STRONG REPRESENTATION OF WEAK CONVERGENCE (U)


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The following result is proved. If \( S_n \) is a separable metric space for \( n \leq \infty \), \( \phi_n : \),

\( S_n \rightarrow S_\infty \) is measurable for \( n \leq \infty \), \( X_n \) is an \( S_n \)-valued random variable for \( n \leq \infty \) and \( \phi_n (X_n) \rightarrow X_\ast \) in \( S_\ast \), then there exist \( S_n \)-valued random variables \( X_n' \) such that \( X_n = X_n' \) for \( n \leq \infty \) and \( \phi_n (X_n') \rightarrow X_\ast' \) w.p.1. Conditions on \( S_n \) and \( \phi_n \) are presented that allow a construction in the context of Polish spaces.
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STRONG REPRESENTATION OF WEAK CONVERGENCE

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The following result is proved. If $S_n$ is a separable metric space for $n \leq \infty$, $q_n$: $S_n \rightarrow S_\infty$ is measurable for $n < \infty$, $X_n$ is an $S_n$-valued random variable for $n \leq \infty$ and $q_n(X_n) \rightarrow_d X_\infty$ in $S_\infty$, then there exist $S_\infty$-valued random variables $X'_n$ such that $X_n \rightarrow_d X'_n$ for $n \leq \infty$ and $q_n(X'_n) \rightarrow X'_\infty$ wpL. Conditions on $S_n$ and $q_n$ are presented that allow a construction in the context of Polish spaces.

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Skorohod's representation theorem - strong representation of weak convergence

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In this paper we prove the following variant of Skorohod's representation theorem for weak convergence. Equality in distribution is denoted by $=_d$, convergence in distribution by $\rightarrow_d$.

**Theorem 1.** Let $S_n$ for $n = 1, 2, \ldots , \infty$ be a separable metric space and let $q_n$ for $n = 1, 2, \ldots$ be a measurable function from $S_n$ into $S_\infty$. If $X_n$ is an $S_n$-valued random variable for $n = 1, 2, \ldots , \infty$ and $q_n(X_n) \rightarrow_d X_\infty$ in $S_\infty$, then there exist $S_\infty$-valued random variables $X'_n$ for $n = 1, 2, \ldots , \infty$ defined on one probability space and such that $X'_n =_d X_n$ in $S_n$ for $n = 1, 2, \ldots , \infty$ and $q_n(X'_n) \rightarrow_d X'_\infty$ wpL in $S_\infty$.
When $S_\infty = S$ (separable) and $\varrho_n = \text{id}_S$ for all $n$, then the above theorem specializes to Dudley's (1968) variant of Skorohod's representation theorem. In Skorohod's (1956) original version $S$ was required to be complete as well. See Wichura (1971) and Blackwell & Dubins (1983) for further extensions. Our proof of the present theorem amounts to the construction of a special metric space $T$ to which Dudley's theorem can be applied.

Theorem 1 turns out to be useful in many instances. It is applied in Bai (1984), Bai & Yin (1986) and Yin (1984). The need for these applications led the first two authors to the present research.

Here is the simplest example of a theorem that can be proved by Theorem 1, but not by the theorem of Skorohod-Dudley in its original form. It is Theorem 4.1 of Billingsley (1968), restricted to separable metric spaces:

**Theorem 2.** If $S$ is a separable metric space with metric $\varrho$, $(X_n, Y_n)$ are $S^2$-valued random variables for $n = 1, 2, \ldots$ and $X$ is an $S$-valued random variable such that $X_n \to X$ in $S$ and $\varrho(X_n, Y_n) \to 0$ in $\mathbb{R}$, then $Y_n \to X$ in $S$.

**Proof.** By Billingsley (1968, Th.4.4) we have $(X_n, \varrho(X_n, Y_n)) \to (X, 0)$ in $S \times \mathbb{R}$. Apply Theorem 1 with $S = S \times \mathbb{R}$, $X_n$, replaced by $(X_n, Y_n)$ and $\varrho(x, y) = (x, \varrho(x, y))$, all for $n \leq \infty$. 

**Proof of Theorem 1.** All statements involving $n$ are supposed to hold for $n = 1, 2, \ldots, \infty$ unless restricted explicitly; limit statements without explicit tendency hold as $n \to \infty$. Let $T$ be the disjoint union of all $S_n$. Let $T \to \{1, 2, \ldots, \infty\}$ be defined by $\delta(x) = n$ if $x \in S_n$. Set $\varrho_n := \text{id}_S$ and define $\varrho : T \to S_n$ by $\varrho(x) := \varrho_n(\chi)$. Let $\varrho_n$ be the metric of $S_n$. Let $\delta_n$ be positive for $n \leq \infty$, decreasing to 0 as $n \to \infty$, and set $\delta := 0$. We now define what is going to be the metric on $T$:

$$\delta(x, y) = \begin{cases} \delta_n(\chi), & \text{if } \delta(x) = \delta(y), \\ \delta_n(\chi), & \text{if } \delta(x) \neq \delta(y). \end{cases}$$

Let us first verify that $\delta$ is indeed a metric. Obviously, $\delta(x, x) = \delta(x, x)$ and $\delta(x, x) = 0$. If $\delta(x, y) = 0$, then $\delta(x) = \delta(y)$ and $\varrho_n(\chi) = 0$, so $x = y$. The triangle inequality can be verified separately for both terms on the right-hand side of (1). We note the following properties of $\delta$-convergence:

1. $\delta = \varrho$, on $S_n \times S_n$.
2. $\delta$ is $\varrho_n$-convergent for $n \leq \infty$.
3. If $x, y \in S_n$, then $\delta(x, y) \to 0$ as $k \to \infty$, if $\varrho_n(x, y) \to 0$ and $\varrho(x, y) \to 0$ as $k \to \infty$.
4. $\delta$ is $\varrho_n$-convergent for $n \leq \infty$.
5. If $x, y \in S_n$ for each $n \leq \infty$, then $x_n \to x$ if $x \in S_n$ and $\varrho_n(x_n) \to x$ in $S_n$.

Having established that $T$ with $\delta$ is a separable metric space, we may apply Dudley's theorem to $S$-valued random variables. However, there is one more barrier to take. We want to identify $S_n$-valued random variables with $T$-valued random variables having range in $S_n$. In the first appearance random variables must be $\varrho_n$-measurable, where $\varrho_n$ is the Borel field in $S_n$ generated by $\varrho_n$. In the
second appearance they must be $\mathcal{B}_n$-measurable, where $\mathcal{B}_n$ is the trace in $S_n$ of $\mathcal{B}$, the Borel field in $T$ generated by $\delta$. So we must prove $\mathcal{B}_n = \mathcal{B}_n$.

From the second clause in (3) it follows that $S_n \subset \mathcal{B}_n$. For the converse inclusion we must do a little more. First note that $\mathcal{B}$ is already generated by the open $\delta$-balls in $T$, since $\delta$ is separable. This can be phrased equivalently by stating that $\mathcal{B}$ is the smallest $\sigma$-field in $T$ which makes the functions $\delta(x, \cdot)$ measurable for all $x \in T$. Consequently, $\mathcal{B}_n$ is the smallest $\sigma$-field in $S_n$ which makes the functions $\delta(x, \cdot)$, restricted to $S_n$, measurable for all $x \in T$. First suppose $y \in S_n$. Then $\delta(x, y) = q_n(q(x, y)) + \varepsilon_n$, for $x \in T$, $y \in S_n$, which, as a function of $x$, is obviously $\mathcal{B}_n$-measurable. So $\mathcal{B}_n \subset \mathcal{B}$.

We now write down the scheme of implications that proves the theorem. We are given $S_n$-valued random variables $X_n$ such that

(1) $q_{n}(X_n) \to Y_n$ in $S_n$.

The major point to be proved below is that this implies

(7) $X_n \to Y_n$ in $T$.

By Dudley's theorem there are $T$-valued random variables $Y_n$, defined on one probability space, such that $Y_n \equiv_{d} X_n$ in $T$ and $Y_n \to Y_n$ in $T$. By (4) we have $S_n \subset \mathcal{B}_n$ for each $n$, so there is a measurable function $f_n : T \to S_n$ such that the restriction of $f_n$ to $S_n$ is the identity map on $S_n$; take $f_n$ to be the identity map on $S_n$ and constant on $T S_n$. Set $X_n := f_n Y_n$. Then $X_n$ has range in $S_n$, and $X_n = Y_n$ in $T$. Since $\mathbb{P}[Y_n \in S_n] = \mathbb{P}[X_n \in S_n] = 1$, and $X_n \equiv_{d} X_n$, it follows that $X_n \equiv_{d} X_n$ in $T$. As $X_n$ has range in $S_n$, this implies $X_n \equiv_{d} X_n$ in $S_n$. From $X_n \equiv_{d} Y_n$ in $T$ and $Y_n \to Y_n$ in $T$ we obtain $X_n \to Y_n$ in $T$. As $X_n$ has range in $S_n$, this implies by (5)

$$q_n(X_n) \to Y_n \text{ wp1 in } S_n.$$

We have arrived at all conclusions of the theorem.

It remains to prove the implication (6) $\Rightarrow$ (7). We will interpret (6) and (7) by convergence of probability distributions on continuity sets, so we must compare the boundaries under $\mathcal{B}_n$ and $\mathcal{B}_n$. By (2) we have for $A \subset T$,

$$Z_n(A \cap S_n) = Z_n(A \cap S_n) = Z_n(A).$$

Let $B(x, r) := \{y \in T : \delta(x, y) < r \}$ and set

$$\mathcal{B} := \{B(x, r) : r > 0, x \in S_n \} \cup \{\emptyset, T\},$$

and let $\mathcal{B}$ consist of the unions of finitely many elements of $\mathcal{B}$. By Billingsley (1968, Corollary 2 on p. 15) it is sufficient for (7) that

$$\mathbb{P}[X_n \in A] \to \mathbb{P}[X_n \in A] \text{ for } A \subset \mathcal{B}.$$

If $B(x, \varepsilon) \in \mathcal{B}$ with $\delta(x, \varepsilon) < \varepsilon$, then $B(x, \varepsilon) = S_{n+1}$, so $\mathbb{P}[X_n \in B(x, \varepsilon)] = 0$ unless $n \geq \delta(x, \varepsilon)$. If $B(x, \varepsilon) \in \mathcal{B}$ with $\varepsilon < \varepsilon$, then
\[ X_n \in B(x, r) \] = [q_n(x, q_n(x_n)) < r - \varepsilon_n] = [q_n(X_n) \in B(x, r - \varepsilon_n) \cap S_x]. \]

So

(11) \[ \{q_n(X_n) \in B(x, r) \cap S_x \} \subset \liminf \{X_n \in B(x, r)\} \subset \limsup \{X_n \in B(x, r)\} \]

\[ \subset \{q_n(X_n) \in B(x, r) \cap S_x\}. \]

By (6), (8) and (9) the outmost sides of (11) have equal probabilities. Combining the previous observations for separate \( B(x, r) \in V \) we arrive at (11) with \( A \in \mathcal{A} \) instead of \( B(x, r) \in V \), again with equal probabilities for the outmost sides. This proves (10), hence (7). The proof of the theorem is complete.

**Remarks.** In general the space \( T \) is not complete under \( \delta \), even if all \( S_n \) are under \( q_n \). To see this, consider the case that all \( x_n (n < x) \) lie in \( S_m \) for one fixed \( m \). Then \( (x_n) \) is \( \delta \)-Cauchy iff \( \lim \{x_n\}_n \) is \( q_m \times q_m \)-Cauchy. If the latter holds, then \( \{(x_n, q_m(x_n))\} \) converges in \( S_m \times S_m \), but not necessarily in graph \( q_m \), unless the latter is closed. This combined with the observation that \( \delta \)-Cauchy sequences \( \{x_n\}_n \) with \( x_n \in S_n \) converge if \( S_n \) is \( q_n \)-complete leads us to the following result.

**Theorem 3.** Let \( S_n \) be separable and \( q_n \)-complete for each \( n \). Then \( T \) is \( \delta \)-complete if graph \( q_n \) is closed in \( S_n \times S_n \) for each \( n \).

It is well-known that graph \( q_n \) is closed if \( q_n \) is continuous, and that \( q_n \) is continuous if graph \( q_n \) is closed and \( S_n \) is compact. Using the fact that a subset of a Polish space is Polish if it is \( G_\delta \) (Dugundji (1966, Th XIV.8.3)), we arrive at the following variation on Theorem 3.

**Theorem 4.** Let \( S_n \) be Polish for each \( n \). Then \( T \) is Polish if graph \( q_n \) is \( G_\delta \) in \( S_n \times S_n \) for each \( n \).

For results on real functions with \( G_\delta \) graphs, see van Rooij & Schikhof (1982, Exerc. 11.Y.Z). Functions of the first class of Baire (pointwise limits of continuous functions) have \( G_\delta \) graphs. \( F_\sigma \) graphs are also \( G_\delta \).

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Strong representation of weak convergence

References


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