A NUMERICAL TREATMENT OF THE FLUID/ELASTIC INTERFACE UNDER RANGE-DEPENDENT (U) VALE.

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ABSTRACT

The technique introduced by McDaniel-Lee for the handling of the fluid/fluid interface boundary under range-dependent environments is extended to handle the horizontal fluid/elastic interface boundary. Representative wave equations of the parabolic type are considered in both fluid and elastic media. The required interface conditions, (1) continuity of vertical components of displacement, (2) continuity of vertical components of stress, and (3) horizontal components of stress vanish on the interface, are satisfied with this numerical treatment. A complete theoretical development is presented along with a test example to demonstrate its validity.

A Numerical Treatment of the Fluid/Elastic Interface under Range–dependent Environments

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I. INTRODUCTION

Modern computational techniques improved much of the efficiency of range-dependent wave propagation models. The efficiency as well as usefulness of the range-dependent model can be further enhanced if a capability can be incorporated to handle the fluid/elastic interface. This paper introduces a numerical treatment of the horizontal fluid/elastic interface boundary. Since the numerical treatment introduced by McDaniel-Lee [1] to handle the fluid/fluid interface in 1982, interest was on the rise in searching for efficient methods to handle the fluid/elastic interface. Some contributions were made in the seismology science to treat the fluid/elastic interface. R. Stephen [2] developed finite difference methods to treat this problem dealing with a hyperbolic wave equation. J. T. Kuo and Y. C. Teng [3] applied the finite element as well as finite difference schemes extensively to solve the same problem as above but dealing with an elliptic wave equation. The above techniques, though workable, are not simple to adapt into any existing range-dependent model without requiring excessive efforts; moreover, these techniques are by no means simple. The numerical treatment we introduced in this paper is based on the standard Parabolic Equation (PE) in the fluid medium introduced by Tappert [4] and on the coupled parabolic equations in the elastic medium derived by McCoy [5]. The fluid/elastic interface requires three conditions to be satisfied on the interface, i.e., (1) the continuity of vertical components of displacement, (2) the continuity of vertical components of
stress, and (3) horizontal components of stress must vanish on the interface. These conditions were derived to be consistent with the PE representation in both fluid and elastic media. McDaniel-Lee's technique was modified to treat these conditions numerically. This modification allows the existing implicit finite difference (IFD) [6] marching scheme to be applied systematically. A test problem with known solution, given by Ewing and Press [7], is used to examine the validity of this development.

II. BACKGROUND SUMMARY

Since the new treatment to the fluid/elastic interface boundary is an extension of the McDaniel-Lee technique, it is desirable that the McDaniel-Lee's treatment to the fluid/fluid interface boundary be briefly reviewed.

We use $u(r,z)$ to indicate the wave field, the pressure, in a 2-dimensional medium, depth and range. Thus, $u(r,z)$ satisfies the parabolic wave equation

$$u_r = a(k_o, r, z) u + b(k_o, r, z) u_{zz}, \quad (2.1)$$

where

$$a(k_o, r, z) = ik_o(n^2(r, z) - 1)/2,$$

and

$$b(k_o, r, z) = i/(2k_o).$$
where $k_0$ is the reference wavenumber, $n(r,z)$ stands for the index of refraction which is defined as a ratio of the reference sound speed $c_0$ and the sound speed $c(r,z)$.

At the ocean bottom, the change of sound speed and density form an interface (see Figure 1). From one medium to the next, at each interface, the interface conditions must be satisfied, i.e., the pressure and normal components of particle velocity are continuous at the interface.

The standard PE, Eq. (2.1), does not contain the density. In order to satisfy the interface conditions, McDaniel–Lee developed a special equation with density variations to represent the wave field on the interface. It turned out that this special equation is again a PE.
In developing this interface PE, McDaniel-Lee applied the Taylor series expansion to the points near the interface and then match the fields on the interface which is denoted by $z_B$. A clear configuration is given in Figure 2.

![Figure 2: The Interface Between Two Media](image)

In carrying out the matching process, let us describe the interface mathematically below.

The continuity of pressure requires that

$$u(r,z_B) = u_2(r,z_B), \quad (2.2)$$

the continuity of normal component of particle velocity requires that

$$\frac{\rho_1 au_1}{\rho_2 a_2} = \frac{\rho_1 u_2}{\rho_2 u_2} \bigg|_{z_B} \bigg|_{z_B} . \quad (2.3)$$

Note that the interface is assumed horizontal.
In every medium the field \( u(r,z) \) satisfies the PE, (2.1). Therefore, in medium 1, \( u_1(r,z) \) satisfies

\[
(u_1)_r = a_1(k_0, r, z) u_1 + b_1(k_0, r, z) (u_1)_{zz},
\]

where

\[
a_1(k_0, r, z) = ik_0 (n^2_1(r, z) - 1)/2.
\]

and

\[
b_1(k_0, r, z) = 1/(2k_0).
\]

Using the first three terms of the Taylor expansion for \( (u_1)_m \) upon \( (u_1)_m \) and solve for \( (u_1)_{zz} \), then substitute them into Eq. (2.4), we find

\[
\frac{3u_1}{az} = \frac{h}{2b_1} (u_1)_r - \frac{h}{2b_1} a_1 u_1 + \frac{1}{h} (u_1 - (u_1)_{m-1})
\]

where \( h = \Delta z \).

Similarly, in medium 2 use a three-term Taylor expansion for \( (u_2)_m+1 \) upon \( (u_2)_m \) and follow the same procedures as carried out in medium 1, we find

\[
\frac{3u_2}{az} = -\frac{h}{2b_2} (u_2)_r + \frac{h}{2b_2} a_2 u_2 + \frac{1}{h} ((u_2)_{m+1} - u_2).
\]
The first interface condition (2.2) allows one to write \( u_1 = u_2 = u \) on the interface. Then, multiply both sides of (2.7) by \( \rho_2 \) and multiply both sides of (2.8) by \( \rho_1 \); then the second interface condition (2.3) allows the above results to be equal. After simplification, the McDaniel-Lee horizontal interface wave field is obtained between two fluid media to be

\[
u_r = \left( \frac{1}{b_1} + \frac{\rho_1}{\rho_2} \frac{1}{b_2} \right)^{-1} \left[ \left( \frac{a_1}{b_1} + \frac{\rho_1}{\rho_2} \frac{a_2}{b_2} \right) u + \frac{2}{h^2} \left( \frac{\rho_1}{\rho_2} (u_2)(m+1) - u \right) - \left( u-(u_1)(m-1) \right) \right]
\]

(2.9)

Note that the density is assumed to remain constant in each medium.

III. FIELD REPRESENTATIONS IN FLUID/ELASTIC MEDIUM

Dealing with the fluid/elastic interface, two media are involved, i.e., a fluid medium and an elastic medium. The field representation in the fluid medium by the parabolic approximation is a scalar PE while in the elastic medium the representative wave equation is a vector PE. A standard derivation of the scalar PE in fluid medium can be found in references 4 [Tappert] and 8 [Lee-Siegmann]. The vector PE has been derived by a few authors, though their derivations, and method of derivation differ from one another. These derivations are
(1) Lander and Claerbout's equation [9], derived using dilatation and rotation,
(2) Hudson's equation [10], derived using displacement, and
(3) McCoy's equation [5], derived using dilational and shear potentials.

In this paper we use potential representation to deal with the fluid/elastic interface, thus, it is appropriate that we adopt the vector PE developed by McCoy.

In the fluid medium the parabolic wave equation was expressed in the 2-dimensional cylindrical coordinates where we use $\phi_1(r,z)$ to indicate the potential; in the elastic medium we use $\phi_2(r,z)$ and $\psi_2(r,z)$ to represent the potentials there. In the above notation, the subscript "1" is used to indicate the fluid medium, and the subscript "2" to indicate the elastic medium. We use $D_1$ to indicate the displacement in the fluid medium and $D_2$ to indicate the displacement in the elastic medium. Their relationships with the potential are given by

$$D_1 (u_1, w_1) = \text{grad} \phi_1 (r,z), \quad (3.1)$$

$$D_2 (u_2, w_2) = \text{grad} \phi_2 (r,z) + \text{rot} \psi_2 (r,z). \quad (3.2)$$

Equivalently we can write the above as

$$\begin{bmatrix} u_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_1}{\partial r} \\ \frac{\partial \phi_1}{\partial z} \end{bmatrix} \quad (3.3)$$
\[
\begin{align*}
\psi_2 &= \frac{a_2}{ar} - \frac{a_2}{az} \\
\phi_2 &= \frac{a_2}{az} + \frac{a_2}{ar} + \frac{1}{r}\psi_2
\end{align*}
\] (3.4)

In the fluid medium, the potential \( \phi_1(r,z) \) satisfies the Helmholtz equation below.

\[
\nabla^2 \phi_1(r,z) + k_1^2(r,z)\phi_1 = 0.
\] (3.5)

Following the derivation of the PE in the fluid medium by Lee-Siegmann, the wave field \( \phi_1(r,z) \) obeys the following decomposition

\[
\phi(r,z) = \psi_1(r,z) \sim A_1(r,z) H_0^{(1)}(k_0r) - A_1(r,z) \sqrt{\frac{2}{\pi k_0 r}} e^{i(k_0r - \frac{\pi}{4})}.
\] (3.6a)

where \( k_0 \) is the reference wavenumber and \( H_0^{(1)}(k_0r) \) is the zeroth order Hankel function of the first kind. In the foregoing development, all Hankel functions, in their asymptotic expansion form, have a common multiplicative constant \( \sqrt{\frac{2}{\pi}} e^{-\frac{\pi}{4}} \); this can be ignored for simple calculation. This treatment allows Eq. (3.6a) to be written as

\[
\phi(r,z) \sim A_1(r,z) \sqrt{\frac{1}{k_0 r}} e^{i k_0 r}.
\] (3.6b)
Assuming that $A_1(r,z)$ is weakly dependent in range, the representative PE in the fluid medium is obtained, by the parabolic approximation

$$\frac{aA_1}{ar} = a_1 A_1 + b_1 \frac{a^2 A_1}{az^2} \quad (3.7)$$

where $a_1 = \frac{1}{2k_0} (k_1^2 (r,z) - k_0^2) \quad (3.8)$

and

$$b_1 = \frac{1}{2k_0} \quad (3.9)$$

In an inhomogeneous elastic medium, $\phi_2(r,z)$ and $\psi_2(r,z)$ no longer satisfy the two independent Helmholtz equations because of the coupling effects. Following the derivation of McCoy, the propagation of time-harmonic stress waves through an inhomogeneous, linear elastic solid medium which is locally isotropic, is governed by the equation

$$\left(\lambda_2 + \mu_2\right) \nabla \cdot D_2 + \mu_2 \nabla^2 D_2 + \bar{\rho} \omega^2 D_2 + F_2 = 0 \quad (3.10)$$

where $\lambda_2, \mu_2$ are Lame parameters and $\bar{\rho}_2$ is the density such that

$$\lambda_2 = \bar{\lambda}_2 [1 + \epsilon_\lambda(r,z)], \quad (3.11)$$

$$\mu_2 = \bar{\mu}_2 [1 + \epsilon_\mu(r,z)], \quad (3.12)$$

$$\bar{\rho}_2 = \bar{\rho}_2 [1 + \epsilon_\rho(r,z)]. \quad (3.13)$$
\[ F_2 = \nabla \cdot [ \bar{\lambda}_2 \epsilon_{\lambda} (\nabla \cdot D_2) \mathbf{e} + \bar{\mu}_2 \, \epsilon_{\mu} (\nabla D_2 + D_2 \nabla)] + \bar{\rho}_2 \, \epsilon \omega^2 D_2, \quad (3.14) \]

where \( \mathbf{e} \) is a unit vector.

In the above equation, the upper bar indicates the spatial average and \( \epsilon \) with a subscript indicates the perturbation with respect to that subscript.

Define
\[ \bar{k}_D = \frac{\omega^2_{\epsilon}}{c_D^2} = \frac{-\rho_2 \omega^2}{\kappa_2 + 2\bar{\mu}_2}, \quad (3.15) \]

and
\[ \bar{k}_s = \frac{\omega^2_{\epsilon}}{c_s^2} = \frac{-\rho_2 \omega^2}{\bar{\mu}_2}, \quad (3.16) \]

assuming that
\[ \phi_2(r,z) = A_2(r,z) \, H_0^{(1)}(\bar{k}_D r) \sim A_2(r,z) \sqrt{\frac{1}{\bar{k}_D r}} \, e^{i \bar{k}_D r}, \quad (3.17) \]

and
\[ \psi_2(r,z) = B_2(r,z) \, H_0^{(1)}(\bar{k}_s r) \sim B_2(r,z) \sqrt{\frac{1}{\bar{k}_s r}} \, e^{i \bar{k}_s r}. \quad (3.18) \]
With the assumptions above, $A_2(r,z)$ and $B_2(r,z)$ satisfy the following PE's, respectively:

$$\frac{aA_2}{ar} = a_2 A_2 + b_2 \frac{a^2 A_2}{az^2} + c_2 \frac{aB_2}{az}, \quad (3.19)$$

and

$$\frac{aB_2}{ar} = a_2' B_2 + b_2' \frac{a^2 B_2}{az^2} + c_2' \frac{aA_2}{az}, \quad (3.20)$$

where $c_2$ and $c_2'$ are coupling coefficients whose definitions along with other symbols are given below:

$$a_2 = \frac{1}{2\bar{k}_D} [k_D^2(r,z) - \bar{k}_D^2], \quad (3.21)$$

$$b_2 = -\frac{i}{2\bar{k}_D}, \quad (3.22)$$

$$c_2 = \frac{-1}{2\bar{k}_D} [i\Delta \bar{k} \bar{e}_{\mu\rho}], \quad (3.23)$$

$$a_2' = \frac{1}{2\bar{k}_S} [k_S^2(r,z) - \bar{k}_S^2], \quad (3.24)$$

$$b_2' = -\frac{i}{2\bar{k}_S}, \quad (3.25)$$

$$c_2' = \frac{-1}{2\bar{k}_S} \left[ i\Delta \bar{k} \bar{e}^*_{\mu\rho} \right], \quad (3.26)$$


IV. FLUID/ELASTIC INTERFACE CONDITIONS

In this section, we derive the fluid/elastic interface conditions associated with the representative PE’s. Furthermore, from these interface conditions, a system of three equations which relate the fluid potential $A_1$ to the elastic potentials $A_2$ and $B_2$ and their derivatives on the fluid/elastic interface will be derived.

We begin by discussing the fluid/elastic interface conditions using the expression of displacement. These conditions, in cylindrical coordinates, are

1. Continuity of vertical components of displacement:

$$w_1 = w_2.$$  \hspace{1cm} (4.1)

2. Continuity of vertical components of stress:
3. The horizontal components of stress must vanish on the interface:

\[ \nu_2 \left( \frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial r} \right) = 0. \]  

(4.3)

In terms of potentials, the equivalent interface conditions to (4.1), (4.2), and (4.3) are

1. Continuity of vertical components of displacement:

\[ \frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} + \frac{1}{r} \frac{\partial \psi_2}{\partial r}. \]  

(4.4)

2. Continuity of vertical components of stress:

\[ \lambda_1 \left( \frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_1}{\partial r} + \frac{1}{az^2} \phi_1 \right) = \lambda_2 \left( \frac{\partial^2 \phi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_2}{\partial r} + \frac{1}{az^2} \phi_2 \right) + 2\nu_2 \left( \frac{\partial^2 \phi_2}{\partial r \partial z} + \frac{1}{r \partial z} \psi_2 \right). \]  

(4.5)

3. The horizontal component of stress must vanish on the interface:

\[ \nu_2 \left( 2 \frac{\partial^2 \phi_2}{\partial r \partial z} - \frac{\partial^2 \psi_2}{az^2} + \frac{\partial^2 \psi_2}{ar^2} + \frac{1}{r} \frac{\partial \psi_2}{\partial r} - \frac{1}{r^2} \psi_2 \right) = 0. \]  

(4.6)

Making use of Eq. (3.6) a for \( \phi_1 \), (3.17) for \( \phi_2 \), (3.18) for \( \psi_2 \), and the PE's (3.7), (3.19), and (3.20), the corresponding interface conditions for the PE's are obtained below.
1. Continuity of vertical components of displacement:

\[
\frac{aA}{az} = \sqrt{\frac{k_o}{k_D}} \frac{aA_2}{az} e^{-i \Delta_D r} + \sqrt{\frac{k_o}{k_s}} \left( b_2 + \frac{1}{k_s} B_2 \right) e^{-i \Delta_s r} \quad (4.7)
\]

2. Continuity of vertical components of stress:

\[
-\rho_1 \omega^2 A_1 = \left[ \lambda_2 \left( \frac{a^2 A_2}{az^2} + 2i k_D \frac{aA_2}{az} - \frac{k_D}{2} A_2 \right) + 2\mu_2 \frac{a^2 A_2}{az^2} \right] \sqrt{\frac{k_o}{k_D}} e^{-i \Delta_D r}
\]

\[
+ 2\mu_2 \left( \frac{aB_2}{araz} + i k_s \frac{aB_2}{az} \right) \sqrt{\frac{k_o}{k_s}} e^{-i \Delta_s r}
\]

(4.8)

3. The horizontal components of stress must vanish on the interface:

\[
2 \frac{a^2 A_2}{az} = \frac{a^2 B_2}{az^2} - 2i k_s \frac{aB_2}{ar} - (ik_s)^2 B_2 \sqrt{\frac{k_D}{k_s}} e^{-i \Delta_s D r}
\]

(4.9)

where \( \Delta_D = k_D - k_o, \Delta_s = k_s - k_o \).

From this point on, since we deal with the potentials and their partial derivatives on the interface boundary and at the present range level, for economy in writing we drop the superscript \( n \) and the subscript \( j \), e.g., \( A_2 \) means \((A_2)^n_j\) unless otherwise specified.

Following McDaniel-Lee's technique in fluid medium, we use the Taylor expansion for \((A_1)^n_j - 1\) upon \((A_1)_j\), solve for \( \frac{a^2 A_1}{az^2} \) and substituting into
Eq. (3.7), we find

\[
\frac{\partial A_1}{\partial z} = h \frac{\partial A_1}{\partial r} - h \frac{\partial A_1}{\partial r} a_1 A_1 + \frac{1}{h} (A_1 - (A_1)_{j-1})
\]  

(4.13)

where \( h \), is defined as the depth increment \( \Delta z \). Later we will use \( k \) to represent the range increment \( \Delta r \).

If we substitute (4.13) into (4.7), we obtain

\[
\frac{h}{2b_1} \frac{\partial A_1}{\partial r} - \frac{h}{2b_1} a_1 A_1 + \frac{1}{h} (A_1 - (A_1)_{j-1}) = k_D \frac{\partial A_2}{\partial z} + k_s \left( \frac{a b_2}{\partial r} + i k_s B_2 \right).
\]  

(4.14)

where

\[
k_D = \sqrt{k_0} e^{i \Delta r}.
\]  

(4.15)

and

\[
k_s = \sqrt{\frac{k_0}{k_s}} e^{i \Delta r}.
\]  

(4.16)

Using the finite difference for the partial derivatives in (4.14), we obtain

\[
\frac{h}{2b_1} \frac{1}{k} \left( A_{1}^{n+1} - A_1 \right) - \frac{h}{2b_1} a_1 A_1 + \frac{1}{h} (A_1 - (A_1)_{j-1}) = k_D \frac{1}{h} \left( (A_2)_{j+1} - A_2 \right)
\]  

\[
+ k_s \left( \frac{1}{k} [B_2^{n+1} - B_2] + i k_s B_2 \right).
\]  

(4.17)
A simplification of (4.17) gives
\[ p_{11} A_{n+1} + p_{12} A_{n+1} + p_{13} B_{n+1} = \text{RHS 1} \]

where
\[ p_{11} = \frac{h}{2b_1}, \]
\[ p_{12} = 0, \]
\[ p_{13} = -\frac{K_S}{k}, \]

and
\[ \text{RHS 1} = \left[ \frac{h}{2b_1} \left( \frac{1}{k} + a_1 \right) - \frac{1}{h} \right] A_1 + \frac{1}{h} (A_{1j})_{j-1} + \frac{K_D}{h} \left( (A_{2j})_{j+1} - A_2 \right) + \frac{K_S}{k} (B_{2j})_{j-1} - \frac{1}{k} B_2. \]

The next two equations to be derived into the system are based on Eqs. (4.8) and (4.9). In those two equations, two terms are involved, namely,
\[ \frac{\partial^2 A_2}{\partial z^2} \quad \text{and} \quad \frac{\partial^2 B_2}{\partial z^2}. \]

We first try to develop explicit expressions for these two partial derivatives.

Applying the finite difference to the partial derivatives in Eq. (3.20) gives
\[ (B_{2j})_{j+1} = (B_{2j})_{j+1} + k a_2 (B_{2j})_{j+1} + k b_2 \left( \frac{\partial^2 B_2}{\partial z^2} \right)_{j+1}. \]
\[ + k c_2 \left( \frac{\partial A_2}{\partial z} \right)_{j+1} \]

\[ = (1 + k a_2)(B_2)_{j+1} + k b_2 \left( \frac{\partial^2 B_2}{\partial z^2} \right)_{j+1} + k c_2 \left( \frac{\partial A_2}{\partial z} \right)_{j+1} \]

(4.23)

\[ \frac{\partial^2 B_2}{\partial r \partial z} = \frac{\partial}{\partial r} \left( \frac{\partial B_2}{\partial z} \right) = \frac{1}{h} \frac{1}{k} \left( (B_2)_{j+1} - B_2 \right) = \frac{1}{h} \left( \frac{(B_2)_{j+1}^n - (B_2)_{j+1}}{k} - \frac{(B_2)_j^n - B_2}{k} \right) \]

\[ = \frac{1}{hk} \left( (1 + k a_2)(B_2)_{j+1} + k b_2 \left( \frac{\partial^2 B_2}{\partial z^2} \right)_{j+1} + k c_2 \left( \frac{\partial A_2}{\partial z} \right)_{j+1} - (B_2)_{j+1}^n - B_2 \right) \]

(4.24)

Applying the finite difference to the partial derivatives in Eq. (3.19), we obtain

\[ (A_2)_{j+1}^n = (A_2)_{j+1} + k a_2 (A_2)_{j+1} + k b_2 \left( \frac{\partial A_2}{\partial z} \right)_{j+1} + k c_2 \left( \frac{\partial B_2}{\partial z} \right)_{j+1} \]

(4.25)
\[ -\frac{1}{h} \left( \frac{(A_2^{n+1})_j - (A_2^1)_j}{k} - \frac{(A_2^{n+1})_j - A_2}{k} \right) \]

\[ = \frac{1}{hk} \left\{ k a_2 (A_2)_j + k b_2 \left( \frac{\partial^2 A_2}{\partial z^2} \right)_j + k c_2 \left( \frac{\partial B_2}{\partial z} \right)_j \right\} - \left( (A_2)_j^2 - A_2 \right) \} \]  

(4.26)

Eq. (4.8) can be written as

\[ -\rho_1 \omega^2 A_1 = \left[ (\lambda_2 + 2\nu_2) \frac{\partial^2 A_2}{\partial z^2} + \lambda_2 \frac{\partial A_2}{\partial r} \frac{\partial A_2}{\partial r} - \lambda_2 \frac{\partial^2 A_2}{\partial r^2} \right] K_0 \]

\[ + 2 \frac{\rho_2 \omega^2}{k_s} \left( \frac{\partial B_2}{\partial z} + \frac{\partial B_2}{\partial z} \right) K_s. \]  

(4.27)

Substituting (4.24) into (4.27), we obtain

\[ -\rho_1 \omega^2 A_1 = \left[ \frac{\rho_2 \omega^2}{K_0} \frac{\partial^2 A_2}{\partial z^2} + \lambda_2 \frac{\partial A_2}{\partial r} \frac{\partial A_2}{\partial r} - \lambda_2 \frac{\partial^2 A_2}{\partial r^2} \right] K_0 \]

\[ + 2 \frac{\rho_2 \omega^2}{k_s^2} \left( \frac{1}{hk} \left\{ k a_2 \left( B_2 \right)_j + k b_2 \left( \frac{\partial B_2}{\partial z} \right)_j \right\} + k c_2 \left( \frac{\partial B_2}{\partial z} \right)_j \right\} - \left( (B_2)_j^{n+1} - B_2 \right) \} + 1 \frac{\partial B_2}{\partial z} \} K_s. \]  

(4.28)
Using the finite difference for all partial derivatives in (4.28), we obtain

\[-\rho_1 \omega^2 A_1 = \left[ \frac{\rho_2 \omega^2}{\kappa_D} - \frac{1}{h^2} \left( (A_2)_{j+2} - 2(A_2)_{j+1} + A_2 \right) + \lambda_2 \, 2i\kappa_D \frac{1}{k} \left( A_2^{n+1} - A_2 \right) \right] K_0 \]

\[-\lambda_2 \, \kappa^2 D_2 A_2 \right] K_0 + 2 \frac{\rho_2 \omega^2}{\kappa_s^2} \left( \frac{1}{h} \left\{ k \, a_2^i (B_2)_{j+1} \right. \right]

\[+ k \, b_2^i \left( \frac{1}{h^2} \left[ (B_2)_{j+2} - 2(B_2)_{j+1} + B_2 \right] + k \, c_2^j \frac{1}{h} \left( (A_2)_{j+1} - A_2 \right) \right) \]

\[-\left( (B_2)^{n+1} - B_2 \right) \right] + 1 \frac{\kappa}{h} \left( [(B_2)_{j+1} - B_2] \right) K_s. \tag{4.29} \]

A simplification of (4.29) gives

\[p_{21} A_1^{n+1} + p_{22} A_2^{n+1} + p_{23} B_2^{n+1} = \text{RHS2} \tag{4.30} \]

where

\[p_{21} = 0, \tag{4.31} \]

\[p_{22} = \lambda_2 \, 2i\kappa_D \frac{1}{k} \, K_0, \tag{4.32} \]

\[p_{23} = -2 \frac{\rho_2 \omega^2}{\kappa_s^2} \frac{1}{hk} \, K_s, \tag{4.33} \]

and

\[\text{RHS2} = -\rho_1 \omega^2 A_1 - \frac{\rho_2 \omega^2}{h^2} \frac{K_0}{\kappa_D^2} (A_2)_{j+2} \]
\[ + \frac{\rho_2 \omega^2}{\hbar} \left( \frac{2K_D}{\bar{k}_D h} + \frac{2K_S k c_2'}{\bar{k}_S} \right) (A_2')_{j+1} \]
\[ + \left( -\frac{\rho_2 \omega^2}{\bar{k}_D h^2} + \lambda_2 \bar{k}_D \frac{1}{k} + \lambda_2 \bar{k}_D \right) K_D A_2 \]
\[ - 2 \frac{\rho_2 \omega^2}{\bar{k}_S} \left( \frac{1}{h k} k b_2' \frac{1}{h^2} - \frac{1}{h} \right) K_S (B_2)_{j+1} \]
\[ - 2 \frac{\rho_2 \omega^2}{\bar{k}_S} \left( \frac{b_2'}{h^2} + \frac{1}{(k h) - 1} \bar{k}_S \frac{1}{h} \right) K_S B_2 - \rho_1 \omega^2 A_1 . \] (4.34)

Next, substitute (4.26) into (4.9), we obtain
\[ \frac{2}{k} \left\{ k a_2 (A_2)_{j+1} + k b_2 \left( \frac{\partial^2 A_2}{\partial z^2} \right)_{j+1} + k c_2 \left( \frac{\partial B_2}{\partial z} \right)_{j+1} \left( (A_2)^{n+1} - A_2 \right) \right\} \]
\[ + 2i \bar{k}_D \frac{a A_2}{\partial z} = \left( \frac{\partial^2 A_2}{\partial z^2} - 2i \bar{k}_S \frac{\partial B_2}{\partial r} - (i \bar{k}_S)^2 B_2 \right) \kappa_{DS}, \] (4.35)

where \( \kappa_{DS} = \sqrt{\frac{K_D}{\bar{k}_S}} e^{i \Delta_{DS} r} \), and
\[ \Delta_{DS} = \Delta_S - \Delta_D . \] (4.36)
Using the finite difference for all partial derivatives in (4.35), we obtain

\[
\frac{2}{h^2k} \left\{ k a_2(A_2)_{j+1} - b_2 \frac{1}{h^2} \left[ (A_2)_{j+2} - 2(A_2)_{j+1} + A_2 \right] + k \frac{c_2}{h} \left[ (B_2)_{j+1} - B_2 \right] \right. \\
- \left. \left( (A_2)_{n+1} - A_2 \right) \right\} + 2iK_D \frac{1}{h} \left( (A_2)_{j+1} - A_2 \right) \\
= \frac{1}{h^2} \left[ (B_2)_{j+2} - 2(B_2)_{j+1} + B_2 \right] - 2iK_s \frac{1}{k} \left[ B_{n+1} - B_2 \right] \\
- (iK_s)^2 B_2) K DS. 
\] (4.37)

A simplification of (4.37) gives

\[
p_{31} A_{n+1} + p_{32} A_{n+1} + p_{33} B_{n+1} = \text{RHS3} 
\] (4.38)

where

\[
p_{31} = 0, 
\] (4.39)

\[
p_{32} = -2/(hk), 
\] (4.40)

\[
p_{33} = 2ik_s (1/k) K DS, 
\] (4.41)

and

\[
\text{RHS3} = - \frac{2}{h^3} b_2 (A_2)_{j+2} \\
- \left[ \frac{2}{hk} \left( k a_2 - 2k b_2 \frac{1}{h^2} \right) + 2iK_D \frac{1}{h} \right] (A_2)_{j+1} \\
+ \left( \frac{2}{hk} k b_2 \frac{1}{h^2} + \frac{2iK_D}{h} \right) A_2 + \frac{1}{h^2} K DS (B_2)_{j+2} 
\]
Equations (4.18), (4.30), and (4.38) constitute a system of 3 equations among which unknowns $A_1^{n+1}$, $A_2^{n+2}$, and $B_2^{n+1}$ are to be solved. From this system we see that $A_1^{n+1}$ only appears in Eq. (4.18).

Actually, we need only to solve a system of 2 equations, i.e., Eqs. (4.30) and (4.38). This system can be written as

$$
\begin{bmatrix}
  p_{22} & p_{23} \\
  p_{32} & p_{33}
\end{bmatrix}
\begin{bmatrix}
  A_2^{n+1} \\
  B_2^{n+1}
\end{bmatrix}
= 
\begin{bmatrix}
  \text{RHS2} \\
  \text{RHS3}
\end{bmatrix}.
$$

After system (4.43) is solved, we substitute $B_2^{n+1}$ into Eq. (4.18) to obtain $A_1^{n+1}$ by means of

$$
A_1^{n+1} = \frac{(\text{RHS} - p_{13} B_2^{n+1})}{p_{11}}.
$$

To examine the existence and uniqueness of the solution of system (4.43), we evaluate the determinant of (4.43),

$$
\text{determinant of (4.43)} = \left(\lambda_2 2i\kappa_0 \frac{1}{k} \right) \left(2i\kappa_s \frac{1}{k} \kappa_{DS} \right)
- \left( - \frac{2}{h\kappa} \right) \left(2 \frac{\rho_2 e^2}{k_s h\kappa} \right).
$$
\[
\begin{align*}
-4\lambda^2 \frac{k_0^2}{k^2} K_{OS} - 4 \frac{1}{h^2 k^2} K_s \frac{\omega^2}{k_s} = 0
\end{align*}
\]

because \( K_{OS} \) is complex while other quantities are real; therefore, the determinant is not singular and the solution exists and is unique.

V. A NEW COMPUTATIONAL APPROACH

Following McDaniel-Lee [1], a uniform partition in the \( z \)-direction is assumed. The \( h = \Delta z \) is used for the depth increment and the integer index \( j \) is the interface boundary. As before, the superscript indicates the range level, and the subscript indicates the depth level. It is also understood that if both the superscript and the subscript are dropped, it denotes the field at \((n\Delta r, j\Delta z)\), i.e., \( A = A^n_j \).

Our approach can be described by the following statement:

Solving the representative parabolic wave equations in the different media by means of an implicit finite difference (IFD) marching scheme, applying the field values on the interface as boundary information from fluid and elastic media.

We proceed to discuss the meaning of the above statement. Solving the representative parabolic wave equation using an implicit finite difference scheme in a marching process requires the initial field values at range \( r_0 \).
plus the surface and bottom boundary information. The surface boundary point at the present level is denoted by \((A_1)_0^n\) and at the advanced level by \((A_1)_0^{n+1}\), the bottom boundary point at the present level is denoted by \((A_1)_j^n\) and at the advanced level by \((A_1)_j^{n+1}\). The IFD scheme predicts the wave field at the advanced level, \(r + \Delta r\), regardless whether the medium is fluid or elastic. If the medium is fluid, there is only one PE to solve; if the medium is elastic, there is a system of 2 PE's to solve, in this case the field is a vector containing components \(A_2\) and \(B_2\), each component is a subvector. Dealing with the solution of the entire problem, the surface remains unchanged, but an interface boundary comes into existence. This interface boundary separates a fluid medium and an elastic medium. One crosses the interface boundary, the density and the sound speeds change. In the elastic medium two sound speeds occur, the speed of P-wave \(c_0\) and the speed of S-wave \(c_s\). Initial field values at \(r_0\) are used along with surface points \((A_1)_0^n\), \((A_1)_0^{n+1}\) and interface boundary points \((A_1)_j^n\), \((A_1)_j^{n+1}\) to predict the wave field \((r_0 + \Delta r)\) in the fluid medium. The \((A_2)_j^n\), \((A_2)_j^{n+1}\), \((B_2)_j^n\) are used as surface points along with initial field values at the same range level. Beyond interface boundary, the boundary points \((A_2)_{\text{bottom}}^n\), \((A_2)_{\text{bottom}}^{n+1}\), \((B_2)_{\text{bottom}}^n\), \((B_2)_{\text{bottom}}^{n+1}\) are generated artificially. This setup allows the same IFD procedure to solve a system of PE's in the elastic medium in the same manner as in the fluid medium. The key treatment is to determine the interface boundary values \((A_1)_j^n\), \((A_2)_j^n\), and \((B_2)_j^n\) which are related by the system
(4.43) and Eq. (4.44). With these interface boundary values, the IFD can march in range to predict the wave field in the fluid at the next range. This is where the McDaniel-Lee interface treatment is extended to handle the fluid/elastic horizontal interface. Note that \((A_1)_j\) is used as a bottom boundary point to solve the PE in the fluid medium while \((A_2)_j\) and \((B_2)_j\) are used as two "surface" points to solve the system of two parabolic equations in the elastic medium.

For better understanding, the diagram below explains our description and will help clear up our early statements.
Let \((\tilde{A}_1)^n_j, (\tilde{A}_1)^{n+1}_j\) be 2 vectors associated with range levels \(n\) and \(n+1\) respectively. These 2 vectors contain only the nonzero components at the interface boundaries, i.e., \((A_1)^n_j\) and \((A_1)^{n+1}_j\).

Then the matrix representation for the fluid/elastic interface problem can be expressed by

\[
\text{Fluid: } D^{n+1} \tilde{A}_1^{n+1} = D^n \tilde{A}_1^n + (\tilde{A}_1)^n_j + (\tilde{A}_1)^{n+1}_j
\]

\[
\text{Elastic: } \begin{pmatrix} E_1 \vert E_1 \vert \tilde{A}_2^{n+1} \vert \tilde{A}_2^n \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^{n+1}}{\tilde{B}_2^{n+1}} \end{pmatrix} = \begin{pmatrix} E_1 \vert E_1 \vert \tilde{A}_2^{n+1} \vert \tilde{A}_2^n \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^{n+1}}{\tilde{B}_2^{n+1}} \end{pmatrix} \begin{pmatrix} \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \end{pmatrix} + \begin{pmatrix} \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \end{pmatrix} + \begin{pmatrix} \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \vert \frac{\tilde{A}_2^n}{\tilde{B}_2^n} \end{pmatrix}
\]

where \((\tilde{A}_2)_j\) and \((\tilde{B}_2)_j\) are 2 vectors having the same structure as \((\tilde{A}_1)_j\) except the nonzero elements are the first components and \(D, E, F\) are tri-diagonal matrices, and \(E_1, F_1\) are sparse matrices whose non-zero elements appear on the diagonal and lower diagonal.

Note that the points which influence the fluid/elastic interface boundary are \((A_1)_j, (A_2)_j,\) and \((B_2)_j\) at all range levels. It is very clear that their relationships are defined by the system (4.43) and the Eq. (4.44). Once these boundary values are obtained, the IFD scheme can march forward in range.
VI. A NUMERICAL EXAMPLE

This section examines the validity of our approach and an example whose solutions [6] are known is used as a demonstration.

First of all, in the case where the elastic waves are absent which implies that \( \nu_2 \) is zero. Under such an environment only one PE is needed to represent the wave propagation in the fluid. Thus, the equation (2.1) is the representative equation. Furthermore, Eq. (4.6) becomes zero identically on both sides. Eq. (4.4) reduces to

\[
\frac{aA_1}{az} = \frac{aA_2}{az} \tag{6.1}
\]

and Eq. (4.5) reduces to

\[
\rho_2 A_1 = \rho_1 A_2 \tag{6.2}
\]

Eqs. (6.1) and (6.2) are equivalent interface conditions to (2.2) and (2.3) in the fluid medium. Then all Taylor expansions can be applied following the McDaniel-Lee technique.
Next we use an example discussed by Ewing et al [7] to computationally demonstrate the validity. This example neglects the branch line integrals and is feasible for large values of range; thus, it is very suitable for our PE's (far-field). The solutions \( \phi_1, \phi_2, \psi_2 \) are given for Helmholtz equations. For consistency we derive the corresponding solutions to the PE.

The exact mathematical expressions for \( \phi_1, \phi_2, \) and \( \psi_2 \) are

\[
\phi_1 = \frac{2\pi}{H} \sum_n \frac{1}{\sqrt{k_n}} e^{i(\omega t - k_n r - \frac{\pi}{4})} \Phi_1(k_n) \sin(\xi_n d) \sin(\xi_n z)
\]

\( 0 \leq z \leq H = z_j, \) \( d = \) source depth

(6.3)

\[
\phi_2 = \frac{2\pi}{H} \sum_n \frac{1}{\sqrt{k_n}} e^{i(\omega t - k_n r - \frac{\pi}{4})} \Phi_2(k_n) \sin(\xi_n d) e^{-n(z-H)}
\]

\( z \geq H \)

(6.4)

\[
\psi_2 = \frac{2\pi}{H} \sum_n \frac{1}{\sqrt{k_n}} e^{i(\omega t - k_n r - \frac{\pi}{4})} \Psi_2(k_n) \sin(\xi_n d) e^{-\xi(z-H)}
\]

\( z \geq H \)

(6.5)
where

\[
\Phi_1(k_n) = -\frac{\rho_1 c^4}{\rho_2 \beta_2^4} \frac{n_n}{\xi_n} k_n H \left( \sqrt{\frac{c^2}{\beta_2^2} - 1} \right) \cos(\xi_n H),
\]

(6.6)

\[
\Phi_2(k_n) = -\frac{\rho_1 c^2}{\rho_2 \beta_2^2} (2 - \frac{c^2}{\beta_2^2}) k_n H \left( \sqrt{\frac{c^2}{\beta_2^2} - 1} \right) \cos(\xi_n H),
\]

(6.7)

\[
\Psi_2(k_n) = -\frac{\rho_1 c^2}{\rho_2 \beta_2^2} \frac{n_n}{\xi_n} \left( \sqrt{\frac{c^2}{\beta_2^2} - 1} \right) \cos(\xi_n H),
\]

(6.8)

\[
\begin{align*}
\{ \} &= \frac{\rho_1 c^4}{\rho_2 \beta_2^4} \left( \frac{\sin(\xi_n H)}{\sqrt{\frac{c^2}{\beta_2^2} - 1}} \right) \left[ 1 + \frac{1 - \frac{c^2}{\alpha_2^2}}{\frac{c^2}{v_1^2} - 1} \right] - \\
&\left[ \frac{k_n H}{\sqrt{1 - \frac{c^2}{\alpha_2^2}}} \right] \\
&\sqrt{\frac{1 - \frac{c^2}{\beta_2^2}}{1 - \frac{c^2}{\alpha_2^2}}} \left[ \sec(\xi_n H) \right] \\
&- 4 \left( \sqrt{1 - \frac{c^2}{\beta_2^2}} \right) + \frac{1 - \frac{c^2}{\alpha_2^2}}{1 - \frac{c^2}{\beta_2^2}} + 2 \left( \frac{c^2}{\alpha_2^2} \right) - \frac{c^2}{\beta_2^2} \\
&- 2(2 - \frac{c^2}{\beta_2^2}) \left( \cos(\xi_n H) \right),
\end{align*}
\]

(6.9)
and  $\xi_n = k_n \sqrt{\frac{c^2}{v_1^2} - 1} \quad (6.10)$

$\eta_n = k_n \sqrt{1 - \frac{c^2}{a_2^2}} \quad (6.11)$

$\zeta_n = k_n \sqrt{1 - \frac{c^2}{b_2^2}} \quad (6.12)$

where $n$ is a subscript which indicates that the quantity is to be evaluated at $k = k_n$, where $k_n$ are the roots of

$$\frac{\rho_1}{\rho_2} \frac{4}{\beta_2} \tan(\xi H) - 4k^2 \left[ \xi^2 - \frac{2k^2}{\beta_2^2} \right] = 0 \quad (6.13)$$

For computational simplicity, we use one mode, i.e., $n = 1$. After separating the $H^{(1)}_0 (k, r)$ ($k = k_0$, $k_s$, or $k_0$) and the time-harmonic, the corresponding PE's $A_1$, $A_2$, and $B_2$ to Eqs. (6.3), (6.4), and (6.5) become

$$A_1 = \frac{2\pi}{z} \sqrt{\frac{k_0}{k_1}} e^{i(k_n - k_0)r} \phi_1(k_1) \sin(\xi_n d) \sin(\xi_n z) \quad (6.14)$$

$$A_2 = \frac{2\pi}{z} \sqrt{\frac{k_0}{k_1}} e^{i(k_n - k_0)r} \phi_2(k_1) \sin(\xi_n d e^{-\eta(z-H)}) \quad (6.15)$$

$$B_2 = \frac{2\pi}{z} \sqrt{\frac{k_0}{k_1}} e^{i(k_n - k_0)r} \psi_2(k_1) \sin(\xi_n d e^{-\zeta(z-H)}) \quad (6.16)$$
and

\{e\} has the same definition as (6.9) but has the subscript \( n = 1 \).

Next select the compressional wave velocity \( v_1 \) in a fluid of 1500 m/s with compressibility \( \lambda_1 = v_1^2 \) and a water density of \( \rho_1 = 1.0 \). \( c \) is the phase velocity such that

\[
f = c \frac{k_n}{2\pi} \Rightarrow \frac{2\pi f}{k_n} = c
\]

We follow the case I of Press-Ewing's to select \( \alpha_2, \beta_2, \) and \( v_1 \) such that

\[
\alpha_2 > \beta_2 \geq c \geq v_1 \quad (6.17)
\]

We make the following choices:

\[
f = 68.03 \text{ Hz}
\]
\[
H = z_j = 100 \text{m}
\]
\[
d = z_s = 25 \text{m}
\]
\[
v_1 = 1500 \text{m/s}
\]
$\theta_2 = 1530.0 \text{ m/s}$

$\alpha_2 = 1725.0 \text{ m/s}$

$c_1 = 1507.5 \text{ m/s}$

$k_1 = 0.283546$

$\lambda_1 = (1500)^2$

$\lambda_2 = 2.5 \cdot (1725.0)^2 - 2 \cdot 2.5 \cdot (1530.0)^2$

$\nu_2 = 2.5 \cdot (1530.0)^2$

$\rho_1 = 1.0 \text{ g/cm}^3$

$\rho_2 = 1.97 \text{ g/cm}^3$

$k_0 = 0.284963$

$k_D = 0.247794$

$k_S = 0.279376$
$\left(A_1\right)^{n+1}_j$ is calculated by means of formula (4.44) where RHS1 is defined by formula (4.22), $p_{13}$ is defined by formula (4.21), $p_{11}$ is defined by formula (4.19), and $B_2^{n+1}$ is calculated by formula (6.16).

As illustrated in the previous section, our main effort is to determine $\left(A_1\right)^{n+1}_j$ and use it as a fluid/elastic boundary information. $\left(A_2\right)^{n+1}_j$ is obtained in such a way that it is related to the information of elastic potentials $\left(A_2\right)^{n+1}_j$ and $\left(B_2\right)^{n+1}_j$ on the fluid/elastic interface. System (4.43) was developed to relate these points. In this test example, we use an accurate $\left(A_2\right)^{n+1}_j$ and $\left(B_2\right)^{n+1}_j$ at every range from a known solution as the accurate boundary interface values. These accurate values are applied at every step when solving system (4.43). Results are tabulated describing the comparison of computed field values against the known solution in dB. Accuracy was carried up to 2 significant digits using the VAX 11/780 computer.
TABLE OF RESULTS

<table>
<thead>
<tr>
<th>Range (m)</th>
<th>1500</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth (m)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>(-0.366E-02,-0.590E-02)</td>
<td>(-0.669E-02,-0.198E-02)</td>
</tr>
<tr>
<td></td>
<td>(-0.366E-02,-0.590E-02)</td>
<td>(-0.662E-02,-0.209E-02)</td>
</tr>
<tr>
<td>60</td>
<td>(0.480E-02,-0.785E-02)</td>
<td>(-0.874E-02,-0.270E-02)</td>
</tr>
<tr>
<td></td>
<td>(0.482E-02,-0.777E-02)</td>
<td>(-0.872E-02,-0.276E-02)</td>
</tr>
<tr>
<td>90</td>
<td>(-0.276E-02,-0.428E-02)</td>
<td>(-0.464E-02,-0.152E-02)</td>
</tr>
<tr>
<td></td>
<td>(-0.269E-02,-0.433E-02)</td>
<td>(-0.487E-02,-0.154E-02)</td>
</tr>
</tbody>
</table>

From this selected test example, it is clearly seen that the numerical results, produced by this model, agree satisfactorily with the exact solution. The results not only demonstrate the validity of this model, but also show the correct computational procedure following the IFO procedure. This also serves as an early indication that this model can be readily incorporated into the IFO code.

VII. CONCLUSIONS

A mathematical model has been developed by means of the parabolic approximation method for handling the fluid/elastic interface. The complete mathematical development plus the numerical example proved the validity of
the model. As it stands now, even though this model is accurate, it is limited to narrow angle propagation only. However, an important feature of this model is that it can handle a range-dependent index of refraction in the elastic medium. Moreover, another attractive feature is that this model is readily adaptable into the existing IFD code without requiring excessive effort.
ACKNOWLEDGMENT

This work was performed when both authors were visiting the Department of Computer Science of Yale University, New Haven, CT, USA in 1987. The present address of the first author is: NOAA/ERL, Wave Propagation Laboratory, Boulder, CO, USA. The present address of the second author is: Naval Underwater Systems Center, New London, CT, USA. Work performed at the Naval Underwater Systems Center was sponsored jointly by ONR Grant No. N0001487WX24234 and NUSC Independent Research Project A65020. The authors would like to thank Drew Drinkard for his technical assistance in producing the numerical results in this paper.
REFERENCES


ADDENDUM

SUBROUTINE UFIELD

This subroutine provides a complex starting field for Gaussian field

PARAMETER MXLYR=101, MXXN=10000, MXSVP=101, MXTRK=101, NIU=1,
NOU=2, NPU=6

REAL*4 K0, K1, KS, KD, MU2, LAMNDA2
COMPLEX PHI1(MXXN), PH12(MXXN), PS12(MXXN), CARG, B1, A1,
KDD, KSS, CARI, CARR
COMMON /USTFLD/ K0, K1, KS, KD, SIGMA1, CPHI1, CPHI2, CPSI2,
B1, DELTAS, DELTAD, A1JM
EQUIVALENCE (H, ZLYR(1)), (D, ZS), (Z, ZI)

COMMON /IFDCOM/ACOFX, ACOFY, ALPH, BCOF, BETAMXLYR, BOTA, BOTY,
BTA(MXXN), CO, CSV(MXXSVP), DR, DR1, DZ, FR, IHNK, ISF, ITYPEB,
K/types, IXSV(MXXLYR), KSVP, N, N1, NLYR, NSVP, NWSVP, R12(MXXN), RA,
RHO(MXXLYR), RSVP, SURX, SURY, THETA, TRACXMTRK, 2, U(MXXN),
XD(MXXN), X0, Y(MXXN), ZA, ZLYR(MXXLYR), ZP, ZS, ZSVP(MXXSVP)
DATA PI/3.141592654/, DEG/57.29578/

DATA C/1507.5/, V1/1500.0/, ALPH2/1725.00/, BETA2/1530.00/,
RHO1/1.0/, RHO2/1.97/, IND/0/

SEC(ANG) = 1.0 / COS(ANG)

** STARTING FIELD - EWING & PRESS

AR = RA
IF (IND .NE. 0) AR = RA - DR
OMEGA = 2.0 * PI * FRQ
K0 = OMEGA / CO
K1 = OMEGA / C
KS = OMEGA / BETAMXLYR
KD = OMEGA / ALPHAMXLYR

1 FORMAT (2X, A4, E12.6, 3X, A4, E12.6, 3X, A4, E12.6, 3X, A4, E12.6)
CARG = CMPLX(0.0, (K1 - K0) * AR)
CARH = CMPLX(0.0, (K1 - KD) * AR)
CAR1 = CMPLX(0.0, (K1 - KS) * AR)
B1 = CMPLX(0.0, 1.0 / (2.0 * K0))

WRITE (NPU, 2) 'CARG: ', CARG, 'B1: ', B1
2 FORMAT (2X, A6, '(', E12.6, 2X, E12.6, ')', 5X, A4, '(', E12.6, 2X, E12.6,
F
')

DELTAS = (KS - K0) * AR
DELTAD = (KD - K0) * AR
DELTADS = (KD - KS) * AR

WRITE (NPU, 3) 'DELTAS: ', DELTAS, 'DELTAD: ', DELTAD, 'DELTADS: ',
C W

DELTA = 3

FORMAT (2X,A8,E12.6,3X,A8,E12.6,3X,A9,E12.6)
MU2 = RH02 / BETA2 ** 2
LAMNDA2 = RH02 * (ALPHA2 ** 2 - 2.0 * BETA2 ** 2)

WRITE (NPU,4) 'MU2: ',MU2,'LAMNDA2: ',LAMNDA2

FORMAT (2X,A5,E12.6,3X,A9,E12.6)

ARG1 = C ** 2 / V1 ** 2 - 1.0
ARG2 = 1.0 - C ** 2 / ALPHA2 ** 2
ARG3 = 1.0 - C ** 2 / BETA2 ** 2
ARG4 = 2.0 - C ** 2 / BETA2 ** 2

WRITE (NPU,5) 'ARG1: ',ARG1,'ARG2: ',ARG2,'ARG3: ',ARG3,'ARG4: ',ARG4

FORMAT (2X,A6,12.6,3X,A6,12.6,3X,A6,12.6)

SIGMA1 = K1 * SQRT(ARG1)
ETA1 = K1 * SQRT(ARG2)
ZETA1 = K1 * SQRT(ARG3)

WRITE (NPU,6) 'SIGMA1: ',SIGMA1,'ETA1: ',ETA1,'ZETA1: ',ZETA1

FORMAT (2X,A8,12.6,3X,A8,12.6,3X,A8,12.6)

BRACE = (RH01 / RH02) * (C ** 4 / BETA2 ** 4)

B = *(SIN(SIGMA1 * H) / (SQR(ARG1) * SQRT(ARG2))
B = *(1.0 + ARG2 / ARG1) - ((K1 * H * SQRT(ARG2)) / ARG1
B = SEC(SIGMA1 * H)) - 4.0 * (SQR(ARG3) / SQRT(ARG2)
B = + SQRT(ARG2) / SQRT(ARG3) + 2.0 * SQRT(ARG2)
B = SQRT(ARG3) - 2.0 * ARG4) * COS(SIGMA1 * H)

CPHI1 = -((RH01 / RH02) * (C ** 4 / BETA2 ** 4) * (ETA1 / SIGMA1)
C = K1 * H) / (1.0 * BRACE * COS(SIGMA1 * H))
CPHI2 = -((RH01 / RH02) * (C ** 2 / BETA2 ** 2) * ARG4 * K1 * H)
C = / (SQR(ARG1) * BRACE)
CPSI2 = -((RH01 / RH02) * (C ** 2 / BETA2 ** 2) * (ETA1 / SIGMA1))
C = BRACE

WRITE (NPU,7) 'BRACE: ',BRACE,'CPHI1: ',CPHI1

FORMAT (2X,A7,12.6,5X,A7,12.6)

DO 10 I=1,N
ZI = I*DZ
IF ((Z .GE. 0.0) .OR. (Z .LE. H)) THEN
  PHI1(I) = ((2.0 * PI) / H) * CEXP(CARG) * CPHI1
  U
   * SIN(SIGMA1 * D) * SIN(SIGMA1 * Z)
  U
   * CSQRT(CMPLX(K0 / K1, 0.0))
  END IF

IF (Z .GE. H) THEN
  PHI2(I) = ((2.0 * PI) / H) * CEXP(CARH) * CPHI2
  U
   * SIN(SIGMA1 * D) * EXP(-ETA1 * (Z - H))
  U
   * CSQRT(CMPLX(KD / K1, 0.0))
PSI2(I) = (2.0 * PI) * CEXP(CARI) * CPSI2
  U
   * CMPLX(0.0,-K1)
  U
   * SIN(SIGMA1 * D) * EXP(-ZETA1 * (Z - H))
  U
   * CSQRT(CMPLX(KS / K1, 0.0))
  END IF
U(I) = PHI1(I)
10 CONTINUE
PHI2(N+1) = ((2.0*PI)/H)*CEXP(CARH)*CPHI2
U
  *SIN(SIGMA1*D)*EXP(-ETA1*((N+1)*DZ-H))
U
  *CSQRT(CMPLX(KD/K1,0.0))
PSI2(N+1) = (2.0*PI)*CEXP(CARI)*CPSI2*CMPLX(0.0,-K1)
U *SIN(SIGMA1*D)*EXP(-ZETA1*((N+1)*DZ-H))
U *CSQRT(CMPLX(KS/K1,0.0))
KDD = CSQRT(CMPLX(K0 / KD, 0.0)) * CEXP(CMPLX(0.0, DELTAD))
KSS = CSQRT(CMPLX(K0 / KS, 0.0)) * CEXP(CMPLX(0.0, DELTAS))
T33 = CMPLX(0.0, DR / (K0 * DZ ** 2))
C WRITE (NPU, 8) 'T33: ',T33
8 FORMAT (2X,A5,'(',EI2.6,2X,E12.6,')')
T32 = (CMPLX(1.0, 0.0) - T33) * PHII(N)
T32 = T32 + T33 * PHII(N-1)
C WRITE (NPU, 8) 'T32: ',T32
T12 = CMPLX(0.0, KS)
T12 = KSS * T33 * DZ * PSI2(N) * (T12 - CMPLX(1.0 / DR, 0.0))
C WRITE (NPU, 8) 'T12: ',T12
CHI = T12 - T32 / T33
C WRITE (NPU, 8) 'CHI: ',CHI
T21 = KSS * T33 * CEXP(CMPLX(0.0, (K1 - KS) * DR))
T * PSI2(N) * DZ / DR
IF (IND .EQ. 0) T21 = KSS * T33 * PSI2(N) * DZ / DR
C WRITE (NPU, 8) 'T21: ',T21
T22 = KDD * T33 * (PHI2(N+1) - PHI2(N))
C WRITE (NPU, 8) 'T22: ',T22
9 FORMAT (2X,A6,'(',E12.6,2X,E12.6,')')
A1 = T32 + T22 + T21 + T12
IF (IND .GT. 0) GO TO 15
U(N) = A1
IND = 1
RETURN
15 A1JM = A1
RETURN
END
SUBROUTINE BCON

******************************************************************************
*** USER PREPARED BOTTOM CONDITION SUBROUTINE  
*** BCON IS CALLED IF INPUT PARAMETER ITYPEB = 1  
*** SEE MAIN PROGRAM FOR DEFINITIONS  
******************************************************************************
*** SUBROUTINE RETURNS:  
** BOTY, BOTX  
******************************************************************************

PARAMETER MXLYR=101, MXN=10000, MXSVP=101, MXTRK=101, NIU=1,
C NOU=2, NPU=6
COMPLEX ACOFX, ACOFY, BCOF, BOTX, BOTY, BTA, HNK, HNKL, SURX, SURY, TEMP,
C U, X, Y
REAL*4 K0, K1, KS, KD
COMPLEX CARG, B1, CKS, CKD, P11, P13, A1JN, A1JM, A2JN, A2JP1N,
C B2JN, B2JNP1, RHS1, CARH, CARI
COMMON /USTFLD/ K0, K1, KS, KD, SIGMA1, CPHI1, CPHI2, CPSI2, B1,
C DELTAS, DELTAD, A1JM
COMMON /IFDCOM/ACOFX, ACOFY, ALPHA, BCOF, BETA(MXLYR), BOTX, BOTY,
C BTA(MXN), C0, CSVPMXSVP, DR, DR1, DZ, FRQ, IHNK, ISF, ITYPEB,
C ITYPES, IXSVP(MXLYR), KSVP, N, N1, NLYR, NSVP, NWSVP, R12(MXN), RA,
C RHO(MXLYR), RSVP, SURX, SURY, THETA, TRACK(MXTRK, 2), U(MXN),
C X(MXN), XK0, Y(MXN), ZA, ZLYR(MXLYR), ZP, ZS, ZSVP(MXSVP)
EQUIVALENCE (H, ZLYR(1)), (D, ZS), (ZJ, ZLYR(1))
DATA PI/3.141592654/, DEG/57.29578/

C IF(THETA) 50, 100, 150
C
** THETA LESS THAN 0.0. BOTTOM SLOPES UP.
50 CONTINUE
BOTY = U(N)
C BOTX = .................
RETURN
C
** THETA = 0.0. BOTTOM IS FLAT.
100 CONTINUE
BOTY = U(N)
CALL UFIELD
BOTX = A1JM
RETURN
C
** THETA GREATER THAN 0.0, BOTTOM SLOPES DOWN.
150 CONTINUE
C BOTY = .................
C BOTX = .................
RETURN
END