A MINIMAX METHOD
FOR A CLASS OF HAMILTONIAN SYSTEMS
WITH SINGULAR POTENTIALS

Abbas Bahri
and
Paul H. Rabinowitz

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

October 1987

(Received August 25, 1987)

Approved for public release
Distribution unlimited

Sponsored by
U. S. Army Research Office
Research Triangle Park
North Carolina 27709

Air Force Office of Scientific Research
Bolling AFB
Washington, DC 20332

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550
This paper presents a minimax method which gives existence and multiplicity results for time periodic solutions of a class of Hamiltonian systems when a singular potential is present. The singularity satisfies the strong force condition of Gordon. When milder singularities are permitted a notion of generalized T-periodic solution is introduced and we get existence and multiplicity results for such solutions.

AMS (MOS) Subject Classifications: 34C25, 58E05, 70F99

Key Words: minimax method, calculus of variations, Hamiltonian systems, singular potential, strong force condition, collision orbits, weak force condition, generalized T-periodic solution.

This research was supported in part by the United State Army under Contract No. DAAL03-87-K-0043, the Air Force Office of Scientific Research under Grant No. AFOSR-87-0202 and the National Science Foundation under Grant No. MCS-8110556.
A MINIMAX METHOD FOR A CLASS OF HAMILTONIAN SYSTEMS WITH SINGULAR POTENTIALS

Abbas Bahri* and Paul H. Rabinowitz**

Introduction

Recently there has been a considerable amount of work on the existence of time periodic solutions of prescribed period for Hamiltonian systems of the form

\[ \ddot{q} + V'(q) = 0. \]

Here \( q = (q_1, \ldots, q_n) \), \( n > 2 \), \( V : \mathbb{R}^n \setminus S + \mathbb{R} \), \( V \) and \( V' \to 0 \) as \( |q| \to \infty \) and \( V \) is singular on \( S \), i.e. \( |V(q)| \to \infty \) as \( q \to S \). The case where \( V \) depends explicitly on \( t \) in a \( T \)-periodic fashion has also been treated. See e.g. Ambrosetti and Coti-Zelati [1-2], Coti-Zelati [3], Degiovanni, Giannoni, and Marino [4], Greco [5-6], and especially the extensive bibliographies of [1] and [3]. These papers were motivated in part by earlier work of Gordon [7] which is mainly for \( n = 2 \). Our own study of singular Hamiltonian systems was a consequence of our interest in [5]; we only learned later of [1-4].

The major focus of this paper is with singular potentials for which \( S \) is a single point which is taken to be the origin. Slight modifications of our methods permit us to treat more general compact sets \( S \). To describe our results assume \( V \) satisfies:

\[ (V_1) \quad V \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}), \quad n > 3. \]

* Department of Mathematics, Ecole Polytechnique, Palaiseau, France.

** Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706.

This research was supported in part by the United States Army under Contract No. DAAL03-87-K-0043, the Air Force Office of Scientific Research under Grant No. AFOSR-87-0202 and the National Science Foundation under Grant No. MCS-8110556.
\((V_2)\) \(V(q) < 0\) and \(V(q), V'(q) \to 0\) uniformly as \(|q| \to \infty\),

\((V_3)\) \(-V(q) \to \infty\) as \(q \to 0\),

\((V_4)\) There is a neighborhood \(W\) of \(0\) in \(\mathbb{R}^n\) and a function

\[ U \in C^1(W \setminus \{0\}, \mathbb{R}) \] such that \(U(q) \to \infty\) as \(q \to 0\) and \(-V(q) > \left|U'(q)\right|^2\)

for \(q \in W \setminus \{0\}\),

\((V_5)\) \(K \equiv \{V(q) \mid V'(q) = 0\}\) is bounded.

Hypothesis \((V_4)\) governs the rate at which \(-V(q) \to \infty\) as \(q \to 0\). It was introduced by Gordon who called it the strong force condition. If e.g.

\[ V(q) = -|q|^{-\beta} \] for \(q\) near \(0\), \((V_4)\) is satisfied if \(\beta > 2\). Thus the Coulomb potential with \(\beta = 2\) satisfies \((V_4)\) but the gravitational potential with \(\beta = 1\) is excluded.

We have two types of results for \((HS)\). First in §1 under hypotheses like \((V_1) - (V_4)\), the existence of \(T\) periodic solutions of \((HS)\) for any \(T > 0\) will be established. Other authors [1-3], [5-6] have obtained similar results. Like them, we will use the calculus of variations to obtain solutions of \((HS)\) as critical points of the corresponding functional

\[ I(q) = \int_{0}^{T} \left[ \frac{1}{2} |q|^2 - V(q) \right] dt. \]

The main novelty in our treatment of \(I\) is our rather geometrical minimax characterization of a corresponding critical value, \(c\). This approach leads to an estimate in §2 relating \(c\) and \(T\) of the form

\[ T < c_{\varphi}(2(2Tc)^{1/2}) \]

where

\[ \varphi(r) = \max_{0 < |q| < r} -(V(q))^{-1}. \]

Inequality \((0.2)\) leads to multiplicity results for \((HS)\) like:
Theorem 0.3: If \( V \) satisfies (\( V_1 \)) - (\( V_5 \)), then for any \( T > 0 \), (HS) possesses infinitely many distinct nonconstant \( T \)-periodic solutions.

Our second type of result for (HS) is also based on inequality (0.2) and involves removing the condition (\( V_4 \)). One of the consequences of (\( V_4 \)) is that if \( q \in W^{1,2} \) and \( V \) satisfies (\( V_1 \)), (\( V_3 \)) - (\( V_4 \)) then \( q(t) \neq 0 \) for all \( t \in [0,T] \). However if (\( V_4 \)) is eliminated, it is possible that a \( W^{1,2} \) solution \( q \) of (HS) can vanish somewhere, i.e. enter the singularity of \( V \). Such "collision" orbits cannot be classical solutions of (HS). Thus a broader notion of solution is needed. See also [8] in this regard. In §3 a notion of generalized \( T \)-periodic solutions of (HS) will be introduced. With the aid of the existence results of §1 - 2, and an approximation argument, it will be shown that (HS) possesses multiple generalized \( T \)-periodic solutions. E.g. we have

Theorem 0.4: If \( V \) satisfies (\( V_1 \)) - (\( V_3 \)), then for each \( T > 0 \), (HS) has infinitely many distinct generalized \( T \)-periodic solutions

§1. A minimax method to solve (HS)

In order to establish the existence of periodic solutions of (HS) via a minimax argument, a few preliminaries are required. Let \( C_T(\mathbb{R}, \mathbb{R}^n) \) denote the Banach space of \( T \)-periodic functions on \( \mathbb{R} \) with values in \( \mathbb{R}^n \) under the usual \( L^\infty \) norm. Let \( E_T \equiv W^{1,2}_T(\mathbb{R}, \mathbb{R}^n) \) denote the space of \( T \)-periodic functions on \( \mathbb{R} \) with values in \( \mathbb{R}^n \) under the norm

\[
\|q\| = \left( \int_0^T |q|^2 \, dt + |q|^2 \right)^{1/2}
\]

where

\[
|q| = \frac{1}{T} \int_0^T q(t) \, dt
\]

When there is no ambiguity, the subscript \( T \) will be omitted. Note that
$E_T \subset C_T(R, R^n)$. Let

$$\Lambda = \{ q \in E_T \mid q(t) \neq 0 \text{ for all } t \in [0, T] \},$$

i.e. $\Lambda$ is the subset of $E$ of loops which avoid the origin. Clearly $\Lambda$ is an open subset of $E$. Standard results and the definition of $\Lambda$ imply that if

$$I(q) = \int_0^T \frac{1}{2} |q|^2 - V(q) \, dt,$$

then $I \in C^1(\Lambda, R)$ and any critical point of $I$ on $\Lambda$ is a classical solution of (HS) [9]. The family of constant loops, $R^n \setminus \{0\}$ lies in $\Lambda$ and will be denoted by $\tilde{\Lambda}$.

Critical points of $I$ will be obtained by minimizing $I$ over certain surfaces that will be introduced next. Let $D^n$ denote the unit ball in $R^n$.

For $n > 3$, let

$$\Gamma = \{ h \in C(D^{n-2}, \Lambda) \mid h\bigg|_{S^{n-3}} \equiv \text{constant} \}$$

while for $n = 3$

$$\Gamma = \{ h \in C([-1, 1], \Lambda) \mid h(1) \text{ and } h(-1) \in \tilde{\Lambda} \}.$$

Identifying $0$ and $T$, and the interval $[0, T]$ with $S^1$, associated with each $h \in \Gamma$ is a map $\tilde{h} \in C(D^{n-2} \times S^1, S^{n-1})$ defined by

$$\tilde{h}(x, t) = \frac{h(x)(t)}{|h(x)(t)|}.$$

For $n > 3$, this map is constant on $S^{n-3} \times S^1$, the boundary of $D^{n-2} \times S^1$. Therefore $\tilde{h}$ can be considered to be a map from $S^{n-2} \times S^1$ into $S^{n-1}$. As such it has a degree which will be denoted by $\deg \tilde{h}$. The situation is a bit different for $n = 3$. Then

$$\tilde{h} : [-1, 1] \times S^1 \rightarrow S^2$$

with $\tilde{h}(1, t) \equiv \tilde{h}(1, T)$ and $\tilde{h}(-1, t) \equiv \tilde{h}(-1, T)$. Collapsing $[-1] \times S^1$ to a
point $\xi$ and $\{1\} \times S^1$ to $-\xi$, $\tilde{h}$ may be viewed as a map of $S^2$ into $S^2$ and therefore has a degree, $\deg \tilde{h}$.

Let

$\Gamma^* = \{ h \in \Gamma \mid \deg \tilde{h} \neq 0 \}$.

**Lemma 1.2:** $\Gamma^* \neq \emptyset$.

**Proof:** This is obvious if $n = 3$. If $n > 3$, more care is needed. The set $S^{n-2} \times S^1$ can be parametrized by $(x, e^{it})$ where $x \in \mathbb{R}^{n-1}$ with $|x|^2 = x_1^2 + \ldots + x_{n-1}^2 = 1$ and $t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Define a map $g : S^{n-2} \times S^1 + S^{n-1}$ as follows:

$$g(x, e^{it}) = \left( x \sqrt{1 - \frac{4t^2}{\pi^2}}, \frac{2t}{\pi} \right), \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

$$= (-\cos t, 0, \ldots, 0, \sin t), \quad t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

Then $g \in C(S^{n-2} \times S^1, S^{n-1})$. To calculate its degree, choose a regular point of $g$, e.g. $(-1,0,\ldots,0)$. The inverse image of this point is $(-1,0,\ldots,0) \times \{1\} \subset S^{n-1} \times S^1$. Therefore $\deg g = 1$ or $-1$ depending on the orientation chosen for $S^{n-2} \times S^1$ and $S^{n-1}$. It is then easy to see that there exists $h \in \Gamma$ such that $g = \tilde{h}$. Therefore $\Gamma^* \neq \emptyset$.

Now a minimax value of $I$ can be defined as

$$(1.3) \quad c = \inf_{h \in \Gamma^*} \max_{x \in S^{n-2}} I(h(x)).$$

We will show that under appropriate conditions on $V$, $c$ is a positive critical value of $I$. A few technical points are required to do this. First:

**Proposition 1.4:** If $V$ satisfies $(V_1) - (V_2)$ and

$(V_6)$ There is an $\alpha > 0$ such that
\[
\lim \inf_{q \to 0} -V(q) > a,
\]
then \( c > 0. \)

Proof: If not, there is a sequence \((h_m) \subset \Gamma^*\) such that \( I(h_m(x)) \to 0\) as \( m \to \infty \) for all \( x \in \mathbb{D}^{n-2} \). By (1.1) and \((V_2)\), as \( m \to \infty \),

\[
(1.5) \quad \frac{d}{dt} \|h_m(x)\|_{L^2} \to 0.
\]

Since for all \( q \in E \),

\[
(1.6) \quad \|q - [q]\|_{L^\infty} < \frac{1}{2} \|q\|_{L^2},
\]

(1.5) shows that

\[
(1.7) \quad \|h_m(x) - [h_m(x)]\|_{L^\infty} \to 0
\]
as \( m \to \infty \). By \((V_2)\) and (1.1) again,

\[
(1.8) \quad \int_0^T V(h_m(x)(t))dt \to 0
\]
as \( m \to \infty \). Consequently by (1.7), \((V_2)\), and \((V_6)\), \([h_m(x)] \to \infty\). It follows that for large \( m \), \( h_m(\cdot) \) is homotopic in \( \Gamma \) to \([h_m(\cdot)]\). Therefore for \( n > 3 \) and large \( m \), \( h_m \) is homotopic to a map \( \varphi \in C(S^{n-2} \times S^1, S^{n-1}) \):

\[
(1.9) \quad \varphi(x,t) = \frac{[h_m(x)]}{\|h_m(x)\|}
\]
while for \( n = 3 \), \( \varphi \in C(S^2, S^2) \) and is defined as in (1.9). For \( n > 3 \), \( \varphi \) factors through the projection map \( S^{n-2} \times S^1 \to S^{n-2} \) since it is independent of \( t \). Therefore \( \deg \varphi = 0 \). Similarly if \( n = 3 \), \( \varphi \) factors through the map \( (x,t) \to t \) and \( \deg \varphi = 0 \). Thus in both cases for large \( m \), \( h_m \notin \Gamma^* \), a contradiction. Hence \( c > 0 \).

Remark 1.10: Suppose \( S \subset \mathbb{R}^3 \) is compact with \( 0 \in S \) and \((V_1)\) is replaced by

\[
(V_1') \quad V \in C^1(\mathbb{R}^3 \setminus S, \mathbb{R}), \ n \geq 3.
\]
Let
\[ \Lambda_S = \{ q \in E \mid q(t) \notin S \text{ for all } t \in [0,T] \} , \]
the set of loops in $E$ which avoid $S$ and let
\[ \bar{\Lambda}_S = \{ \xi \in \mathbb{R}^n \mid \xi \notin S \} . \]
Finally set
\[ \Gamma_S = \{ h \in C(D^{n-2}, \Lambda_S) \mid h|_{S^{n-3}} \equiv \text{constant} \} . \]
Then the proof of Lemma 1.4 yields: If $V$ satisfies $(V_1)$, $(V_2)$, and

\[ (V_5) \]
\[ \lim_{q \nearrow S} -V(q) > a > 0 , \]
then $c > 0$.

Next it will be shown that $I$ satisfies the $(PS)^+$ condition on $\Lambda$, i.e.

$(PS)^+$: For any $s > 0$, if $(q_m) \subseteq \Lambda$, $I(q_m) \to s$ and $I'(q_m) \to 0$, then

$q_m$ possesses a subsequence converging to some $q \in \Lambda$.

Note that by taking a sequence $q_m \in \bar{\Lambda}$ with $q_m \to \infty$, $I(q_m) \to 0$. Thus $I$

does not satisfy $(PS)$ at level 0 and does not attain its infimum. Indeed this

is one of the difficulties in treating $I$ variationally. Our argument is a

variant of that of Greco [5]. It is easy to check that:

**Lemma 1.11:** Let $V$ satisfy $(V_1)$, $(V_3)$ - $(V_4)$. If $(q_m) \subseteq \Lambda$ and $q_m$

converges weakly in $E$ and strongly in $L^\infty$ to $q \in \partial \Omega$, then

\[ - \int_0^T V(q_m(t))dt \to \infty \]

(and therefore $I(q_m) \to \infty$).

**Proof:** See [5, Lemma 2.1].

**Proposition 1.12:** If $V$ satisfies $(V_1)$ - $(V_4)$, $I$ satisfies $(PS)^+$ on $\Lambda$.

**Proof:** Let $s > 0$ and $(q_m) \subseteq \Lambda$ with $I(q_m) \to s$ and $I'(q_m) \to 0$. By

$(1.1)$, $(q_m)$ is bounded in $L^2$. We claim $(q_m)$ is also bounded in $\mathbb{R}^n$ and
therefore \((q_m)\) is bounded in \(E\). If not, \(|q_m| \to \infty\) along a subsequence. Hence by (1.6) and \((V_2)\),

\[
(1.13) \quad \int_0^T V(q_m(t))dt \to 0
\]

and

\[
(1.14) \quad \int_0^T V'(q_m) \cdot (q_m - [q_m])dt \to 0
\]
as \(m \to \infty\). Therefore by (1.13) - (1.14),

\[
I(q_m) = \frac{1}{2} I'(q_m)(q_m - [q_m]) + \frac{1}{2} \int_0^T V'(q_m) \cdot (q_m - [q_m])dt
\]

But \(I(q_m) \to s\), strictly positive, a contradiction. Now the boundedness of \(q_m\) in \(E\) and standard embedding theorems imply along a subsequence \(q_m\) converges weakly in \(E\) and strongly in \(L^\infty\) to \(q \in E\). Since \(I(q_m) \to s\), Lemma 1.11 shows \(q \in A\). Hence the form of \(I'\) shows \(q_m \to q\) in \(E\).

**Remark 1.16:** In the spirit of Remark 1.10, if \((V_1)\) is replaced by \((V_1^1)\) and \((V_3) - (V_4)\) by

\((V_1^1)\) \lim_{q \to S} V(q) = -\infty,

\((V_4)\) There is a neighborhood \(W\) of \(S\) in \(\mathbb{R}^n\) and \(U \in C^1(W \setminus S, \mathbb{R})\) such that \(U(q) \to \infty\) as \(q \to S\) and \(-V(q) > |U'(q)|^2\) for \(q \in W \setminus S\), then the above proof shows \((PS)^+\) holds for \(A_S\).

The next step in showing that \(c\) as given by (1.3) is a critical value of \(I\) is a version of a standard "Deformation Theorem" that is appropriate for our setting. For \(\sigma \in \mathbb{R}^k\), let \(A_{\sigma} = \{q \in A \mid I(q) < \sigma\}\) and \(K_{\sigma} = \{q \in A \mid I(q) = \sigma\}\) and \(I'(q) = 0\).

**Proposition 1.17:** Let \(V\) satisfy \((V_1) - (V_4)\). Suppose \(s\) is not a critical value of \(I\). Then for all \(-c > 0\) there is an \(\varepsilon > 0\) and \(\eta \in C([0,1] \times A, A)\) such that
\[ \eta(1,q) = q \text{ if } I(q) \notin (s-\varepsilon, s+\varepsilon), \]

2° \[ I(\eta(s,q)) < I(q) \text{ for } s > 0 \]

3° \[ \eta(1,\alpha_s + \varepsilon) < \alpha_s - \varepsilon, \]

4° \[ \eta(1,\cdot) : \tilde{\Lambda} \to \tilde{\Lambda}, \]

5° \[ \eta(1,\cdot) : \Gamma^* \to \Gamma^*. \]

Proof: The proof of a "standard" version of the Deformation Theorem can be found in [9, Appendix A]. We will only indicate the minor modifications in its proof needed to handle the differences in structure encountered here. The map \( \eta \) is a solution of a differential equation of the form

\[ \frac{d\eta}{dt} = \omega(\eta)\Psi(\eta), \quad \eta(0,q) = q \]

where \( 0 < \omega < 1 \) is a cut-off function and \( \Psi \) is a pseudogradient vector field for \( I' \). The choice of \( \omega \) guarantees that 1° holds and the choice of \( \Psi \) gives 2°. Since \( I(q) + \alpha \to q + 2\alpha \), 2° and Lemma 1.11 show that \( \eta \in C([0,1] \times \Lambda, \Lambda) \).

Property 3° follows as in [9]. Note that \( I \) is invariant under a natural \( S^1 \) symmetry, namely

\[ I(q(t+\theta)) = I(q(t)) \]

for all \( \theta \in \mathbb{R}/[0,T] \). Hence \( \omega \) can be chosen so that it is also invariant under the \( S^1 \) action and \( \Psi \) so that it is equivariant with respect to this action. The fixed point set of the action in \( \Lambda \) is \( \tilde{\Lambda} \). Hence \( \eta(t,\cdot) : \tilde{\Lambda} \to \tilde{\Lambda} \) and \( \eta(t,\cdot) \) is also \( S^1 \) equivariant. Finally if \( h \in \Gamma^* \), \( h_s \equiv \eta(s,h) \in \Gamma \) via 4° and the associated map \( \tilde{h}_s \) from \( S^2 \to S^2 \) if \( n = 3 \) or from \( S^{n-2} \times S^1 \) to \( S^{n-1} \) for \( n > 3 \) is clearly homotopic to \( \tilde{h} \). Therefore \( \deg \tilde{h}_s = \deg \tilde{h} \neq 0 \) and 5° holds.

Theorem 1.20: If \( V \) satisfies (V1) - (V4), then for each \( T > 0 \), \( I \) has a critical value \( c \) given by (1.3) with a corresponding critical point \( q \in \Lambda \)
which is a classical solution of (HS).

Proof: Since (V₃) holds, c > 0 via Proposition 1.4. Using 3° and 4° in a standard way - see e.g. [9] - shows that c is a critical value of I in A. Lastly a simple regularity argument shows q is a classical solution of (HS).

Remark 1.21: If V satisfies (V¹), (V²), (V³), and (V₄), Remark 1.10 and 1.16 and a slightly modified proof of Theorem 1.20 shows the conclusions of Theorem 1.20 also hold for this setting. See also [1].

Remark 1.22: Since Theorem 1.20 holds for all T > 0, if (HS) has no equilibrium solutions, a sequence of T periodic solutions of (HS), (distinct in $E_π$), can be constructed as follows: For each $k \in \mathbb{N}$, let $q_k(t)$ be a solution of (HS) of period $T/k$ given by Theorem 1.20. Say $q_1(t)$ has minimal period $T/k_1$. Then for $k > k_1$, $q_k(t)$ is distinct from $q_1(t)$. Similar reasoning shows that infinitely many of the functions $q_k(t)$ are distinct. However if (HS) has equilibrium solutions, a more careful argument is needed to get multiple solutions and thus will be carried out in §2.

§2. A lower bound for $c$ and its consequences

In this section, the lower bound (0.2) for $c$ will be derived and it will be used to get better existence and multiplicity results for (HS) than in §1.

Proposition 2.1: Let V satisfy (V₁) - (V₃) and let c be as in (1.3). If $φ(0) = 0$ and for $r > 0$,

$$φ(r) = \max_{0<|q|<r} -(V(q))^{-1},$$

then
(2.3) \( T < c_\varphi(2(2Tc)^{1/2}) \).

Proof: \((V_1) - (V_3)\) imply that \( \varphi \) is a continuous monotone nondecreasing function with \( \varphi(r) \to \infty \) as \( r \to \infty \). For \( \theta > 0 \), set

(2.4) \( \varphi_\theta(r) = \theta r + \varphi(r) \).

Then \( \varphi_\theta \) has the same properties as \( \varphi \) and is strictly monotone increasing.

Let \( \xi \in \Lambda \). It can be written as

(2.5) \( q(t) = \xi + Q(t) \)

where \( \xi = [q] \) and \( Q \) is orthogonal in \( E \) to \( R^n \subset E \). Suppose

(2.6) \( I(q) < \beta \).

Then (1.1), (1.6), and (2.6) imply

(2.7) \( T_\xi < \left(2\beta T\right)^{1/2} \).

Now by \((V_2)\), (1.1), and (2.4),

\[
T = \int_0^T \left( T - V(q) \right)^{1/2} dt < \left( \int_0^T - V(q) dt \right)^{1/2} \left( \int_0^T - (V(q))^{-1} dt \right)^{1/2}
\]

(2.8)

\[
< \frac{1}{2} \int_0^T \varphi_\theta(||q||) dt.
\]

Substituting (2.5) into (2.8) and using (2.7) and the monotonicity of \( \varphi_\theta \) then shows

(2.9) \( T_\xi < T \cdot \varphi_\theta(||\xi|| + ||Q||) < T \cdot \varphi_\theta(||\xi|| + \left(2\beta T\right)^{1/2}) \).

Consequently

(2.10) \( \varphi_\theta^{-1}(T/b) - (2\beta T)^{1/2} < ||\xi|| \).

Let \( s \in [0,1] \). Then by (2.10) and (2.7),

(2.11) \( ||\xi + sQ(t)|| > ||\xi|| - s||Q|| \Rightarrow \varphi_\theta^{-1}(T/b) - 2(2\beta T)^{1/2} \).

Suppose that

(2.12) \( \varphi_\theta^{-1}(T/b) - 2(2\beta T)^{1/2} > 0 \).

Then (2.11) - (2.12) show those \( q \in \Lambda \) satisfying (2.6) are homotopic in \( \Lambda \) to their mean values. If (2.12) held for \( b = c + \epsilon \) for some \( \epsilon > 0 \), then
there would be an \( h \in \Gamma^* \) such that \( h(x) \) satisfies (2.6) for all \( x \in \mathbb{D}^{n-2} \). Consequently the map \( h(x) \) would be homotopic to its mean value for all \( x \in \mathbb{D}^{n-2} \) and therefore \( \deg \tilde{h} = 0 \), contrary to \( h \in \Gamma^* \). Hence (2.12) cannot hold for \( b = c + \epsilon \). Since this is the case for all \( \epsilon > 0 \), it follows that

\[
(2.13) \quad T < \varphi(\frac{2(2cT)^{1/2}}{2})
\]

and letting \( \theta \to 0 \) yields (2.3).

Remark 2.14: (i) Suppose that \( V \) also depends on \( t \) in a \( T \)-periodic fashion. Then the above argument goes through virtually unchanged to yield (2.3) for this case.

(ii) Suppose that \( (V_1) \) and \( (V_3) \) are replaced by \( (V'_1) \) and \( (V'_3) \) (with \( 0 \leq S \)) and the definition of \( \varphi \) is replaced by \( \varphi(0) = 0 \) and for \( r > 0 \),

\[
\varphi(r) = \sup_{|q| < r, q \in S} - (V(q))^{-1}.
\]

If further \( \Lambda \) and \( \Gamma^* \) are replaced by \( \Lambda_S \) and \( \Gamma^*_S \), it is easy to see that the argument of Proposition 2.1 carries over unchanged yielding (2.3) for this situation.

Two kinds of applications of (2.3) will be given. The first is to forced versions of (HS):

\[
(2.15) \quad q + V(t, q) = 0.
\]

For this setting we interpret \( (V_1) - (V_4) \) to mean the natural extension of these hypotheses to reflect the further dependence of \( V \) on \( t \) in a \( T \)-periodic fashion.

Theorem 2.16: Suppose \( V \) is \( T \)-periodic in \( t \) and satisfies \( (V_1) - (V_4) \). Then (2.15) possesses a \( T \)-periodic solution.

Proof: Let \( c \) be defined as in (1.3). By Remark 2.14(i), the estimate (2.3) holds. If \( c' \) is defined as a minimax of \( I \) over a subclass of \( \Gamma^* \), then
\(c_1 > c\). Let

\[
\hat{\Gamma} = \{ h \in \Gamma^* \mid I(h(\pm 1)) < \frac{c}{2} \}
\]

if \(n = 3\) or

\[
\hat{\Gamma} = \{ h \in \Gamma^* \mid I(h(x)) < \frac{c}{2} \text{ for } x \in S^{n-3} \}
\]

if \(n > 3\). Proposition 1.17 no longer holds in its entirety since \(V\) depends on \(t\) so \(I\) is no longer \(S^1\) invariant. However the proof holds up to (1.19). Since \(c_1 > c > \frac{c}{2}\), choosing \(\varepsilon = c/3\), by 1º of Proposition 1.17, it follows that \(\eta(1,\varepsilon) : \hat{\Gamma} \to \hat{\Gamma}\). Hence the reasoning of the proof of Theorem 1.20 shows \(c_1\) is a critical value of \(I\) and the corresponding critical point is a solution of (2.15).

Combining the proof of Theorem 2.16 with Remarks 1.10, 1.16, and 2.14(ii) immediately gives:

**Theorem 2.17:** If \(V\) is \(T\)-periodic in \(t\) and satisfies \((V_1), (V_2), (V_3) - (V_4)\), then (2.15) possesses a \(T\)-periodic solution.

As a second application of (2.3), some multiplicity results will be obtained in the setting of Theorem 1.20.

**Theorem 2.18:** If \(V\) satisfies \((V_1) - (V_4)\), \(I\) possesses an unbounded sequence of critical values.

**Proof:** For each \(k \in \mathbb{N}\), Theorem 1.20 can be applied with \(T\) replaced by \(T/k\) to the functional

\[
I_k(q) = \int_0^{T/k} \frac{1}{2} |\dot{q}|^2 - V(q) dt
\]

obtaining a critical value \(b_k\) and critical point \(q_k \in W_{T/k}^{1,2} \cap \Lambda\). By (2.3),

\[
T/k < b_k \varphi(2(T/k - b_k)^{1/2})
\]

If \(k b_k\) is bounded along some subsequence, (2.20) shows

\[
T < 0
\]

-13-
which is impossible. Hence \( k b_k \to \infty \) as \( k \to \infty \). Considering \( q_k \) as an element of \( W_T^{1,2} \), we have

\[
(2.22) \quad c_k = I(q_k) = k b_k \to \infty
\]
as \( k \to \infty \) and the Theorem follows.

**Corollary 2.23:** Under the hypotheses of Theorem 2.18, if further \((V_5)\) holds,\( (HS) \) possesses infinitely many distinct nonconstant \( T \)-periodic solutions.

**Proof:** Note that for \( q \in K, I(q) = -TV(q) \). By \((V_5)\), \( I \) is bounded on \( K \). Hence the result follows.

**Corollary 2.24:** Along a subsequence of \( k \to \infty \), either

\[
(2.25) \quad \min_{t \in [0,T]} |q_k(t)| + 0
\]
or

\[
(2.26) \quad \|q_k\|_\infty \to \infty
\]

**Proof:** By \((2.22)\), \( I(q_k) \to \infty \) as \( k \to \infty \). The form of \( I \) and \((V_1)-(V_4)\) imply either \((2.24)\) holds or

\[
(2.27) \quad \|q_k\|_2 \to \infty
\]
along some subsequence of \( k \to \infty \). \((HS)\) implies

\[
(2.28) \quad \|q_k\|^2 = \int_0^T V'(q_k) \cdot q_k dt
\]
so if neither \((2.25)\) nor \((2.26)\) were valid, by \((2.28)\) and \((V_1)-(V_4)\), \( q_k \)
would be bounded in \( L^2 \), contrary to \((2.27)\).

**Remark 2.29:** Corollary 2.24 implies infinitely many of the loops \( q_k(t) \) are geometrically distinct.
Once again by applying the reasoning of Theorem 2.18 together with Remarks 1.10, 1.16, and (2.14)(ii), we get an analogous result where there is a more general singular set.

**Theorem 2.30:** If $V$ satisfies $(V_1'), (V_2'), (V_3') - (V_4)$, then for each $T > 0$, (HS) possesses infinitely many distinct $T$-periodic solutions. If further $(V_5)$ holds, (HS) possesses infinitely many distinct nonconstant solutions.

**Remark 2.31:** Ambrosetti and Coti-Zelati [1, 3] have obtained analogues of Theorems 2.18 and 2.30 when $V$ depends on $t$ in a $T$-periodic fashion. Their arguments are less direct than ours. It is unclear as to whether they extend to treat cases like those of the next section.

§3. A weak force condition

In this section will show how dropping $(V_4)$ still leads to the existence of multiple $T$-periodic solutions of (HS). As was seen in Lemma 1.11, under hypothesis $(V_3) - (V_4)$, if $q \in W^{1,2}_{T}(\mathbb{R}, \mathbb{R}^n)$ and $I(q) < \infty$, then $q \in \Lambda$. If $(V_3)$ merely holds, this is no longer the case. Functions $q \in E$ can be constructed so that $q(0) = 0$. Thus critical points of $I$ under $(V_3)$ may enter the singularity, i.e. may be "collision orbits". This leads us to the following notion of generalized solution of (HS).

**Definition 3.1:** A function $q \in C_T(\mathbb{R}, \mathbb{R}^n)$ is a generalized $T$-periodic solution of (HS) if

1. $q$ vanishes on a set $\mathcal{D}$ of measure 0
2. $q \in C^2_T(\mathbb{R} \setminus \mathcal{D}, \mathbb{R}^n)$
3. $q$ satisfies (HS) on $\mathbb{R} \setminus \mathcal{D}$
4. $q \in E_T$ and $I(q) < \infty$
5. $\frac{1}{2} |\dot{q}(t)|^2 + V(q(t))$ is constant for $t \in \mathbb{R} \setminus \mathcal{D}$. 

-15-
Note that energy is conserved for \((HS)\) in each component of \(\mathbb{R}^d\). Condition (v) in Definition 3.1 further requires that the constant is the same for each component.

By using an approximation argument based on the results of §1—2, we will show that \((HS)\) possesses multiple generalized \(T\)-periodic solutions. More precisely we have:

**Theorem 3.2:** Let \(V\) satisfy \((V_1)-(V_3)\). Then for each \(T > 0\) \((HS)\) possesses infinitely many distinct generalized \(T\)-periodic solutions in \(E_T\).

**Proof:** For each \(\delta > 0\), let \(V_\delta(q)\) be a potential satisfying \((V_1)-(V_4)\),

\[
V_\delta = V \text{ if } |q| > \delta, \quad \text{and} \quad -V_\delta > -V.
\]

Let \(k \in \mathbb{N}\) and set

\[
I_{k, \delta}(q) = \int_0^{T/k} \left[ \frac{1}{2} |\dot{q}|^2 - V_\delta(q) \right] dt.
\]

We claim there exist constants \(a_k, b_k\) independent of \(\delta\) such that if \(q_{k, \delta} \in W_{1/2}^{1/2} \cap \Lambda\) is the critical point of \(I_{k, \delta}\) obtained via Theorem 2.18, then

\[
0 < a_k < I_{1, \delta}(q_{k, \delta}) < b_k.
\]

Furthermore \(a_k\) can be chosen so that

\[
a_k \to \infty \text{ as } k \to \infty.
\]

To verify these assertions, let

\[
\Lambda_k = \{ q \in \Lambda \mid q \text{ is } T/k \text{ periodic} \}
\]

and

\[
\Gamma_k = \{ h \in C(D^{n-2}, \Lambda_k) \mid \deg h \neq 0 \}.
\]

Choose \(h_1 \in \Gamma_1\). Therefore for the associated map \(\mathcal{h}_1\), there is a constant \(\sigma > 0\) such that

\[
\min_{x \in D^{n-2}, t \in [0,T]} |\mathcal{h}_1(t)(t)| > \sigma > 0.
\]

Define \(h_k\) by
(3.8) \[ h_k(x)(t) = h_1(x)(kt) \]

Hence \( h_k \in \Gamma_k \) and by (3.8),

(3.9) \[ \min_{x \in D^{n-2}, t \in [0, T/k]} |\tilde{h}_k(x)(t)| > \sigma \]

Thus if \( \delta < \sigma \),

(3.10) \[ V_\delta(h_k(x)(t)) = V(h_k(x)(t)) \]

Therefore for \( \delta < \sigma \),

\[
  c_k(\delta) \equiv I_{1, \delta}(q_k, \delta) = k \inf_{h \in \Gamma_k} \max_{x \in D^{n-2}} I_{k, \delta}(h(x))
\]

(3.11) \[ \leq k \max_{x \in D^{n-2}} I_{k, \delta}(h_k(x)) = k \max_{x \in D^{n-2}} I_k(h_k(x)) = \beta_k \]

To get the lower bound in (3.5), let \( \varphi(0) = 0 \) and for \( s > 0 \),

(3.12) \[ \varphi_s(s) = \max_{0 < |q| < s} \left( V_\delta(q) \right)^{-1} \]

Then by (3.3),

(3.13) \[ \varphi_s(s) < \varphi(s) \]

for \( s > 0 \). Now \( c_k(\delta) = k I_{k, \delta}(q_k, \delta) = k b_k(\delta) \) and by (2.3),

\[
  T/k < b_k(\delta) \varphi(2(2T_0 b_k(\delta))^{1/2})
\]

or using (3.13),

(3.14) \[ T < c_k(\delta) \varphi(2(2Tk^{-2} c_k(\delta))^{1/2}) \]

Define

\[ a_k = \inf_{0 < \delta < \sigma} c_k(\delta) \]

Passing to a limit in (3.14) shows

(3.15) \[ T < a_k \varphi(2(2Tk^{-2} a_k)^{1/2}) \]

Hence the form of \( \varphi \) shows \( a_k > 0 \) yielding (3.5). Finally if \( (a_k) \) were bounded in \( k \) along a subsequence, passing to a limit in (3.15) shows \( T < 0 \) which is impossible. Therefore (3.6) holds.
Now the existence of generalized $T$-periodic solutions of (HS) can be established. Fixing $k$ and letting $\delta \to 0$, (3.5) provides upper bounds for

$$I_{k,\delta} = \|q_k,\delta\|^2_L - \int_0^T V_\delta(q_k,\delta(t))\,dt.$$  

Note that $\xi_{k,\delta} = [q_k,\delta]$ must be bounded independently of $\delta$ for otherwise $V_\delta(q_k,\delta(t))$ and $V_{k,\delta}'(q_k,\delta(t)) \to 0$ uniformly for $t \in [0,T]$ and then as in (1.13) - (1.15), $a_k = 0$. Hence $(q_k,\delta \mid \delta < \sigma)$ are bounded in $E$. Consequently a subsequence of these functions converges weakly in $E$ and strongly in $L^\infty$ to $q_k \in E$. Moreover

$$-\int_0^T V(q_k(t))\,dt < b_k.$$  

Indeed by (3.5), for all $\delta < \sigma$

$$-\int_0^T V_\delta(q_k,\delta(t))\,dt < b_k.$$  

Let $\varepsilon > 0$ and $\chi_\varepsilon(s) = 0$ if $s < \varepsilon$ and $\chi_\varepsilon(s) = 1$ if $s > \varepsilon$. Then (3.17) implies

$$-\int_0^T \chi_\varepsilon(|q_k,\delta(t)|) V_\delta(q_k,\delta(t))\,dt < b_k.$$  

Since $q_k,\delta \to q_k$ in $C_T(\mathbb{R},\mathbb{R}^n)$, (3.18) shows

$$-\int_0^T \chi_\varepsilon(|q_k(t)|) V(q_k(t))\,dt < b_k.$$  

Letting $\varepsilon \to 0$ in (3.19) yields (3.17). By (3.17) and (V3), the set $D_k$ where $q_k$ vanishes must have measure 0. Let $\tau \in [0,T] \setminus k$. Then there is an $\varepsilon, \omega > 0$ such that $|t-\tau| < \omega$ implies $-V(q_k(t)) > \varepsilon$. Since $q_k,\delta \to q_k$ uniformly in $C^2$ on $|t-\tau| < \omega$ (along a subsequence) and $q_k,\delta$ satisfies (HS) for $V_\delta$, it readily follows that $q_k$ is a classical solution of (HS) on $\mathbb{R}\setminus D_k$. Next observe that (v) of Definition 3.1 holds for $q_k,\delta$ with $D = \emptyset$. Hence by passing to a limit we get it for $q_k$ on $\mathbb{R}\setminus D_k$.

At this point for each $k \in \mathbb{N}$, we have constructed a $T/k$ periodic solution $q_k$ of (HS). Note that if $q_k$ is an equilibrium solution of (HS), $|q_k| \neq 0$ and therefore $q_k,\delta \to q_k$ uniformly in $C^2_T(\mathbb{R},\mathbb{R}^n)$ as $\delta \to 0$. Consequently

$$I_{1,\delta}(q_k,\delta) + I(q_k) > a_k.$$  

-18-
This observation shows that infinitely many of the functions $q_k$ are distinct. Indeed if a subsequence of the $q_k$ were constant solutions, by (3.20) and (3.6), $I(q_k) > a_k \to \infty$ as $k \to \infty$ along this subsequence. On the other hand if there were only finitely many constant solutions, since the period $T/k$ of $q_k$ approaches 0 as $k \to \infty$, infinitely many of the (nonconstant) $q_k$ must be distinct.

Corollary 3.21: If $V$ satisfies $(V_1) - (V_3)$ and $(V_5)$, then for each $T > 0$, (HS) possesses infinitely many distinct nonconstant generalized $T$-periodic solutions.

Proof: If not, by Theorem 3.2, $q_k$ is a constant solution for all $k \in \mathbb{N}$ and

$$(3.22) \quad I(q_k) = -TV(q_k) > a_k$$

so $I(q_k) \to \infty$ as $h \to \infty$ by (3.20). But by $(V_5)$, $I$ is bounded on $K$, a contradiction.

Next consider (2.15)

Definition 3.23: A function $q \in C_T(\mathbb{R}, \mathbb{R}^3)$ is a generalized $T$-periodic solution of (2.15) if $q$ satisfies (i) - (iv) of Definition 3.1.

With this definition of solution and our understanding of the meaning of $(V_1) - (V_3)$ in the time dependent case, Theorem 2.16 and the argument of Theorem 3.2 show that

Theorem 3.24: If $V$ satisfies $(V_1) - (V_3)$, then (2.15) possesses a generalized $T$-periodic solutions.

Remark 3.25: As in §2, by replacing $(V_1)$ and $(V_3)$ by $(V_1')$ and $(V_3')$, we get analogues of Theorems 3.2 and 3.24 for this setting.
Remark 3.26: The definition of a generalized $T$-periodic solution allows such a solution to pass through the origin, indeed for possibly infinitely many values of $t \in [0,T]$. We suspect that this is not possible for a minimax solution. An interesting question to pursue is the regularity of the solutions our minimax procedure produces. Can such a solution actually be a collision orbit? Alternatively what further conditions on $V$ guarantee classical solutions?
REFERENCES


END
DATE
FILMED 7-88
BTIC