COMBINED REFLECTION AND DIFFRACTION
BY A VERTICAL WEDGE

by

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This report presents an analytical solution for the combined wave reflection and diffraction by a vertical wedge, of arbitrary wedge angle, subject to the excitation of a plane simple harmonic wave train coming from a far field. Results of two special cases are calculated: one a thin semi-infinite breakwater and the other a wedge of 90-deg angle. The results are presented in amplification factor diagrams. A subroutine WEDGE is also documented in the appendix. The subroutine can be used to calculate the case of arbitrary wedge angle.
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1. The boundary value problem of linear wave reflection and diffraction by a vertical wedge of arbitrary wedge angle has been well formulated and presented by Stoker (1957) among many other investigators. The technique to obtain an analytical solution for the problem is also depicted in the cited book. However, analytical solutions are not available for the problem, except for the special case of wave diffraction by a thin semi-infinite breakwater, that is, a wedge with wedge angle equal to zero.

2. The solution of the thin semi-infinite breakwater was presented in the dimensionless diffraction diagrams by Wiegel (1962). The diagrams have been especially useful in preliminary engineering design and have been included in the Shore Protection Manual (SPM) (1984). Although equally useful, the combined reflection and diffraction diagrams are not available, perhaps because of the complexity of the diagrams which makes them difficult to create without using modern high-speed computers for computation and graphing.

3. The objectives of the present study are (a) to obtain an analytical solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle subject to excitation of a plane simple harmonic wave train coming from infinity and (b) to provide the combined reflection and diffraction diagrams. The diagrams included in this report have two cases: one for a thin semi-infinite breakwater and the other for a 90-deg vertical wedge. Subroutine WEDGE for computing the combined reflection and diffraction by a vertical wedge of arbitrary wedge angle is also documented in the report (Appendix A).
PART II: BOUNDARY VALUE PROBLEM

Mathematical Formulation

4. In this study our primary interest is the wave reflection and diffraction by a vertical wedge of arbitrary wedge angle in a constant water depth $h$ subject to the excitation of monochromatic incident waves of infinitesimal amplitude coming from infinity. Let $(r, \theta, z)$ be cylindrical coordinates, with $z = 0$ representing the undisturbed water free surface and upward direction representing the positive $z$-axis. The tip of the wedge is chosen to be the origin of the coordinates and two rigid walls of the wedge to coincide with $\theta = 0$ and $\theta = \theta_0$, respectively, as illustrated in Figure 1. Cartesian coordinates $(x, y, z)$, corresponding to the cylindrical coordinates, are also occasionally used and shown in the same figure. Therefore, the wedge

* For convenience, symbols and abbreviations are listed in the notation (Appendix B).
angle is $2\pi - \theta_0$, and the water region is defined by $\theta_0 \geq \theta \geq 0$ and $0 \geq z \geq -h$.

5. The velocity field for the wave reflection and diffraction in an ideal fluid can be represented by the velocity potential function $\phi(r, \theta, z, t)$ which must satisfy the Laplace equation, where $t$ is the temporal coordinate. We assume that the waves are sinusoidal in time with radian frequency $\omega$. Water depth is constant, and the bottom is rigid and impermeable. Therefore, the vertical and temporal components of the velocity potential function, which follow from separation of variables, can be factored out and the velocity potential written as

$$\phi(r, \theta, z, t) = A_o \frac{\cosh k(z + h)}{\cosh kh} \phi(r, \theta) e^{i\omega t} \tag{1}$$

where

- $A_o = -iga_o/\omega$
- $i = \sqrt{-1}$
- $g$ = gravitational acceleration
- $a_o$ = incident wave amplitude
- $k$ = wave number
- $\phi$ = horizontal component of the velocity potential function

6. Substituting Equation 1 into the Laplace equation and using both the kinematic and dynamic boundary conditions at the free surface, the Laplace equation is then reduced to the Helmholtz equation which is written in polar coordinates as follows:

$$r^2 \frac{\partial^2 \phi}{\partial r^2} + r \frac{\partial \phi}{\partial r} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi}{\partial \theta^2} + k^2 r^2 \phi = 0 \tag{2}$$

where $k$ must be a real number and satisfy the dispersion relationship

$$\omega^2 = gk \tanh kh \tag{3}$$

7. The free surface displacement $\eta$ from the mean water level $z = 0$ can be obtained from linear wave theory and is represented as

$$\eta(r, \theta, t) = \frac{1}{g} \frac{\partial \phi}{\partial t} = a_o \phi(r, \theta) e^{i\omega t} \tag{4}$$
8. Thus only the horizontal part of the velocity potential function $\phi$ is needed to be determined as a solution of Equation 2 in the water region $\theta_o \geq \theta \geq 0$, with the following boundary conditions at the rigid and impermeable walls of the wedge:

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta_o \quad (5)$$

9. A condition at infinity is also required to ensure a unique solution. The classic approach is to use the Sommerfeld radiation condition at infinity which states that the scattered wave $\phi_s$ must behave like a cylindrical outgoing progressing wave at infinity such that

$$i r \sqrt{\frac{\partial \phi_s}{\partial r} + i k \phi_s} = 0 \quad (6)$$

The total wave represented by $\phi$ is the linear superposition of an incident wave $\phi_i$, a reflected wave from the the $\theta = 0$ wall of the wedge $\phi_r$, and the scattered wave $\phi_s$ from the tip of the wedge.

$$\phi = \phi_i + \phi_r + \phi_s \quad (7)$$

Equation 6 can be satisfied if

$$\phi_s \sim \frac{e^{-ikr}}{\sqrt{kr}} \quad \text{at} \quad r \rightarrow \infty \quad (8)$$

10. The incident wave coming from a large distance from the tip of the wedge is assumed to be a plane progressive wave of amplitude $a_o$ and incident angle $\alpha$ to the x-axis as given by

$$\phi_i = e^{ikr \cos(\theta - \alpha)} \quad (9)$$

Consequently, the perfectly reflected wave from the $y = 0$ wall of the wedge is

$$\phi_r = e^{ikr \cos(\theta + \alpha)} \quad (10)$$
Thus the boundary value problem (in which the governing equation is Equation 2, the boundary condition is Equation 5, and the radiation condition is Equation 6) is completely formulated.

**Analytical Solution**

11. Analytical solution to the problem formulated in the preceding section is obtained by following the solution technique by Stoker (1957). To obtain the solution, the water region is divided into three subregions—I, II, and III—by the incident wave ray passing through the tip of the wedge and the reflected wave ray reflected away from the tip of the wedge, as shown in Figure 2. Obviously, the total wave in subregion I is the sum of the incident, reflected, and scattered waves; the total wave in subregion II, where the reflected wave does not exist, is the sum of the incident and scattered waves; and the total wave in subregion III, where the incident and reflected waves have been shaded out, is only the scattered wave. For certain combinations of

![Figure 2. Three subregions and the wedge](image_url)
the wedge angle and incident wave angle, subregions II and III may not exist at all. In general, the solution function can be written as

$$\phi = \phi_o(r, \theta) + \phi_s(r, \theta)$$  \hspace{1cm} (11)

where

$$\phi_o(r, \theta) = \begin{cases} 
\phi_i + \phi_r & \pi - \alpha > \theta > 0 \\
\phi_i & \pi + \alpha > \theta > \pi - \alpha \\
0 & \theta_o > \theta > \pi + \alpha 
\end{cases}$$  \hspace{1cm} (12)

The equation reveals that $\phi_o$ is the sum of the incident and reflected waves $\phi_i$ and $\phi_r$ and is a known function. The scattered wave $\phi_s$ is the only unknown function to be determined in the problem. Nevertheless, the total wave $\phi$ instead of the scattered wave $\phi_s$ is the desired solution to be obtained in this study.

12. The solution for the total wave $\phi$ is pursued. The finite cosine transform of $\phi$, denoted by $\overline{\phi}$, is introduced by the formula

$$\overline{\phi}(kr, n) = \int_0^\pi \phi(kr, \theta) \cos \frac{n\theta}{\nu} \, d\theta$$  \hspace{1cm} (13)

where $n = 0, 1, 2, \ldots$ are integers, and $\nu$ is related to the wedge angle as defined by

$$\theta_o = \frac{\nu}{\pi}$$  \hspace{1cm} (14)

Applying the finite cosine transform and using the boundary condition in Equation 5, Equation 2 becomes

$$r^2 \frac{\partial^2 \Phi}{\partial r^2} + r \frac{\partial \Phi}{\partial r} + \left[ (kr)^2 - \left( \frac{n}{\nu} \right)^2 \right] \Phi = 0$$  \hspace{1cm} (15)

Equation 15 is a form of the Bessel equation for which general solutions are the Bessel functions of the first and second kinds, $J_{n/\nu}(kr)$ and $Y_{n/\nu}(kr)$, respectively. Since $Y_{n/\nu}(kr)$ are singular at the origin, the solution is chosen to be
\[ \phi(kr, n) = a_n \frac{J_n}{n/(kr)} \]  

(16)

where \( a_n \) are constants to be determined.

13. Taking the finite cosine transform of Equation 11 and using Equation 16, we have

\[ \int_0^{\pi} \phi_s \cos \frac{n\theta}{\nu} \, d\theta = a_n \frac{J_n}{n/(kr)} - \int_0^{\pi} \phi_o \cos \frac{n\theta}{\nu} \, d\theta \]  

(17)

or

\[ \phi_s = a_n \frac{J_n}{n/(kr)} - \phi_o \]  

(18)

Then applying the operation \( \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial}{\partial r} + ik \right) \) to both sides of Equation 18, and using the Sommerfeld radiation condition (Equation 6) we have

\[ \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial}{\partial r} + ik \right) \left[ a_n \frac{J_n}{n/(kr)} - \int_0^{\pi} \phi_o \cos \frac{n\theta}{\nu} \, d\theta \right] = 0 \]  

(19)

14. Equation 19 can be asymptotically evaluated to determine \( a_n \).

Firstly, the first term involving the Bessel function is evaluated. The function \( J_{n/(kr)} \) at \( r \to \infty \) behaves asymptotically (Abramowitz and Stegun 1964) as follows:

\[ J_{n/(kr)} \sim \sqrt{\frac{2}{k \pi r}} \cos \left( kr - \frac{n\pi}{2\nu} - \frac{\pi}{4} \right) \]  

(20)

Hence, we have

\[ \lim_{r \to \infty} \sqrt{r} \left( \frac{\partial}{\partial r} + ik \right) J_{n/(kr)} \sim \sqrt{\frac{2k}{\pi}} e^{i(kr-n\pi/2\nu+\pi/4)} \]  

(21)
Secondly, the second term involving the integral of $\phi_0$ is evaluated. The asymptotic behavior of the integral over $\theta = (0, \pi)$ and at large distance $r \to \infty$ can be found by the method of stationary phase. The integral, after substituting $\phi_0$ from Equations 9, 10, and 12, can be written as

$$\int_0^{\pi} \phi_0 \cos \frac{n_0}{\nu} d\theta = \int_0^{\pi-\alpha} \left[ e^{-ikr \cos(\theta-\alpha)} + e^{ikr \cos(\theta+\alpha)} \right] \cos \frac{n_0}{\nu} d\theta$$

$$+ \int_{\pi-\alpha}^{\pi+\alpha} e^{i1kr \cos(\theta-\alpha)} \cos \frac{n_0}{\nu} d\theta$$

(22)

In the integrals, there are three points of stationary phase at $\theta = \alpha$ and $\theta = \pi \pm \alpha$. If the same argument as that of Stoker (1957) is followed, of the three contributions only the first one $\theta = \alpha$ furnishes a nonvanishing contribution for $r \to \infty$ when the operator $\sqrt{r}(\partial/\partial r + ik)$ is applied to it. The physical significance of this statement is that only the incident wave is effective in determining the cosine coefficients of the solution. Therefore,

$$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial}{\partial r} + ik \right) \int_0^{\pi} \phi_0 \cos \frac{n_0}{\nu} d\theta = 2\sqrt{2\pi k} \cos \frac{n\alpha}{\nu} e^{i(kr+\pi/4)}$$

(23)

Substituting Equations 21 and 23 into Equation 19, we obtain the unknown coefficients $a_n$:

$$a_n = 2\pi \cos \frac{n\alpha}{\nu} e^{in\pi/2\nu}$$

(24)

15. Since the solution $\phi$ in the cosine series expression is

$$\phi(r, \theta) = \frac{1}{\nu} \bar{\phi}(r, 0) + \frac{2}{\nu} \sum_{n=1}^{\infty} \bar{\phi}(r, n) \cos \frac{n_0}{\nu}$$

(25)
the solution is obtained by substituting Equations 16 and 24 into Equation 25 as follows:

\[
\phi(r, \theta) = \frac{2}{\nu} \left[ J_0(kr) + 2 \sum_{n=1}^{\infty} e^{in\pi/2\nu} J_{n/\nu}(kr) \cos \frac{n\alpha}{\nu} \cos \frac{n\theta}{\nu} \right]
\]  

(26)

Equation 26 is the solution for the combined wave reflection and diffraction by a vertical wedge of arbitrary wedge angle and is considered to be extended from the solution by Stoker (1957) who only solved the problem of a thin semi-infinite breakwater. The solution in Equation 26 and the one by Stoker are not only in nonclosed form but also in terms of Bessel functions. It seems that the calculations of the solutions are very difficult without using a modern high-speed computer. This is probably the reason why Stoker arrived at his solution expressed in the same cosine series but did not use it to calculate the result. Instead, he further transformed the expression into a very complex integral form for further approximation in calculating the result.

16. Notably, the solution at the origin point is obtained by simply substituting \( r = 0 \) into Equation 26 to arrive at

\[
\phi(0, \theta) = \frac{2}{\nu}
\]

(27)

Therefore, wave response at the origin point depends only on the wedge angle and does not depend on the incident wave angle.

Two Special Cases

17. The solutions for two special cases are used to verify Equation 26: one for the case of a thin semi-infinite breakwater and the other for the case of an infinite wall extending from \( x = -\infty \) to \( \infty \).

18. The vertical wedge should reduce to a thin semi-infinite breakwater as the wedge angle reduces to 0 deg. Therefore, solution of the combined wave reflection and diffraction by a thin semi-infinite breakwater is obtained by substituting \( \nu = 2 \) (that is, \( \theta = 2\pi \)) into Equation 26 which then becomes
\[
\phi(r, \theta) = J_0(kr) + 2 \sum_{n=1}^{\infty} e^{in\pi/4} J_{n/2}(kr) \cos \frac{n\alpha}{2} \cos \frac{n\theta}{2}
\] (28)

Equation 28 is precisely the same one obtained by Stoker (1957).

19. The vertical wedge should also become an infinite wall extending from \(x = -\infty\) to \(\infty\) with the water occupying only the half plane of \(y \geq 0\) as the wedge angle increases to 180 deg. In this situation the scattered wave is absent from the solution, and the total wave is only the sum of the incident and reflected waves as follows:

\[
\phi(r, \theta) = e^{ikr \cos(\theta-\alpha)} + e^{ikr \cos(\theta+\alpha)}
\] (29)

After expansion of the exponential functions in terms of Bessel functions (Abramowitz and Stegun 1964), Equation 29 becomes

\[
\phi(r, \theta) = 2 \left[ J_0(kr) + 2 \sum_{n=1}^{\infty} i^n J_n(kr) \cos \frac{n\alpha}{2} \cos \frac{n\theta}{2} \right]
\] (30)

Equation 30 is the same equation reduced from Equation 26 by substituting \(v = 1\) into it.
PART III: CALCULATION AND RESULTS

20. Results of the combined reflection and diffraction by a wedge of arbitrary wedge angle can be calculated from Equation 26. Since the solution is not only in terms of Bessel functions but also in a nonclosed form, the computer program WEDGE is therefore written to calculate the solution.

21. In the program the subroutine BESJ for calculating Bessel function of fractional or integer order was used. The subroutine was originally written by Amos, Daniel, and Weston in 1975 (Morris 1984) and is collected in the Naval Surface Weapons Center Library of Mathematics Subroutines (Morris 1984).

22. In the calculation the summation of the infinite terms in Equation 26 was carried out to the term which is preceded by eight successive terms of the absolute value of the Bessel function, all equal to or less than $10^{-8}$. The solution has a truncation error less than $10^{-8}$, and it is of the order of one.

23. In this study, results of the combined wave reflection and diffraction for the wedge are calculated for two cases: one for a vertical wedge of 0-deg wedge angle and the other for a vertical wedge of 90-deg wedge angle.

**Vertical Wedge of 0-Deg Wedge Angle**

24. When the wedge angle is equal to zero, the wedge is actually a thin semi-infinite breakwater extending from $x = 0$ to $\infty$. Figure 3 shows the thin semi-infinite breakwater along with the polar coordinates. In this case the diffraction results for various incident wave angles in the water region from $\theta = \pi$ to $2\pi$ and $r/\lambda \leq 10$, where $\lambda$ is the incident wave length, have already been presented by Wiegel (1962) and are shown in the SPM, Volume I (1984). The present results combine reflection and diffraction effects and cover the water region from $\theta = 0$ to $2\pi$ and $r/\lambda \leq 10$. Therefore, the present results for this particular case can be considered to be a complementary and extended version to the ones in the SPM.

25. In this study wave response was calculated at 1,460 grid points intersected by $r/\lambda = 0.5, (0.5), 10$, which means that the values of $r/\lambda$ are from 0.5 to 10.0 with each value increment being 0.5. Hereafter, all similar expressions are to be interpreted in the same way (e.g., $\theta = 0, (\pi/36), 2\pi$ for...
Figure 3. Thin semi-infinite breakwater and polar coordinates

the incident wave angle $\alpha = 0, (\pi/12), \pi$ . The wave response at the origin point is obtained substituting $v = 2$ into Equation 27, as follows:

$$\Phi(0, \theta) = 1$$  \hspace{1cm} (31)

Those calculated values were used to interpret the value for each non-overlapping pixel of size $0.1r/\lambda$ by $0.1r/\lambda$ in the area within the $10r/\lambda$ radius from the origin. A diagram was then constructed by patching those pixels over the entire area. The wave response diagrams for each incident wave angle are shown in Figures 4 through 15. Notably, the values in the diagrams constitute the amplification factor which is defined as the ratio of the total wave height to the incident wave height. Therefore, in subregions II
and III (as defined in Figure 2) where the reflected wave is absent, the amplification factor is essentially the diffraction coefficient as defined in the SPM.

26. Figures 4 through 15 reveal that the amplification factors in sub-region I change very rapidly between 0 and 2.35 over the subregion, and the diagram patterns become very complex because of the interesting superposition of the incident, reflected, and scattered waves. (In the legend of Figures 4 through 15, the width of the pixel is one incident wave length, and the values are amplification factors.) Such patterns would be very difficult to construct without using a high-speed computer and computer graphics. In sub-regions II and III, the amplification factors change smoothly from 1.15 roughly along the reflected wave ray reflected from the origin point to nearly
Figure 5. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 15 deg
Figure 6. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 30 deg
Figure 7. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 45 deg
Figure 8. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 60 deg
Figure 9. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 75 deg
Figure 10. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 90 deg
Figure 11. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 105 deg
Figure 12. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 120 deg
Figure 13. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 135 deg
Figure 14. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 150 deg
Figure 15. Amplification factor diagram for the thin semi-infinite breakwater for incident wave angle = 165 deg
0.00 at the back wall of the wedge in the shadow zone. The diagram patterns are relatively smooth and simple.

27. Notably, the diagrams do not include phase information of the wave response which is usually unimportant in most engineering practice. Should the phase of the wave response need to be known, one can always use the computer program WEDGE to calculate it.

28. The contour diagrams of the amplification factor, similar to the ones presented by Wiegel (1962) and shown in the SPM (1984), were also plotted as typically shown in Figure 16. Examination of those contour diagrams indicates that, in subregion III, the results are identical to Wiegel's results. But in subregion II the contour patterns for the amplification factor (or the diffraction coefficient $K'$ used in the SPM (1984)) equal to 1.0; thus the present results are far more complicated than Wiegel's. The author believes that Wiegel's results may lose accuracy because of insufficient resolution of the computational tools during the late fifties and early sixties. Nevertheless, such inaccuracies are usually either tolerable or immaterial in most engineering practice.

29. The contour patterns in subregion I are very complex, and it is difficult to track specific contours. Therefore, for clarity only the patched diagrams are presented, and the contour diagrams are omitted in this report.

**Vertical Wedge of 90-Deg Wedge Angle**

30. When the wedge angle is equal to $\pi/2 (\theta_o = 3\pi/2)$, the vertical wedge occupies the entire fourth quadrant of the space as shown in Figure 17. Wave response was calculated at 1,100 grid points intersected at $r/\lambda = 0.5, (0.5), 10.0$ and $\theta = 0, (\pi/36), 3\pi/2$ for the incident wave angle $\alpha = 0, (\pi/12), \pi$. The wave response at the origin is obtained by substituting $v = 1.5$ into Equation 27 as follows:

$$\phi(0, \theta) = \frac{4}{3}$$

Those calculated results were used to construct the amplification factor diagrams by following the same procedures described for the case of the thin semi-infinite breakwater. The diagrams are shown in Figures 18 through 27. Because of symmetry of the results, the diagrams for the incident wave angle $\alpha > 3\pi/4$ can be obtained from those for an incident wave angle of $\pi - \alpha$. 27
Therefore, the diagrams for the incident wave angles $\alpha = \pi/6, (\pi/12), \pi$ are omitted in this report.

31. The diagrams indicate that, for each corresponding incident wave angle, the results in subregion I are very similar to those obtained for the vertical wedge of 0-deg wedge angle. However, the results in subregions II and III from both wedges are discernibly different.
Figure 17. A 90-deg wedge and polar coordinates
Figure 18. Amplification factor diagram for the 90-deg wedge for incident wave angle = 0 deg
Figure 19. Amplification factor diagram for the 90-deg wedge for incident wave angle = 15 deg
Figure 20. Amplification factor diagram for the 90-deg wedge for incident wave angle = 30 deg
Figure 21. Amplification factor diagram for the 90-deg wedge for incident wave angle = 45 deg
Figure 22. Amplification factor diagram for the 90-deg wedge for incident wave angle = 60 deg
Figure 23. Amplification factor diagram for the 90-deg wedge for incident wave angle = 75 deg
Figure 24. Amplification factor diagram for the 90-deg wedge for incident wave angle = 90 deg
Figure 25. Amplification factor diagram for the 90-deg wedge for incident wave angle = 105 deg
Figure 26. Amplification factor diagram for the 90-deg wedge for incident wave angle = 120 deg
Figure 27. Amplification factor diagram for the 90-deg wedge for incident wave angle = 135 deg
PART IV: CONCLUSION

32. An analytical solution for the combined wave reflection and diffractive by a vertical wedge of arbitrary wedge angle is obtained and expressed in Equation 26. The analytical solution is in terms of Bessel functions and in nonclosed form. The computer subroutine WEDGE, written for calculating the solution, is documented in Appendix A.

33. The amplification factor diagrams for a vertical wedge of 0-deg wedge angle and a vertical wedge of 90-deg wedge angle are calculated and presented. The calculated results indicate that the wave response in subregion I, where the incident, reflected, and scattered waves all exist, is in a very complex pattern with the amplification factor varying from 2.35 to 0.0 over the subregion. The wave response in subregions II and III is a relatively simple pattern with the amplification factor decreasing from 1.15 roughly along the reflected wave ray reflected from the origin point to nearly 0.00 at the back wall of the wedge in the shadow zone.

34. Diagrams of the special case of a vertical wedge of 0-deg wedge angle can be considered complementary and extended versions to the ones presented in the SPM (1984).
REFERENCES


APPENDIX A: SUBROUTINE WEDGE

1. Subroutine WEDGE is used to calculate the value of $\phi$ in Equation 26 which is generally a complex number. Its absolute value is the amplification factor, and its phase is the phase indicator from the phase of the incident wave. As mentioned in the main text, $\phi$ is a function of the Bessel function of either fractional or integer order, depending on the wedge angle, and is the summation of a series of infinite terms. The subroutine BESJ, documented in the Naval Surface Weapons Center (NSWC) Library of Mathematics Subroutines (Morris 1984)* is used in the WEDGE subroutine. The programming of the WEDGE subroutine is very straightforward if a truncation term in the series in Equation 26 is determined. The program is written in FORTRAN language and is listed in this appendix.

Description

2. The following subroutine is available for computing $\phi$ in Equation 26:

CALL WEDGE(F,FABS,FPHA,XRL,XTH,WEDGEA,WAVEA,IDX)

where the arguments are all real values except F which is a complex value.

Input arguments are as follows:

a. (XRL,XTH)=(r/$\lambda$,\theta) where (r,\theta) are polar coordinates of the location where $\phi$ is to be computed, and $\lambda$ is the incident wave length. Therefore, XRL is the radius vector or radius distance normalized by the incident wave length. XTH is the vectorial angle in degree.

b. WEDGEA = wedge angle in degree.

c. WAVEA = incident wave angle in degree.

d. IDX = an index (set to 0 in this subroutine).

Output arguments are as follows:

* References cited in the Appendix can be found in the References at the end of the main text.
a. \( F = \phi \) in Equation 26 (wave response normalized by incident wave amplitude).

b. \( \text{FABS} \) = amplification factor, the absolute value of \( \phi \).

c. \( \text{FPHA} \) = phase difference, the phase of \( \phi \).

Example and Test Run

3. To serve as an example as well as to ensure the subroutine is the correct one, the user should run the test program listed in Figure A1 and make sure the output is the same as that listed in Table A1.

```fortran
PROGRAM WEDGE1(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT) TEST1
COMPLEX F TEST2
PI=3.141592654 TEST3
WRITE(6,4) TEST4
4 FORMAT('\(/3X, WEDG-ANG WAV-ANG LOCATION WAVE RESPONSE TEST5
1 \text{ABS-VAL PHASE'/'/3X, (DEG) (DEG) XRL XTH(DEG) (NORMALIZED) AMP-FAC (RAD)'/3X́, (/) TEST6
2ALIZED) \) TEST7
WEDGEA=90. TEST8
WAVEA=135. TEST9
XRL=2.0 TEST10
XTH=30. TEST11
IDX=0 TEST12
CALL WEDGE(F,FABS,FPHA,XRL,XTH,WEDGEA,WAVEA,IDX) TEST13
WRITE(6,40) WEDGEA,WAVEA,XRL,XTH,F,FABS,FPHA TEST14
40 FORMAT(1X,BF9.2) TEST15
STOP TEST16
END TEST17
```

Figure A1. Computer program list 1

Table A1

Sample Output of the Test Program

<table>
<thead>
<tr>
<th>WEDG-ANG</th>
<th>WAV-ANG</th>
<th>LOCATION</th>
<th>WAVE RESPONSE</th>
<th>ABS-VAL</th>
<th>PHASE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(DEG)</td>
<td>(DEG)</td>
<td>XRL</td>
<td>XTH(DEG)</td>
<td>(NORMALIZED)</td>
<td>(RAD)</td>
</tr>
<tr>
<td>90.00</td>
<td>135.00</td>
<td>2.00</td>
<td>30.00</td>
<td>-.41</td>
<td>.68</td>
</tr>
</tbody>
</table>

A2
4. The input arguments in the test programs are as follows:
   a. WEDGEA = 90; the wedge angle is 90 deg.
   b. WAVEA = 135; the incident wave angle is 135 deg.
   c. XRL = 2.0 and XTH = 30; the location of the wave response \( \phi \) to be computed is at radial vector of two incident wave length distances and vectorial angle of 30 deg in polar coordinates.
   d. IDX = 0; the index IDX is set to 0.

This case is shown in Figure A2. The outputs are given in Table A1 which is self-explanatory.

5. If the location of the wave response \( \phi \) to be computed is a very large distance, for example XRL greater than 18 or so, the output might print the message that the number of terms is insufficient for summation in computing \( \phi \). In this situation, the user must replace the integer 200 in PARAMETER(NN=200), listed in Card WEDGE15 in the SUBROUTINE WEDGE list, by a larger integer to ensure the accuracy.

![Figure A2. Example problem](image-url)
Subroutine WEDGE Listing

6. Subroutine WEDGE, which is listed in this section, calls subroutine BESJ which, in turn, calls subroutine JAIRY and function GAMLN. Subroutines BESJ, JAIRY and function GAMLN are borrowed from the NSWC Library of Mathematics Subroutines (Morris 1984). Including these borrowed subroutines here in the list is only for the purpose of allowing subroutine WEDGE to be self-contained and complete. The computer program of subroutine WEDGE is listed in Figure A3.
SUBROUTINE WEDGE(F,FABS,FPHA,XRL,XTH,WEDGEA,WAVEA,IDX)

C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *NOTICE
C * THIS COMPUTER PROGRAM WAS WRITTEN BY H.S. CHEN OF CERC IN
C * 1996 UNDER CORP. CIVIL WORK R&D PROGRAM. NEITHER ANY OF
C * AGENCIES NOR ANY INDIVIDUAL ASSUMES ANY LEGAL LIABILITY OR
C * RESPONSIBILITY FOR THE ACCURACY OF THE PROGRAM. *NOTICE5
C * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *NOTICE6

C -----------------------------------
WEDGE7
C WAVE REFLECTION AND DIFFRACTION BY A VERTICAL WEDGE.

C INPUT :   XRL = R/L, (RADIUS VECTOR)/(WAVE LENGTH)
C          XTH = VECTORIAL ANGLE IN DEGREE
C          WEDGEA = WEDGE ANGLE IN DEGREE
C          WAVEA = INCIDENT WAVE ANGLE IN DEGREE

C OUTPUT :  F = PHI, WAVE RESPONSE NORMALIZED BY INCIDENT
C          WAVE AMPLITUDE
C          FABS = ABSOLUTE VALUE OF PHI, THE AMPLIFICATION
C          FACTOR
C          FPHA = PHASE OF PHI IN Radian.

C -----------------------------------
PARAMETER(NN=200)
DIMENSION BJ(NN),W(NN),XM(NN)
COMPLEX F,TM
DATA TOLR/1.E-8/,ITER/B/
PI=3.141592654
CPI=PI/180.
XKR=XRL*2.0*PI
TH=XTH*CPI
WA=WAVEA*CPI
XNU=(360.-WEDGEA)/180.
XM(1)=1.0
IF(IDX.NE.0) GOTO 14
CALL BESJ(XKR,0.0,1,W,NZ)
BJ(1)=W(1)
ICOUNT=0
DO 10 N=I,NN
   M=INT(XM(NI))
   ALPHA=XM(NI)-M
   CALL BESJ(XKR,ALPHA,M,1,W,NZ)
   IF(NI.EQ.NN) WRITE(6,9) XKR,ALPHA,M,1,W(NZ)
   BJ(NI)=W(M)
10   INCOUNT=ICOUNT+I
   IF(ICOUNT.LE.ITER) GOTO 8
   NNN=N
   GOTO 14
8  XM(NI)=FLOAT(N)/XNU
    M=INT(XM(NI))
    ALPHA=XM(NI)-M
    M=M+1
    CALL BESJ(XKR,ALPHA,M,1,W,NZ)
    IF(NI.EQ.NN) WRITE(6,9) XKR,ALPHA,M,1,W(M),NZ
    9 FORMAT(/' **** NO. OF TERMS FOR SUMMATION IS INSUFFICIENT ****,
                ',2F10.4,15,E15.6,I5/)}
   BJ(NI)=W(M)

Figure A3. Computer program list 2 (Sheet 1 of 25)
IF(ABS(BJ(N1)).GT.TOLR) ICONT=0
10 CONTINUE
14 CONTINUE
F=BJ(1)/2.
DO 20 N=1,NNN
N1=N+1
XMN=XM(NI)
TM=(0.0,1.0)**XMN*BJ(NI)*COS(XMN*WA)*COS(XMN*TH)
F=F+TM
20 CONTINUE
F=4./XMU*F
FR=REAL(F)
FI=AIMAG(F)
IF(WA.LE.1.E-8) FABS=FABS/2.
IF(FABS.LT.TOLR) SOTO 70
FPHA=ATAN2(FI,FR)
RETURN
30 FPHA=0.0
RETURN END
SUBROUTINE BESJ(X,ALPHA,N,Y,NZ)
C--------------------------------------BESJ
C
C ABSTRACT
C BESJ COMPUTES AN N MEMBER SEQUENCE OF J BESSEL FUNCTIONS J/(ALPHA+K-1)/(X), K=1,...,N FOR NON-NEGATIVE ALPHA AND X.
C A COMBINATION OF THE POWER SERIES, THE ASYMPTOTIC EXPANSION FOR X TO INFINITY AND THE UNIFORM ASYMPTOTIC EXPANSION FOR NU TO INFINITY ARE APPLIED OVER SUBDIVISIONS OF THE (NU,X) PLANE. FOR VALUES OF (NU,X) NOT COVERED BY ONE OF THESE FORMULAE, THE ORDER IS INCREMENTED OR DECREMENTED BY INTEGER VALUES INTO A REGION WHERE ONE OF THE FORMULAE APPLY. BACKWARD RECURSION IS APPLIED TO REDUCE ORDERS BY INTEGER VALUES EXCEPT WHERE THE ENTIRE SEQUENCE LIES IN THE OSCILLATORY REGION. IN THIS CASE FORWARD RECURSION IS STABLE AND VALUES FROM THE ASYMPTOTIC EXPANSION FOR X TO INFINITY START THE RECURSION WHEN IT IS EFFICIENT TO DO SO. LEADING TERMS OF THE SERIES AND THE UNIFORM EXPANSION ARE TESTED FOR UNDERFLOW. IF A SEQUENCE IS REQUESTED AND THE LAST MEMBER WOULD UNDERFLOW, THE RESULT IS SET TO ZERO AND THE NEXT LOWER ORDER TRIED, ETC., UNTIL A MEMBER COMES ON SCALE OR ALL MEMBERS ARE SET TO ZERO. OVERFLOW CANNOT OCCUR. BESJ CALLS SUBROUTINE Jairy AND FUNCTION GAMLN.
C
C DESCRIPTION OF ARGUMENTS
C
C INPUT
C X - X.GE.0
C ALPHA - ORDER OF FIRST MEMBER OF THE SEQUENCE, ALPHA.GE.0
C N - NUMBER OF MEMBERS IN THE SEQUENCE, N.GE.1
C
C OUTPUT
C Y - A VECTOR WHOSE FIRST N COMPONENTS CONTAIN VALUES FOR J/(ALPHA+K-1)/(X), K=1,...,N
C
Figure A3. (Sheet 2 of 25)
NZ - ERROR INDICATOR

NZ=0  NORMAL RETURN - COMPUTATION COMPLETED
NZ=-1  X IS LESS THAN 0.0
NZ=-2  ALPHA IS LESS THAN 0.0
NZ=-3  N IS LESS THAN 1
NZ.GT.0  LAST NZ COMPONENTS OF Y SET TO 0.0, BECAUSE OF UNDERFLOW

ERROR CONDITIONS

IMPROPER INPUT ARGUMENTS - A FATAL ERROR
UNDERFLOW - A NON-FATAL ERROR (NZ.GT.0)

DOUBLE PRECISION DX,TRX,DTM,DFN
DIMENSION Y(N)
DIMENSION C(11,10),ALFA(26,4),BETA(26,5)
DIMENSION A1(52),A2(52),B1(52),B2(52),B3(26)
DIMENSION GAMA(5),AR(5),BR(10),UPOL(10)
DIMENSION FNULIM(2),PP(4)
DIMENSION CR(10),DR(10)

EQUIVALENCE (C(1,1),C1(1))
EQUIVALENCE (C(1,9),C2(1))
EQUIVALENCE (ALFA(1,1),A1(1))
EQUIVALENCE (ALFA(1,3),A2(1))
EQUIVALENCE (BETA(1,1),B1(1))
EQUIVALENCE (BETA(1,3),B2(1))
EQUIVALENCE (BETA(1,5),B3(1))

DATA ELIM1,ELIM2,TOL / 667. 644. 1.E-15 /
DATA PP(1)/8.7290915393555E+00/, PP(2)/2.656973226503E+01/
  PP(3)/1.2457857686559E+01/, PP(4)/7.701377464039E+04/
DATA TOLS=LN(1.E-3)
DATA CE=-ALOG(TOL),TCE=-0.75J*ALOG(TOL)
DATA INLIM /150 /
DATA AR(1)/8.3550347222222E-02/, AR(2)/1.2822657455633E-01/
Figure A3. (Sheet 4 of 25)
<table>
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<th>DATA</th>
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<th>A1 (2) / 1.1001714026925E+02/</th>
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<td>C2 (20) / 2.7856181280866E+06/</td>
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**Figure A3. (Sheet 5 of 25)**
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**Figure A3.** (Sheet 6 of 25)
DATA B(1) /7.3646581057258E-04/, B(2) /8.7279080514619E-04/, BESJ213
1 B(3) /6.2261482657314E-04/, B(4) /2.8599815419430E-04/, BESJ215
2 B(5) /3.8473767287937E-06/, B(6) /1.8790600363697E-04/, BESJ215C
3 B(7) /-2.976036459456E-04/, B(8) /-3.4599812683267E-04/, BESJ215D
4 B(9) /-3.53528427091604E-04/, B(10) /-3.3571563577505E-04/, BESJ215E
5 B(11) /-3.0432112478904E-04/, B(12) /-2.667272304761E-04/, BESJ215F
6 B(13) /-2.276542142282E-04/, B(14) /-1.899226118854E-04/, BESJ215G
7 B(15) /-1.5505891859990E-04/, B(16) /-1.2377824076118E-04/, BESJ215H
8 B(17) /-6.292614771764E-05/, B(18) /-7.2517832771442E-05/, BESJ215I
9 B(19) /-5.2207002889563E-05/, B(20) /-3.5034775051190E-04/, BESJ215J
1 B(21) /-2.0648976103555E-05/, B(22) /-8.7010609687977E-06/, BESJ215K
2 B(23) /1.1369866667510E-06/, B(24) /9.1642647412278E-06/, BESJ215L
3 B(25) /1.5647778542887E-05/, B(26) /2.0822362948247E-05/, BESJ215M
C DATA GAMA(1) /6.2996052494744E-01/, GAMA(2) /2.519842099798E-01/, GAMA(3) /1.5479030041566E-01/, GAMA(4) /1.0710306241616E-01/, GAMA(5) /8.57309359552740E-02/, GAMA(6) /6.9716131695868E-02/, GAMA(7) /5.860567189371E-02/, GAMA(8) /5.0469887356361E-02/, GAMA(9) /4.4260058068916E-02/, GAMA(10) /3.9372066154351E-02/, GAMA(11) /3.5428319592444E-02/, GAMA(12) /3.218188750210E-02/, GAMA(13) /2.9464624079116E-02/, GAMA(14) /2.7158167711293E-02/, GAMA(15) /2.5176827297386E-02/, GAMA(16) /2.3457075530608E-02/, GAMA(17) /2.195083903491E-02/, GAMA(18) /2.062108228356E-02/, GAMA(19) /1.9438224089788E-02/, GAMA(20) /1.8381663380062E-02/, GAMA(21) /1.7429321323196E-02/, GAMA(22) /1.6558853778661E-02/, GAMA(23) /1.5786582598792E-02/, GAMA(24) /1.5072950149410E-02/, GAMA(25) /1.4419325083996E-02/, GAMA(26) /1.3819480573534E-02/
C TEST INPUT ARGUMENTS
C
NZ=0
KT=1
IF(N-1) 92,108,109
100 KT=2
109 NN=N
110 IF(ALPHA) 91,114,116
114 Y(I)=1.
116 IF(N.EQ.1) RETURN
118 I=2
120 GO TO 118
116 I=1
118 GO TO 119 I=I,N
119 Y(I)=0.
120 RETURN
122 CONTINUE
124 IF(ALPHA.LT.0.) GO TO 91
C

Figure A3. (Sheet 7 of 25)

All
DFN=DBLE(FLOAT(N))+DBLE(ALPHA)-1.0

FNU=DFN
X02=X*0.5
SX02=X02*X02

C

DECISION TREE FOR REGION WHERE SERIES, ASYMPTOTIC EXPANSION FOR X TO INFINITY AND ASYMPTOTIC EXPANSION FOR NU TO INFINITY ARE APPLIED.

IF(SX02.LE.(FNU+1.)) GO TO 850
TA=AMAX1(20.,FNU)
IF(X.GT.TA) GO TO 880
IF(X.GT.12.) GO TO B60
X02L=ALOG(X02)
NS=SX02-FNU
GO TO 852

850 FN=FNU
FNP1=FN+1.
X02L=ALOG(X02)
IS=KT
IF(X.LE.0.5) GO TO 134
NS=0

852 DFN=DFN+DBLE(FLOAT(NS))
FN=DFN
FNP1=FN+1.
IS=KT
IF(N-1+NS.GT.0) IS=3
GO TO 134

860 NS=AMAX1(36.-FNU,0.)
DFN=DFN+DBLE(FLOAT(NS))
FN=DFN
IS=KT
IF(N-1+NS.GT.0) IS=3
GO TO 130

880 CONTINUE
RTX=SQRT(X)
TAU=RTWO*RTX
TA=TAU+FNULIM(KT)
IF(FNU.LE.TA) GO TO 500

129 FN=FNU
IS=KT

C UNIFORM ASYMPTOTIC EXPANSION FOR NU TO INFINITY

C

130 CONTINUE
XX=X/FN
W2=1.-XX*XX
ABW2=ABS(W2)
RA=SQRT(ABW2)
IF(ABW2.GT.0.2775) GO TO 200

C CASES NEAR X=FN, ABS(1.-X/FN)**2).LE.0.2775
C COEFFICIENTS OF ASYMPTOTIC EXPANSION BY SERIES
C

Figure A3. (Sheet 8 of 25)
C ZETA AND TRUNCATION FOR A(ZETA) AND B(ZETA) SERIES BEJ295
C BEJ296
C KMAX IS TRUNCATION INDEX FOR A(ZETA) AND B(ZETA) SERIES=MAX(2,SA) BEJ297
C BEJ298
SA=0.
IF(ABW2.EQ.0.) GO TO 21
SA=TOLS/ALOG(ABW2)  BEJ299
21 SB=SA
DO 22 1=1,5
KMAX(I)=AMAX1(SA,2.)
SA=SA+SB
22 CONTINUE
KB=KMAX(5)
KLAST=KB-1
SA=GAMA(KB)
DO 24 K=L,KLAST
KB=KB-1
SA=SA+W2+GAMA(KB)
24 CONTINUE
Z=W2*SA
AZ=ABS(Z)
RTZ=SQRT(AZ)
FN13=FN**CON2
RTARY=RTZ*FN13
ARY=-RTARY*RTARY
AZ32=AZ*RTZ*CON1
ACZ=FN*AZ32
IF(Z.LE.0.) GO TO 27
C TEST FOR UNDERFLOW, 1.E-2180=EXP(-644.), ONE WORD LENGTH C BEJ324
C UP FROM UNDERFLOW LIMIT OF CDC 6600 C BEJ325
IF(ACZ.GT.ELIM&.60) GO TO 180
ARY=-ARY
27 PHI=SQRT(SQRT(SA+SA+SA+SA)) C BEJ329
C B(ZETA) FOR S=0 C BEJ330
KB=KMAX(5)
KLAST=KB-1
SB=BETA(KB,1)
DO 23 K=1,KLAST
KB=KB-1
SB=SB+W2+BETA(KB,1)
23 CONTINUE
KSPI=1
FN2=FN*FN
RFN2=1./FN2
RDEN=1.
ASUM=1.
RELB=TOL*ABS(SB)
BSUM=SB
DO 25 KS=1,4
KSPI=KSPI+1
RDEN=RDEN*RFN2

Figure A3. (Sheet 9 of 25)
A(ZETA) AND B(ZETA) FOR S=1,2,3,4

KB=KMAX(5-KS)
KLAST=KB-1
SA=ALFA(KB,KS)
SB=BETA(KB,KSP1)
DO 26 K=1,KLAST
KB=KB-1
SA=SA+W2+ALFA(KB,KS)
SB=SB+W2+BETA(KB,KSP1)
26 CONTINUE
TA=SA*RDEN
TB=SB*RDEN
ASUM=ASUM+TA
BSUM=BSUM+TB
IF(ABS(TA).LE.TOL.AND.ABS(TB).LE.RELB) GO TO 152
25 CONTINUE
152 CONTINUE
BSUM=BSUM/(FN*FN13)
GO TO 400

200 CONTINUE
TAU=1./RA
T2=1./W2
IF(W2.GE.0.) GO TO 30
CASES FOR (X/FN).GT.SQRT(1.2775)
AZ32=ABS(RA-ATAN(RA))
ACZ=AZ32*FN
CZ=-ACZ
Z32=1.5*AZ32
RTZ=Z32**CON2
FN13=FN**CON2
RTARY=RTZ*FN13
ARY=-RTARY*RTARY
GO TO 150
30 CONTINUE
CASES FOR (X/FN).LT.SQRT(0.7225)
AZ32=ABS(ALOG((1.+RA)/XX) -RA)
TEST FOR UNDERFLOW, 1.E-280 = EXP(-644.), ONE WORD LENGTH
UP FROM UNDERFLOW LIMIT OF CDC 6600
ACZ=AZ32*FN
CZ=ACZ
IF(ACZ.GT.ELIM2) GO TO 180
Z32=1.5*AZ32
RTZ=Z32**CON2
FN13=FN**CON2
RTARY=RTZ*FN13
ARY=RTARY*RTARY

Figure A3. (Sheet 10 of 25)
150 CONTINUE
PHI=SQRT((RTZ+RTZ)*TAU)  
TB=1.  
ASUM=1.  
TFN=TAU/FN  
UPOL(2)=(C(1,1)*T2+C(2,1))*TFN  
RCZ=CONI/CZ  
CRZ32=CON548*RCZ  
BSUM=UPOL(2)+CRZ32  
RELB=TOL*ABS(BSUM)  
AP=TFN  
KS=0  
KPI=2  
RZDEN=RCZ  
DO 155 LR=2,8,2  
C
C  COMPUTE TWO U POLYNOMIALS FOR NEXT A(ZETA) AND B(ZETA)
C
  LRP1=LR+1  
  DO 101 K=LR,LRP1  
  KS=KS+1  
  KPI=KPI+1  
  S1=C(I,K)  
  DO 102 J=2,KP1  
  S1=S1*T2+C(J,K)  
  102 CONTINUE  
  AP=AP*TFN  
  UPOL(KP1)=AP*S1  
  CR(KS)=BR(KS)*RZDEN  
  RZDEN=RZDEN*RCZ  
  DR(KS)=AR(KS)*RZDEN  
101 CONTINUE  
  SUMA=UPOL(LRP1)  
  SUMB=UPOL(LR+2)+UPOL(LRP1)*CRZ32  
  JU=LRP1  
  DO 151 JR=1,LR  
  JU=JU-1  
  SUMA=SUMA+CR(JR)*UPOL(JU)  
  SUMB=SUMB+DR(JR)*UPOL(JU)  
151 CONTINUE  
  TB=-TB  
  IF(W2.GT.0.) TB=ABS(TB)  
  ASUM=ASUM+SUMA*TB  
  BSUM=BSUM+SUMB*TB  
  IF(ABS(SUMA).LE.TOL.AND.ABS(SUMB).LE.RELB) GO TO 165  
155 CONTINUE  
165 TB=RTARY  
  IF(W2.GT.0.) TB=-TB  
  BSUM=BSUM/TB  
C
400 CONTINUE  
  CALL JAIRY(ARY,RTARY,ACZ,AI,DAI)  
  TEMP(IS)=PHI*(AI*ASUM+DAI*BSUM)/FN13  
  GO TO (401,202,650), IS  
402 TEMP(1)=TEMP(3)

Figure A3. (Sheet 11 of 25)
KT=1
401 IS=2
DFN=DFN-1.D+0
FN=DFN
GO TO 130
C
C SERIES FOR (X/2)**2.LE.NU+1
C
134 CONTINUE
GLN=GAMLN(FNPI)
ARG=FN*XO2L-GLN
IF(ARG.LT.-ELIMI) GO TO 123
EARG=EXP(ARG)
300 CONTINUE
S=1.
AK=3.
T2=1.
T=1.
S1=FN
DO 125 K=1,17
S2=T2+S1
T=-T+SX02/S2
S=S+T
IF(ABS(T).LT.TOL) GO TO 127
T2=T2+AK
AK=AK+2.
S1=S1+FN
125 CONTINUE
127 CONTINUE
TEMP(IS)=S*EARG
GO TO (301,202,600), IS
301 EARG=EARG*FN/X02
DFN=DFN-1.D+0
FN=DFN
IS=2
GO TO 300
C
C SET UNDERFLOW VALUE AND UPDATE PARAMETERS
C
180 Y(NN)=0.
NN=NN-1
DFN=DFN-1.D+0
FN=DFN
IF (NN-1) 170,171,130
171 KT=2
IS=2
GO TO 130
123 Y(NN)=0.
NN=NN-1
FNPI=FN
DFN=DFN-1.D+0
FN=DFN
IF (NN-1) 170,172,173
172 KT=2
IS=2

Figure A3. (Sheet 12 of 25)
173 IF(SXO2.LE.FNPI) GO TO 133
    GO TO 130
133 ARG=ARG-XO2L+ALOG(FNPI)
    IF(ARG.LT.-ELIMI) GO TO 123
    GO TO 134
170 NZ=N-NN
    RETURN
C
C   BACKWARD RECURSION SECTION
C
202 CONTINUE
    NZ=N-NN
    IF(KT.EQ.2) GO TO 250
203 CONTINUE
C   BACKWARD RECUR FROM INDEX ALPHA+NN-1 TO ALPHA
C
250 CONTINUE
    IN-ALPHA-TAU+2.
    IF(IN.LE.0) GO TO 502
    INPI=IN+I
    DALPHA=ALPHA-FLOAT(INPI)
    KT=1
    GO TO 511
502 DALPHA=ALPHA
    IN=0
511 IS=KT
512 ARG=X-PIDT+DALPHA-PDF
    SA=SIN(ARG)
    SB=COS(ARG)
    RA=RTTP/RTX
    ETX=B.*X
703 DX=DALPHA
    DX=DX+DX
    DTM=DX+DX

C   ASYMPTOTIC EXPANSION FOR X TO INFINITY WITH FORWARD RECURSION IN
C   OSCILLATORY REGION X.GT.MAX(20, NU), PROVIDED THE LAST MEMBER
C   OF THE SEQUENCE IS ALSO IN THE REGION.
C
500 CONTINUE
    IN=ALPHA-TAU+2.
    IF(IN.LE.0) GO TO 502
    INPI=IN+1
    DALPHA=ALPHA-FLOAT(INPI)
    KT=1
    GO TO 511
502 DALPHA=ALPHA
    IN=0
511 IS=KT
512 ARG=X-PIDT+DALPHA-PDF
    SA=SIN(ARG)
    SB=COS(ARG)
    RA=RTTP/RTX
    ETX=B.*X
703 DX=DALPHA
    DX=DX+DX
    DTM=DX+DX

Figure A3. (Sheet 13 of 25)
T2=DTM-1.D+0  
T2=T2/ETX  
S2=T2  
RELB=TOL*ABS(T2)  
T1=ETX  
S1=1.  
FN=1.  
AK=8.  
DO 504 K=1,13  
T1=T1+ETX  
FN=FN+AK  
DX=FN  
TRX=DTM-DX  
AP=TRX  
T2=-T2*AP/T1  
S1=S1+T2  
T1=T1+ETX  
AK=AK+8.  
FN=FN+AK  
DX=FN  
TRX=DTM-DX  
AP=TRX  
T2=-T2*AP/T1  
S2=S2+T2  
IF(ABS(T2).LE.RELB) GO TO 505  
AK=AK+8.  
504 CONTINUE  
505 TEMP(IS)=RA*(S1*SB-S2*SA)  
GO TO (506,507),IS  
506 DALPHA=DALPHA+1.  
IS=2  
TB=SA  
SA=-SB  
SB=TB  
GO TO 503  
C  
C  FORWARD RECURSION SECTION  
C  
507 IF(KT.EQ.2) GO TO 250  
S1=TEMP(1)  
S2=TEMP(2)  
TX=2./X  
TM=DALPHA*TX  
IF(IN.EQ.0) GO TO 520  
C  
C  FORWARD RECURL TO INDEX ALPHA  
C  
DO 510 I=1,IN  
S=S2  
S2=TM*S2-S1  
TM=TM+TX  
S1=S  
510 CONTINUE  
IF(NN.EQ.1) GO TO 535  
S=S2  

Figure A3.  (Sheet 14 of 25)
S2 = TM - S2 - S1
TM = TM + TX
S1 = S

520 CONTINUE

C
C FORWARD RECUR FROM INDEX ALPHA TO ALPHA+N-1
C
Y(1) = S1
Y(2) = S2
IF(NN.EQ.2) RETURN
DO 530 I = 3, NN
Y(I) = TM*Y(I-1) - Y(I-2)
TM = TM + TX
530 CONTINUE
RETURN

535 Y(1) = S2
RETURN

C
C BACKWARD RECURSION WITH NORMALIZATION BY
C ASYMPTOTIC EXPANSION FOR NU TO INFINITY OR POWER SERIES.
C
600 CONTINUE

C COMPUTATION OF LAST ORDER FOR SERIES NORMALIZATION
KM = AMAX1(3.-FN, 0.)
TFN = FN + FLOAT(KM)
TA = (GLN + TFN - 0.9189385352 - 0.08333333333/TFN)/(TFN + 0.5)
TA = XQ2L - TA
TB = -(1. - 1.5/TFN)/TFN
IN = CE/(-TA + SQRT(TA*TA - CE*TB) + 1.5)
IN = IN + KM
GO TO 602

650 CONTINUE

C COMPUTATION OF LAST ORDER FOR ASYMPTOTIC EXPANSION NORMALIZATION
GLN = A232*AR
IF(ARY.GT.30.) GO TO 675
RDNEN = PP(4)*ARY + FP(3)*ARY + 1.
RZDEN = PP(1) + PP(2)*ARY
TA = RZDEN/RDEN
IF(W2.LT.0.10) GO TO 651
TB = GLN/RTARY
GO TO 677

651 TB = (1.259921049 + 0.1679824730*W2)/FN13
GO TO 677

675 CONTINUE

TA = CON1*TE/ACZ
TA = ((0.0493827160*TA - 0.1111111111)*TA + 0.6666666667)*TA*ARY
IF(W2.LT.0.10) GO TO 651
TB = GLN/RTARY

677 IN = TA/TB + 1.5
IF(IN.GT.INLIM) GO TO 402

603 DX = FLOAT(IN)
DTM = DFN + DX
DX = X
TRX = 2.0 + 0.1679892427160
DTM = DTM*TRX

Figure A3. (Sheet 15 of 25)
TM=DTM
TA=0.
TB=TOL
KK=1

605 CONTINUE

C
C BACKWARD RECUR UNINDEXED
C
DO 601 I=1,IN
S=TB
TB=TM*TB-TA
TA=S
DTM=DTM-TRX
TM=DTM

601 CONTINUE
C
NORMALIZATION
IF(KK.NE.1) GO TO 604
TA=(TA/TB)*TEMP(3)
TB=TEMP(3)
KK=2
IN=NS
IF(NS.NE.0) GO TO 605

604 Y(NN)=TB
615 NZ=N-NN
IF(NN.EQ.1) RETURN
S=TB
TB=TM*TB-TA
TA=S
DTM=DTM-TRX
TM=DTM
K=NN-1
Y(K)=TB
IF(NN.EQ.2) RETURN
KM=K-1

C
C BACKWARD RECUR INDEXED
C
DO 602 I=1,KM
Y(K-1)=TM*Y(K)-Y(K+1)
DTM=DTM-TRX
TM=DTM
K=K-1

602 CONTINUE
RETURN

C
C
91 CONTINUE
NZ=-2
RETURN
92 CONTINUE
NZ=-3
RETURN
93 CONTINUE
NZ=-1

Figure A3. (Sheet 16 of 25)
SUBROUTINE JAIRY(XRX,C,AI,DAI)
CJAIRY COMPUTES THE AIRY FUNCTION AI(X)
AND ITS DERIVATIVE DAI(X) FOR JBESS
C
INPUT: X - ARGUMENT, COMPUTED BY JBESS, X UNRESTRICTED
RX = SQRT(ABS(X)), COMPUTED BY JBESS
C - C=2.*ABS(X)**1.5/3., COMPUTED BY JBESS
C
OUTPUT: AI - VALUE OF FUNCTION AI(X)
DAI - VALUE OF THE DERIVATIVE DAI(X)

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DATA Ni,N2,N3,N4/I4,23,19,15/
DATA Mi,M2,M3,M4/12,21,17,13/
DATA FPI,12,CON1,CON2,CON3,CON4,CON5/
DATA 1.3089969389958E+00, 6.6666666666667E-01, 5.0315471619678E+00,
2.3800045896629E-01, 3.3333333333333E-01, 8.6602540378444E-01/
DATA AK1(14),AK2(23),AK3(14)
DATA AJP(19),AJN(19),A(5),B(I5)
DATA DAK1(14),DAK2(24),DAK3(14)
DATA DAJP(19),DAJN(19),DA(15),DB(15)
DATA Ni,N2,N3,N4/I4,23,19,15/
DATA Mi,M2,M3,M4/12,21,17,13/
DATA FPI,12,CON1,CON2,CON3,CON4,CON5/
DATA 1.3089969389958E+00, 6.6666666666667E-01, 5.0315471619678E+00,
2.3800045896629E-01, 3.3333333333333E-01, 8.6602540378444E-01/
DATA AK1(14),AK2(23),AK3(14)
DATA AJP(19),AJN(19),A(5),B(I5)
DATA DAK1(14),DAK2(24),DAK3(14)
DATA DAJP(19),DAJN(19),DA(15),DB(15)

Figure A3. (Sheet 17 of 25)
Figure A3. (Sheet 18 of 25)
CIF(X.LT.0.) GD TO

%WYDB(15)/-7.1179333,7757953-E-16/

-- - - - - - -

6.865735378444E-17/

DAJN(13)/-1.682314651092E-15/,

DAJN(14)/2.5537477509706E-16/

DATA DAJP(12) / 2.2612065309577E-13/,

DAJN(17)/1.3963176533104E-12/

DAJN(9) / 1.0129732689135E-14/,

DAJN(7)

DAJN(3) /-3.122770681356E-01/, AIRY155

DAJN(11)/-4.8348113033798E-05/,

DAJN(4) / 1.6749842950500E-02/,

DAJN(5) /-1.9714614018213E-03/, AIRY120

DAJN(6) /-9.4556029509887E-04/,

DAJN(7) / 9.4228962070198E-05/, AIRY122

DAJN(8) / 2.258286094548E-05/,

DAJN(9) /-2.9967870915999E-06/, AIRY125

DAJN(10) /-3.763499113692E-07/,

DAJN(11)/3.4566392355956E-08/, AIRY124

DAJN(12) / 4.2961132200301E-09/,

DAJN(13)/-3.5867369121499E-10/, AIRY127

DAJN(14) /-5.725488136190E-10/, DAJN(15)/2.728690106634E-12/, AIRY126

DAJN(16) / 2.2612065309577E-13/, DAJN(17)/-1.5863205253850E-14/, AIRY129

DAJN(18) /-1.1260437448512E-15/, DAJN(19)/7.3132752951537E-17/, AIRY128

DAJN(1) / 1.0859453981329E-02/, AIRY129

DAJN(2) /-8.5331319485709E-02/,

DAJN(3) /-3.1527706811506E-01/, AIRY130

DAJN(4) /-8.784207259426E-02/, DAJN(5)/5.5325190679605E-02/, AIRY131

DAJN(6) / 9.4167460650324E-03/, DAJN(7)/-3.3218702601900E-03/, BEJS132

DAJN(8) /-4.115734315683E-04/, DAJN(9)/1.0129732689135E-04/, AIRY133

DAJN(10) / 9.8763368220840E-06/, DAJN(11)/-1.873129681239E-06/, AIRY134

DAJN(12) / 1.5079850013147E-07/, DAJN(13)/2.3268766952539E-08/, AIRY135

DAJN(14) / 1.595999174192E-09/, DAJN(15)/-2.078652268683E-10/, AIRY136

DAJN(16) /-1.241035050030E-11/, DAJN(17)/1.3963175633104E-12/, AIRY137

DAJN(18) /-7.3940097115574E-14/, DAJN(19)/7.328747562750E-15/, AIRY137

DATA DAJN(1)/ 4.916273211046E-01/, DAJN(2)/3.1161493042749E-03/, AIRY139

DAJN(3) /-8.2314076285408E-05/,

DAJN(4) /-4.6167977617214E-06/, AIRY140

DAJN(5) /-6.131588053463E-08/, DAJN(6)/2.8729580465652E-08/, AIRY141

DAJN(7) /-1.8195971573212E-09/, DAJN(8)/1.447528264204E-10/, AIRY142

DAJN(9) / 4.5372404342042E-11/, DAJN(10)/-3.9965506584722E-12/, AIRY143

DAJN(11) /-5.240891189032E-13/, DAJN(12)/1.620989526874E-13/, AIRY144

DAJN(13) /-2.4076524797406E-14/, DAJN(14)/1.6938481126449E-16/, AIRY145

DAJN(15) / 8.179007847470E-16/;

DATA DB(1) /-2.7757135694423E-01/, DB(2)/4.4421283341992E-03/, AIRY146

DB(3) /-8.422825219009E-05/, DB(4)/-2.5804031841871E-06/, AIRY147

DB(5) / 3.4238972021762E-07/, DB(6)/-6.242869479078E-09/, AIRY149

DB(7) /-2.3657783844585E-09/, DB(8)/3.1699104285667E-10/, AIRY150

DB(9) /-4.4099569165819E-12/, DB(10)/-5.186542209358E-12/, AIRY151

DB(11) / 9.6487401573702E-13/, DB(12)/-4.8901595668671E-14/, AIRY152

DB(13) /-1.772534067811E-14/, DB(14)/5.5959061044266E-15/, AIRY153

DB(15) /-7.1179333757953E-16/;

C

IF(X.LT.0.) GO TO 300

IF(C.GT.5.) GO TO 200

IF(X.GT.1.2) GO TO 150

T=(X+X-1.2)*CON4

T= T + T

J=N1

F1=AK1(J)

F2=0.

Figure A3. (Sheet 19 of 25)
DO 105 I=1,M1
J=J-1
TEMP1=F1
F1=TT*F1-F2+AK1(J)
F2=TEMP1
105 CONTINUE
A1=T*F1-F2+AK1(1)
C
J=N1D
F1=DAK1(J)
F2=0.
DO 106 I=1,M1D
J=J-1
TEMP1=F1
F1=TT*F1-F2+DAK1(J)
F2=TEMP1
106 CONTINUE
DAI=-T*F1-F2+DAK1(1))
RETURN
C
150 CONTINUE
T=(X+X-CON2)*CON3
TT = T + T
J=N2
F1=AK2(J)
F2=0.
DO 155 I=1,M2
J=J-1
TEMP1=F1
F1=TT*F1-F2+AK2(J)
F2=TEMP1
155 CONTINUE
RTRX=SQRT(RX)
EC=EXP(-C)
AI=EC*(T*F1-F2+AK2(1)) /RTRX
J=N2D
F1=DAK2(J)
F2=0.
DO 156 I=1,M2D
J=J-1
TEMP1=F1
F1=TT*F1-F2+DAK2(J)
F2=TEMP1
156 CONTINUE
DAI=-EC*(T*F1-F2+DAK2(1))*RTRX
RETURN
C
200 CONTINUE
T=10./C-1.
TT=T+T
J=N1
F1=AK3(J)
F2=0.
DO 205 I=1,M1
J=J-1
Figure A3. (Sheet 20 of 25)
TEMP1 = F1
F1 = TT * F1 - F2 + AJP(1)
F2 = TEMP1
205 CONTINUE
RTRX = SQRT(RX)
EC = EXP(-C)
A1 = EC * (T * F1 - F2 + AJP(1)) / RTRX
J = N3
F1 = DAI(P(J))
F2 = 0.
DO 206 I = 1, M3
J = J - 1
TEMP1 = F1
F1 = TT * F1 - F2 + DAI(P(J))
F2 = TEMP1
206 CONTINUE
DAI = - RTRX * EC * (T * F1 - F2 + DAI(P(1)))
RETURN
C
300 CONTINUE
IF(C.GT.5.) GO TO 350
T = 4 * C - 1.
TT = T + T
J = N3
F1 = AJP(J)
E1 = AJN(J)
F2 = 0.
E2 = 0.
DO 305 I = 1, M3
J = J - 1
TEMP1 = F1
TEMP2 = E1
F1 = TT * F1 - F2 + AJP(J)
E1 = TT * E1 - E2 + AJN(J)
F2 = TEMP1
E2 = TEMP2
305 CONTINUE
A1 = (T * E1 - E2 + AJN(1)) - X * (T * F1 - F2 + AJP(1))
J = N3
F1 = DAI(P(J))
E1 = DAI(N(J))
F2 = 0.
E2 = 0.
DO 306 I = 1, M3
J = J - 1
TEMP1 = F1
TEMP2 = E1
F1 = TT * F1 - F2 + DAI(P(J))
E1 = TT * E1 - E2 + DAI(N(J))
F2 = TEMP1
E2 = TEMP2
306 CONTINUE
DAI = X * X * (T * F1 - F2 + DAI(P(1))) + (T * E1 - E2 + DAI(N(1))
RETURN
C
350 CONTINUE
T=10./C-1.
TT=T+T
J=N4
F1=A(J)
E1=B(J)
F2=0.
E2=0.
DO 310 I=1,N4
J=J-1
TEMP1=F1
TEMP2=E1
F1=TT*F1-F2+A(J)
E1=TT*E1-E2+B(J)
F2=TEMP1
E2=TEMP2
310 CONTINUE
TEMP1=T*F1-F2+A(I)
TEMP2=T*E1-E2+B(I)
RTRX=SQRT(RX)
CV=C-PI12
CCV=COS(CV)
SCV=SIN(CV)
A1=(TEMP1*CCV-TEMP2*SCV)/RTRX
J=N4D
F1=DA(J)
E1=DB(J)
F2=0.
E2=0.
DO 311 I=1,N4D
J=J-1
TEMP1=F1
TEMP2=E1
F1=TT*F1-F2+DA(J)
E1=TT*E1-E2+DB(J)
F2=TEMP1
E2=TEMP2
311 CONTINUE
TEMP1=T*F1-F2+DA(I)
TEMP2=T*E1-E2+DB(I)
E1=CCV*CON5+.5*SCV
E2=SCV*CON5-.5*CCV
DA1=(TEMP1*E1-TEMP2*E2)*RTRX
RETURN
END

FUNCTION GAMLN(X)
C------------------------------------GLN2
C WRITTEN BY D. E. AMOS, SEPTEMBER, 1977. GLN3
C REFERENCES
C * SAND-77-1518 GLN4
C * COMPUTER APPROXIMATIONS BY J.F.HART, ET AL., SIAM SERIES IN GLN5
C APPLIED MATHEMATICS, WILEY, 1968, P.135-136. GLN6
C * NBS HANDBOOK OF MATHEMATICAL FUNCTIONS, AMS 55, BY GLN7
C M. ABRAMOWITZ AND I.A. STEGUN, DECEMBER, 1955, P.257. GLN8
C ABSTRACT
C
C Figure A3. (Sheet 22 of 25)
GAMLN COMPUTES THE NATURAL LOG OF THE GAMMA FUNCTION FOR X.GT.0. A RATIONAL CHEBYSHEV APPROXIMATION IS USED ON 0.LT.X.LT.1000., THE ASYMPTOTIC EXPANDSION FOR X.GE.1000. AND A RATIONAL CHEBYSHEV APPROXIMATION ON 2.LT.X.LT.3. FOR 0.LT.X.LT.8. AND X NON-INTEGRAL, FORWARD OR BACKWARD RECURSION FILLS IN THE INTERVALS 0.LT.X.LT.2 AND 3.LT.X.LT.8. FOR X=1.,2.,...,100., GAMLN IS SET TO NATURAL LOGS OF FACTORIALS.

DESCRIPTION OF ARGUMENTS

INPUT
X - X.GT.0

OUTPUT
GAMLN - NATURAL LOG OF THE GAMMA FUNCTION AT X

------------

DIMENSION GLN(100),P(5),Q(2),PCOE(9),QCOE(4)

DATA XLIM1,XLIM2,RTWPIL/8. ,1000. , 9.189385332047E-01/

DATA P(1)/7.663451880000E-04/, P(2)/-5.940956105200E-04/, P(3)/7.936431104845E-04/, P(4)/-2.777777756577E-03/, P(5)/8.333333333333332/0-02/

DATA Q(1)/-2.777777777777778E-03/, Q(2)/8.3333333333333333E-02/

DATA PCOE(1)/2.973786644810E-05/,PCOE(2)/9.238194595028E-03/,PCOE(3)/1.093115956710E-01/,PCOE(4)/3.980671310240E-01/,PCOE(5)/2.159943128461E+00/,PCOE(6)/6.338067999387E00/,PCOE(7)/2.07827253179E01/,PCOE(8)/3.603677253002E01/,PCOE(9)/6.200383800713E01/

DATA QCOE(1)/1.000000000000E00/, QCOE(2)/8.906016594898E00/, QCOE(3)/9.822521104714E00/, QCOE(4)/6.200383800713E01/

DATA GLN(1)/0.00/, GLN(2)/0.00/, GLN(3)/6.93141705599E-01/, GLN(4)/1.791759462298E00/, GLN(5)/3.178053593048E00/, GLN(6)/4.787491742782E00/, GLN(7)/6.579251212010E00/, GLN(8)/8.525161361065E00/, GLN(9)/1.060460290274E01/, GLN(10)/1.280182748008E01/, GLN(11)/1.510441257308E01/, GLN(12)/1.750230784587E01/, GLN(13)/1.998721449566E01/, GLN(14)/2.255216355121E01/, GLN(15)/2.519122118274E01/, GLN(16)/2.789927135384E01/, GLN(17)/3.067186016608E01/, GLN(18)/3.350507345014E01/, GLN(19)/3.635445208035E01/, GLN(20)/3.933988418720E01/, GLN(21)/4.233561646075E01/, GLN(22)/4.538013898984E01/, GLN(23)/4.847118135184E01/, GLN(24)/5.160667555776E01/, GLN(25)/5.478472979011E01/, GLN(26)/5.80360522298E01/, GLN(27)/6.126170176100E01/, GLN(28)/6.455753622710E01/, GLN(29)/6.788974313718E01/, GLN(30)/7.125703896717E01/, GLN(31)/7.465823634838E01/, GLN(32)/7.809222353532E01/, GLN(33)/8.155795945612E01/, GLN(34)/8.505446701758E01/, GLN(35)/8.858082752202E01/, GLN(36)/9.213617560396E01/, GLN(37)/9.571969453241E01/, GLN(38)/9.933601245479E01/, GLN(39)/1.039681986145E02/, GLN(40)/1.086531760260E02/, GLN(41)/1.130203697148E02/, GLN(42)/1.140342117815E02/, GLN(43)/1.177718813997E02/, GLN(44)/1.215330815154E02/, GLN(45)/1.253172711494E02/, GLN(46)/1.291239326591E02/, GLN(47)/1.329525750356E02/, GLN(48)/1.368027226573E02/, GLN(49)/1.406739253648E02/, GLN(50)/1.445657439463E02/, GLN(51)/1.484777669518E02/

Figure A3. (Sheet 23 of 25)
GLN(52)/1.524095925845E+02/, GLN(53)/1.567608363031E+02/, GLN(54)/1.603311282166E+02/, GLN(55)/1.643201126232E+02/, GLN(56)/1.681274454846E+02/, GLN(57)/1.723527971392E+02/, GLN(58)/1.765954840709E+02/, GLN(59)/1.804562914175E+02/, GLN(60)/1.845358288614E+02/

GLN(61)/1.888291734237E+02/, GLN(62)/1.927390472878E+02/, GLN(63)/1.967361947298E+02/, GLN(64)/2.007331472520E+02/, GLN(65)/2.047302102862E+02/, GLN(66)/2.087272733204E+02/, GLN(67)/2.127243363547E+02/, GLN(68)/2.167213993890E+02/, GLN(69)/2.207184634233E+02/, GLN(70)/2.247155264574E+02/, GLN(71)/2.287125894916E+02/, GLN(72)/2.327096525257E+02/, GLN(73)/2.367067155598E+02/, GLN(74)/2.407037785939E+02/, GLN(75)/2.447008416281E+02/, GLN(76)/2.486979046622E+02/, GLN(77)/2.526949676964E+02/, GLN(78)/2.566920307305E+02/, GLN(79)/2.606890937647E+02/, GLN(80)/2.646861568088E+02/, GLN(81)/2.686832198529E+02/, GLN(82)/2.726802829070E+02/, GLN(83)/2.766773459511E+02/, GLN(84)/2.806744089942E+02/, GLN(85)/2.846713709283E+02/, GLN(86)/2.886683329224E+02/, GLN(87)/2.926652949165E+02/, GLN(88)/2.966622559506E+02/, GLN(89)/3.006592169847E+02/, GLN(90)/3.046561779688E+02/, GLN(91)/3.086531389529E+02/, GLN(92)/3.126501000471E+02/, GLN(93)/3.166468610312E+02/, GLN(94)/3.206439220253E+02/, GLN(95)/3.246408830194E+02/, GLN(96)/3.286378440135E+02/, GLN(97)/3.326348050076E+02/, GLN(98)/3.366317660017E+02/, GLN(99)/3.406287270958E+02/, GLN(100)/3.446256880999E+02/

---

5  NDX=X
6  T=X-FLOAT(NDX)
7  IF(T.EQ.0.0) GO TO 51
8  DX=XLIM1-X
9  IF(DX.LT.0.0) GO TO 40

C RATIONAL CHEBYSHEV APPROXIMATION ON 2.LT.X.LT.3 FOR GAMMA(X)
C
C NXM=NDX-2
10 PX=PCOE(1)
11 DO 10 I=2,9
       PX=P*X+PCOE(I)
15 QX=QCOE(1)
16 DO 15 I=2,4
       QX=Q*X+QCOE(I)
18 DGAM=PX/QX
19 IF(NXM.GT.0) GO TO 22
20 IF(NXM.EQ.0) GO TO 25

C BACKWARD RECURSION FOR 0.LT.X.LT.2
C
C DGAM=DGAM/(1.+T)
21 IF(NXM.EQ.-1) GO TO 25
22 DGAM=DGAM/T
23 GAMLN=ALOG(DGAM)
24 RETURN

C FORWARD RECURSION FOR 3.LT.X.LT.8
C
C Figure A3. (Sheet 24 of 25)
22 XX=X+T
   DO 24 I=1,NXM
   DGAM=DGAM*X
24  XX=XX+1.
25 GAMLN=ALOG(DGAM)
   RETURN
C
   X.GT.XLIM1
   GLN127
   GLN128
   GLN129
C
40 RX=X/X
   RXX=RXX*RX
   IF((X-XLIM2).LT.0.) GO TO 41
   PX=Q(1)*RXX+Q(2)
   GAMLN=P*RX*(X-.5)*ALOG(X)-X+RTWPIL
   RETURN
C
   X.LT.XLIM2
   GLN130
   GLN131
C
41 PX=P(1)
   SUM=(X-.5)*ALOG(X)-X
   DO 42 I=2,5
   PX=PX*RXX+P(I)
42 CONTINUE
   GAMLN=PX*RX+SUM+RTWPIL
   RETURN
C
   TABLE LOOK UP FOR INTEGER ARGUMENTS LESS THAN OR EQUAL 100.
   GLN134
   GLN135
   GLN136
C
51 IF(NDX.GT.100) GO TO 40
   GAMLN=GLN(NDX)
   RETURN
END

Figure A3. (Sheet 25 of 25)
APPENDIX B: NOTATION

\( a_0 \)  
Incident wave amplitude

\( A_0 \)  
\(-\text{i}ga_0/\omega\)

\( g \)  
Gravitational acceleration

\( h \)  
Water depth

\( i \)  
\( \sqrt{-1} \)

\( J \)  
Bessel function of the first kind

\( k \)  
Wave number

\( n \)  
Non-negative integers

\( r \)  
Polar coordinate

\( t \)  
Temporal coordinate

\( x \)  
Horizontal coordinate

\( y \)  
Horizontal coordinate

\( Y \)  
Bessel function of the second kind

\( z \)  
Vertical coordinate

\( a \)  
Incident wave angle

\( \eta \)  
Free surface displacement

\( \theta \)  
Polar coordinate

\( \theta_0 \)  
Angle related to wedge angle

\( \nu \)  
Value related to wedge angle

\( \phi \)  
Velocity potential function

\( \phi \)  
Horizontal component of the velocity potential function

\( \phi_0 \)  
Defined in Equation 12

\( \phi_i \)  
Incident wave velocity potential function

\( \phi_r \)  
Reflected wave velocity potential function

\( \phi_s \)  
Scattered wave velocity potential function

\( \phi \)  
Finite cosine transform of \( \phi \)

\( \omega \)  
Wave radian frequency
END
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