_o and no energy is transferred beyond this point. The penetration distance δ is analogous to the boundary layer thickness in fluid mechanics. The solution method is analogous to the momentum integral method in that the basic equations are satisfied on average over the volume of thickness $\delta(t)$, rather than at each point. The conduction equation is

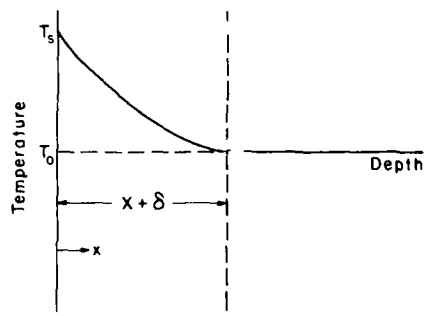

Figure B1. Temperature penetration depth.

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad . \tag{B1}$$

Now, this equation is integrated over the distance $\delta(t)$. Thus

$$\int_0^{\delta(t)} \alpha \frac{\partial^2 T}{\partial x^2} dx = \int_0^{\delta(t)} \frac{\partial T}{\partial t} dx \quad .$$

The properties are assumed constant and

$$\alpha \int_0^{\delta} \frac{\partial^2 T}{\partial x^2} dx = \alpha \left[\frac{\partial T(\delta,t)}{\partial x} - \frac{\partial T(0,t)}{\partial x}\right] \quad . \tag{B2}$$

Leibniz's rule for a general function is

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t)dx = f(b,t) \frac{db}{dt} - f(a,t) \frac{da}{dt} + \int_a^b \frac{\partial f}{\partial t} (x,t)dx \quad .$$

Then

$$\int_0^{\delta(t)} \frac{\partial T}{\partial t}\, dx = \frac{d}{dt} \int_0^{\delta(t)} T(x,t)dx - T(\delta,t) \frac{d\delta}{dt} \quad .$$

Let

$$\theta = \int_0^{\delta} T(x,t)\, dx \ . \tag{B3}$$

Then the heat balance integral equation is

$$\frac{d\theta}{dt} + \alpha \frac{\partial T(0,t)}{\partial x} - T_o \frac{d\delta}{dt} = 0 \ . \tag{B4}$$

This equation is valid if there is no phase change.

EXAMPLE

Consider a semi-infinite region, initially at T_o, with a constant heat flux Q at the surface as shown in Figure B2. The temperature of the surface increases as the temperature disturbance penetrates into the solid. The exact solution to this problem is known (Carslaw and Jaeger (1959) but the heat balance integral (HBI) will be used to illustrate the method.

The system of equations to solve is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \qquad t > 0 \tag{B5}$$

$$k \frac{\partial T(0,t)}{\partial x} = -Q \tag{B5a}$$

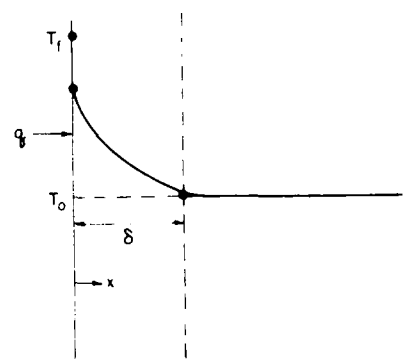

Figure B2. Heat flux at surface of semi-infinite region.

At the thermal penetration depth, the temperature is T_o, and the heat flux is zero; then

$$T(\delta,t) = T_o \tag{B5b}$$

$$\frac{\partial T(\delta,t)}{\partial x} = 0 \ . \tag{B5c}$$

The temperature is assumed to have a quadratic profile satisfying eq 5a,b,c:

$$T = T_o + \frac{Q}{2\delta k}(x-\delta)^2 \quad . \tag{B6}$$

From eq B3, B4 and B6, the differential equation for δ is

$$\frac{d\delta^2}{dt} = 6\alpha \tag{B7}$$

with the solution

$$\delta = \sqrt{6\alpha t} \quad . \tag{B8}$$

The HBI technique has reduced the partial differential equation to an ordinary differential equation with a simple solution. The values of δ and t, when the surface temperature reaches the value T_f, are given by

$$\delta_m = \frac{2k(T_f - T_o)}{Q}$$

$$t_m = \frac{2k^2(T_f - T_s)^2}{3\alpha Q^2} \quad .$$

The exact value for the time t_m is

$$t_m = 0.785\,\frac{k^2(T_f - T_o)^2}{\alpha Q^2} \quad .$$

If a quartic temperature profile is used, additional smoothness relations at δ can be used:

$$\frac{\partial^2 T(\delta,t)}{\partial x^2} = \frac{\partial^3 T(\delta,t)}{\partial x^3} = 0 \quad .$$

Then

$$T = T_o + \frac{Q}{4k\delta^3}(x-\delta)^4$$

$$\delta_m = \frac{4k(T_f - T_o)}{Q}$$

$$t_m = 0.8\,\frac{k^2(T_f - T_o)^2}{\alpha Q^2} \quad .$$

In general, the accuracy of the approximate solution can be improved by using higher order temperature profiles; however, the algebraic work also increases. Unfortunately the accuracy may decrease when a polynomial above a certain order is chosen. For example, a 10th order polynomial for the temperature is no more accurate than the quadratic profile. The convergence of the method for a given situation is unpredictable, as has been discussed by Langford (1973).

PHASE CHANGE INTEGRAL EQUATIONS

Consider the case of phase change where the properties of the frozen region differ from those of the thawed region as shown in Figure B3. There will then be two integral equations as follows;

$$\frac{d\theta_1}{dt} - T_f \frac{dX}{dt} - \alpha_1 \left[\frac{\partial T_1\ (X,t)}{\partial x} - \frac{\partial T_1\ (0,t)}{\partial x}\right] = 0 \qquad \text{(B9)}$$

$$\frac{d\theta_2}{dt} - T_o \frac{d\delta}{dt} + T_f \frac{dX}{dt} + \alpha_2 \frac{\partial T_2\ (X,t)}{\partial x} = 0 \qquad \text{(B10)}$$

where

$$\theta_1 = \int_0^X T_1\ (x,t)dx \qquad \text{(B11)}$$

$$\theta_2 = \int_X^\delta T_2\ (x,t)dx \qquad \text{(B12)}$$

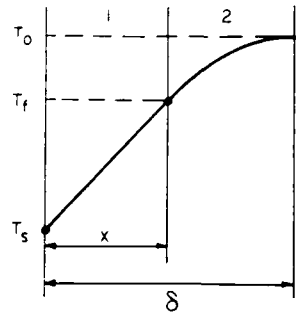

Figure B3. Heat balance integral geometry.

The following boundary conditions have been used:

$$T_1(X,t) = T_2(X,t) = T_f$$

$$T_2(\delta,t) = T_o.$$

The solution of a general problem with superheating or subcooling (the initial temperature not at the freezing temperature) will involve two coupled parameters X and δ. The solution will often be difficult. However, assume that the initial temperature is T_f. Then the problem reduces to only one differential equation since the penetration distance δ is now identical to the phase change depth X:

$$\frac{d\theta_1}{dt} - T_f \frac{dX}{dt} = \alpha_1 \left[\frac{\partial T_1(X,t)}{\partial x} - \frac{\partial T_1(0,t)}{\partial x}\right] = 0 \qquad \text{(B13)}$$

$$\theta_1 = \int_0^X T_1(x,t)dx \ .$$

The derivation of the method and some applications have been described by Goodman (1958, 1964).

The heat balance integral equations in cylindrical coordinates are

$$\alpha_1\left[R \frac{\partial T_1}{\partial r}(R,t) - r_o \frac{\partial T_1}{\partial r}(r_o,t)\right] = \frac{d}{dt} \int_{r_o}^{R} rT_1 \, dr - R \, T_f \frac{dR}{dt}$$

$$\alpha_2 \left[\delta \frac{\partial T_2(\delta,t)}{\partial r} - R \frac{\partial T_2(R,t)}{\partial r}\right] = \frac{d}{dt} \int_{\delta}^{R} r \, T_2 \, dr - T_2(\delta,t) \, \delta \frac{d\delta}{dt} + T_f \, R \frac{dR}{dt} \ .$$

For spherical coordinates

$$\alpha_1 \left[R^2 \frac{\partial T_1(R,t)}{\partial r} - r_o^2 \frac{\partial T_1(r_o,t)}{\partial r}\right] = \frac{d}{dt} \int_{r_o}^{R} r^2 T_1 dr - T(R,t) \, R^2 \frac{dR}{dt}$$

$$\alpha_2 \left[\delta^2 \frac{\partial T_2(\delta,t)}{\partial r} - R^2 \frac{\partial T_2(R,t)}{\partial r}\right] = \frac{d}{dt} \int_{R}^{\delta} r^2 T_2 dr - T_2(\delta,t) \, \delta^2 \frac{d\delta}{dt} - T_f \, R^2 \frac{dR}{dt} \ .$$

An alternative form of the spherical heat balance integral equations can be written as

$$\alpha_i \left[\left(\frac{\partial f_i}{\partial r}\right)_{b_i} - \left(\frac{\partial f_i}{\partial r}\right)_{a_i}\right] = \frac{d}{dt} \int_{a_i}^{b_i} f_i dr - f_i \, (b_i,t) \frac{db_i}{dt} + f_i(a_i,t) \frac{da_i}{dt}$$

where $f_i = r \, T_i$ and a_i, b_i are the limits of the regions to be integrated.

THE STEFAN PROBLEM

As an example of the method the solution of the Stefan Problem will be given. Goodman (1958) has solved this problem as well as a number of others. The heat balance integral is eq 13:

$$\frac{d\theta_1}{dt} = \alpha_1 \left[\frac{\partial T_1(X,t)}{\partial x} - \frac{\partial T_1(0,t)}{\partial x}\right] + T_f \frac{dX}{dt}$$

$$\theta_1 = \int_0^X T_1 \, dx \quad .$$

The boundary conditions are

$$T_1(X,t) = T_f \tag{B13a}$$

$$T_1(0,t) = T_s \tag{B13b}$$

$$k_1 \frac{\partial T_1(X,t)}{\partial x} = \rho_1 \ell \frac{dX}{dt} \quad . \tag{B13c}$$

Assume that T_1 is a quadratic function of x:

$$T_1 = T_f + a(x - X) + b\,(x - X)^2 \ . \tag{B14}$$

In order to avoid a second-order differential equation for X, boundary condition eq B13c can be written in the form given by eq 1.33:

$$- k_1 \left[\frac{\partial T_1(X,t)}{\partial x}\right]^2 = \rho_1 \ell \alpha_1 \frac{\partial^2 T_1(X,t)}{\partial x^2} \quad . \tag{B15}$$

The temperature is then given by

$$T_1 = T_f + a(x - X)\frac{x}{X} - \frac{\ell S_T}{c_1}\frac{(x - X)^2}{x^2} \tag{B16}$$

where $S_T = \dfrac{c_1\,(T_f - T_s)}{\ell}$

$$\frac{\partial T_1(X,t)}{\partial x} = a \tag{B17}$$

$$\frac{\partial T_1(0,t)}{\partial x} = - a + \frac{2\ell S_T}{c_1 X} \quad . \tag{B18}$$

The coefficient a is given by the solution to

$$a^2 + \frac{2\ell a}{c_\ell X} - 2\left(\frac{\ell}{c_\ell}\right)^2 \frac{S_T}{X^2} = 0 \quad . \tag{B19}$$

The positive root of this equation is chosen since $a > 0$ for $\frac{\partial T\,(X,t)}{\partial x} > 0$.

Then

$$a = \frac{\ell}{c_\ell X}\left[-1 + \sqrt{1 + 2\,S_T}\right] \tag{B20}$$

The equation for X is

$$X\,\frac{dX}{dt} = 6\alpha_\ell \left(\frac{1 - \sqrt{1 + 2\,S_T} + 2\,S_T}{5 + 2\,S_T + \sqrt{1 + 2\,S_T}}\right) \quad . \tag{B21}$$

The solution is then

$$X = \sqrt{3}\left(\frac{1 - \sqrt{1 + 2\,S_T} + 2\,S_T}{5 + 2\,S_T + \sqrt{1 + 2\,S_T}}\right)^{1/2} 2\sqrt{\alpha_\ell t} \quad . \tag{B22}$$

The solution is in the same form as the exact solution for the Neumann problem (see Section 2.1). The Stefan solution (Section 2.1.3) can also be put into this form:

$$X = 1/2\,\sqrt{2S_T} \quad 2\sqrt{\alpha_\ell t} \quad . \tag{B23}$$

For $T_f - T_s = 5°C$ and using the properties of ice, $c_\ell = 0.465$ cal/g-°C, $\ell = 79.71$ cal/g, then $S_T = 0.0292$

Solution	Exact	Quasi-Static	Integral Method
γ	0.124	0.1208	0.1207

The integral solution is virtually identical to the much simpler quasi-steady solution and both are -2.7% in error. Of course, the integral solution is not limited to small Stefan numbers.

This example illustrates a problem with the integral method for a quadratic temperature approximation. By using different combinations of the equations, different solutions can be obtained. Equation B22 was ob-

tained by using eq B13c and B18.

When eq B13c and B17 are used, then

$$X = \left(\frac{\sqrt{1 + 2S_T} - 1}{2}\right)^{1/2} 2\sqrt{\alpha_1 t} \quad . \qquad \text{(B24)}$$

This solution is a generalization of the quasi-steady solution, with a quadratic temperature in place of the linear temperature profile.
If eq B17 and B18 are used, then

$$X = \sqrt{6} \left(\frac{1 + S_T - \sqrt{1 + 2\,S_T}}{2S_T - 1 + \sqrt{1 + 2\,S_T}}\right)^{1/2} \quad . \qquad \text{(B25)}$$

Let us compare these Equations for $S_T = 0.9205$:

Solution	Exact	Eq B23	Eq B22	Eq B24	Eq 25
Value of γ	0.600	0.678	0.638	0.586	0.747

Apparently, eq B24 is the most accurate, with eq B22 close behind. This quandary can be eliminated if a cubic temperature profile is used, but the computational work is greatly increased; refer to Goodman (1958) for further discussion.

EXTENSION OF HEAT BALANCE INTEGRAL METHOD

The problem of the proper approximation to use for the assumed temperature profile can be eased with a refinement to the heat balance integral method suggested by Noble (1975) and carried out by Bell (1978). Instead of following only the phase change and initial temperature penetration depths, any number of isotherms can be followed by writing the heat balance integral for an arbitrary number of single phase Neumann problem shown in Figure B4. The frozen region will be divided into two parts where

$$T = T_a = (T_f + T_s)/2 \text{ at } x = X_1 .$$

$$T = T_f \text{ at } x = X.$$

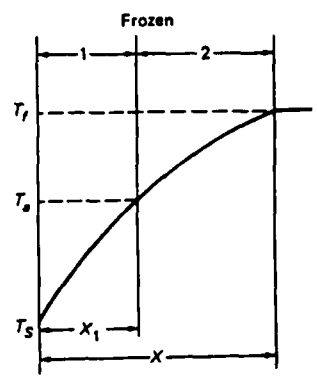

Figure B4. Two-zone frozen region.

The accuracy increases with an increasing number of subdivisions, but the computational work also increases. The heat balance integral for region 1 is

$$\alpha\left[\frac{\partial T_1(X_1,t)}{\partial x} - \frac{\partial T_1(0,t)}{\partial x}\right] = \frac{d}{dt}\int_0^{X_1} T_1 dx - T_a \frac{dX_1}{dt} \tag{B26}$$

while that for region 2 is

$$\alpha\left[\frac{\rho \ell}{k}\frac{dX}{dt} - \frac{\partial T_2(X_1,t)}{\partial x}\right] = \frac{d}{dt}\int_{X_1}^{X} T_2\, dx - T_f \frac{dX}{dt} + T_a \frac{dX_1}{dt} \quad . \tag{B27}$$

Quadratic temperature approximations are used in each region with

$$T_1 = ax^2 + bx + c \tag{B28}$$

$$T_2 = dx^2 + ex + f \ . \tag{B29}$$

The conditions to be satisfied by T_1 are

$$T_1(X_1,t) = T_a \tag{B30a}$$

$$T_1(0,t) = T_s \ . \tag{B30b}$$

and the energy flux continuity relation

$$\frac{\partial T_1(X_1,t)}{\partial x} = \frac{\partial T_2(X_1,t)}{\partial x} \quad . \tag{B30c}$$

The conditions to be satisfied by T_2 are

$$T_2(X,t) = T_f \tag{B30d}$$

$$T_2(X_1,t) = T_a \tag{B30e}$$

$$\left(\frac{\partial T_2(X)}{\partial x}\right)^2 = -\frac{\ell\ \partial^2 T_2(X)}{c\ \partial x^2} \quad . \tag{30f}$$

The equations are solved simultaneously for the six unknowns in eq B28 and B29. With these relations eq B26 and B27 can then be solved for X_1 and X.

Bell (1978) showed that the error can be reduced from 6.5% with one region to 1.2% for two subdivisions. Bell (1979, 1982) also noted that

calculations suggested that the approximate solution asymptotically approached the exact solution as the number of intervals increased. The refinement described here can reduce the errors on the heat balance integral method, but it also tends to negate the simplicity of the method. For more than two subdivisions or for more complicated boundary conditions it is likely that numerical solution of the set of simultaneous differential equations will be required.

Bell and Abbas (1985) used a subdivision of the penetration depth and solved for the temperatures at each subdivision. They proved that the solution converges to the exact value, at least for a simple problem (without phase change).

COLLOCATION METHOD

The usual heat balance integral equations for two-phase problems, eq B9 and B10, are coupled and the solution can be difficult. A slight variation of the heat balance integral method can be used to find an explicit functional relation between δ and X (following Lunardini 1983). This will uncouple the equations and simplify the solution.

The energy balance at the phase change interface is

$$k_1 \frac{\partial T_1(X,t)}{\partial x} - k_2 \frac{\partial T_2(X,t)}{\partial x} = -\rho_1 \ell \frac{dX}{dt} \quad . \tag{B31}$$

If eq B9, B10 and B31 are added together the result will be the overall energy balance for the entire volume of interest:

$$\frac{d}{dt} [\rho_1 c_1 \theta_1 + \rho_2 c_2 \theta_2 + \rho_1 \ell X + (\rho_2 c_2 - \rho_1 c_1) T_f X$$

$$- \rho_2 c_2 T_o (X + \delta)] = -k_1 \frac{\partial T_1(0,t)}{dx} \quad . \tag{B32}$$

The term $(\rho_2 c_2 - \rho_1 c_1)\, T_f dX/dt$, in eq B32, is the net sensible flux of enthalpy at the phase change interface resulting from the sudden jump in the specific heats of the frozen and thawed volumes. This term was omitted in a study by Yuen (1980), since Yuen's derivations implicitly assumed that $\rho_2 c_2 = \rho_1 c_1$ at the phase change interface. The retention of the sensible enthalpy term gives better numerical comparisons to exact solutions.

Equation B31 can be rewritten as two collocation equations (see Lunardini 1981):

$$-k_1 \frac{\partial T_1(X,t)}{\partial x} + k_2 \frac{\partial T_2(X,t)}{\partial x} = -\rho_1 \ell \alpha_1 \frac{\partial^2 T_1(X,t)}{\partial x^2} \Big/ \frac{\partial T_1(X,t)}{\partial x} \qquad \text{(B33)}$$

$$-k_1 \frac{\partial T_1(X,t)}{\partial x} + k_2 \frac{\partial T_2(X,t)}{\partial x} = -\rho_2 \ell \alpha_2 \frac{\partial^2 T_2(X,t)}{\partial x^2} \Big/ \frac{\partial T_2(X,t)}{\partial x} \quad . \qquad \text{(B34)}$$

For semi-infinite solids the following temperature approximations can be used:

$$T_1 = T_f + a_1(x - X) + a_2(x - X)^2 \qquad \text{(B35)}$$

$$T_2 = T_f - 2 \frac{(T_f - T_o)}{\delta} (x - X) + \frac{(T_f - T_o)}{\delta^2} (x - X)^2 \quad . \qquad \text{(B36)}$$

Equation B35, the temperature in the region which has changed phase, contains two unknown coefficients. One of these can be found from the specified boundary condition at $x = 0$. Combining eq B33-B36 yields

$$\frac{a_2}{a_1} = - \frac{\alpha_{21}}{2\delta} \quad . \qquad \text{(B37)}$$

Equation B34 can be used, with eq B35 and B36 to find an explicit relation between δ and X.

NOMENCLATURE - APPENDIX B

c	specific heat
k	thermal conductivity
ℓ	latent heat of fusion
Q	specified surface heat flux
r	radial coordinate
r_o	radius of sphere or cylinder
R	phase change radius, cylindrical or spherical coordinates
S_T	$\frac{c_1}{\ell} (T_f - T_s)$
t	time

t_m	time for surface temperature to reach T_f
T	temperature
T_a	$\frac{T_f + T_s}{2}$
T_f, T_o, T_s	fusion, initial, surface temperatures
x	Cartesian coordinate
X	phase change depth
α	thermal diffusivity
δ	thermal disturbance depth or radius
δ_m	value of δ when surface temperature reaches T_f
ρ	density
θ	integrated temperature

Subscripts

1,2	different phases of material

APPENDIX C. BIOT'S VARIATIONAL PRINCIPLE

We will consider the restricted principle for conduction heat transfer only. Consider a solid with isotropic thermal properties is shown in Figure C1.

The conservation of energy for the solid may be written in the familiar form

$$\frac{\partial C\theta}{\partial t} = -\operatorname{div} \dot{\underline{H}} \qquad \text{(C1)}$$

where C = heat capacity per unit volume
θ = scalar temperature field, defined as $T - T_i$
$\dot{\underline{H}}$ = vector field representing the rate of the local flux of energy
t = time.

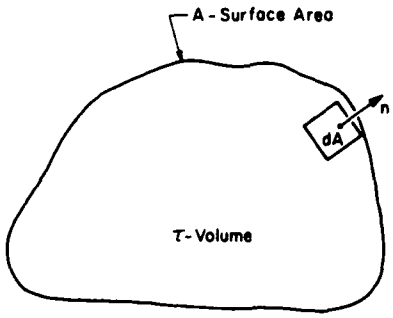

Figure C1. Solid with isotropic properties.

Fourier's law of conduction is

$$\dot{\underline{H}} = -k \operatorname{grad} \theta \qquad \text{(C2)}$$

where k is the thermal conductivity (in general a tensor quantity but usually reduces to a scalar). Combining eq C1 and C2 yields the well-known energy equation

$$\operatorname{div} (k \operatorname{grad} \theta) = \frac{\partial C\theta}{\partial t} \quad . \qquad \text{(C3)}$$

Equation C1 is integrated over time as follows:

$$\int \frac{\partial C\theta}{\partial t} dt = -\int \operatorname{div} \dot{\underline{H}} \, dt = -\operatorname{div} \int \dot{\underline{H}} \, dt \ .$$

Thus

$$C\theta = -\operatorname{div} \underline{H} \qquad \text{(C4)}$$

where $\underline{H} = \int \dot{\underline{H}} \, dt = \int \frac{\partial \underline{H}}{\partial t} dt$ is a vector field called the heat displacement.

Clearly eq C4 is an alternative form of the energy conservation equation. Equations C2 and C4 make up the fundamental relations for the variational principle. The advantage of the two separate equations, rather than the usual single eq C3, is that it is possible to satisfy approximately the heat conduction law, eq C2, while maintaining exact energy conservation.

The temperature and the heat displacement are analogous to the force (stress) and the displacement (strain) of classical mechanics.

The variational principle is as follows. A variation of the heat displacement of $\delta\underline{H}$ associated with a variation of the temperature, $\delta\theta$, is

$$C\delta\theta = -\operatorname{div}(\delta\underline{H}) \tag{C5}$$

Equation C2 is multiplied by $\delta\underline{H}$ and integrated over the volume τ:

$$\int_\tau (\operatorname{grad}\theta + \frac{1}{k}\dot{\underline{H}})\,\delta\underline{H}\,d\tau = 0\ . \tag{C6}$$

Using the formula for the divergence of the product of a scalar and a vector,

$$\int_\tau \operatorname{grad}\theta \cdot \delta\underline{H}\,d\tau = \int_\tau \operatorname{div}(\theta\delta\underline{H})\,d\tau - \int_\tau \theta \operatorname{div}\delta\underline{H}\,d\tau\ .$$

From the divergence theorem,

$$\int_\tau (\operatorname{grad}\theta \cdot \delta\underline{H}\,d\tau = \int_A \theta\delta\underline{H}\cdot\underline{n}\,dA - \int_\tau \operatorname{div}\delta\underline{H}\,d\tau\ .$$

Equation C6 is then

$$\int_\tau (-\theta \operatorname{div}\delta\underline{H} + \frac{1}{k}\dot{\underline{H}}\cdot\delta\underline{H})\,d\tau = \int_A \theta\delta\underline{H}\cdot\underline{n}\,dA$$

where $\underline{n}$ is the outward normal from the boundary surface A.
From eq C5

$$-\int_\tau \theta \operatorname{div}\delta\underline{H}\,d\tau = \int_\tau C\theta\delta\theta d\tau = \delta\int_\tau \frac{C\theta^2}{2}\,d\tau\ .$$

A thermal potential is defined as

$$V = 1/2\int_\tau C\theta^2\,d\tau\ . \tag{C7}$$

Then

$$\delta V + \int_\tau \frac{1}{k}\dot{\underline{H}}\cdot\delta\underline{H}\,d\tau = -\int_A \theta\,\underline{n}\cdot\delta\underline{H}\,dA\ . \tag{C8}$$

The variation principle, eq C8, is a statement of Fourier's conduction law with energy conservation satisfied automatically as a constraint.

The derivation is also valid for time-dependent thermal conductivity and moving boundaries (Biot 1970).

The heat displacement field may be written as

$$\underline{H} = \underline{H}\ (\varepsilon_1,\ \varepsilon_2\ .\ .\ .\ \varepsilon_n,\ x,\ y,\ z,\ t)\ . \tag{C9}$$

The parameters ε_i are unknown functions of time and are regarded as generalized coordinates representing the field $\underline{H}$. The variations in the field quantities can thus be related to the arbitrary variations in the generalized coordinates, $\delta\varepsilon_i$,

$$\delta\underline{H} = \sum_i \frac{\partial \underline{H}}{\partial \varepsilon_i}\ \delta\varepsilon_i \tag{C10}$$

$$\delta V = \sum_i \frac{\partial V}{\partial \varepsilon_i}\ \delta\varepsilon_i\ . \tag{C11}$$

Equation C8 can now be written as

$$\sum_i \left\{ \int \frac{\partial V}{\partial \varepsilon_i} + \int_\tau \frac{1}{k}\ \dot{\underline{H}} \cdot \frac{\partial \underline{H}}{\partial \varepsilon_i}\ d\tau + \int_A \theta\ \underline{n} \cdot \frac{\partial \underline{H}}{\partial \varepsilon_i}\ dA \right\} \delta\varepsilon_i = 0\ . \tag{C12}$$

Since $\delta\varepsilon_i$ is arbitrary this leads to

$$\frac{\partial V}{\partial \varepsilon_i} + \int_\tau \frac{1}{k}\ \dot{\underline{H}} \cdot \frac{\partial \underline{H}}{\partial \varepsilon_i}\ d\tau = -\int_A \theta\ \underline{n} \cdot \frac{\partial \underline{H}}{\partial \varepsilon_i}\ dA\ . \tag{C13}$$

The total change of $\dot{\underline{H}}$ with respect to time can be written as

$$\dot{\underline{H}} = \frac{d\underline{H}}{dt} = \sum_i \frac{\partial \underline{H}}{\partial \varepsilon_i}\ \frac{\partial \varepsilon_i}{dt} + \frac{\partial \underline{H}}{dt}\ . \tag{C14}$$

Since the generalized coordinates are functions of time,

$$\dot{\underline{H}} = \sum_i \frac{\partial \underline{H}}{\partial \varepsilon_i}\ \varepsilon_i + \frac{\partial \underline{H}}{\partial t} \tag{C15}$$

It follows directly, from eq C14, that

$$\frac{\partial \dot{\underline{H}}}{\partial \dot{\varepsilon}_i} = \frac{\partial \underline{H}}{\partial \varepsilon_i}\ . \tag{C16}$$

Let a dissipation function be defined by

$$\frac{\partial D}{\partial \dot{\varepsilon}_i} = \int_\tau \frac{1}{k} \dot{\underline{H}} \cdot \frac{\partial \underline{H}}{\partial \varepsilon_i} \, d\tau \quad . \tag{C17}$$

Using eq C16 this can be written as

$$\frac{\partial D}{\partial \dot{\varepsilon}_i} = \int_\tau \frac{1}{k} \dot{\underline{H}} \cdot \frac{\partial \dot{\underline{H}}}{\partial \dot{\varepsilon}_i} \, d\tau = \frac{\partial}{\partial \dot{\varepsilon}_i} 1/2 \int_\tau \frac{1}{k} (\dot{\underline{H}})^2 \, d\tau \quad .$$

Then

$$D = 1/2 \int_\tau \frac{1}{k} \dot{\underline{H}}^2 \, d\tau \quad . \tag{C18}$$

Finally eq C13 may be written as

$$\frac{\partial V}{\partial \varepsilon_i} + \frac{\partial D}{\partial \dot{\varepsilon}_i} = Q_i \tag{C19}$$

where

$$Q_i = -\int_A \theta \frac{\partial \underline{H}}{\partial \varepsilon_i} \, d\underline{A} \quad . \tag{C20}$$

From eq C15, if $\underline{H}$ is a function of only one generalized coordinate and not an explicit function of time,

$$\dot{\underline{H}} = \frac{\partial \underline{H}}{\partial \varepsilon} \dot{\varepsilon} \quad .$$

Then

$$\frac{\partial \underline{H}}{\partial \varepsilon} = \frac{\dot{\underline{H}}}{\dot{\varepsilon}} \quad . \tag{C21}$$

Equation C20 can then be written in the form

$$Q_i = -\int_A \theta \frac{\dot{\underline{H}}}{\dot{\varepsilon}_i} \, d\,\underline{A} \tag{C22}$$

where

$$d\underline{A} = \underline{n}\, dA \quad .$$

Equation C22 is often more convenient to use than eq C20.

If there are n generalized coordinates then eq C19 will lead to a system of n differential equations for the unknown q_i. It is clear that the derivation follows exactly that for the well known Lagrangran equations in classical mechanics. The physical interpretation is also analogous.

EXAMPLE

Consider a semi-infinite medium, initially at T_o, which has a surface heat flux q imposed upon it at $t \geq o$ (see Fig. C2, the same example as in App. B).

The heat transfer problem is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \qquad \text{(C23)}$$

$$k\,\frac{\partial\theta(o,t)}{\partial x} = -q \qquad \text{(C24)}$$

$$\theta\,(\delta,t) = 0 \qquad \text{(C25)}$$

$$\frac{\partial\theta}{\partial x}\,(\delta,t) = 0 \qquad \text{(C26)}$$

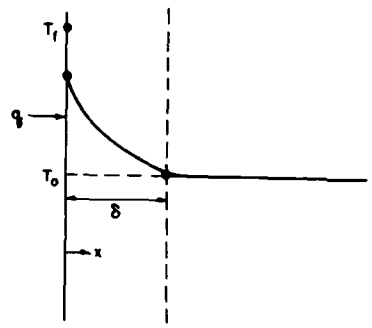

Figure C2. Surface heat flux imposed on semi-infinite medium.

where

$$\theta = T - T_o$$

The thermal penetration distance, δ, is chosen for the generalized coordinate. A quadratic temperature can be assumed as

$$\theta = \frac{q}{2k\delta}\,(x - \delta)^2 \quad . \qquad \text{(C27)}$$

From eq C4, using $\dot{H}(o) = q$,

$$H = \frac{-q}{6\alpha\delta}\,(x - \delta)^3 + q\,(t - \frac{1}{6}\,\frac{\delta^2}{\alpha}) \qquad \text{(C28)}$$

$$\dot{H} = \frac{q}{6\alpha\delta^2}\,\dot{\delta}\,(x - \delta)^2\,(x + 2\delta) + q\,(1 - \frac{\delta\dot{\delta}}{3\alpha}) \quad . \qquad \text{(C29)}$$

For a one-dimensional problem, eq C20 can be written as

$$Q = -\int_A \theta(o,t) \left(\frac{\partial \underline{H}}{\partial \delta}\right)_{x=0} \underline{n}_o \, dA - \int_A \theta(\delta,t) \left(\frac{\partial \underline{H}}{\partial \delta}\right)_{x=\delta} \underline{n}_\delta \, dA \quad .$$

Note that $\underline{n}_o = -\underline{e}_x$ while $\underline{n}_\delta = +\underline{e}_x$. Then for a unit area of material

$$Q = \theta(o,t) \left(\frac{\partial H}{\partial \delta}\right)_o - \theta(\delta,t) \left(\frac{\partial H}{\partial \delta}\right)_\delta \quad .$$

Since $\theta(\delta,t) = o$,

$$Q = \theta(o,t) \left(\frac{\partial H}{\partial \delta}\right)_{x=o} = 0 \quad . \tag{C30}$$

From eq C7

$$V = \frac{C}{2} \int_o^\delta \frac{q^2}{4k^2} \frac{(x-\delta)^4}{\delta^2} dx \quad .$$

Then

$$V = \frac{Cq^2\delta^3}{40k^2} \quad . \tag{C31}$$

The dissipation function, eq C17, leads to

$$D = \frac{1}{2k} \left[\frac{33}{35} a^2 \delta^7 + \frac{3}{2} ab \, \delta^4 + b^2 \delta\right] \tag{C32}$$

where

$$a = \frac{q\dot{\delta}}{6\alpha\delta^2}$$

$$b = q \left(1 - \frac{\delta\dot{\delta}}{3\alpha}\right) \quad .$$

Then

$$\frac{\partial D}{\partial \dot{\delta}} = \frac{q^2\delta^2}{12\alpha k} \left[\frac{68}{35} \frac{\delta\dot{\delta}}{3\alpha} - \frac{5}{2}\right] \quad . \tag{C33}$$

Equation C18, the variational equation, may now be written as

$$\frac{q^2\delta^2}{12\alpha k}\left[\frac{68}{35}\frac{\delta\dot{\delta}}{3\alpha} - \frac{5}{2}\right] = \frac{3}{40}\frac{q^2\delta^2}{k\alpha} \quad . \tag{C34}$$

The solution is

$$\delta^2 = \frac{84}{17}\alpha t \quad . \tag{C35}$$

The time it takes for the surface temperature to reach T_f, or $\theta_o = T_f - T_o$ is

$$t_m = P(n)\frac{k^2\theta_o^2}{\alpha q^2} \tag{C36}$$

where P(2) = 0.8095. The exact solution for t_m (from Carslaw and Jaeger 1959) is P = 0.785. A heat balance integral solution for this problem, see Appendix B, yields P(2) = 0.667.

The solution accuracy can be improved by a higher order polynominal for the assumed temperature.

The following table notes the effect of the polynominal order on the solution accuracy.

n	P(n) Variational	Error (%)	Heat balance integral	Error (%)
2	0.8095	3.1	0.667	-15.0
4	0.8039	2.4	0.800	1.9
6	0.8442			
∞	1.0000			

Note that the heat balance integral is more accurate at higher n values. Since the H.B.I. method is somewhat easier than the variational method it may be preferable for semi-infinite systems. For finite systems, however, the variational method has a distinct advantage due to the nature of the generalized coordinates.

This example points out a characteristic of the variational method. The conduction law, eq C2, cannot be satisfied at both x = 0 and x = δ. Since the choice was made to satisfy eq C2 at x = 0, it follows that

$$\dot{H}(\delta) = q\left(1 - \frac{\delta\dot{\delta}}{32}\right) = 0.1765\ q \quad .$$

Ideally $\dot{H}(\delta) = 0$ since $\frac{\partial \theta}{\partial x}\Big|_{x=\delta} = 0$. For this problem it is preferable to satisfy eq C2 at x = 0 since the heat flux is applied there. It is possible to solve this problem using $\dot{H}(\delta) = 0$ but the accuracy is significantly lower. For a quadratic temperature the value of P(2) is in error by 27% but improves to 4% at n = 4.

NOMENCLATURE - APPENDIX C

A	surface area
C	heat capacity per unit volume
D	dissipation (thermal) function (Btu-°F)/hr
$\underline{e}_x$	unit vector in x direction
$\underline{H}$	heat displacement vector Btu/ft^2
$\dot{\underline{H}}$	$\frac{d\underline{H}}{dt}$, local heat flux rate
k	thermal conductivity
$\underline{n}$	outward unit normal vector
q	heat flux rate
Q_i	thermal force (Btu-°F)/ft
t	time
T	temperature
T_i	reference temperature
T_o	initial temperature
V	thermal potential Btu-°F
x,y,z	Cartesian coordinates
α	thermal diffusivity
δ	thermal penetration distance
ε_i	generalized coordinates
θ	$T - T_i$, temperature
τ	volume

APPENDIX D. ERROR FUNCTION AND ERROR INTEGRAL FAMILY

The error functon is defined as

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} \, d\beta \tag{D1}$$

Thus erf ∞ = 1 and erf (-x) = -erf x.

The complementary error function is

$$\text{erfc } x = 1 - \text{erf } x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\beta^2} \, d\beta \tag{D2}$$

$$\text{erfc } (-x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^0 e^{-\beta^2} \, d\beta = 1 + \text{erf } x = 2 - \text{erfc } x \quad . \tag{D3}$$

For small values of x the error function is approximated by

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \sum_{n=o}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n + 1) \, n!} = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} + \ldots\right) . \tag{D4}$$

For large values of x the following approximation is acceptable (Carslaw and Jaeger 1959):

$$\text{erf } x = 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \left(\frac{1}{x} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdots (2n + 1)}{2^{n+1} \, x^{2n+3}}\right) \tag{D5}$$

DERIVATIVES

The derivative of the error function is

$$\Phi_n (x) = \frac{d^n}{dx^n} \text{ erf } x \quad . \tag{D6}$$

Thus

$$\Phi_1 (x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$\Phi_2 (x) = - \frac{4}{\sqrt{\pi}} x \, e^{-x^2} \quad .$$

INTEGRALS

The integrals of the error function are often useful. Let

$$i^n \operatorname{erfc} x = \int_x^\infty i^{n-1} \operatorname{erfc} \xi \, d\xi \qquad n=1,2, \ldots \tag{D7}$$

with

$$i^o \operatorname{erfc} x = \operatorname{erfc} x \quad . \tag{D8}$$

Then

$$i^1 \operatorname{erfc} x = i\operatorname{erfc} x = \int_x^\infty \operatorname{erfc} x \, dx \tag{D9}$$

$$i^2 \operatorname{erfc} x = \int_x^\infty \left(\int_x^\infty \operatorname{erfc} x \, dx \right) dx$$

and so on. Also

$$i^n \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(t-x)^n}{n!} e^{-t^2} \, dt \quad . \tag{D10}$$

It can be shown using integration by parts that

$$i \operatorname{erfc} x = \frac{1}{\sqrt{\pi}} e^{-x^2} - x \operatorname{erfc} x$$

$$i^2 \operatorname{erfc} x = \frac{1}{4} (\operatorname{erfc} x - 2x \, i\operatorname{erfc} x)$$

$$i^3 \operatorname{erfc} x = \frac{1}{6\sqrt{\pi}} e^{-x^2} (1 + x^2) - \frac{x}{6} \left(\frac{3}{2} + x^2\right) \operatorname{erfc} x$$

and in general

$$2n \, i^n \operatorname{erfc} x = i^{n-2} \operatorname{erfc} x - 2 x \, i^{n-1} \operatorname{erfc} x. \tag{D11}$$

It also follows that

$$i^n \operatorname{erfc} o = \frac{1}{2^n \, \Gamma\left(\frac{n}{2} + 1\right)} \tag{D12}$$

where Γ is the gamma or fractional function defined by

$$\Gamma\ (p + 1) = \int_o^\infty t^p\ e^{-t}\ dt \qquad p \geq 0 \tag{D13}$$

$$\Gamma\ (n + 1) = n! \qquad \text{if } n \text{ is a positive integer.}$$

$$i\ \text{erfc}\ (-x) = 2x + i\text{erfc}\ x$$

$$i^2\ \text{erfc}\ (-x) = x^2 + \frac{1}{2} - i^2\ \text{erfc}\ x$$

$$i^n\ \text{erfc}\ (-x) + (-1)^n\ i^n\ \text{erfc}\ x = \sum_o^{[\frac{n}{2}]} \frac{x^{n-2m}}{2^{2m-1}\ m!\ (n-2m)!} \tag{D14}$$

$$i^{-m}\ \text{erfc}\ \xi = (-1)^{m+1}\ \Phi_m\ (\xi) \qquad m > 0 \tag{D15}$$

$$i^{-n}\ \text{erfc}\ x = (-1)^{n+1}\ \frac{2}{\sqrt{\pi}}\ \frac{d^{n-1}}{dx^{n-1}}\ (e^{-x^2}) \tag{D16}$$

$$i^{-1}\ \text{erfc}\ x = 2\ i\text{erfc}\ x + 2\ x\ \text{erfc}\ x = \frac{2}{\sqrt{\pi}}\ e^{-x^2}$$

$$i^{-2}\ \text{erfc}\ x = \frac{4x}{\sqrt{\pi}}\ e^{-x^2}$$

$$i^{-3}\ \text{erfc}\ x = \frac{4}{\sqrt{\pi}}\ (2\ x^2 - 1)\ e^{-x^2}$$

Also let

$$I^n\ \text{erf}\ x = \int_o^x I^{n-1}\ \text{erf}\ \xi\ d\xi \tag{D17}$$

$$I^o\ \text{erf}\ x = \text{erf}\ x\ .$$

Then

$$I^1\ \text{erf}\ x = \int_o^x \text{erf}\ \xi\ d\xi = x + i\ \text{erfc}\ x - i\ \text{erfc}\ o$$

$$= x + i\ \text{erfc}\ x - \frac{1}{\sqrt{\pi}}\ .$$

Table D1 The error function and its derivatives and integrals.

x	erf x	erfc x	$-\Phi_2(x)$ $4\pi^{-1/2}xe^{-x^2}$	$\Phi_1(x)$ $2\pi^{-1/2}e^{-x^2}$	$2\,i\mathrm{erfc}\,x$	$4\,i^2\mathrm{erfc}\,x$	$6\,i^3\mathrm{erfc}\,x$	$8\,i^4\mathrm{erfc}\,x$	i^1 erf x	i^2 erf x
0	0	1.0	0	1.1284	1.1284	1.0	0.5642	0.25	.00001	0.0
0.05	0.056372	0.943628	0.1126	1.1256	1.0312	0.8921	0.4933	0.2148	.00140	0.000016
0.1	0.112463	0.887537	0.2234	1.1172	0.9396	0.7936	0.4301	0.1841	.0056	0.00018
0.15	0.167996	0.832004	0.3310	1.1033	0.8537	0.7040	0.3740	0.1573	.0127	0.00060
0.2	0.222703	0.777297	0.4336	1.0841	0.7732	0.6227	0.3243	0.1341	.0224	0.0015
0.25	0.276326	0.723674	0.5300	1.0600	0.6982	0.5491	0.2805	0.1139	.0349	0.0029
0.3	0.328627	0.671373	0.6188	1.0313	0.6284	0.4828	0.2418	0.0965	.0500	0.0050
0.35	0.379382	0.620618	0.6988	0.9983	0.5639	0.4233	0.2079	0.0816	.0678	0.0080
0.45	0.475482	0.524518	0.8294	0.9215	0.4495	0.3223	0.1522	0.0577	.1106	0.0168
0.5	0.520500	0.479500	0.8788	0.8788	0.3993	0.2799	0.1297	0.0484	.1355	0.0229
0.55	0.563323	0.436677	0.9172	0.8338	0.3535	0.2423	0.1101	0.0404	.1626	.0304
0.6	0.603856	0.396144	0.9447	0.7872	0.3119	0.2090	0.0932	0.0336	.1918	.0392
0.65	0.642029	0.357971	0.9614	0.7395	0.2742	0.1798	0.0787	0.0279	.2229	.0496
0.7	0.677801	0.322199	0.9678	0.6913	0.2402	0.1541	0.0662	0.0231	.2559	.0615
0.75	0.711156	0.288844	0.9644	0.6429	0.2097	0.1316	0.0555	0.0190	.2907	.0752
0.8	0.742101	0.257899	0.9520	0.5950	0.1823	0.1120	0.0464	0.0156	.3270	.0906
0.85	0.770668	0.229332	0.9314	0.5479	0.1580	0.0950	0.0386	0.0128	.3648	.1079
0.9	0.796908	0.203092	0.9035	0.5020	0.1364	0.0803	0.0321	0.0104	.4040	.1272
0.95	0.820891	0.179109	0.8695	0.4576	0.1173	0.0677	0.0265	0.0085	.4445	.1483
1.0	0.842701	0.157299	0.8302	0.4151	0.1005	0.0568	0.0218	0.0069	.4861	.1716
1.1	0.880205	0.119795	0.7403	0.3365	0.0729	0.0396	0.0147	0.0045	.5723	.2245
1.2	0.910314	0.089686	0.6416	0.2673	0.0521	0.0272	0.0097	0.0029	.6619	.2862
1.3	0.934008	0.065992	0.5413	0.2082	0.0366	0.0184	0.0063	0.0019	.7541	.3570
1.4	0.952285	0.047715	0.4450	0.1589	0.0253	0.0122	0.0041	0.0011	.8485	.4371
1.5	0.966105	0.033895	0.3568	0.1189	0.0172	0.0080	0.0026	0.0007	.9444	.5267
1.6	0.976348	0.023652	0.2791	0.0872	0.0115	0.0052	0.0016	0.0004	1.0416	.6260
1.7	0.983790	0.016210	0.2132	0.0627	0.0076	0.0033	0.0010	0.0003	1.1396	.7351
1.8	0.989091	0.010909	0.1591	0.0442	0.0049	0.0021	0.0006	0.0002	1.2383	.8539
1.9	0.992790	0.007210	0.1160	0.0305	0.0031	0.0013	0.0003	0.0001	1.3374	.9827
2.0	0.995322	0.004678	0.0827	0.0207	0.0020	0.0008	0.0002	0.0001	1.4368	1.1214
2.1	0.997021	0.002979	0.0576	0.0137	0.0012	0.0005	0.0001		1.5364	1.2701
2.2	0.998137	0.001863	0.0393	0.0089	0.0007	0.0003			1.6362	1.4287
2.3	0.998857	0.001143	0.0262	0.0057	0.0004	0.0002			1.7360	1.5973
2.4	0.999311	0.000689	0.0171	0.0036	0.0002	0.0001			1.8359	1.7759
2.5	0.999593	0.000407	0.0109	0.0022	0.0001					
2.6	0.999764	0.000236	0.0068	0.0013	0.0001					
2.7	0.999866	0.000134	0.0042	0.0008						
2.8	0.999925	0.000075	0.0025	0.0004						
2.9	0.999959	0.000041	0.0015	0.0003						
3.0	0.999978	0.000022	0.0008	0.0001						

$$I^2 \text{ erf } x = \int_0^x \left(\int_0^x \text{erf } \xi \, d\xi\right) d\xi = \frac{x^2}{2} - x \text{ ierfc } 0 - i^2 \text{ erfc } x + i^2 \text{ erfc } 0$$

$$= \frac{x^2}{2} - \frac{x}{\sqrt{\pi}} + \frac{1}{4} - i^2 \text{ erfc } x \, .$$

The functions i^n erfc x are tabulated in Table D1.

The error function, y = erf η, satisfies the equation

$$\frac{d^2y}{d\eta^2} + 2\eta \frac{dy}{d\eta} = 0. \tag{D18}$$

The function $y = i^n$ erfc η is a solution to the differential equation

$$\frac{d^2y}{d\eta^2} + 2\eta \frac{dy}{dy} - 2\eta \, y = 0 \quad . \tag{D19}$$

The modified Hermite polynomials are

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n} \qquad n = 0,1,2, \ldots \tag{D20}$$

$$H_n(x) = \sum_{\gamma=0}^{[\frac{n}{2}]} \frac{(-1)^\gamma n! \, (2x)^{n-2\gamma}}{\gamma! \, \Gamma(n-2\gamma+1)}$$

$$E_n(x) = \sum_{r=0}^{[\frac{n}{2}]} \frac{n! \, (2x)^{n-2r}}{r! \, \Gamma(n-2r+1)}$$

$$H_n(ix) = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m n! \, (2ix)^{n-2m}}{m! \, (n-2m)!} = n! \, (2i)^n G_n(x) \tag{D21}$$

$$G_n(x) = \sum_{m=0}^{[\frac{n}{2}]} \frac{x^{n-2m}}{2^{2m} m! \, (n-2m)!} \tag{D22}$$

$$G_n(x) = \frac{1}{n!\sqrt{\pi}} \int_{-\infty}^{\infty} (x-\lambda)^n e^{-\lambda^2} d\lambda \tag{D23}$$

$$G_0(x) = 1 \qquad G_2 = \frac{1}{4}(1 + 2x^2)$$

$$G_1(x) = x \qquad G_3 = \frac{1}{12}(3x + 2x^3)$$

$$G_{2n}(x) = \sum_{m=0}^{n} \frac{x^{2n-2m}}{2^{2m}\, m!\, (2n-2m)!} \tag{D24}$$

$$G_{2n+1} = \sum_{m=0}^{n} \frac{x^{2n+1-2m}}{2^{m} m!\, (2n+1-2m)!} \tag{D25}$$

$$G_{2n}(0) = \frac{1}{2^{2n} n!}$$

$$G_{2n+1}(0) = 0$$

$$G_{2n}(x) = \frac{1}{2^{2n}} \sum_{m=0}^{n} \frac{(2x)^{2m}}{(n-m)!(2m)!} \tag{D26}$$

$$G_{2n+1}(x) = \frac{x}{2^{2n}} \sum_{m=o}^{n} \frac{(2x)^{2m}}{(n-m)!(2m+1)!} \tag{D27}$$

$$G_n(-x) = (-1)^n\, G_n(x)$$

$$E_n(-x) = E_n(x)$$

$$F_n(-x) = -F_n(x)$$

$$E_{2n} = G_{2n}$$

$$F_{2n+1} = G_{2n+1}$$

$$\frac{d^{2m}F_n}{d\,x^{2m}} = F_{n-2m} \qquad \frac{d^{2m}E_n}{d\,x^{2m}} = E_{n-2m}$$

$$\frac{d^{2m+1}E_n}{d\,x^{2m+1}} = \frac{d^{2m}F_{n-1}}{d\,x^{2m}} = F_{n-2m-1}$$

$$\frac{d^{2m+1}F_n}{d\,x^{2m+1}} = \frac{d^{2m}E_{n-1}}{d\,x^{2m}} = E_{n-2m-1}$$

$$E_{2n+1} = G_{2n+1} + i^{2n+1} \operatorname{erfc} x \tag{D28}$$

$$F_{2n} = G_{2n} - i^{2n} \operatorname{erfc} x$$

$$\frac{dE_n}{dx} = F_{n-1} \qquad \frac{dF_n}{dx} = E_{n-1}$$

$$F_n(x) = \frac{1}{2}\left[i^n \operatorname{erfc}(-x) - i^n \operatorname{erfc}(x)\right] \tag{D29}$$

$$F_n(x) = G_n(x) - \frac{1}{2}\left[1 + (-1)^n\right] i^n \operatorname{erfc} x \tag{D30}$$

$$F_o(x) = \operatorname{erf} x$$

$$F_1(x) = x$$

$$F_2(x) = \frac{1}{4}(2x^2 + 1) \operatorname{erf} x + \frac{x}{2\sqrt{\pi}} e^{-x^2}$$

$$F_{-1} = G_{-1} = \frac{dG_o}{dx} = 0$$

$$E_o = G_o = 1$$

$$E_{-1} = G_{-1} + i^{-1} \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$E_1 = G_1 + i\operatorname{erfc} x = x + i\operatorname{erfc} x \tag{D31}$$

$$E_{2n+1}(0) = \frac{1}{2^{2n+1}\,\Gamma\left(\frac{2n+1}{2} + 1\right)}$$

$$F_{2n}(0) = G_{2n+1}(0) = \phi_{2n}(0) = 0$$

$$F_n(0) = 0$$

$$(4\alpha t)^{\frac{n}{2}} G_n\left(\frac{x}{\sqrt{4\alpha t}}\right)\Big|_{t=0} = \frac{x^n}{n!} \tag{D32}$$

$$(4\alpha t)^{\frac{n}{2}} \, i^n \operatorname{erfc}(-\xi) \Big|_{t=0} = \frac{2x^n}{n!} \tag{D33}$$

$$(4\alpha t)^{\frac{n}{2}} \, F_n(\xi) \Big|_{t=0} = \frac{x^n}{n!} \tag{D34}$$

$$(4\alpha t)^{\frac{n}{2}} \, i^n \operatorname{erfc} \xi \Big|_{t=0} = 0 \tag{D35}$$

APPENDIX E. EXPONENTIAL INTEGRAL AND RELATED FUNCTIONS

The following definitions apply:

$$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t}\, dt \qquad (|\arg z| < \pi) \tag{E.1}$$

$$Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt = \int_{-\infty}^{x} \frac{e^{t}}{t}\, dt \qquad (x > 0) \tag{E.2}$$

$$li(x) = \int_0^x \frac{dt}{\ln t} = Ei(\ln x) \qquad (x > 1) \tag{E.3}$$

$$E_n(z) = \int_1^{\infty} \frac{e^{-zt}}{t^n}\, dt \qquad (n = 0,1,2 \ldots ;\ \text{Real } z > 0)\ . \tag{E.4}$$

In eq E1 it is assumed that the path of integration excludes the origin and does not cross the negative real axis.

Analytic continuation of the functions for $n > 0$ yields multi-valued functions with branch points at $z = 0$ and $z = \infty$. The function $li(z)$, the logarithmic integral, has an additional branch point at $z = 1$. They are single-valued functions in the z-plane cut along the negative real axis:

$$Ei(x) = \gamma_e + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{nn!}$$

$$E_1(x) = -Ei(-x) = -\gamma_e - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{nn!}\ .$$

These functions are plotted as Figures E1 and E2. A complete set of values for the exponential integrals is given in Table E1.

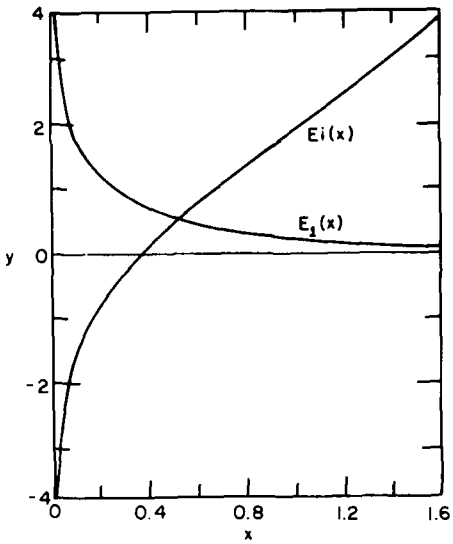

Figure E1. Graph of Ei(x) and $E_1(x)$.

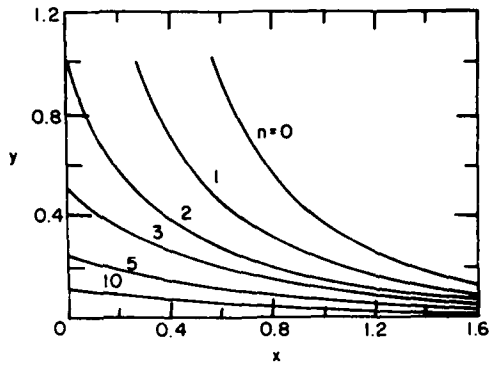

Figure E2. Graph of $E_n(x)$, n = 0,1,2,3,5,10.

Table E1. Exponential integral and related functions.

x	$x^{-1}[Ei(x)-\ln x-\gamma_e]$		$x^{-1}[E_1(x)+\ln x+\gamma_e]$	
0.00	1.00000	0000	1.00000	00000
0.05	1.01264	0202	0.98763	75971
0.10	1.02556	6141	0.97554	53033
0.15	1.03878	6018	0.96371	56702
0.20	1.05230	8298	0.95214	14833
0.25	1.06614	1726	0.94081	57528
0.30	1.08029	5334	0.92973	17075
0.35	1.09477	8451	0.91888	27858
0.40	1.10960	0714	0.90826	26297
0.45	1.12477	2082	0.89786	50778
0.50	1.14030	2841	0.88768	41584

Table E1. contd.

x	Ei(x)		$E_1(x)$	
0.50	0.45421	9905	0.55977	3595
0.55	0.61529	0657	0.50336	4081
0.60	0.76988	1290	0.45437	9503
0.65	0.91938	6468	0.41151	6976
0.70	1.06490	7195	0.37376	8843
0.75	1.20733	2816	0.34034	0813
0.80	1.34739	6548	0.31059	6579
0.85	1.48571	4176	0.28401	9269
0.90	1.62281	1714	0.26018	3939
0.95	1.75914	5612	0.23873	7524
1.00	1.89511	7816	0.21938	3934
1.05	2.03108	7184	0.20187	2813
1.10	2.16737	8280	0.18599	0905
1.15	2.30428	8252	0.17155	5354
1.20	2.44209	2285	0.15840	8437
1.25	2.58104	7974	0.14641	3373
1.30	2.72139	8880	0.13545	0958
1.35	2.86337	7453	0.12541	6844
1.40	3.00720	7464	0.11621	9313
1.45	3.15310	6049	0.10777	7440
1.50	3.30128	5449	0.10001	9582
1.55	3.45195	4503	0.09288	2108
1.60	3.60531	9949	0.08630	8334
1.65	3.76158	7569	0.08024	7627
1.70	3.92096	3201	0.07465	4644
1.75	4.08365	3659	0.06948	8685
1.80	4.24986	7557	0.06471	3129
1.85	4.41981	6080	0.06029	4967
1.90	4.59371	3687	0.05620	4378
1.95	4.77177	8785	0.05241	4380
2.00	4.95423	4356	0.04890	0511

x	$xe^{-x}Ei(x)$		$xe^{x}E_1(x)$	
2.0	1.34096	5420	0.72265	7234
2.1	1.37148	6802	0.73079	1502
2.2	1.39742	1992	0.73843	1132
2.3	1.41917	1534	0.74562	2149
2.4	1.43711	8315	0.75240	4829
2.5	1.45162	5159	0.75881	4592
2.6	1.46303	3397	0.76488	2722
2.7	1.47166	2153	0.77063	6987
2.8	1.47780	8187	0.77610	2123
2.9	1.48174	6162	0.78130	0252

Table E1. contd.

x	$xe^{-x}Ei(x)$		$xe^{x}E_1(x)$	
3.0	1.48372	9204	0.78625	1221
3.1	1.48398	9691	0.79097	2900
3.2	1.48274	0191	0.79548	1422
3.3	1.48017	4491	0.79979	1408
3.4	1.47646	8706	0.80391	6127
3.5	1.47178	2389	0.80786	7661
3.6	1.46625	9659	0.81165	7037
3.7	1.46003	0313	0.81529	4342
3.8	1.45321	0902	0.81878	8821
3.9	1.44590	5765	0.82214	8967
4.0	1.43820	8032	0.82538	2600
4.1	1.43020	0557	0.82849	6926
4.2	1.42195	6813	0.83149	8602
4.3	1.41354	1719	0.83439	3794
4.4	1.40501	2424	0.83718	8207
4.5	1.39641	9303	0.83988	7144
4.6	1.38780	5263	0.84249	5539
4.7	1.37920	9093	0.84501	7971
4.8	1.37066	3313	0.84745	8721
4.9	1.36219	6054	0.84982	1778
5.0	1.35383	1278	0.85211	0880
5.1	1.34558	9212	0.85432	9519
5.2	1.33748	6755	0.85648	0958
5.3	1.32953	7845	0.85856	8275
5.4	1.32175	3788	0.86059	4348
5.5	1.31414	3566	0.86256	1885
5.6	1.30671	4107	0.86447	3436
5.7	1.29947	0536	0.86633	1399
5.8	1.29241	6395	0.86813	8040
5.9	1.28555	3849	0.86989	5494
6.0	1.27888	3860	0.87160	5775
6.1	1.27240	6357	0.87327	0793
6.2	1.26612	0373	0.87489	2347
6.3	1.26002	4184	0.87647	2150
6.4	1.25411	5417	0.87801	1816
6.5	1.24839	1155	0.87951	2881
6.6	1.24284	8032	0.88097	6797
6.7	1.23748	2309	0.88240	4955
6.8	1.23228	9952	0.88379	8662
6.9	1.22726	6684	0.88515	9176
7.0	1.22240	8053	0.88648	7675
7.1	1.21770	9472	0.88778	5294
7.2	1.21316	6264	0.88905	3119
7.3	1.20877	3699	0.89029	2173
7.4	1.20452	7026	0.89150	3440
7.5	1.20042	1500	0.89268	7854
7.6	1.19645	2401	0.89384	6312
7.7	1.19261	5063	0.89497	9666
7.8	1.18890	4881	0.89608	8737
7.9	1.18531	7334	0.89717	4302
8.0	1.18184	7987	0.89823	7113

Table E1. contd.

x	$xe^{-x}Ei(x)$		$xe^{x}E_1(x)$	
8.1	1.17849	2509	0.89927	7888
8.2	1.17524	6676	0.90029	7306
8.3	1.17210	6376	0.90129	6033
8.4	1.16906	7617	0.90227	4695
8.5	1.16612	6526	0.90323	3900
8.6	1.16327	9354	0.90417	4228
8.7	1.16052	2476	0.90509	6235
8.8	1.15785	2390	0.90600	0459
8.9	1.15526	5719	0.90688	7415
9.0	1.15275	9209	0.90775	7602
9.1	1.15032	9724	0.90861	1483
9.2	1.14797	4251	0.90944	9530
9.3	1.14568	9889	0.91027	2177
9.4	1.14347	3855	0.91107	9850
9.5	1.14132	3476	0.91187	2958
9.6	1.13923	6185	0.91265	1897
9.7	1.13720	9523	0.91341	7043
9.8	1.13524	1130	0.91416	8766
9.9	1.13332	8746	0.91490	7418
10.0	1.13147	0205	0.91563	3339

x^{-1}	xe^{-x} Ei(x)		xe^{x} $E_1(x)$		<x>
0.100	1.13147	021	0.91563	33394	10
0.095	1.12249	671	0.91925	68286	11
0.090	1.11389	377	0.92293	15844	11
0.085	1.10564	739	0.92665	90998	12
0.080	1.09773	775	0.93044	09399	13
0.075	1.09014	087	0.93427	87466	13
0.070	1.08283	054	0.93817	42450	14
0.065	1.07578	038	0.94212	92486	15
0.060	1.06896	548	0.94614	56670	17
0.055	1.06236	365	0.95022	55126	18
0.050	1.05595	591	0.95437	09099	20
0.045	1.04972	640	0.95858	41038	22
0.040	1.04366	194	0.96286	74711	25
0.035	1.03775	135	0.96722	35311	29
0.030	1.03198	503	0.97165	49596	33
0.025	1.02635	451	0.97616	46031	40
0.020	1.02085	228	0.98075	54965	50
0.015	1.01547	157	0.98543	08813	67
0.010	1.01020	625	0.99019	42287	100
0.005	1.00505	077	0.99504	92646	200
0.000	1.00000	000	1.00000	00000	∞