The \( \varepsilon \) Function

By

J.D.A. Walker and R.K. Scharnhorst

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This report is a compendium of properties of a mathematical function (the $\Xi$ function) which arises in the modeling of turbulent boundary-layer flows near solid walls. The $\Xi$ function is formally defined as a triple integral which cannot be easily expressed in terms of elementary or known special functions. Thus series and asymptotic expansions are developed here as well as a table of integrals (most of which are indefinity). The function and its derivative are tabulated correct to ten significant figures. A FORTRAN function routine for the calculation of $\Xi(x)$ and $\Xi'(x)$ is also given.
THE $\Xi$ FUNCTION

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ABSTRACT

This report is a compendium of properties of a mathematical function (the Ξ function) which arises in the modeling of turbulent boundary-layer flows near solid walls. The Ξ function is formally defined as a triple integral which cannot be easily expressed in terms of elementary or known special functions. Thus series and asymptotic expansions are developed here as well as a table of integrals (most of which are indefinite). The Ξ function and its derivative are tabulated correct to ten significant figures. A FORTRAN function routine for the calculation of Ξ(x) and Ξ'(x) is also given.
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Table B.2. Tabulated values of $\Xi(x)$ and $\Xi'(x)$. 27
1. Introduction

The diffusion equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} , \]  

(1.1)

arises in a wide variety of applications in engineering including fluid mechanics and heat and mass transfer. In many situations, the relevant solution of equation (1.1) is a function of the similarity variable,

\[ n = \frac{y}{2\sqrt{t}} . \]  

(1.2)

When the solution of equation (1.1) is required for \( y \) (or \( n \)) in the semi-infinite range \( 0 \leq y < \infty \), a boundary condition is normally known at \( y=0 \) as well as some particular type of asymptotic behavior as \( y \to \infty \). Specific types of asymptotic conditions which can occur might include the following situations:

i) \( u \) must approach zero or a constant exponentially, viz.

\[ u \sim A_0 , \text{ as } y \to \infty , \]  

(1.3)

where \( A_0 \) is a constant (or zero);

ii) \( u \) must approach zero or a constant algebraically, viz.

\[ u \sim A_1 + A_2 f(t) n^{-\alpha} \text{ as } n \to \infty , \]  

(1.4)

where \( f(t) \) is a function of \( t \), \( A_1 \) and \( A_2 \) are constants and \( \alpha > 0 \); and

iii) \( u \) must become large algebraically for large \( y \), viz.

\[ u \sim A_3 g(t) n^\beta + ... \text{ as } n \to \infty , \]  

(1.5)

where \( g(t) \) is a function of \( t \), \( A_3 \) is a constant and \( \beta > 0 \).
For the types of asymptotic conditions described by equations (1.3), (1.4) and (1.5), the solution to equation (1.1) can often be written in terms of parabolic cylinder functions or alternatively in terms of repeated integrals of the error function (Abramowitz and Stegun, 1964).

A rather different type of asymptotic condition for large \( y \) arises in the study of velocity and temperature profiles in the near-wall region of turbulent boundary-layer flows (Black, 1968; Scharnhorst, Walker and Abbott, 1977; Scharnhorst, 1978; Weigand, 1978; Walker, Scharnhorst and Weigand, 1986); here a solution of the diffusion equation is required whose time-average over a finite interval of time is logarithmic for large \( y \). It is in this context that the \( \Xi \) function arises.

The details of the turbulence model discussed by Walker, Scharnhorst and Weigand (1986) are complex and will not be discussed here. However an essential feature of the analysis is that a solution of equation (1.1) is required subject to the asymptotic boundary condition,

\[
    u - B \ln y + C \quad \text{as} \quad y \rightarrow \infty,
\]

where \( B \) and \( C \) are known constants. The solution of equation (1.1) satisfying,

\[
    u(0,t) = A,
\]

where \( A \) is constant, is given by,
\[ u = \left[ C + \frac{B}{2} \{ 2 \ln(2) - \gamma + \ln(t) \} \right] \text{erf}(n) + A\text{erfc}(n) \]
\[ + \frac{4B}{\sqrt{\pi}} \Xi(n) . \]  

(1.8)

Here \( \gamma \) is Euler's constant and the \( \Xi \) function is formally defined as the triple integral,

\[ \Xi(n) = \int_0^n e^{-\xi^2} \int_0^\xi e^{\zeta^2} \int_0^\zeta e^{-t^2} \, dt \, d\xi . \]

(1.9)

The function defined by equation (1.9) cannot be easily expressed in terms of elementary or known special functions. Consequently because the function appears in velocity and temperature profile representations of the flow in the near wall region of turbulent boundary layers (Walker, Scharnhorst and Weigand, 1986), it is worthwhile to develop a compendium of useful properties of this function and this is the purpose of this report.

The plan of the report is as follows. In section 2, a number of general properties of the \( \Xi \) function are given and \( \Xi(x) \) and its derivative \( \Xi'(x) \) are plotted in figures 1 and 2 respectively. The notation is generally consistent with Abramowitz and Stegun (1964) and is defined in a separate section on page iv; generally \( x \) is used to denote a real variable and \( a \) denotes a positive real constant. In section 3, a number of indefinite and definite integrals are listed; for the indefinite integrals, the constants of integration have been omitted. In three of the integrals given in section 3, a new function arises which has been denoted by \( \text{erfei}(x) \); some properties of this
function are given in Appendix A. Finally a FORTRAN program for
the evaluation of $\Xi$ and $\Xi'$ is given in Appendix B as well as tabu-
lated values of these functions correct to ten significant figures.
2. GENERAL PROPERTIES

2.1 Definitions

\[
\Xi(x) = \int_0^x e^{-\xi^2} \left( \int_0^\xi e^{-t^2} \int_0^t dtd\xi \right), \tag{2.1}
\]

\[
\Xi(x) = \frac{\sqrt{\pi}}{2} \int_0^x e^{-\xi^2} \left( \int_0^\xi e^{-t^2} \epsilon\delta(\xi) d\xi \right), \tag{2.2}
\]

\[
\Xi'(x) = e^{-x^2} \int_0^x e^{\xi^2} \left( \int_0^\xi e^{-t^2} dtd\xi \right), \tag{2.3}
\]

\[
\Xi'(x) = \frac{\sqrt{\pi}}{2} e^{-x^2} \int_0^x e^{\xi^2} \epsilon\delta(\xi) d\xi. \tag{2.4}
\]

2.2 Differential Equation

\[
\Xi'' + 2x\Xi' = \frac{\sqrt{\pi}}{2} \epsilon\delta(x), \quad \Xi(0) = \Xi'(0) = 0. \tag{2.5}
\]

2.3 Integral Representation

\[
\Xi(x) = -\frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-p^2/4} \frac{\sin(px)}{p} \left[ \ln(p) + \frac{\gamma}{2} \right] dp. \tag{2.6}
\]

2.4 Symmetry Relations

\[
\Xi(-x) = -\Xi(x), \tag{2.7}
\]

\[
\Xi'(-x) = \Xi'(x). \tag{2.8}
\]
2.5 Series Expansions

\[ \Xi(x) = e^{-x^2} \sum_{j=1}^{\infty} \frac{2j d_j}{(2j + 1)!!} x^{2j+1}, \quad |x| < \infty. \quad (2.9) \]

Here \( d_j = \gamma + \psi(j + 1) = \frac{1}{2} k^{-1}. \)

\[ \Xi'(x) = e^{-x^2} \sum_{j=1}^{\infty} \frac{2j}{j(2j - 1)!!} x^{2j}, \quad |x| < \infty, \quad (2.10) \]

\[ \Xi(x) = \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} c_j}{(2j + 1)!!} x^{2j+1}, \quad |x| < \infty, \quad (2.11) \]

\[ \Xi'(x) = \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} c_j}{j!!} x^{2j}, \quad |x| < \infty. \quad (2.12) \]

In (2.11) and (2.12), \( c_j = \ln(2) + \frac{\gamma}{2} + \frac{1}{2} \psi(j + 1/2) = \frac{1}{2} (2i - 1)^{-1}. \)

2.6 Asymptotic expansions

\[ \Xi(x) \sim \frac{\sqrt{\pi}}{4} \left[ \ln(x) + \frac{\gamma}{2} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{(2j - 1)!!}{j^2 x^{2j}} \right], \quad x \to \infty \quad (2.13) \]

\[ \Xi'(x) \sim \frac{\sqrt{\pi}}{4} \sum_{j=0}^{\infty} \frac{(2j - 1)!!}{2^j x^{2j+1}}, \quad x \to \infty. \quad (2.14) \]

2.7 Expression as a series of incomplete gamma functions

\[ \Xi(x) = \frac{\sqrt{\pi}}{8} \sum_{j=1}^{\infty} \frac{1}{j^r (j + 1/2)} \gamma(j + 1/2, x^2). \quad (2.15) \]

\[ ^{\dagger}\text{for negative argument the double factorial is defined to be identically equal to one.} \]
2.8 Relation to Dawson's Integral

\[ \Xi'(x) = \frac{\sqrt{\pi}}{2} e^{x^2} \text{erf}(x) F(x) - e^{-x^2} \int_0^x F(t) dt. \]  

(2.16)

2.9 Maximum of \( \Xi'(x) \)

\[ \Xi'(1.379005630\ldots) = 0.3048918974\ldots, \]

(2.17)

\[ \Xi(1.379005630\ldots) = 0.2257293172\ldots. \]

(2.18)
Figure 1. Graphs of $\Xi(x)$ and $\Xi'(x)$ on a linear x-scale.
Figure 2. Graphs of $\Xi(x)$ and $\Xi'(x)$ on a logarithmic x-scale.
3. INTEGRALS

3.1 Combination of $\Xi$ and $\Xi'$ with Powers

1. \[ \int \Xi(ax)\,dx = x\Xi(ax) + \frac{1}{2} \Xi'(ax) - \frac{\sqrt{\pi}}{4} x e^a (ax) - \frac{1}{4a} e^{-a^2 x^2}, \]

2. \[ \int x\Xi(ax)\,dx = \frac{(2a^2 x^2 - 1)}{4a^2} \Xi(ax) + \frac{x}{4a} \Xi'(ax) - \frac{\sqrt{\pi} \, 2}{32a^2} (2a^2 x^2 - 1) e^a (ax) \]
\[- \frac{x}{16a} e^{-a^2 x^2}, \]

3. \[ \int x^n \Xi(ax)\,dx = p_1(a,x) \Xi(ax) + p_2(a,x) \Xi'(ax) + p_3(a,x) e^a (ax) \]
\[ + p_4(a,x) e^{-a^2 x^2}, \]

In 3.1 3 for $n$ a positive even integer, viz. $n = 2k$ with $k = 0,1,2,\ldots$:

\[ p_1 = \frac{x^{2k+1}}{2k+1}, \quad p_2 = \frac{k!}{2(2k+1)a^{2k+1}} \sum_{j=0}^{k} \frac{1}{j!} (ax)^{2j} \]

\[ p_3 = -\frac{\sqrt{\pi} \, k!}{4(2k+1)a^{2k}} \sum_{j=0}^{k} \frac{a^{2j}}{(2j+1)j!} x^{2j+1}, \]

\[ p_4 = -\frac{k!}{4(2k+1)a^{2k+1}} \sum_{j=0}^{k} \frac{1}{j!} (ax)^{2j} (c_{k+1} - c_j), \]

where $c_j$ is defined in (2.12) for $j \geq 1$ and $c_0 = 0$. In 3.1 3 for $n$ a positive odd integer, viz. $n = 2k + 1$ with $k = 0,1,2,3,\ldots$:
\[ P_1 = \frac{1}{2(k+1)} \left[ x^{2k+2} - \frac{(2k+1)!!}{2^{k+1}a^{2k+2}} \right], \]
\[ P_2 = \frac{(2k+1)!!}{2^{k+2}a^{2k+2}(k+1)} \sum_{j=0}^{k} \frac{2^j}{(2j+1)!!} (ax)^{2j+1}, \]
\[ P_3 = -\frac{\sqrt{\pi}}{2k+4} \frac{(2k+1)!!}{a^{2k+2}(k+1)} \left[ \sum_{j=0}^{k} \frac{2^j}{(j+1)(2j+1)!!} (ax)^{2j+2} - \frac{1}{2} d_{k+1} \right], \]
\[ P_4 = -\frac{(2k+1)!!}{2^{k+4}a^{2k+2}(k+1)} \sum_{j=0}^{k} \frac{2^j}{(2j+1)!!} (ax)^{2j+1} (d_{k+1} - d_j), \]

where \( d_j \) is defined in (2.9) for \( j \geq 1 \) and here \( d_0 = 0 \).

4. \( \int x^2 dx = -\frac{1}{2a^2} x^2 - \frac{\sqrt{\pi}}{2a} x e^{-\frac{a^2}{4}} + \frac{1}{4a^2} e^{-a^2 x^2} \),

5. \( \int x^n dx = \frac{x^{n+1}}{a} \),

for \( n \) integer and \( n \geq 1 \).

6. \( \int_0^x \frac{t}{t} \ dt = \frac{1}{4} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} c_j}{j!} (ax)^{2j}, \quad |ax| < \infty, \)

where \( c_j \) is defined in (2.12).

7. \( \int_0^\infty x dx = \frac{2}{16} \),

8. \( \int_0^x \frac{t}{t} \ dt = \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} c_j}{(2j+1)^2} (ax)^{2j+1}, \quad |ax| < \infty, \)
where $c_j$ is defined in (2.12),

9. \[ \int \Xi(ax) \, dx = - \frac{1}{x} \Xi(ax) + a \int \Xi'(ax) \, dx, \]

10. \[ \int \Xi(ax) \, dx = - \left[ a^2 + \frac{1}{2x^2} \right] \Xi(ax) - \frac{a}{2x} \Xi'(ax) + \frac{\sqrt{\pi}}{4} a^2 \text{erf}e(ax), \]

where the function $\text{erf}e(x)$ is discussed in Appendix A.

11. \[ \int \Xi'(ax) \, dx = - \frac{1}{x} \Xi'(ax) - 2a\Xi(ax) + \frac{\sqrt{\pi}}{2} a \text{erf}e(ax), \]

12. \[ \int \Xi(ax) \, dx_n = \frac{1}{n-1} \left[ \frac{1}{x^{n-1}} \Xi'(ax) + \frac{a\sqrt{\pi}}{2(n-2)x^{n-2}} \text{erf}e(ax) \right. \\
\left. + \frac{a^{n-1}}{2(n-2)} \Gamma(\frac{3-n}{2}, a^2x^2) + 2a^2 \int \Xi'(ax) \, dx_n \right], \]

for $n \geq 3$. In 3.1 12, for $n$ an even integer, viz. $n = 2k+2$ with $k = 1,2,3,...$

\[ \Gamma(\frac{3-n}{2}, a^2x^2) = \frac{(-1)^{k+1}}{\Gamma(2k+1)} \left[ e^{-a^2x^2} \sum_{i=1}^{k} \frac{(-1)^i \Gamma(2i-1)}{(ax)^{2i-1}} + \pi\text{erf}e(ax) \right], \]

while for $n$, an odd integer, viz. $n = 2k+1$ with $k = 1,2,3,...$

\[ \Gamma(\frac{3-n}{2}, a^2x^2) = \frac{(-1)^{k}}{(k-1)!} \left[ e^{-a^2x^2} \sum_{i=1}^{k-1} \frac{(-1)^i \Gamma(i)}{(ax)^{2i}} + E_1(a^2x^2) \right] \]

13. \[ \int \Xi(ax) \, dx_n = - \frac{1}{(n-1)} \left[ \frac{\Xi(ax)}{x^{n-1}} - a \int \Xi'(ax) \, dx_n \right], \quad n \geq 2, \]

\[ \text{for } k = 1 \text{ the summation term is identically zero.} \]
3.2 Combinations of $\Xi$ and $\Xi'$ with Exponentials and Powers

1. $\int_0^\infty e^{-\Xi'(t)} dt = \frac{1}{2} \sum_{j=1}^{\infty} \frac{a_j}{(2j + 1)} x^{2j+1}$, \quad x, < \infty.

Here

$$a_j = \frac{(-1)^j + 1}{j!} \sum_{i=1}^j \binom{j}{i} c_i,$$

where $c_i$ is defined in (2.12).

2. $\int_0^\infty e^{-\Xi^2/2} (\alpha x) dx = \frac{1}{8} \sqrt{\frac{\pi}{\alpha^2 + 1}} \ln(\alpha^2 + 1),$

3. $\int xe^{-\Xi^2/2} \Xi'(x) dx = -\frac{1}{4} \frac{\Xi'}{\Xi} + \frac{\pi}{32} e \Xi_0(x),$

4. $\int x^2 e^{-\Xi^2/2} \Xi'(x) dx = -\frac{x}{4} e^{-\Xi^2/2} \Xi'(x) - \frac{\sqrt{\pi}}{16} e^{-x^2/2} e \Xi_0(x) + \frac{\sqrt{2\pi}}{32} e \Xi_0(\sqrt{2x})$
   $+ \frac{1}{4} \int e^{-\Xi^2/2} \Xi'(x) dx,$

5. $\int x^3 e^{-\Xi^2/2} \Xi'(x) dx = \frac{1}{8} (2x^2 + 1) e^{-\Xi^2/2} \Xi'(x) + \frac{\pi}{32} e \Xi_0^2(x)$
   $- \frac{\sqrt{\pi}}{16} x e^{-x^2/2} e \Xi_0(x) - \frac{1}{32} e^{-2x^2},$

6. $\int x^n e^{-\Xi^2/2} \Xi'(x) dx = -\frac{x}{4} e^{-\Xi^2/2} \Xi'(x) + \frac{\sqrt{\pi}}{8} \int x^{n-1} e^{-x^2/2} e \Xi_0(x) dx$
   $+ \frac{(n-1)}{4} \int x^{n-2} e^{-x^2/2} \Xi'(x) dx.$

For the evaluation of the second term on the right side of equation 3.2

6 the following reduction formulae are useful:
\[ \int x^{n-1} e^{-x^2} \varphi\delta(x)\,dx = -\frac{x^{n-2}}{2} e^{-x^2} \varphi\delta(x) + \frac{1}{\sqrt{\pi}} \int x^{n-2} e^{-2x^2} \,dx \]
\[ + \left(\frac{n-2}{2}\right) \int x^{n-3} e^{-x^2} \varphi\delta(x)\,dx, \]
\[ \int x^{n-1} e^{-2x^2} \,dx = -\frac{x^{n-2}}{4} e^{-2x^2} + \left(\frac{n-2}{4}\right) \int x^{n-3} e^{-2x^2} \,dx, \]
\[ 7. \int_{0}^{\infty} e^{-t^2} \varphi(t) \,dt = \sum_{j=1}^{\infty} b_j x^{j+2}, \]

where
\[ b_j = \frac{(-1)^j+1}{4(j+1)!} \sum_{i=1}^{j} \binom{j}{i} \frac{c_i}{2i+1}, \]

and \( c_i \) is defined in (2.12).

\[ 8. \int_{0}^{\infty} e^{-t^2} \varphi(t) \,dt = \frac{G}{8} - \frac{\pi}{32} \ln(2), \]

where \( G \) is Catalan's constant.

\[ 9. \int x e^{-x^2} \varphi(x) \,dx = -\frac{e^{-x^2}}{2} \varphi(x) + \frac{1}{2} \int e^{-x^2} \varphi'(x) \,dx, \]
\[ 10. \int x^n e^{-x^2} \varphi(x) \,dx = -\frac{x^{n-1}}{2} e^{-x^2} \varphi(x) + \frac{1}{2} \int x^{n-1} e^{-x^2} \varphi'(x) \,dx \]
\[ + \left(\frac{n-1}{2}\right) \int x^{n-2} e^{-x^2} \varphi(x) \,dx. \]

For the evaluation of the second term on the right side of 3.2 10 see 3.2 6.
11. \[
\int_0^x e^{-t^2} \Xi'(t) \, \frac{dt}{t} = \frac{1}{4} \sum_{j=1}^{\infty} a_j x^{2j}, \quad |x| < \infty,
\]

where \(a_j\) is defined in 3.2 1.

12. \[
\int_0^x e^{-t^2} \Xi'(at) \, \frac{dt}{t} = \frac{\pi^2}{16} - \frac{1}{4} \arccot(a)[\pi - \arccot(a)], \quad -\infty < a < \infty,
\]

13. \[
\int_0^x e^{-t^2} \Xi'(t) \, \frac{dt}{t^2} = -\frac{e^{-x^2}}{x} \Xi'(x) - 4 \int_0^x e^{-t^2} \Xi'(t) \, dt + \frac{\pi}{8} \operatorname{erf}^2(x)
\]

\[+ \frac{\pi}{8} \int_0^x \operatorname{erf}^2(t) \, \frac{dt}{t^2},\]

14. \[
\int_0^x e^{-t^2} \Xi(t) \, \frac{dt}{t^2} = \frac{\sqrt{\pi}}{2} \ln(1 + \sqrt{2}) - \frac{\sqrt{2\pi}}{4} \ln(\sqrt{2})
\]

15. \[
\int_0^x e^{-t^2} \Xi'(t) \, \frac{dt}{t^n} = \frac{e^{-x^2}}{(n-1)x^{n-1}} \Xi'(x) - \frac{4}{(n-1)} \int_0^x e^{-t^2} \Xi'(t) \, \frac{dt}{t^{n-2}}
\]

\[+ \frac{\sqrt{\pi}}{2(n-1)} \int_x^\infty e^{-t^2} \operatorname{erf}(t) \, \frac{dt}{t^{n-1}},\]

for \(n > 2\).

16. \[
\int_0^x e^{-t^2} \Xi(t) \, \frac{dt}{t} = \sum_{j=1}^\infty \frac{2(j+1)}{2j+1} b_j x^{2j+1}, \quad |x| < \infty,
\]

where \(b_j\) is defined in 3.2 7.

17. \[
\int_0^\infty e^{-t^2} \Xi(t) \, \frac{dt}{t} = 0.04712894918,
\]
18. \[
\int e^{-x^2} \Xi(x) \frac{dx}{x^n} = -\frac{e^{-x^2}}{(n-1)x^{n-1}} \Xi(x) - \frac{2}{(n-1)} \int e^{-x^2} \Xi(x) \frac{dx}{x^{n-2}}
+ \frac{1}{(n-1)} \int e^{-x^2} \Xi'(x) \frac{dx}{x^{n-1}},
\]
for \( n \geq 2 \).

3.3 Combinations of \( \Xi \) and \( \Xi' \) with the Error function and Powers

1. \[
\int er\delta(x) \Xi'(x) dx = er\delta(x) \Xi(x) - \frac{2}{\sqrt{\pi}} \int e^{-t^2} \Xi'(t) dt,
\]
2. \[
\int er\delta(x) \Xi(x) dx = \left[ x er\delta(x) + \frac{e^{-x^2}}{\sqrt{\pi}} \Xi(x) + \frac{1}{2} er\delta(x) \Xi'(x) \right.
- \frac{2}{\sqrt{\pi}} \int e^{-t^2} \Xi'(t) dt
- \frac{\sqrt{\pi}}{4} \left[ x er\delta^2(x) + \frac{2}{\sqrt{\pi}} e^{-x^2} er\delta(x) - \sqrt{\frac{2}{\pi}} er\delta(\sqrt{2}x) \right],
\]
3. \[
\int x er\delta(x) \Xi'(x) dx = -\frac{1}{2} er\delta(x) \Xi'(x) + \frac{1}{\sqrt{\pi}} \int e^{-t^2} \Xi'(t) dt
+ \frac{\sqrt{\pi}}{4} x er\delta^2(x) + \frac{1}{2} e^{-x^2} er\delta(x) - \frac{\sqrt{2}}{4} er\delta(\sqrt{2}x),
\]
4. \[
\int x er\delta(x) \Xi(x) dx = \left[ \frac{(2x^2-1)}{4} er\delta(x) + \frac{1}{2\sqrt{\pi}} xe^{-x^2} \right] \Xi(x)
+ \frac{1}{4} \left[ x er\delta(x) + \frac{e^{-x^2}}{\sqrt{\pi}} \right] \Xi'(x) - \frac{\sqrt{\pi}}{16} x^2 er\delta^2(x)
- \frac{1}{8} xe^{-x^2} er\delta(x) - \frac{1}{16\sqrt{\pi}} e^{-2x^2},
\]

-16-
5. \[ \int x^n e \tilde{\delta}(x) \Xi'(x) \, dx = x^n e \tilde{\delta}(x) \Xi(x) - \frac{2}{\sqrt{\pi}} \int x^n e^{-x^2} \Xi(x) \, dx \]

\[ - n \int x^{n-1} e \tilde{\delta}(x) \Xi(x) \, dx. \]

For evaluation of the second term on the right side of 3.3 5 see 3.2 10.

3.4 Combinations of \( \Xi \) and \( \Xi' \) with Logarithms and Powers

1. \[ \int_0^x \ln(t) \Xi'(t) \, dt = \ln(x) \Xi(x) - \int_0^x \frac{\Xi(t)}{t} \, dt, \]

2. \[ \int_0^x \ln(t) \Xi(t) \, dt = \left[ x \Xi(x) + \frac{1}{2} \Xi'(x) - \frac{\sqrt{\pi}}{4} x e \tilde{\delta}(x) - \frac{e^{-x^2}}{4} + \frac{1}{4} \right]. \]

\[ \frac{\left[ \ln(x) - 1 \right]}{4} x e \tilde{\delta}(x) + \frac{1}{4} \left[ e^{-x^2} - 1 \right] - \frac{1}{8} E_1(x^2) \]

\[ - \frac{1}{4} \ln(x) - \frac{\gamma}{8} - \frac{1}{2} \int_0^x \Xi'(t) \, \frac{dt}{t}, \]

3. \[ \int_0^x t \ln(t) \Xi'(t) \, dt = - \left[ \frac{1}{2} \Xi'(x) - \frac{\sqrt{\pi}}{4} x e \tilde{\delta}(x) - \frac{1}{4} e^{-x^2} + \frac{1}{4} \right] \ln(x) \]

\[ + \frac{1}{2} \int_0^x \Xi'(t) \, \frac{dt}{t} - \frac{\sqrt{\pi}}{4} x e \tilde{\delta}(x) - \frac{1}{4} \left( e^{-x^2} - 1 \right) \]

\[ + \frac{1}{8} \left[ E_1(x^2) + 2 \ln(x) + \gamma \right], \]

4. \[ \int_0^x t \ln(t) \Xi(t) \, dt = \left[ \frac{(2x^2-1)}{4} \Xi(x) + \frac{x}{4} \Xi'(x) - \frac{\sqrt{\pi}}{32} \left( 2x^2 - 1 \right) e \tilde{\delta}(x) \right] \]

\[ - \frac{x}{16} e^{-x^2} \] \[ \cdot \left[ \ln(x) - \frac{1}{2} \right] - \frac{1}{4} \Xi(x) + \frac{1}{4} \int_0^x \Xi(t) \, \frac{dt}{t} \]

\[ + \frac{\sqrt{\pi}}{32} \left[ x^2 e \tilde{\delta}(x) + \frac{1}{\sqrt{\pi}} x e^{-x^2} + \frac{1}{2} e \tilde{\delta}(x) - e \tilde{\delta} e \tilde{\delta}(x) \right]. \]
3.5 Miscellaneous Integrals

1. \( \int_{0}^{x} e^{a^2t^2} \text{er}_6(at) \, dt = \frac{2}{a\sqrt{\pi}} e^{a^2x^2} \Xi'(ax), \)

2. \( \int_{0}^{x} e^{t^2} \text{er}_6^2(t) \, dt = \frac{2}{\sqrt{\pi}} \, e^{x^2} \text{er}_6(x) \Xi'(x) - \frac{4}{\pi} \, \Xi(x), \)

3. \( \int_{0}^{x} e^{t^2} \text{er}_6^3(t) \, dt = \frac{2}{\sqrt{\pi}} \, e^{x^2} \text{er}_6^2(x) \Xi'(x) - \frac{8}{\pi} \, \text{er}_6(x) \Xi(x) \)
   \[ + \frac{16}{\pi\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \Xi(t) \, dt, \]

4. \( \int_{0}^{x} e^{t^2} \text{er}_6c^2(t) \text{er}_6(t) \, dt = \frac{2}{\sqrt{\pi}} \, \text{er}_6c^2(x) e^{x^2} \Xi'(x) + \frac{8}{\pi} \, \text{er}_6c(x) \Xi(x) \)
   \[ + \frac{16}{\pi\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \Xi(t) \, dt, \]

5. \( \int_{0}^{x} e^{t^2} \text{er}_6c(\sqrt{2}t) \text{er}_6(t) \, dt = \frac{2}{\sqrt{\pi}} \, \text{er}_6c(\sqrt{2}x) e^{x^2} \Xi'(x) \)
   \[ + \frac{4\sqrt{2}}{\pi} \int_{0}^{x} e^{-t^2} \Xi'(t) \, dt. \]
REFERENCES


APPENDIX A: THE FUNCTION ERFEI(x)

In integrals 3.1 10, 3.1 11 and 3.4 4, the function defined by equation (A.1) arises. In this Appendix, some useful properties of this function are given; the function is plotted in figure A.1 and tabulated in table A.1.

A.1 Definition

\[ \text{erfei}(x) = \int_0^x \frac{erf(t)}{t} \, dt. \]

A.2 Integral Representation

\[ \text{erfei}(x) = -\frac{2}{\sqrt{\pi}} x \int_0^1 e^{-x^2 t^2} \ln(t) \, dt, \quad |x| < \infty. \]

A.3 Symmetry Relation

\[ \text{erfei}(-x) = -\text{erfei}(x). \]

A.4 Series Expansion

\[ \text{erfei}(x) = \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j + 1)^{2j+1}} x^{2j+1}, \quad x < \infty. \]

A.5 Asymptotic Expansions

\[ \text{erfei}(x) \sim \ln(2x) + \frac{x}{2} + \frac{e^{-x^2}}{2\sqrt{\pi} x^3} \left[ 1 - \frac{2}{x^2} + \frac{23}{4x^4} - \frac{22}{x^6} + \frac{1689}{16x^8} - \frac{4881}{8x^{10}} + \ldots \right], \quad x \to \infty. \]

-20-
A.6 Indefinite Integrals

1. \[ \int e^{\text{erf}i(x)} \, dx = x \text{erf}i(x) - x \text{erf}(x) - \frac{1}{\sqrt{\pi}} e^{-x^2}, \]

2. \[ \int x^n e^{\text{erf}i(x)} \, dx = \frac{1}{n+1} x^{n+1} \text{erf}i(x) - \frac{1}{(n+1)^2} x^{n+1} \text{erf}(x) \]
   \[ + \frac{1}{(n+1)^2 \sqrt{\pi}} \gamma(1 + \frac{n}{2}, x^2). \]
Figure A.1. The function $erfi(x)$
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<th>x</th>
<th>( e^{\text{erf}^{-1}(x)} )</th>
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\( ^{+} \text{Values of } e^{\text{erf}^{-1}(x)} \text{ for } x > 4 \text{ (correct to ten significant figures) may be obtained by using the first two terms of A.5.} \)
APPENDIX B: FORTRAN FUNCTION ROUTINE AND TABULATED VALUES OF $\Xi(x)$ AND $\Xi'(x)$

In Table B.1, the listing of a function subprogram is given for the numerical evaluation of $\Xi(x)$ and $\Xi'(x)$. The functions $\Xi(x)$ and $\Xi'(x)$ are accessed from the calling program by,

$$Y = \Xi(x, \text{EPSI, IERR}), \quad Y = \Xi'(x, \text{EPSI, IERR}),$$

respectively. In these calling statements

- $X \equiv$ the argument of either $\Xi(x)$ or $\Xi'(x)$,
- $\text{EPSI} \equiv$ the assigned tolerance which is dependent of the number of significant figures desired for $\Xi(x)$ or $\Xi'(x)$,
- $\text{IERR} \equiv$ an integer variable returned by the function subprogram. ($\text{IERR} = 0$ if the tolerance EPSI is met; $\text{IERR} = 1$ otherwise.)

The subprogram accepts only zero or positive values of $x$ and returns a zero value if $x < 0$. If the evaluation of $\Xi(x)$ or $\Xi'(x)$ is required for $x < 0$, the calling statement should reflect the appropriate symmetry relation (either (2.7) or (2.8)).

For $x < 5.0$ the routine evaluates $\Xi(x)$ and $\Xi'(x)$ using the power series expansions (2.9) and (2.11) in the DO loops commencing with cards 20 and 29, respectively. The power series in (2.9) and (2.10) are summed until they have converged to the number of significant figures dictated by the assigned value of the tolerance EPSI; once convergence occurs control is transferred out of the DO loops to statements 3 or 5, respectively. The DO loops commencing
with cards 20 and 29 are initially set (by card 11) so that at most 100 terms in either power series will be evaluated. If convergence occurs before the DO loops are satisfied, a reliable result has been obtained and IERR = 0 is returned to the calling program. If the DO loops are satisfied, the returned value of $\Xi(x)$ or $\Xi'(x)$ may not be reliable to the specified tolerance (EPSI) and IERR = 1 is returned to the calling program.

For $x \geq 5.0$, the asymptotic expansions (2.13) and (2.14) are used for the evaluation of $\Xi$ and $\Xi'$. The sums in (2.13) and (2.14) are evaluated in the DO loops commencing with cards 38 and 46, respectively. Control is transferred out of these DO loops when the last term added to these sums is less than the desired exit tolerance EPSI; again a returned value of IERR = 1 indicates this condition has not been met after 100 terms in the series have been summed.

Computed values of $\Xi$ and $\Xi'$ are given in Table B.2; these values were evaluated with EPSI = $10^{-12}$ and are correct to ten significant figures (the last digit is rounded off). The CPU time required to produce the entire table was 0.842 seconds on a CDC 6500. The storage required for XI is 171 words. The program was written for a CDC machine; for computers which operate with less precision, XI may easily be modified for double precision if a high degree of accuracy is needed.
FUNCTION XI(X*EPSI, IERR)
DATA SRP1, GAM0/1.77245385095516, 0.57721566490153286/
X1=0.1 IF=1
IF(X.LE.0.) RETURN
GO TO 1
ENTRY XIP
X1=0.1 IF=2
IF(X.LE.0.) RETURN
X2=X*X
FAC=2.*X2
M=100
IERR=0
SUM=0.
SUMT=0.
TERM=1.
IF(X.GE.5.) GO TO 110
IF(IF.EQ.2) GO TO 100
TERM=X
ALPHA=1.
DO 2 I=1,M
TERM=TERM*FAC/FLOAT(2*I-1)
SUMT=SUM+TERM*ALPHA
IF(ABS((SUM-SUMT)/SUM).LT.EPSI) GO TO 3
ALPHA=ALPHA1./FLOAT(I+1)
2 SUMT=SUM
IERR=1
3 XI=0.25*EXP(-X2)*SUM
RETURN
100 DO 4 I=1,M
TERM=TERM*FAC/FLOAT(2*I-1)
SUMT=SUM+TERM/FLOAT(I)
IF(ABS((SUM-SUMT)/SUM).LT.EPSI) GO TO 5
4 SUMT=SUM
IERR=1
5 XI=0.25*EXP(-X2)*SUM
RETURN
110 IF(IF.EQ.2) GO TO 120
DO 6 I=1,M
TERM=TERM*FLOAT(2*I-1)/FAC
TERM=TERM/FLOAT(I)
IF(TERM.LT.EPSI) GO TO 7
6 SUMT=SUM+TERM
IERR=1
7 XI=SRPI*ALOG(X2)+GAM0/SUM)/8.
RETURN
120 DO 8 I=1,M
TERM=TERM*FLOAT(2*I-1)/FAC
IF(TERM.LT.EPSI) GO TO 9
8 SUMT=SUM+TERM
IERR=1
9 XI=SRPI*(1.*SUM)/((4.*X)
RETURN
END

Table B.1. FORTRAN function routine for the numerical evaluation of \( \Xi(x) \) and \( \Xi'(x) \)

-26-
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