OFFICE OF NAVAL RESEARCH
Final Report
for
Contract N00014-85-K-0495

Research in Enumeration and Graph Theory
with Applications to Computer Science

Edward A. Bender
Department of Mathematics
University of California at San Diego
La Jolla, CA 92093

Reproduction in whole, or in part, is permitted for any purpose of the United States Government.

* This document has been approved for public release and sale; its distribution is unlimited.
Section 1: Overview

In addition to being at UCSD, I attended a month long working meeting in Vancouver, B.C., and visited the University of Georgia and the University of Auckland, N.Z. At the meeting I learned a considerable amount and began various lines of research, some leading nowhere and others that reached fruition in Georgia.

During the contract, almost half of my effort has been directed toward studying maps on surfaces other than the sphere, an area in which very little progress had been made. I was very pleasantly surprised by what I and my colleagues were able to accomplish. The concluded research is described in [1–4]. Canfield and I are currently pursuing additional research.

Other completed research was on convex polyhedra, reported in [5] and [6], and the size of PLA's, the conclusion of work begun earlier and reported in [7].

At present, research is underway on three problems in graph theory. The current state is described in the next section. One of these, [C], involves the use of a PC purchased under the grant.

The visit to the University of Georgia lasted for two quarters. It involved some teaching, paid for by UGA, and some research, paid for partially by UGA but primarily by ONR. (The UGA sponsored research is not reported here.) During this time, [1], [2], [4] and [6] were completed, [A] was started, [B] was worked on and issue of the feasibility of [C] was raised.

The visit to the University of Auckland lasted about two months. During this time [3] and [5] were completed and some unfruitful lines of research were pursued. For part of this visit, Brendan McKay was also visiting and [B] was begun.

At UCSD, [7] was finished and serious study of [C] was initiated.

Section 2: Completed Research

The enumeration of rooted maps by number of edges on the torus and projective plane is studied. Explicit expressions for the generating functions are obtained.

1
Asymptotic results, recurrence relations and numerical tables are also presented. (Some of this material is preparatory for [2].)

We asymptotically enumerate two classes of maps by edges on a fixed but arbitrary surface: rooted maps and rooted smooth maps. This is done by obtaining a complicated set of recursive equations that expression generating functions for maps with distinguished faces on a given surface in terms of similar generating functions on surfaces of lower genus. Asymptotics are obtained directly from the recursions without solving them.

We obtain asymptotics for the number of rooted nonseparable maps on a fixed but arbitrary surface. This is done by first enumerating a more restrictive class of maps, called nonsingular, by using the results in [2]. Then it is shown that almost all nonseparable maps are nonsingular.

Tree rooted maps were studied by Walsh and Lehman on orientable surfaces. Asymptotic formulas for tree–rooted maps and smooth tree–rooted maps can be obtained from their results. We also obtain results on nonorientable surfaces using the methods in [2].

Using Lagrange inversion, we obtain a previously overlooked single summation formula with decreasing terms for the number of rooted convex polyhedra. This leads to the pretty asymptotic formula for the number of convex polyhedra with \( i+1 \) vertices and \( j+1 \) faces:

\[
\frac{1}{4 \times 3^i i! j! (i+j)! (j+3) (i+3)} (\binom{2i}{j+3} \binom{2j}{i+3}).
\]

We study the root face valency of random convex polyhedra as the number of vertices gets large.

7. (with J.T. Butler) On the size of PLA’s required to realize binary and multiple–
We study the average number of product terms required to realize binary and
multiple-valued functions as a function of the number of nonzero output values.
This improves on previously known bounds. Furthermore, we study the variance,
which had previously been intractable. This allows us to obtain estimates of the
probability that a function with relatively few nonzero outputs will be realizable
with a given PLA.

Section 3: Research in Progress

[A] (with E.R. Canfield and R.W. Robinson) We are studying labeled planar graphs
(a map is embedded, a planar graph can be embedded). The embedding of a
graph is unique if and only if it is 3-connected. This fact and [6] together pro-
vide a way to go from 3-connected maps to 3-connected labeled planar graphs.
Methods developed by Walsh (and others) allow one to then go to 2-connected
and to connected planar maps. Unfortunately, this is complicated by the presence
of parametrizations. We are able to obtain exact numbers and are working on
asymptotics. (This was tabled after I left Georgia.)

[B] (with E.R. Canfield and B.D. McKay) Assume that the probability of that a ran-
dom labeled, $n$-vertex, $q$-edge graph is connected approached the form $A(x)B(x)^n$
where $x = q/n$. McKay and I obtained an equation for $B(z)$ and began attempts
at proving that the nth root of the probability approached $B(z)$. While McKay
obtained strong numerical evidence for the conjecture, Canfield and I obtained an
equation for $A(z)$ and worked on proving the conjecture. We are continuing our
efforts while Canfield is visiting UCSD. The proof is nearly complete. It involves
a double induction on $n$ and $q$ and utilizes Wright's results for small $q - n$. So far
we have written up nearly fifty pages.

[C] (with E.R. Canfield) Although the results in [2] allow one, in principle, to de-
terminate the generating function for maps on any given surface, the amount of
calculation involved is tremendous. As stated, the calculations involve intermedi-
ate equations with $g + 2$ variables, where $g$ is the genus of the surface; however, we
can show that the final result must have a relatively simple form. An attempt to
use a symbolic language on a supercomputer bogged down due to lack of memory.
Further study has shown ways to reduce the amount of calculations considerably.
It should be possible to obtain some generating functions if one tailors a symbolic
language to the problem. Canfield and I plan to return to this shortly. I have
already done some programming for it using a PC.
END
DATE
FILMED
6-1988
DTIC