Experiments indicate that a multigrid-type cycle can be used as an efficient preconditioner in the iterative solution of the discrete problem corresponding to a singularly perturbed elliptic boundary value problem. Motivated by a report of Goldstein, we explore the theoretical basis for the efficiency of such a preconditioner when applied to a model problem. The techniques developed are also used to analyze a multigrid V-cycle when used alone as a fast iterative solver.
INSTRUCTIONS FOR PREPARATION OF REPORT DOCUMENTATION PAGE

GENERAL INFORMATION

The accuracy and completeness of all information provided in the DD Form 1473, especially classification and distribution limitation markings, are the responsibility of the authoring or monitoring DoD activity. Because the data input on this form will be what others will retrieve from DTIC's bibliographic data base or may determine how the document can be accessed by future users, care should be taken to have the form completed by knowledgeable personnel. For better communication and to facilitate more complete and accurate input from the originators of the form to those processing the data, space has been provided in Block 22 for the name, telephone number, and office symbol of the DoD person responsible for the input cited on the form.

All information on the DD Form 1473 should be typed. Only information appearing on or in the report, or applying specifically to the report in hand, should be reported. If there is any doubt, the block should be left blank.

Some of the information on the forms (e.g., title, abstract) will be machine indexed. The terminology used should describe the content of the report or identify it as precisely as possible for future identification and retrieval.

NOTE: Unclassified abstracts and titles describing classified documents may appear separately from the documents in an unclassified context, e.g., in DTIC announcement bulletins and bibliographies. This must be considered in the preparation and marking of unclassified abstracts and titles.

The Defense Technical Information Center (DTIC) is ready to offer assistance to anyone who needs and requests it. Call Data Base Input Division, Autovon 284-7044 or Commercial (202) 274-7044.

SECURITY CLASSIFICATION OF THE FORM

In accordance with DoD 5200.1-R, Information Security Program Regulation, Chapter IV Section 2, paragraph 4-200, classification markings are to be stamped, printed, or written at the top and bottom of the form in capital letters that are larger than those used in the text of the document. See also DoD 5220.22-M, Industrial Security Manual for Safeguarding Classified Information, Section II, paragraph 11a(2). This form should be unclassified, if possible.

SPECIFIC BLOCKS

Block 1a Report Security Classification: Designate the highest security classification of the report. (See DoD 5220.1-R, Chapters I, IV, VII, XI, Appendix A.)

Block 1b Restricted Marking: Enter the restricted marking or warning notice of the report (e.g., CNNWDI, RD, NATO).

Block 2a Security Classification Authority: Enter the commonly used markings in accordance with DoD 5200.1-R, Chapter IV, Section 4, paragraph 4-400 and 4-402. Indicate classification authority.

Block 2b Declassification / Downgrading Schedule: Indicate specific date or event for declassification or the notation, “Originating Agency Determination Required” or “OADR.” Also insert (when applicable) downgrade to On (e.g., Downgrade to Confidential on 6 July 1983) (see also DoD 5220.22-M. Industrial Security Manual for Safeguarding Classified Information, Appendix II.)

NOTE: Entry must be made in Blocks 2a and 2b except when the original report is unclassified and has never been upgraded.

Block 3 Distribution/Availability Statement of Report: Insert the statement as it appears on the report. If a limited distribution statement is used, the reason must be one of those given by DoD Directive 5200.20: Distribution Statements on Technical Documents, as supplemented by the 18 OCT 1983 SECDEF Memo, “Control of Unclassified Technology with Military Application.” The Distribution Statement should provide for the broadest distribution possible within limits of security and controlling office limitations.

Block 4 Performing Organization Report Number(s): Enter the unique alphanumeric report number(s) assigned by the organization originating or generating the report from its research and whose name appears in Block 6. These numbers should be in accordance with ANSI STD 239.21-74, “American National Standard Technical Report Number.” If the Performing Organization is also the Monitoring Agency, enter the report number in Block 4.

Block 5 Monitoring Organization Report Number(s): Enter the unique alphanumeric report number(s) assigned by the Monitoring Agency. This should be a number assigned by a DoD or other government agency and should be in accordance with ANSI STD 239.21-74. If the Monitoring Agency is the same as the Performing Organization, enter the report number in Block 4 and leave Block 5 blank.

Block 6a Name of Performing Organization: For in-house reports, enter the name of the performing activity. For reports prepared under contract or grant, enter the contractor or the grantee who generated the report and identify the appropriate corporate division, school, laboratory, etc., of the author.

Block 6b Office Symbol: Enter the office symbol of the Performing Organization.

Block 6c Address: Enter the address of the Performing Organization. List city, state, and ZIP code.

Block 7a Name of Monitoring Organization: This is the agency responsible for administering or monitoring a project, contract, or grant. If the monitor is also the Performing Organization, leave Block 7a blank. In the case of joint sponsorship, the Monitoring Organization is determined by advance agreement. It can be either an office, a group, or a committee representing more than one activity, service, or agency.

Block 7b Address: Enter the address of the Monitoring Organization. Include city, state, and ZIP code.

Block 8a Name of Funding/Sponsoring Organization: Enter the full official name of the organization under whose immediate funding the document was generated, whether the work was done in-house or by contract. If the Monitoring Organization is the same as the Funding Organization, leave Block 8a blank.

Block 8b Office Symbol: Enter the office symbol of the Funding/Sponsoring Organization.

Block 8c Address: Enter the address of the Funding/Sponsoring Organization. Include city, state and ZIP code.

DD FORM 1473, 84 MAR
| Block 9 | Procurement Instrument Identification Number: For a contractor grantee report, enter the complete contract or grant number(s) under which the work was accomplished. Leave this block blank for in-house reports. |
| Block 10 | Source of Funding (Program Element, Project, Task Area, and Work Unit Numbers): These four data elements relate to the DoD budget structure and provide program and/or administrative identification of the source of support for the work being carried on. Enter the program element, project, task area, work unit accession number, or their equivalents which identify the principal source of funding for the work required. These codes may be obtained from the applicable DoD forms such as the DF Form 1498 (Research and Technology Work Unit Summary) or from the fund citation of the funding instrument. If this information is not available to the authoring activity, these blocks should be filled in by the responsible DoD official designated in Block 22. If the report is funded from multiple sources, identify only the Program Element and the Project, Task Area, and Work Unit Numbers of the principal contributor. |
| Block 11 | Title: Enter the title in Block 11 in initial capital letters exactly as it appears on the report. Titles on all classified reports, whether classified or unclassified, must be immediately followed by the security classification of the title enclosed in parentheses. A report with a classified title should be provided with an unclassified version if it is possible to do so without changing the meaning or obscuring the contents of the report. Use specific, meaningful words that describe the content of the report so that when the title is machine-indexed, the words will contribute useful retrieval terms. |

If the report is in a foreign language and the title is given in both English and a foreign language, list the foreign language title enclosed in parentheses. If the report is in a foreign language and the title is given in English, list the English title first followed by the foreign language title enclosed in parentheses. If the title is given in more than one foreign language, use a title that reflects the language of the text. If both the text and titles are in a foreign language, the title should be translated, if possible, unless the title is a name of a foreign periodical. Transliterations of often used foreign alphabets (see Appendix of a foreign periodical) Transliterations of often used foreign alphabets (see Appendix of a foreign periodical) |

For reference on standard terminology the "DTIC Retrieval and Indexing Terminology" DRIT-1979, AD-A068 500, and the DoD "Thesaurus of Engineering and Scientific Terms" (TEST) 1968, AD-672 000, may be useful. |

| Block 12 | Personal Author(s): Give the complete name(s) of the author(s) in this order: last name, first name, and middle name. In addition, list the affiliation of the authors if it differs from that of the performing organization. |

List all authors if the document is a compilation of papers. It may be more useful to list the authors with the titles of their papers as a contents note in the abstract. If appropriate, the names of editors and compilers may be entered in this block. |

| Block 13a | Type of Report: Indicate whether the report is summary, final, annual, progress, interim, etc. |
| Block 13b | Time Covered: Enter the inclusive dates (year, month, day) of the period covered, such as the life of a contract in a final contractor report. |
| Block 14 | Date of Report: Enter the year, month, and day, or the year and the month the report was issued as shown on the cover. |
| Block 15 | Page Count: Enter the total number of pages in the report that contain information, including cover, preface, table of contents, distribution lists, partial pages, etc. A chart in the body of the report is counted even if it is unnumbered. |
| Block 16 | Supplementary Notation: Enter useful information about the report in hand, such as Prepared in cooperation with: "Translation at (or by): "Symposium," if there are report numbers for the report which are not noted elsewhere on the form (such as internal series numbers or participating organization report numbers) enter in this block. |
| Block 17 | COSATI Codes: This block provides the subject coverage of the report for announcement and distribution purposes. The categories are to be taken from the "COSATI Subject Category List" (DoD Modified), Oct 65, AD-624 000 A copy is available on request to any organization generating reports for DoD. At least one entry is required as follows: Block 17. COSATI Codes: This block provides the subject coverage of the report for announcement and distribution purposes. The categories are to be taken from the "COSATI Subject Category List" (DoD Modified), Oct 65, AD-624 000. A copy is available on request to any organization generating reports for DoD. At least one entry is required as follows: |
| Field | to indicate subject coverage of report |
| Group | to indicate greater subject specificity of information in the report. |
| Sub-Group | if specificity greater than that shown by Group is required, use further designation as the numbers after the period. |

Example: The subject "Solid Rocket Motors" is Field 21 Group 08, Subgroup 2 (page 32, AD-624 000). |

| Block 18 | Subject Terms: These may be descriptors, keywords, posting terms, identifiers, open-ended terms, subject headings, acronyms, code-words, or any words or phrases that identify the principal subjects covered in the report, and that conform to standard terminology and are exact enough to be used as subject index entries. Certain acronyms or "buzz words" may be used if they are recognized by specialists in the field and have a potential for becoming accepted terms. "Laser" and "Reverse Osmosis" were once such terms. |

It possible, this set of terms should be selected so that the terms individually and as a group will remain UNCLASSIFIED without losing meaning. However, priority must be given to specifying proper subject terms rather than making the set of terms appear "UNCLASSIFIED." Each term on classified reports must be immediately followed by its security classification, enclosed in parentheses. |

For further information on preparing abstracts, employing scientific symbols, verbalizing, etc., see paragraphs 2 1(e) and 2 3(b) in MIL-STD-847B. |

Block 19 | Abstract: The abstract should be a pithy, brief (preferably not to exceed 300 words), factual summary of the most significant information contained in the report. However, since the abstract may be machine-searched, all specific and meaningful words and phrases which express the subject content of the report should be included, even if the word limit is exceeded. If possible, the abstract of a classified report should be unclassified and consist of publicly releasable information (U), but in no instance should the report content description be sacrificed for the security classification. |

NOTE: An unclassified abstract describing a classified document may appear separately from the document in an unclassified context e.g., in DTIC announcement or bibliographic products. This must be considered in the preparation and marking of unclassified abstracts. |

For further information on preparing abstracts, employing scientific symbols, verbalizing, etc., see paragraphs 2 1(e) and 2 3(b) in MIL-STD-847B. |

Block 20 | Distribution / Availability of Abstract: This block must be completed for all reports. Check the applicable statement: "unclassified/unclassified" Sample as required. If the report is available to DTIC registered users, "DTIC users." |

Block 21 | Abstract Security Classification: To ensure proper safeguarding of information, this block must be completed for all reports to designate the classification level of the entire abstract. For CLASSIFIED abstracts, each paragraph must be preceded by its security classification code in parentheses. |

Block 22 | Name, Telephone, and Office Symbol of Responsible Individual: Give name, telephone number, and office symbol of DoD person responsible for the accuracy of the completion of this form. |
THE K-GRID FOURIER ANALYSIS
OF MULTIGRID-TYPE ITERATIVE METHODS
by
Naomi H. Decker
Computer Sciences Technical Report #703
July 1987
THE K-GRID FOURIER ANALYSIS
OF MULTIGRID-TYPE ITERATIVE METHODS

by

Naomi H. Decker

Computer Sciences Technical Report #703

(1) Sponsored by the Air Force Office of Scientific Research under Contracts No. AFOSR-82-0275 and 86-0163.
ABSTRACT

Experiments indicate that a multigrid-type cycle can be used as an efficient preconditioner in the iterative solution of the discrete problem corresponding to a singularly perturbed elliptic boundary value problem. Motivated by a report of Goldstein, we explore the theoretical basis for the efficiency of such a preconditioner when applied to a model problem. The techniques developed are also used to analyze a multigrid V-cycle when used alone as a fast iterative solver.
1. Introduction

This work is motivated by a report of Charles Goldstein [7] in which the author discusses the task of numerically solving the following elliptic boundary value problem:

\[
\begin{cases}
-\varepsilon^2 \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( a_i(x) \frac{\partial u(x)}{\partial x_i} \right) + \varepsilon \sum_{i=1}^{2} b_i(x) \frac{\partial u(x)}{\partial x_i} + a_0(x) u(x) = f(x) \quad \text{in } \Omega \subset \mathbb{R}^2 \\
u(x) = g(x) \quad \text{on } \partial \Omega
\end{cases}
\]  

(1.1)

where \( x = (x_1, x_2) \in \Omega \), \( 0 < \varepsilon << 1 \), the coefficients and data are sufficiently smooth, and \( a_i(x) > c_0 > 0 \) in \( \Omega \), \( i = 0, 1, 2 \).

The discrete problem arising from a typical discretization of (1.1) on a uniform grid of mesh size \( h \), \( h < \varepsilon \), is a large system of linear equations. For the solution of this system to approximate the solution of the boundary value problem (1.1) with a fixed accuracy, we must choose the mesh size small for small \( \varepsilon \), specifically, it is sufficient to keep the ratio \( h/\varepsilon \) fixed [1], [11]. In doing so, we not only get a much larger system, but the resulting system is also more poorly conditioned.

With the goal of trying to solve this type of system, we use the conjugate gradient algorithm as our iterative solver. It is known (e.g., [2], [9]) that if we apply the method of conjugate gradients to the problem \( Bv = F \) where \( B \) is symmetric, positive definite, then the number of iterations, \( N_B \), required to solve the system to within a given relative error, \( \|v - v^i\|/\|v - v^0\| < \eta \), is given by

\[
N_B(\eta) \leq C \ln(2/\eta) \sqrt{K(B)}
\]

(1.2)

where \( K(B) = \lambda_{\max}(B)/\lambda_{\min}(B) \), \( v^0 \) is the initial guess and \( v^i \) is the \( i \)th approximant to the solution, \( v \). Our goal is to precondition the system so that the condition number, \( K(B') \), of the new system, \( B'v' = F' \), is much smaller than \( K(B) \) and behaves nicely (bounded or slowly increasing) as \( \varepsilon \) and \( h \) decrease to zero.

It has been observed experimentally that a certain multigrid-type cycle is an inexpensive preconditioner for this system. The effectiveness of this preconditioner is quite sensitive to the choice of the number of grids, \( k \), used in the multigrid process. Fourier
analysis was used in [7] in an attempt to prove that a careful choice of the number of grids does guarantee a good preconditioner in the case where \( \Omega \) is a rectangle. Although Fourier analysis is routinely used to study 2-grid multigrid cycles, the \( k \)-grid analysis, for \( k > 2 \), is quite unwieldy and is not usually attempted. The difficulty arises from the use of coarser grids on which certain modes "alias" (see [3]) or are "not visible" (see [12]). Unfortunately, this "aliasing" was ignored in [7]. The experimental evidence is so striking, however, that it seemed worth trying to complete the analysis.

We examine the effectiveness of the multigrid preconditioner by considering a special case of the boundary value problem (1.1) with \( a_i(x) \equiv 1, \ i = 0, 1, 2 \), \( b_i(x) \equiv 0, \ i = 1, 2 \), \( \Omega = (0, 1) \times (0, 1) \) and \( \varepsilon \) real and small. It is for this model operator, \( A_L^e = -\varepsilon^2 \Delta + I \), that we prove our basic results. More general singularly perturbed problems such as variable coefficient and/or non-symmetric with positive definite symmetric part can be analyzed using the properties of the multigrid preconditioner acting on \( A_L^e \) together with such ideas as spectral or norm equivalence, see [5] and [7].

Let \( h = 2^{-n} \) for a positive integer, \( n \). Discretizing this model problem on a uniform grid, \( \Omega_h = \{(lh, mh) : l, m = 1, 2, \ldots, 2^n - 1 \} \), with mesh size, \( h \), using a standard 5-point discretization of the Laplacian (see Section 2.1), we obtain the linear system

\[
A_h^e u_h := (-\varepsilon^2 \Delta + I) u_h = f_h.
\]

In Section 3.1 we define a symmetric linear operator, \( M_k \), based on multigrid ideas, using \( k - 1 \) auxiliary grids of larger mesh sizes, \( 2^p h \), for \( p = 1, 2, \ldots, k - 1 \). In fact, the vector \( M_k w_h \) is essentially one "partial" multigrid V-cycle applied as if to solve the problem:

\[
A_h v_h = w_h,
\]

starting with initial guess = 0, where \( A_h \) is the matrix resulting from the corresponding discretization of the Dirichlet boundary value problem for Poisson's equation. In order to obtain a symmetric operator, we take symmetric smooths. I.e., if \( r_p \) smooths are done on the \( p \)-th grid in the fine to coarse part of the cycle, then \( r_p \) smooths must be done on the \( p \)-th grid in the coarse to fine part. We take a fixed \( r_p = r \) for all \( p = 0, \ldots, k - 1 \). The adjective "partial" refers to the following property of this particular V-cycle: instead of solving for the coarse grid correction exactly on the coarsest grid, \( 2r \) iterations of the
smoother are applied. We choose the smoother to be a damped Jacobi iteration with
damping parameter, \( \omega \), where \( 0 < \omega < 1 \). Taking \( \omega = 1 \) would correspond to an
undamped Jacobi iteration, but we exclude this choice. The choice \( \omega = .5 \) corresponds to
a Richardson iteration. Using \( M_k \) as a preconditioner for (1.3), we claim:

If the mesh size on the coarsest grid is chosen to be approximately equal
to the singular perturbation parameter, \( \epsilon \), then the condition number of
the preconditioned system is bounded independent of \( \epsilon \) and \( h \).

Defining \( M_k^\epsilon = M_k \), where \( k \) is chosen so that \( h_1 \approx \epsilon \), we justify this claim in 3 steps:

1. In Section 3.2 we reduce the problem to finding appropriate upper and lower
bounds for the eigenvalues of \( M_k^\epsilon A_h^\epsilon \). Let \( q : \Omega_h \to \{1,2,\ldots,(2^n - 1)^2\} : \)
\((i_1 h,i_2 h) \mapsto q_i,i = (i_1,i_2)\), be a given ordering of the \((2^n - 1)^2\) points of
\( \Omega_h \), and let \( \{\alpha_i\} \) be a (given) complete set of eigenvectors of \( A_h \). Define a
\((2^n - 1)^2 \times (2^n - 1)^2\) matrix, \( M \), by

\[(M)_{q_i,q_j} = \mu_{ij}\]

where

\[\mu_{ij} := (M_k^\epsilon A_h^\epsilon \alpha_i, \alpha_j)\]

for each \( i = (i_1,i_2), j = (j_1,j_2) \) where \( 1 \leq i_1,i_2,j_1,j_2 \leq 2^n \) and \( \langle \cdot, \cdot \rangle \) is the
discrete - \( L^2 \) inner product. Using this eigenfunction analysis (Fourier analysis),
the problem reduces to finding bounds on the eigenvalues of \( M \). The off-diagonal
elements of \( M \) represent the "aliasing".

2. In Section 3.3 we obtain a formula for a bound, \( C_{h,k,r,\omega}^i \), such that, for every \( i \),

\[\sum_{j \neq i} |\mu_{ij}| \leq C_{h,k,r,\omega}^i |\mu_{ii}|.\]

Therefore we have diagonal dominance of the matrix, \( M \), provided \( \tilde{C}_{h,k,r,\omega} \),
where

\[\tilde{C}_{h,k,r,\omega} := \sup_i C_{h,k,r,\omega}^i.\]
can be shown to be less than one. The constant $\tilde{C}_{h,k,r,\omega}$ is calculated for $r = 1, 2, 3, 4$, $\omega = .5, .6, .7, .8, .9$, $h = 1/2, 1/4, 1/8, \ldots, 1/8192$ and all possible corresponding values of $k$. All computed values of $\tilde{C}_{h,k,r,\omega}$ are less than one with the exception of the case where only one smoothing is used and $\omega < .7$.

3. In Section 3.5 we restate the bounds given in [7] on the diagonal entries of the matrix. These bounds are used, combined with the diagonal dominance, to show that:

$$ c_1 \varepsilon^2 \leq \lambda_{\min}(M_h^s A_h^s) \leq \lambda_{\max}(M_h^s A_h^s) \leq c_2 \varepsilon^2, $$

for constants $c_1, c_2 > 0$. The diagonal dominance of $M$ is needed only to guarantee the positivity of the lower bound.

In Section 4 we describe some experiments which illustrate the efficiency of using the optimal number of grids in the multigrid preconditioner. Experimental comparisons are made between three different solvers for the model problem. In a preconditioned conjugate gradient routine, two preconditioners are used, first the preconditioner analyzed in this paper, namely the preconditioner based on the Laplacian with smoothing on the coarsest grid, and secondly a preconditioner which is based on the model operator itself, solving on the coarse grid. The third solver used in the comparison is a symmetric multigrid V-cycle.

The techniques used in the analysis of “multigrid-as-a-preconditioner” can also be used to analyze “multigrid-as-a-solver”. This analysis is simpler than the preconditioner analysis since we don’t need diagonal dominance (and we don’t have it), see Section 5. In Section 6 we show how the $k$-grid convergence bounds obtained in this way compare to the experimentally observed convergence rates and to V-cycle convergence bounds obtained by other methods.
2.1 Notation

Consider the two-dimensional Dirichlet problem

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega = (0,1) \times (0,1) \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\] (2.1)

where \(\Delta = \sum_{j=1}^{2} \partial^2 / \partial x_j^2\). We discretize this problem on a family of grids. Let \(h = 2^{-n}\), as in Section 1. Choose a positive integer \(k, k < n\). Define a coarse grid mesh size \(h_1 = 2^{k-1}h\). In \(\Omega\) we define \(k\) intermediate grids, \(\Omega^p, p = 1, 2, \ldots, k\) with mesh sizes \(h_p = 2^{1-p}h_1\). Clearly \(h = h_k\) and

\[
\Omega^p = \{(x_i, y_m) = (lh_p, mh_p) : l, m = 1, 2, \ldots, N_p - 1\}
\] (2.2)

where \(N_p = 1/h_p\) and \(p = 1, 2, \ldots, k\).

We define the discrete operator, \(A_p\), which is the negative of the discrete five point Laplacian, on the grid \(\Omega^p\), using the standard five-point discretization of the differential operator, \(-\Delta\) (see e.g., [6]). Each \(A_p\) is a sparse \((N_p - 1)^2 \times (N_p - 1)^2\) matrix with a complete set of eigenvectors, \(\alpha^{(p)}_i\), given by:

\[
\alpha^{(p)}_i(m, n) = 2 \sin (i_1 \pi m h_p) \sin(i_2 \pi n h_p) \quad m, n = 1, \ldots, N_p - 1.
\] (2.3)

where \(i = (i_1, i_2)\), and \(i_1, i_2 = 1, 2, \ldots, N_p - 1\). The corresponding eigenvalues are:

\[
\nu^{(p)}_i = \frac{4 - 2 \cos (i_1 \pi h_p) - 2 \cos (i_2 \pi h_p)}{h_p^2}.
\] (2.4)

As usual, the multigrid operators we consider are constructed from smoothers, \(G_p\), \(p = 1, 2, \ldots, k\) and intergrid transfer operators, \(I^p_{p-1}\) and \(I^1_p\), \(p = 2, 3, \ldots, k\).

To simplify the analysis we choose \(G_p(\cdot, \cdot)\) to be a damped Jacobi smoother, defined by

\[
G_p(u_p, f_p) = (I - 2\omega c_p A_p)u_p + 2\omega c_p f_p
= \bar{G}_p u_p + (I - \bar{G}_p) A_p^{-1} f_p
\] (2.5)

where \(c_p = h_p^2/8\), \(p = 1, \ldots, k\), and \(\bar{G}_p\) is the linear part of \(G_p\). We require that \(0 < \omega < 1\). We do not allow \(\omega = 1\), which would correspond to a Jacobi iteration. The constant, \(c_p\), is approximately equal to the inverse of the spectral radius, \(\rho(A_p)\). In fact, \(c_p\rho(A_p) = 1 - O(h_p^2)\), and therefore \(\bar{G}_p\) is a contraction, i.e.,

\[
\rho(I - 2\omega c_p A_p) < 1.
\] (2.6)
We define inner products and norms by:

\[ \langle u^p, v^p \rangle_p = h_p^2 \sum_{x \in \Omega_p} u^p(x) v^p(x) \]  
(2.7a)

and

\[ \| u^p \|^2 = \langle u^p, u^p \rangle_p , \]  
(2.7b)

for \( u^p, v^p \) defined on \( \Omega^p \).

For the projection and weighting operators we take \( I^p_{p-1} \) to be linear interpolation:

\[ I^p_{p-1} = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \]  
(2.8a)

and \( I^{-1}_p \) to be the adjoint of \( I^p_{p-1} \) relative to the discrete \(-L^2\) inner products defined by (2.7a):

\[ I^{-1}_p = \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \]  
(2.8b)

where we have used the "distribution" and "collection" stencils as in [10].

In the eigenfunction analysis we need some notation and simple formulas. Let \( i = (i_1, i_2) \). Define

\[ \xi_i^{(p)} = \cos^2 \left( \frac{i_1 \pi h_p}{2} \right) \]  
(2.9a)

and

\[ \eta_i^{(p)} = \cos^2 \left( \frac{i_2 \pi h_p}{2} \right) . \]  
(2.9b)

A simple trigonometric identity gives us

\[ \xi_i^{(p-1)} = (1 - 2 \xi_i^{(p)})^2 \]  
(2.10a)

and

\[ \eta_i^{(p-1)} = (1 - 2 \eta_i^{(p)})^2 . \]  
(2.10b)
The eigenvalues of $A_p$ can be written as

$$\nu_i(p) = \frac{4(2 - \xi_i(p) - \eta_i(p))}{h_p^2}.$$  \hspace{1cm} (2.11)

A simple calculation shows us that the effect of the projection on the eigenvectors of $A_p$ can be expressed as (2)

$$I_p^{p-1} \alpha_i(p) = \zeta_i(p) \eta_i(p) \alpha_i(p-1).$$  \hspace{1cm} (2.12)

The corresponding formulas for interpolation is

$$I_p^{p-1} \alpha_i(p-1) = \xi_i(p) \eta_i(p) \alpha_i(p) - (1 - \xi_i(p)) \eta_i(p) \alpha_{(N_p-i_1,i_2)} - \xi_i(p) (1 - \eta_i(p)) \alpha_{(i_1,N_p-i_2)} + (1 - \xi_i(p))(1 - \eta_i(p)) \alpha_{(N_p-i_1,N_p-i_2)}.$$  \hspace{1cm} (2.13)

Note that eigenvectors of $A_p$ are also eigenvectors of $G_p$. The eigenvalue, $g_i(p)$, of $G_p$, corresponding to $\alpha_i(p)$, is given by

$$g_i(p) = 1 - 2\omega c_p \nu_i(p),$$  \hspace{1cm} (2.14)

where the constants $c_p$ are related by

$$c_{p-1} = 4c_p.$$  \hspace{1cm} (2.15)

When we apply the multigrid algorithm, we transfer vectors to coarser grids. In the process we lose information. In this two-dimensional problem with an $(h \cdot 2h)$ grid structure the four (if $i_1 \neq N_p/2$ and $i_2 \neq N_p/2$) eigenvectors $\alpha_i^{(p)}$, $-\alpha_i^{(p)}$, $-\alpha_{(N_p-i_1,i_2)}^{(p)}$, $-\alpha_{(i_1,N_p-i_2)}^{(p)}$, defined on $\Omega^p$, are indistinguishable on $\Omega^{p-1}$. There are also $2N_p - 3$ eigenvectors as defined on $\Omega^p$ which are indistinguishable from the null vector as defined on $\Omega^{p-1}$. This phenomenon is what is referred to as aliasing.

This aliasing plays an important role in the analysis of the multigrid process and we introduce the following notation. Given two multi-indices $i = (i_1, i_2)$ and $j = (j_1, j_2)$, consider $\alpha_i^{(k)}$ and $\alpha_j^{(k)}$. If $\alpha_i^{(p)} = \pm \alpha_j^{(p)}$ then we write $i \sim j(p)$. If $\alpha_i^{(p)}$ and $\alpha_j^{(p)}$ are not linearly dependent then $i \not\sim j(p)$.

---

(2) In the cases where $|i| = \max(i_1, i_2) \geq 1/N_p$, one should replace $\alpha_i^{(p-1)}$ by its proper (unique) representation, $\alpha_i^{(p-1)}$, where $|i| < N_p - 1$. However, Formula (2.12) is also correct in this form.
2.2 Intergrid Operator Identities

A multigrid cycle consists of smoothings and intergrid transfers. The smoother is applied to reduce the high frequency (rough) components of the error. The residual is transferred to a coarser grid where solving exactly for the error correction is less expensive. By solving and then interpolating this coarse grid correction back to the fine grid, the low frequency (smooth) components of the error are reduced. In the boundary value problem (2.1), the eigenfunctions are easily identifiable as rough or smooth, being products of sine functions. The same is true for the discrete operators, \( A_p \), \( 1 \leq p \leq k \). To gain insight into the properties of the multigrid process we study the effect of a multigrid cycle on the eigenvectors of \( A_k \).

Using formulas (2.12) and (2.13) it is clear that transferring \( \alpha_i^{(p)} \) from \( \Omega^p \) to \( \Omega^{p-1} \) and then interpolating back, results in a linear combination of the four eigenvectors which alias from \( \Omega^p \) to \( \Omega^{p-1} \). A 'smooth' eigenvector, i.e. \( \zeta_i^{(p)} \) and \( \eta_i^{(p)} \) close to zero, picks up 'rougher' components. In the full \( k \)-grid problem where there are \( 4^{k-1} \) vectors aliasing from \( \Omega^k \) to \( \Omega^1 \), keeping track of the aliasing is difficult. Fortunately, there are a few simplifying features. The second of the following three Lemmas, in particular, simplifies the analysis. Define

\[
I_{p_2}^{p_1} = I_{p_1+1}^{p_1} I_{p_1+2}^{p_1} \cdots I_{p_2}^{p_1-1}, \quad 1 \leq p_1 < p_2 \leq k. \tag{2.16}
\]

Lemma 2.1

If \( j \sim i (n) \) and \( j \neq i (n+1) \) for some \( 0 \leq n < k \), then

\[
\langle \alpha_i^{(p)}, I_k^p \alpha_j^{(k)} \rangle = \begin{cases} 
0 & \text{if } n < p \leq k; \\
\prod_{m=p+1}^{n} \zeta_i^{(m)} \eta_i^{(m)} \langle \alpha_i^{(n)}, I_k^n \alpha_j^{(k)} \rangle_n \langle \alpha_i^{(p)}, \alpha_i^{(p)} \rangle_p & \text{if } p \leq n
\end{cases} \tag{2.17}
\]

Proof of Lemma 2.1

Let \( j \sim i (n) \) and \( j \neq i (n+1) \) for \( n \), \( 0 \leq n < k \).

For \( p > n \), the orthogonality of the \( \alpha_i^{(p)} \) gives
\[ \langle \alpha_i^{(p)}, I^p_k \alpha_j^{(k)} \rangle_p = 0. \] \hfill (2.18)

For \( p \leq n \) and \( i \neq (0,0) \) \( (p) \),
\[ \langle I^p_n \alpha_i^{(p)}, \alpha_i^{(n)} \rangle_n = \left( \prod_{m=p+1}^{n} \xi_{i_{m}}^{(m)} \eta_{i_{m}}^{(m)} \right) \langle \alpha_i^{(n)}, \alpha_i^{(n)} \rangle_n \] \hfill (2.19)
\[ = \left( \prod_{m=p+1}^{n} \xi_{i_{m}}^{(m)} \eta_{i_{m}}^{(m)} \right) \langle \alpha_i^{(p)}, \alpha_i^{(p)} \rangle_p. \]

Since \( I^p_k = I^p_n I^p_k \), then
\[ \langle \alpha_i^{(p)}, I^p_k \alpha_j^{(k)} \rangle_p = \langle I^p_n \alpha_i^{(p)}, I^p_k \alpha_j^{(k)} \rangle_n. \] \hfill (2.20)

Using \( j \sim i \) \( (n) \) and (2.19) gives
\[ \langle \alpha_i^{(p)}, I^p_k \alpha_j^{(k)} \rangle_p = \left( \prod_{m=p+1}^{n} \xi_{i_{m}}^{(m)} \eta_{i_{m}}^{(m)} \right) \langle \alpha_i^{(n)}, I^p_k \alpha_j^{(k)} \rangle_n \langle \alpha_i^{(p)}, \alpha_i^{(p)} \rangle_p. \] \hfill (2.21)

If \( i \sim (0,0) \) \( (p) \), then (2.21) is trivially true. \( \blacksquare \)

**Lemma 2.2**

For any \( n \), \( 1 \leq n \leq k \), and \( i \neq (0,0) \) \( (n) \),
\[ \sum_{j \sim i \, (n)} | \langle \alpha_i^{(n)}, I^p_k \alpha_j^{(k)} \rangle_n | = 1. \] \hfill (2.22)

**Proof of Lemma 2.2**

If \( n = k \) then \( j \sim i \) \( (1) \) implies \( j = i \). Since \( \langle \alpha_i^{(k)}, \alpha_i^{(k)} \rangle_k = 1 \), (2.22) holds for \( n = k \).

Assume \( \sum_{j \sim i \, (s+1)} | \langle \alpha_i^{(s+1)}, I^{s+1}_k \alpha_j^{(k)} \rangle_{s+1} | = 1 \) for \( s \), where \( s < k \).

Define
\[ i^1 = i = (i_1, i_2), \] \hfill (2.23)
\[ i^2 = (N_{s+1} - i_1, i_2), \]
\[ i^3 = (N_{s+1} - i_1, N_{s+1} - i_2), \]
\[ i^4 = (i_1, N_{s+1} - i_2). \]
Figure 2.1: A splitting of the $j$, $j \sim i(s)$, where $s = k - 2$.

The set $\{j \mid j \sim i(s)\}$ can be split into four disjoint subsets corresponding to all $j \sim i^1(s + 1)$, $j \sim i^2(s + 1)$, $j \sim i^3(s + 1)$ and $j \sim i^4(s + 1)$. Figure 2.1 shows this schematically for the case $s = k - 2$. Therefore the summation can be split as:

$$
\sum_{j \sim i(s)} | \langle \alpha_i^{(s)}, I_k \alpha_j^{(k)} \rangle_s | = \sum_{j \sim i(s)} | \langle I_s^{s+1} \alpha_i^{(s)}, I_k^{s+1} \alpha_j^{(k)} \rangle_{s+1} |
$$

(2.24)

$$
= \left( \sum_{j \sim i^1(s + 1)} + \sum_{j \sim i^2(s + 1)} + \sum_{j \sim i^3(s + 1)} + \sum_{j \sim i^4(s + 1)} \right) | \langle I_s^{s+1} \alpha_i^{(s)}, I_k^{s+1} \alpha_j^{(k)} \rangle_{s+1} |.
$$
Using (2.13) and the orthogonality of the $\alpha_i^{(s+1)}$, the summation can be written as:

\[
\sum_{j \sim i} | \langle \alpha_i^{(s)}, I_k^s \alpha_j^{(k)} \rangle_n | = \zeta_i^{(s+1)} \eta_i^{(s+1)} \sum_{j \sim i} | \langle \alpha_i^{(s+1)}, I_k^{s+1} \alpha_j^{(k)} \rangle_{s+1} |
\]

\[
+ (1 - \xi_i^{(s+1)}) \eta_i^{(s+1)} \sum_{j \sim i^2} | \langle \alpha_i^{(s+1)}, I_k^{s+1} \alpha_j^{(k)} \rangle_{s+1} |
\]

\[
+ (1 - \xi_i^{(s+1)})(1 - \eta_i^{(s+1)}) \sum_{j \sim i^3} | \langle \alpha_i^{(s+1)}, I_k^{s+1} \alpha_j^{(k)} \rangle_{s+1} |
\]

\[
+ (\xi_i^{(s+1)})(1 - \eta_i^{(s+1)}) \sum_{j \sim i^4} | \langle \alpha_i^{(s+1)}, I_k^{s+1} \alpha_j^{(k)} \rangle_{s+1} |
\]

By the inductive hypothesis, each summation on the right hand side is equal to one and the coefficients also sum to one. 

Lemma 2.3

For all $n$, $1 \leq n \leq k$, and $i \neq (0,0)$, the following identity holds:

\[
\sum_{j \sim i} | \langle \alpha_i^{(n)}, I_k^n \alpha_j^{(k)} \rangle_n | = 1 - \zeta_i^{(n+1)} \eta_i^{(n+1)}
\]

(2.25)

Proof of Lemma 2.3

Identity (2.25) follows directly from Lemma 2.2 since

\[
\sum_{j \sim i} | \langle \alpha_i^{(n)}, I_k^n \alpha_j^{(k)} \rangle_n | = \sum_{j \sim i} | \langle \alpha_i^{(n)}, I_k^n \alpha_j^{(n)} \rangle_n |
\]

\[
- \zeta_i^{(n+1)} \eta_i^{(n+1)} \sum_{j \sim i} | \langle \alpha_i^{(n+1)}, I_k^{n+1} \alpha_j^{(k)} \rangle_{n+1} |
\]

\[
= 1 - \zeta_i^{(n+1)} \eta_i^{(n+1)} .
\]
3.1 Definition of the Preconditioner

The multigrid preconditioner is based on the discrete five point Laplacian. $M_k$ is one standard multigrid symmetric V-cycle starting with zero as the initial guess, except that the coarse grid correction is obtained by smoothing instead of by solving exactly on the coarsest grid. Having choosen a fixed number of grids, $k$, the multigrid preconditioner is defined recursively. Choose a positive (integer) number of smoothings, $r$. Then $M_k f_k := \bar{u}_k$ where $\bar{u}_p (= M_p f_p)$, for $f_p$ defined on $\Omega^p$, $p = 1, \ldots, k$, is given by:

1.) Smooth $r$ times starting with initial guess $= 0$:

$$\bar{u}_p = G^r_p (0, f_p). \quad (3.1a)$$

2.) Compute the residual and transfer to the coarse grid:

$$r_p = f_p - A_p \bar{u}_p, \quad f_{p-1} = I^{p-1}_p r_p. \quad (3.1b)$$

3.) Compute the coarse grid correction:

$$\begin{align*}
\text{If } & \quad p = 2, \quad \bar{u}_{p-1} = \bar{u}_1 = G^2_1 (0, f_1) \quad (3.1c) \\
\text{If } & \quad p > 2, \quad \bar{u}_{p-1} = M_{p-1} f_{p-1}. \quad (3.1d)
\end{align*}$$

4.) Add the coarse grid correction:

$$\bar{u}_p = \bar{u}_p + I^{p-1}_p \bar{u}_{p-1}. \quad (3.1e)$$

5.) Smooth $r$ times starting with initial guess $= \bar{u}_p$:

$$\bar{u}_p = G^r_p (\bar{u}_p, f_p). \quad (3.1f)$$

Because we have started with an initial guess of zero, the multigrid preconditioner is a linear operator acting on $f_k$. This definition of $M_k$ can be rewritten as:

$$M_p = (I - G^2_p A^{-1}_p) A^{-1}_p + G^r_p I^{p-1}_p M_{p-1} I^{p-1}_p G^r_p \quad p = 2, \ldots, k \quad (3.2)$$

and

$$M_1 = (I - G^2_1 A^{-1}_1).$$

These identities rely on the commutivity of $G_p$ and $A_p$, $p = 1, 2, \ldots, k$. 

12
3.2 The Problem

As remarked in the introduction, it is sufficient to examine the effectiveness of the multigrid preconditioner by considering the model problem (1.3). We take \( \Omega = (0.1) \times (0, 1) \) and \( \varepsilon \) real and small. It is for this model operator, \( A_f = -\varepsilon^2 \Delta + I \), that we prove our basic results.

Define

\[
A_h^\varepsilon = \varepsilon^2 A_k + I.
\]

Writing the symmetric preconditioner as \( M_k = Q_k^* Q_k \), the preconditioned system is \( A_h^\varepsilon v' = F' \) where \( A_h^\varepsilon = Q_k A_f Q_k^* \). Experimental evidence suggests the following:

**Conjecture:**

Let \( r > 0, 0 < \omega < 1, h > 0 \) and \( \varepsilon > h \). Choose the number of grid levels, \( k \), so that \( h_1 = 2^{k-1} h \approx \varepsilon \). Define \( M_h^\varepsilon = M_k \). Then there exist constants \( c_1, c_2 > 0 \) such that

\[
c_1 \varepsilon^2 \leq \lambda_{\text{min}}(M_h^\varepsilon A_h^\varepsilon) \leq \lambda_{\text{max}}(M_h^\varepsilon A_h^\varepsilon) \leq c_2 \varepsilon^2.
\]

What we prove in this paper is:

**Theorem 3.1**

Let \( r = 1, 2, 3, 4 \) and \( \omega = .7, .8, .9 \) or \( r = 2, 3, 4 \) and \( \omega = .5, .6 \). Let \( h \geq 1/8192 \) and \( \varepsilon > h \). Choose \( k \) so that \( h_1 = 2^{k-1} h \approx \varepsilon \). Then there exist constants \( c_1(h), c_2(h) > 0 \) such that

\[
c_1(h) \varepsilon^2 \leq \lambda_{\text{min}}(M_h^\varepsilon A_h^\varepsilon) \leq \lambda_{\text{max}}(M_h^\varepsilon A_h^\varepsilon) \leq c_2(h) \varepsilon^2.
\]

**Remark 3.1**

For fixed \( \varepsilon, r \) and \( \omega \), numerical evidence indicates that, as \( h \to 0 \),

\[
c_1(h) \to c_1 > 0
\]

\[
c_2(h) \to c_2 > 0.
\]

**Remark 3.2:**

Since \( A_h^\varepsilon \) is similar to \( M_h^\varepsilon A_h^\varepsilon \), (3.4) implies that \( K(A_h^\varepsilon) \) is bounded independent of \( \varepsilon \).
Proof of Theorem 3.1:

Define

\[
\mu_{ij} = (M_k (\varepsilon^2 A_k + I) \alpha_i^{(k)}, \alpha_j^{(k)})_k.
\] (3.5)

Because of the aliasing, \(\mu_{ij}\) can be nonzero for \(j \neq i\) (i.e. \(\alpha_i^{(k)}\) and \(\alpha_j^{(k)}\) are distinguishable on the coarsest grid) then \(\mu_{ij} = 0\).

Choose \(m = (m_1, m_2)\) where \(|m| := \max(m_1, m_2) < N_k\).

Let \(j_1, j_2, \ldots, j_{4^k-1}\) be some ordering of the \(j \sim m(1)\).

We now define \(M_m\) to be a \(4^{k-1} \times 4^{k-1}\) matrix given by

\[
(M_m)_{p,q} = \mu_{j_p j_q}.
\] (3.6)

We consider the subspaces

\[
S_m := \text{linear span} (\{\alpha_j^{(k)} : j \sim m(1)\}),
\] (3.7)

where \(|m| < N_k\). The \(S_m\) are orthogonal (with respect to the inner product defined by (2.7a)) subspaces and invariant under \(M_k (\varepsilon^2 A_k + I)\). Therefore if we show that

\[
c_1 \varepsilon^2 \leq \lambda_{\min} (M_m) \leq \lambda_{\max} (M_m) \leq c_2 \varepsilon^2
\] (3.8)

for each \(m\), then (3.4) will be proved.

By the Gershgorin theorem, any eigenvalue, \(\lambda\), of \(M_m\) must satisfy

\[
|\lambda - \mu_{ii}| \leq \sum_{j \sim i (1) \atop j \neq i} |\mu_{ij}|
\] (3.9)

for some \(i \sim m(1)\).

We show that \(M_m\) is diagonally row dominant and therefore we can use information about the behaviour of the diagonal entries of \(M_m\) to prove (3.8). Specifically, in Section 3.3 we give a computable formula, (3.22), for a quantity \(C_{h,k,r,m}\), independent of \(\varepsilon\), such that

\[
\sum_{j \sim i (1) \atop j \neq i} |\mu_{ij}| \leq C_{h,k,r,m} \mu_{ii}
\] (3.10)
For certain choices of $r$ and $\omega$, $C_{h,k,r,\omega}$ has been computed, for every $i$, showing that $ar{C}_{h,k,r,\omega} := \sup_i C_{h,k,r,\omega} < 1$ for the $k = 2,3,\ldots,12$ grid problems, using $h = 2^{-1}$ to $h = 2^{-13}$. See Section 3.4. In Section 3.5 it is shown that $\exists \xi, \bar{c} > 0$ such that

$$\xi^2 \leq \min_{|i| < N_h} \mu_{ii} \leq \max_{|i| < N_h} \mu_{ii} \leq \bar{c} \xi^2. \tag{3.11}$$

Combining (3.9),(3.10) and (3.11) we have, for any eigenvalue, $\lambda$, of $\mathcal{M}_m$,

$$(1 - \bar{C}_{h,k,r,\omega}) \xi^2 \leq \lambda \leq (1 + \bar{C}_{h,k,r,\omega}) \bar{c} \xi^2, \tag{3.12}$$

which verifies (3.8) with $c_1 = (1 - \bar{C}_{h,k,r,\omega}) \xi$ and $c_2 = (1 + \bar{C}_{h,k,r,\omega}) \bar{c}$.

Note that a common factor, $\xi^2 v_i^{(k)} + 1$, appears in all the $\mu_{ij}$, $j \sim i$ (1), therefore (3.10) is equivalent to

$$\sum_{j \sim i \not= i} (M_k \alpha_i^{(k)}, \alpha_j^{(k)}) \leq C_{h,k,r,\omega} (M_k \alpha_i^{(k)}, \alpha_i^{(k)}). \tag{3.13}$$

Let

$$D_i := (M_k \alpha_i^{(k)}, \alpha_i^{(k)}). \tag{3.14}$$

### 3.3 Bounds on the Off-Diagonal Elements of $\mathcal{M}_m$.

When applying a multigrid-type cycle to an eigenvector, $\alpha_i^{(k)}$, of $A_k$, the resulting vector, $M_k \alpha_i^{(k)}$, is a linear combination of $\alpha_i^{(k)}$ and all of the other eigenvectors, $\alpha_j^{(k)}$, which alias with $\alpha_i^{(k)}$ on the coarsest grid. In this section we give a formula for a bound on this aliasing. Specifically, we find an expression, $C_{h,k,r,\omega}$, where

$$J_i := \sum_{j \sim i \not= i} (M_k \alpha_i^{(k)}, \alpha_j^{(k)}) \leq C_{h,k,r,\omega} (M_k \alpha_i^{(k)}, \alpha_i^{(k)}). \tag{3.15}$$

Let $i = (i_1, i_2), h, k, r$ and $\omega$ be fixed.

Define

$$\xi_p = \cos^2 \left( \frac{i_1 \pi h_p}{2} \right), \tag{3.16a}$$

$$\eta_p = \cos^2 \left( \frac{i_2 \pi h_p}{2} \right). \tag{3.16b}$$
\[ g_p = \langle G_p^{r_p} \alpha_i^{(p)}, \alpha_i^{(p)} \rangle_p \]  
(3.16c)

\[ e_p = \langle \left( \sum_{\sigma=0}^{2r-1} G_p^{\sigma} \right) \alpha_i^{(p)}, \alpha_i^{(p)} \rangle_p \]  
(3.16d)

and

\[ \nu_p = \langle A_p \alpha_i^{(p)}, \alpha_i^{(p)} \rangle_p, \]  
(3.16e)

where the \( i, r, h \) and \( \omega \) dependence has been suppressed in the notation and only the grid level is displayed.

The following lemma gives a formula for any entry in the row of \( M_m \) corresponding to \( i \), where \( i \sim m (1) \).

**Lemma 3.1**

For any \( j \sim i \) (1),

\[ \langle M_k \alpha_i^{(k)}, \alpha_j^{(k)} \rangle_k = 2\omega c_k \sum_{p=1}^k e_p \left( \prod_{m=p+1}^k 4g_m \xi_m \eta_m \right) \langle \alpha_i^{(p)}, I_{p+1}^p G_{p+1}^{r_p} \cdots I_{k-1}^{k-1} G_{k}^{r_k} \alpha_j^{(p)} \rangle_p. \]  
(3.17)

**Proof of Lemma 3.1**

A proof by induction shows that for every \( s, 2 \leq s \leq k \),

\[ \langle M_s \alpha_i^{(s)}, \alpha_j^{(s)} \rangle_s = 2\omega c_s \sum_{p=1}^s e_p \left( \prod_{m=p+1}^s 4g_m \xi_m \eta_m \right) \cdot \langle \alpha_i^{(s)}, I_{p+1}^p G_{p+1}^{r_p} \cdots I_{s-1}^{s-1} G_{s}^{r_s} \alpha_j^{(s)} \rangle_p. \]  
(3.18)

Taking \( s = k \) gives (3.17).

For \( s = 2 \), (3.4), (3.16) and (2.12) give

\[ \langle M_2 \alpha_i^{(2)}, \alpha_j^{(2)} \rangle_2 = \langle (I - G_2^{2r}) A_2^{-1} \alpha_i^{(2)}, \alpha_j^{(2)} \rangle_2 \]

\[ + \langle (I - G_1^{2r}) A_1^{-1} I_2^1 G_2^{r_2} \alpha_i^{(2)}, I_2^1 G_2^{r_2} \alpha_j^{(2)} \rangle_1 \]

\[ = 2\omega c_2 e_2 \langle \alpha_i^{(2)}, \alpha_j^{(2)} \rangle_2 + 2\omega c_1 e_1 g_2 \xi_2 \eta_2 \langle \alpha_i^{(1)}, I_2^1 G_2^{r_2} \alpha_j^{(2)} \rangle_1. \]  
(3.19)

Substituting \( 4c_2 = c_1 \), proves (3.18) for \( s = 2 \).
Assume (3.18) is true for \( s - 1 \) grids, \( s > 2 \). For the \( s \)-grid problem, (3.4), (3.16) and (2.12) give

\[
\langle M_s \alpha_i^{(s)}, \alpha_j^{(s)} \rangle_s = \langle (I - G_s^{2r}) A_s^{-1} \alpha_i^{(s)}, \alpha_j^{(s)} \rangle_s
\]

\[
+ \langle M_{s-1} I_s^{r-1} G_s^{r} \alpha_i^{(s)}, I_s^{r-1} G_s^{r} \alpha_j^{(s)} \rangle_{s-1}
\]

\[
= 2\omega c_s \epsilon_s \langle \alpha_i^{(s)}, \alpha_j^{(s)} \rangle_s
\]

\[
+ \xi_s \eta_s g_s \langle M_{s-1} \alpha_i^{(s)}, I_s^{r-1} G_s^{r} \alpha_j^{(s)} \rangle_{s-1}.
\]

Using the inductive hypothesis with \( I_s^{r-1} G_s^{r} \alpha_j^{(s)} \) replacing \( \alpha_j^{(s-1)} \), and using \( 4c_s = c_{s-1} \) proves (3.18).

Lemma 3.1 can be used to get an expression for \( J_i \), but the summation over all \( j \sim i \) (1) would be difficult to compute. Theorem 3.2 shows that \( J_i \) can be bound by an expression which is no more complicated than the expression for \( D_i = \langle M_k \alpha_i^{(k)}, \alpha_i^{(k)} \rangle \).

We claim that the \( J_i \) can be bounded by an expression which is no more complicated than the expression for \( D_i = \langle M_k \alpha_i^{(k)}, \alpha_i^{(k)} \rangle \):

**Theorem 3.2**

\[
\text{a.) } D_i = 2\omega c_k \sum_{p=1}^{k} \epsilon_p \left( \prod_{m=p+1}^{k} \frac{4g_m^2 \xi_m^2 \eta_m^2}{\gamma_m \gamma_{m+1}} \right).
\]

\[
\text{b.) } J_i \leq 2\omega c_k \sum_{p=1}^{k-1} \epsilon_p \left( 1 - \left( \prod_{m=p+1}^{k} \xi_m \eta_m \right) \right) \left( \prod_{m=p+1}^{k} 4g_m \xi_m \eta_m \right).
\]
Proof of Theorem 3.2

a.) Using Lemma 3.1 with \( j = i \), combined with equations (3.16c) and (2.12) proves (3.21a).

b.) To prove (3.21b), split the grid levels by partitioning the \( j \sim i \), \( j \neq i \). See Figure 3.1 for a schematic illustration for \( k = 3 \). For each \( n = 1, 2, \ldots, k-1 \) consider the \( j \)'s such that \( j \sim i(n) \) but \( j \not\sim i(n+1) \). Lemma 3.1, Lemma 2.1, Lemma 2.3 and (2.6) lead to the following bound:

\[
J_i = \sum_{n=1}^{k-1} \sum_{\substack{j \sim i(n) \atop j \neq i(n+1)}} \left| (M^{-1}_k \alpha_{i}^{(k)}, \alpha_{j}^{(k)})_k \right| \tag{3.22}
\]

\[
\leq 2w_c k \sum_{n=1}^{k-1} (1 - \xi_{n+1} \eta_n + 1) \sum_{p=1}^{n} e_p \left( \prod_{m=p+1}^{n} \xi_m \eta_m \right) \left( \prod_{m=p+1}^{k} 4 |g_m| \xi_m \eta_m \right) .
\]

Changing the order of summation gives

\[
J_i \leq 2w_c \sum_{p=1}^{k-1} e_p \left[ \sum_{n=p}^{k-1} (1 - \xi_{n+1} \eta_n + 1) \prod_{m=p+1}^{n} |g_m| \xi_m \eta_m \right] \tag{3.23}
\]

Observe that the quantity in square brackets can be simplified to:

\[
1 - \prod_{m=p+1}^{k} \xi_m \eta_m .
\]
Figure 3.1: A splitting of the $j, j \sim i (s), j \neq 1$.

- $\times$ $j \sim i (1), j \neq i (2)$
- $\circ$ $j \sim i (2), j \neq i (3)$
- $\Delta$ $j \sim i (3)$.

Remark 3.3

The constants $\bar{C}_{h,k,r,\omega}$ can now be expressed as

$$\bar{C}_{h,k,r,\omega} = \sup_i (C^i_{h,k,r,\omega}).$$

where

$$C^i_{h,k,r,\omega} = \frac{\sum_{p=1}^{k-1} e_p \left( 1 - \prod_{m=p+1}^{k} \xi_m \eta_m \right) \left( \prod_{m=p+1}^{k} g_m |\xi_m \eta_m| \right)}{\sum_{p=1}^{k} e_p \left( \prod_{m=p+1}^{k} 4g_m^2 \xi_m^2 \eta_m^2 \right)} \quad (3.24)$$

Note that the denominator has one more term in the sum than does the numerator.
3.4 Computed Values of the Off-Diagonal Bounds

Ideally, one would like to find analytic bounds for $C_{h,k,r,\omega}^i$, independent of $i, h$ and $k$. On the other hand, bounds are easily computed for any given $h, k, r$ and $\omega$.

Figures 3.2–3.5 indicate the dependence of $C_{h,k,r,\omega}^i$ on $i = (i_1, i_2)$ for $h = 1/64, r = 1, \omega = .8$ and $k = 2, 3, 4$ and 5 grids. The maximum is taken on the boundaries $i_1 = 1$ or $i_2 = 1$. Along the boundary $i_2 = 1$ there are $2^{k-2}$ relative maxima for the $k$-grid problem. (For all values of $h, k, r$ and $\omega$ tried, the maximum of $C^i$ was attained at $(1, i_2)$ and $(i_2, 1)$ for some $i_2$.) Figures 3.6–3.9 show the dependence on $r$ for $k = 4$ grids.

Tables 3.1–3.8 give the calculated bounds, $\sup_{|i| < 1/h} (C_{h,k,r,\omega}^i)$, for $\omega = .5, .8$ and $r = 1, 2, 3$ and 4. The multi-index at which the supremum was attained is listed below the bound.

To find bounds for $\omega = .8$ and $r = 1, 2, 3, 4$, independent of $h$ and $k$, we used $h = 1/8192$ (which means > 67 million points on the fine grid). These numbers are bounds for all $h > 1/8192$ and all $k$ corresponding to these meshsizes. Observing the asymptotic behaviour leads one to believe that they are also bounds for all $h < 1/8192$ and any number of grids, $k$. See Tables 3.9–3.10.
Figure 3.2: $C_{h,k,r,\omega}^i$ for $h = 1/64, r = 1, \omega = .8$

Figure 3.3: $C_{h,k,r,\omega}^i$ for $h = 1/64, r = 1, \omega = .8$
Figure 3.4: $C^i_{h,k,r,\omega}$ for $h = 1/64$, $r = 1$, $\omega = .8$ 

Figure 3.5: $C^i_{h,k,r,\omega}$ for $h = 1/64$, $r = 1$, $\omega = .8$
Figure 3.6: $C_{h,k,r,\omega}^1$ for $h = 1/64$, 4 grids, $\omega = .8$

Figure 3.7: $C_{h,k,r,\omega}^1$ for $h = 1/64$, 4 grids, $\omega = .8$
Figure 3.8: $C_{h,k,r,w}^i$ for $h=1/64$, 4 grids, $\omega = .8$

Figure 3.9: $C_{h,k,r,w}^i$ for $h=1/64$, 4 grids, $\omega = .8$
Table 3.1 \( \tilde{C}_{h,k,r,w} \quad \omega = .5 \ , \ r = 1 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.4640</td>
<td>.7165</td>
<td>.8026</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,9)</td>
<td>(1,11)</td>
<td>(1,11)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.4688</td>
<td>.7530</td>
<td>.9484</td>
<td>1.025</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,19)</td>
<td>(1,22)</td>
<td>(1,21)</td>
<td>(1,11)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.4707</td>
<td>.7632</td>
<td>.9942</td>
<td>1.149</td>
<td>&gt; 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,37)</td>
<td>(1,45)</td>
<td>(1,41)</td>
<td>(1,21)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.4712</td>
<td>.7669</td>
<td>1.004</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,74)</td>
<td>(1,89)</td>
<td>(1,81)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.4712</td>
<td>.7669</td>
<td>1.006</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
</tr>
<tr>
<td></td>
<td>(1,149)</td>
<td>(1,178)</td>
<td>(1,163)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/512</td>
<td>.4712</td>
<td>.7671</td>
<td>1.007</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
</tr>
<tr>
<td></td>
<td>(1,298)</td>
<td>(1,357)</td>
<td>(1,325)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2 \( \tilde{C}_{h,k,r,w} \quad \omega = .5 \ , \ r = 2 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.3115</td>
<td>.4296</td>
<td>.4680</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,9)</td>
<td>(1,5)</td>
<td>(1,5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.3196</td>
<td>.4574</td>
<td>.5441</td>
<td>.5573</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,17)</td>
<td>(1,10)</td>
<td>(1,11)</td>
<td>(1,11)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.3215</td>
<td>.4658</td>
<td>.5700</td>
<td>.6084</td>
<td>.6142</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,35)</td>
<td>(1,19)</td>
<td>(1,22)</td>
<td>(1,21)</td>
<td>(1,11)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.3220</td>
<td>.4680</td>
<td>.5771</td>
<td>.6277</td>
<td>.6591</td>
<td>.6643</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,70)</td>
<td>(1,39)</td>
<td>(1,44)</td>
<td>(1,23)</td>
<td>(1,21)</td>
<td>(1,21)</td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.3221</td>
<td>.4685</td>
<td>.5790</td>
<td>.6349</td>
<td>.6741</td>
<td>.6856</td>
<td>.6876</td>
</tr>
<tr>
<td></td>
<td>(1,139)</td>
<td>(1,77)</td>
<td>(1,88)</td>
<td>(1,45)</td>
<td>(1,41)</td>
<td>(1,43)</td>
<td>(1,43)</td>
</tr>
<tr>
<td>1/512</td>
<td>.3221</td>
<td>.4688</td>
<td>.5795</td>
<td>.6368</td>
<td>.6782</td>
<td>.6923</td>
<td>.6997</td>
</tr>
<tr>
<td></td>
<td>(1,278)</td>
<td>(1,155)</td>
<td>(1,177)</td>
<td>(1,91)</td>
<td>(1,82)</td>
<td>(1,86)</td>
<td>(1,43)</td>
</tr>
</tbody>
</table>
Table 3.3  $\bar{C}_{h,k,r,\omega}$  $\omega = .5$, $r = 3$

<table>
<thead>
<tr>
<th>$h$</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.2200</td>
<td>.2998</td>
<td>.3102</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,7)</td>
<td>(1,5)</td>
<td>(1,5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.2271</td>
<td>.3283</td>
<td>.3484</td>
<td>.3581</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,15)</td>
<td>(1,9)</td>
<td>(1,5)</td>
<td>(1,5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.2285</td>
<td>.3346</td>
<td>.3694</td>
<td>.3961</td>
<td>.3981</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,31)</td>
<td>(1,18)</td>
<td>(1,10)</td>
<td>(1,11)</td>
<td>(1,11)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.2289</td>
<td>.3362</td>
<td>.3756</td>
<td>.4093</td>
<td>.4165</td>
<td>.4170</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,61)</td>
<td>(1,36)</td>
<td>(1,20)</td>
<td>(1,22)</td>
<td>(1,21)</td>
<td>(1,21)</td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.2290</td>
<td>.3367</td>
<td>.3773</td>
<td>.4130</td>
<td>.4243</td>
<td>.4297</td>
<td>.4302</td>
</tr>
<tr>
<td></td>
<td>(1,123)</td>
<td>(1,71)</td>
<td>(1,39)</td>
<td>(1,44)</td>
<td>(1,23)</td>
<td>(1,21)</td>
<td>(1,21)</td>
</tr>
<tr>
<td>1/512</td>
<td>.2290</td>
<td>.3368</td>
<td>.3777</td>
<td>.4139</td>
<td>.4279</td>
<td>.4356</td>
<td>.4366</td>
</tr>
<tr>
<td></td>
<td>(1,246)</td>
<td>(1,142)</td>
<td>(1,78)</td>
<td>(1,88)</td>
<td>(1,46)</td>
<td>(1,41)</td>
<td>(1,42)</td>
</tr>
</tbody>
</table>

Table 3.4  $\bar{C}_{h,k,r,\omega}$  $\omega = .5$, $r = 4$

<table>
<thead>
<tr>
<th>$h$</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.1689</td>
<td>.2253</td>
<td>.2368</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,7)</td>
<td>(1,3)</td>
<td>(1,3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.1732</td>
<td>.2481</td>
<td>.2659</td>
<td>.2684</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,14)</td>
<td>(1,7)</td>
<td>(1,5)</td>
<td>(1,5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.1747</td>
<td>.2550</td>
<td>.2823</td>
<td>.2876</td>
<td>.2880</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,27)</td>
<td>(1,15)</td>
<td>(1,9)</td>
<td>(1,10)</td>
<td>(1,9)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.1749</td>
<td>.2567</td>
<td>.2868</td>
<td>.2945</td>
<td>.3013</td>
<td>.3016</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,54)</td>
<td>(1,31)</td>
<td>(1,18)</td>
<td>(1,20)</td>
<td>(1,11)</td>
<td>(1,11)</td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.1750</td>
<td>.2570</td>
<td>.2880</td>
<td>.2971</td>
<td>.3081</td>
<td>.3089</td>
<td>.3090</td>
</tr>
<tr>
<td></td>
<td>(1,109)</td>
<td>(1,63)</td>
<td>(1,35)</td>
<td>(1,19)</td>
<td>(1,23)</td>
<td>(1,19)</td>
<td>(1,19)</td>
</tr>
<tr>
<td>1/512</td>
<td>.1750</td>
<td>.2571</td>
<td>.2883</td>
<td>.2982</td>
<td>.3099</td>
<td>.3125</td>
<td>.3131</td>
</tr>
<tr>
<td></td>
<td>(1,217)</td>
<td>(1,126)</td>
<td>(1,70)</td>
<td>(1,38)</td>
<td>(1,47)</td>
<td>(1,25)</td>
<td>(1,26)</td>
</tr>
</tbody>
</table>

26
Table 3.5 \( \tilde{c}_{h,k,r,\omega} \) \( \omega = .8 \), \( r = 1 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1/8 )</td>
<td>.2801 (1.5)</td>
<td>.3473 (1.3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 1/16 )</td>
<td>.3448 (1.9)</td>
<td>.4724 (1.5)</td>
<td>.5071 (1.5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 1/32 )</td>
<td>.3565 (1.17)</td>
<td>.5080 (1.9)</td>
<td>.5894 (1.11)</td>
<td>.6015 (1.11)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 1/64 )</td>
<td>.3581 (1.34)</td>
<td>.5166 (1.19)</td>
<td>.6179 (1.22)</td>
<td>.6647 (1.21)</td>
<td>.6698 (1.21)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 1/128 )</td>
<td>.3586 (1.68)</td>
<td>.5188 (1.37)</td>
<td>.6256 (1.44)</td>
<td>.6857 (1.41)</td>
<td>.7243 (1.21)</td>
<td>.7293 (1.21)</td>
<td></td>
</tr>
<tr>
<td>( 1/256 )</td>
<td>.3587 (1.136)</td>
<td>.5194 (1.75)</td>
<td>.6278 (1.87)</td>
<td>.6916 (1.45)</td>
<td>.7423 (1.41)</td>
<td>.7531 (1.43)</td>
<td>.7551 (1.43)</td>
</tr>
<tr>
<td>( 1/512 )</td>
<td>.3587 (1.273)</td>
<td>.5195 (1.150)</td>
<td>.6283 (1.175)</td>
<td>.6938 (1.89)</td>
<td>.7471 (1.82)</td>
<td>.7609 (1.86)</td>
<td>.7695 (1.43)</td>
</tr>
</tbody>
</table>
Table 3.6 \( \hat{C}_{h,k,r,\omega} \) \( \omega = .8 \), \( r = 2 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.1954</td>
<td>.2552</td>
<td>.2640</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,7)</td>
<td>(1,3)</td>
<td>(1,3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.2001</td>
<td>.2840</td>
<td>.2993</td>
<td>.3013</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,14)</td>
<td>(1,7)</td>
<td>(1,5)</td>
<td>(1,5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.2013</td>
<td>.2932</td>
<td>.3223</td>
<td>.3252</td>
<td>.3268</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,28)</td>
<td>(1,15)</td>
<td>(1,9)</td>
<td>(1,10)</td>
<td>(1,9)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.2016</td>
<td>.2956</td>
<td>.3285</td>
<td>.3337</td>
<td>.3387</td>
<td>.3389</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1,55)</td>
<td>(1,31)</td>
<td>(1,17)</td>
<td>(1,20)</td>
<td>(1,11)</td>
<td>(1,11)</td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.2017</td>
<td>.2961</td>
<td>.3299</td>
<td>.3387</td>
<td>.3473</td>
<td>.3499</td>
<td>.3500</td>
</tr>
<tr>
<td></td>
<td>(1,111)</td>
<td>(1,63)</td>
<td>(1,34)</td>
<td>(1,18)</td>
<td>(1,21)</td>
<td>(1,19)</td>
<td>(1,19)</td>
</tr>
<tr>
<td>1/512</td>
<td>.2018</td>
<td>.2963</td>
<td>.3304</td>
<td>.3402</td>
<td>.3497</td>
<td>.3537</td>
<td>.3538</td>
</tr>
<tr>
<td></td>
<td>(1,222)</td>
<td>(1,127)</td>
<td>(1,69)</td>
<td>(1,36)</td>
<td>(1,42)</td>
<td>(1,39)</td>
<td>(1,38)</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>1/8192</td>
<td>.2018</td>
<td>.2963</td>
<td>.3305</td>
<td>.3407</td>
<td>.3505</td>
<td>.3550</td>
<td>.3581</td>
</tr>
</tbody>
</table>
Table 3.7  \( C_{h,k,r,w} \quad \omega = .8 \ , \ r = 3 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.1353</td>
<td>.1880</td>
<td>.1891</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.6)</td>
<td>(1.3)</td>
<td>(1.3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.1385</td>
<td>.1993</td>
<td>.2077</td>
<td>.2088</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.12)</td>
<td>(1.7)</td>
<td>(1.3)</td>
<td>(1.3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.1393</td>
<td>.2032</td>
<td>.2233</td>
<td>.2240</td>
<td>.2243</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.24)</td>
<td>(1.13)</td>
<td>(1.7)</td>
<td>(1.7)</td>
<td>(1.7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.1395</td>
<td>.2044</td>
<td>.2267</td>
<td>.2296</td>
<td>.2303</td>
<td>.2307</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.47)</td>
<td>(1.27)</td>
<td>(1.15)</td>
<td>(1.7)</td>
<td>(1.7)</td>
<td>(1.7)</td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.1396</td>
<td>.2046</td>
<td>.2278</td>
<td>.2339</td>
<td>.2341</td>
<td>.2353</td>
<td>.2353</td>
</tr>
<tr>
<td></td>
<td>(1.95)</td>
<td>(1.54)</td>
<td>(1.29)</td>
<td>(1.15)</td>
<td>(1.14)</td>
<td>(1.14)</td>
<td>(1.14)</td>
</tr>
<tr>
<td>1/512</td>
<td>.1396</td>
<td>.2047</td>
<td>.2281</td>
<td>.2348</td>
<td>.2357</td>
<td>.2369</td>
<td>.2370</td>
</tr>
<tr>
<td></td>
<td>(1.189)</td>
<td>(1.108)</td>
<td>(1.58)</td>
<td>(1.30)</td>
<td>(1.15)</td>
<td>(1.27)</td>
<td>(1.29)</td>
</tr>
</tbody>
</table>

Table 3.8  \( \tilde{C}_{h,k,r,w} \quad \omega = .8 \ , \ r = 4 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.1039</td>
<td>.1441</td>
<td>.1482</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.5)</td>
<td>(1.3)</td>
<td>(1.1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.1057</td>
<td>.1531</td>
<td>.1628</td>
<td>.1629</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.10)</td>
<td>(1.6)</td>
<td>(1.3)</td>
<td>(1.3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.1067</td>
<td>.1557</td>
<td>.1705</td>
<td>.1716</td>
<td>.1716</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.21)</td>
<td>(1.12)</td>
<td>(1.6)</td>
<td>(1.6)</td>
<td>(1.6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.1068</td>
<td>.1563</td>
<td>.1736</td>
<td>.1761</td>
<td>.1762</td>
<td>.1762</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.42)</td>
<td>(1.24)</td>
<td>(1.13)</td>
<td>(1.7)</td>
<td>(1.6)</td>
<td>(1.7)</td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.1069</td>
<td>.1565</td>
<td>.1742</td>
<td>.1788</td>
<td>.1795</td>
<td>.1797</td>
<td>.1797</td>
</tr>
<tr>
<td></td>
<td>(1.84)</td>
<td>(1.48)</td>
<td>(1.26)</td>
<td>(1.13)</td>
<td>(1.13)</td>
<td>(1.13)</td>
<td>(1.13)</td>
</tr>
<tr>
<td>1/512</td>
<td>.1069</td>
<td>.1565</td>
<td>.1744</td>
<td>.1795</td>
<td>.1803</td>
<td>.1808</td>
<td>.1810</td>
</tr>
<tr>
<td></td>
<td>(1.168)</td>
<td>(1.95)</td>
<td>(1.51)</td>
<td>(1.27)</td>
<td>(1.25)</td>
<td>(1.13)</td>
<td>(1.13)</td>
</tr>
</tbody>
</table>
Table 3.9  \( \tilde{C}_{h,k,r,\omega} \quad \omega = .8 , \ h = 1/8192 \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.3587</td>
<td>.5196</td>
<td>.6284</td>
<td>.6945</td>
<td>.7487</td>
<td>.7638</td>
</tr>
<tr>
<td>2</td>
<td>.2018</td>
<td>.2963</td>
<td>.3305</td>
<td>.3407</td>
<td>.3505</td>
<td>.3550</td>
</tr>
<tr>
<td>3</td>
<td>.1396</td>
<td>.2047</td>
<td>.2282</td>
<td>.2351</td>
<td>.2370</td>
<td>.2375</td>
</tr>
<tr>
<td>4</td>
<td>.1069</td>
<td>.1566</td>
<td>.1745</td>
<td>.1798</td>
<td>.1812</td>
<td>.1818</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( r )</th>
<th>8 grids</th>
<th>9 grids</th>
<th>10 grids</th>
<th>11 grids</th>
<th>12 grids</th>
<th>13 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.7800</td>
<td>.7896</td>
<td>.7933</td>
<td>.7953</td>
<td>.7948</td>
<td>.7951</td>
</tr>
<tr>
<td>2</td>
<td>.3581</td>
<td>.3589</td>
<td>.3590</td>
<td>.3592</td>
<td>.3592</td>
<td>.3952</td>
</tr>
<tr>
<td>3</td>
<td>.2390</td>
<td>.2394</td>
<td>.2398</td>
<td>.2398</td>
<td>.2398</td>
<td>.2398</td>
</tr>
<tr>
<td>4</td>
<td>.1821</td>
<td>.1824</td>
<td>.1825</td>
<td>.1825</td>
<td>.1825</td>
<td>.1825</td>
</tr>
</tbody>
</table>

Table 3.10  \( C_{h,k,r,\omega} \quad k = 12 , \ h = 1/8192 \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \omega = 5 )</th>
<th>( \omega = .6 )</th>
<th>( \omega = .7 )</th>
<th>( \omega = .8 )</th>
<th>( \omega = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>&gt; 1</td>
<td>&gt; 1</td>
<td>.980</td>
<td>.795</td>
<td>.648</td>
</tr>
<tr>
<td>2</td>
<td>.721</td>
<td>.554</td>
<td>.439</td>
<td>.359</td>
<td>.305</td>
</tr>
<tr>
<td>3</td>
<td>.444</td>
<td>.345</td>
<td>.282</td>
<td>.240</td>
<td>.210</td>
</tr>
<tr>
<td>4</td>
<td>.318</td>
<td>.254</td>
<td>.212</td>
<td>.183</td>
<td>.161</td>
</tr>
</tbody>
</table>
3.5 Bounds on the Diagonal Elements of $\mathcal{M}_m$

Recall that the diagonal elements, $\mu_{ii}$, of $\mathcal{M}_m$ where $i \sim m(1)$, are given by,

$$\mu_{ii} = (M_k A_h^{(k)} \alpha_i^{(k)}, \alpha_i^{(k)})_k. \quad (3.25)$$

Since $A_h^{(k)} = A_k + I$ and hence

$$\mu_{ii} = \left(\varepsilon^2 \nu_i^{(k)} + 1\right) D_i, \quad (3.26)$$

the bounds on the $\mu_{ii}$ can be obtained from suitable information about the $D_i$'s. The following characterization of the effect of the preconditioner on smooth and rough eigenvectors of $A_k$ is central to the analysis and was given by Goldstein in [7].

**Theorem 3.3**

For $r > 1$, $\omega$ suitably chosen and $h$ sufficiently small, the $D_i$'s are positive real numbers such that:

a.) $D_i = 0(h_i^2)$ for $\nu_i^{(k)} < d/h_i^2 \quad (3.27a)$

b.) $D_i = \left(1 - \eta \right) \nu_i^{(k)}$ for $\nu_i^{(k)} \geq d/h_i^2 \quad (3.27b)$

where $0 < \eta < 1$ and $\eta$ is independent of $h$ and $d$ is a constant.

We prove a more explicit version of the same result:

**Theorem 3.4**

For $r \geq 1$, $0 < \omega < 1$ and a fixed constant, $d$, where $\frac{1}{2} < d \leq 2$.

a.) $\frac{\omega(1 - \omega)}{\max(2, d(1 + r\omega))} h_i^2 \leq D_i \leq \frac{2\rho \omega}{3} h_i^2$ for $\nu_i^{(k)} < \frac{d}{h_i^2} \quad (3.28a)$

b.) $\frac{d\omega(1 - \omega)}{8(1 + r\omega)\nu_i^{(k)}} \leq D_i \leq \frac{1}{\nu_i^{(k)}}$ for $\nu_i^{(k)} \geq \frac{d}{h_i^2}. \quad (3.28b)$

**Proof:** in appendix.
These theorems give us bounds on the $\mu_{ii}$, and, for example, Theorem 3.4 leads to the following bounds:

For $\nu_i^{(k)} < \frac{d}{h_1}$,

$$\frac{\omega(1 - \omega)h_1^2}{\max(2, d(1 + r\omega))} \leq \mu_{ii} \leq \frac{2r\omega d}{3} \left( \epsilon^2 + \frac{h_1^2}{d} \right)$$  \hspace{1cm} (3.29a)

For $\nu_i^{(k)} \geq \frac{d}{h_1}$,

$$\frac{d\omega(1 - \omega)\epsilon^2}{8(1 + r\omega)} \leq \mu_{ii} \leq \epsilon^2 + \frac{h_1^2}{d}.$$  \hspace{1cm} (3.29b)

Therefore, taking $h_1 \cong \epsilon$, we prove (3.12).

Using the diagonal dominance of the matrices, $M_m$, we can estimate the dependence of the condition number of $M_k(\epsilon^2 A_k + I)$ on the ratio $\alpha = h_1^2/\epsilon^2$ from the behaviour of the diagonal elements, $\mu_{ii}$. From the inequalities (3.29) we get an estimate for the choice of $\alpha$ which minimizes the condition number:

$$\alpha_{\text{optimal}} \leq \frac{1}{8} \left( \frac{3}{2r\omega} \right)^2.$$  \hspace{1cm} (3.30)

This predicts that the optimal number of grids decreases as the quantity $r\omega$ increases. One can also use (3.29) to show that it is better to choose too many grids, ($\alpha > \alpha_{\text{opt}}$), rather than too few, ($\alpha < \alpha_{\text{opt}}$), (see Figure 3.10). These observations all accurately describe the experimental results — see the next section.
Figure 3.10: The condition number estimated from the diagonal terms.
4. Multigrid Preconditioner — Experimental Results.

Our numerical computations were carried out with three objectives in mind:

i) Observe the optimality of taking the meshsize on the coarsest grid, $h_1$, to approximate the singular perturbation parameter, $\varepsilon$.

ii) Check the boundedness of the condition number of the multigrid-preconditioned system as $\varepsilon$ and the fine grid meshsize, $h$, decrease.

iii) Compare the efficiency to other fast solvers, in particular, the corresponding multigrid algorithm used as an iterative solver.

We discretize the boundary value problem:

\[
\begin{aligned}
A_k^* u &= (-\varepsilon^2 \Delta + I)u = f \quad &\text{in } \Omega = (0,1) \times (0,1) \\
u &= 0 \quad &\text{on } \partial \Omega,
\end{aligned}
\]  

(4.1)

on a grid of uniform meshsize, $h$, as in Section 2.1. Using the multigrid preconditioner, $M_h^k$, as defined in Section 3.1, we iteratively solve the discrete problem using a preconditioned conjugate gradient algorithm. Recall that $k$ is the number of grids used in the multigrid algorithm, $h_k = h$, and the smoothers, $G_p$, $1 \leq p \leq k$, used to define $M_h^k$, depend on the damping parameter, $\omega$, and a fixed number of smooths per iteration, $r$. We solve

\[
(\varepsilon^2 A_k + I)u_k = F_k,
\]  

(4.2)

starting with initial guess, $u_k^0$. We call this iterative solver $\text{PCCG}(-\Delta, \text{sm})$. The "\(\Delta\)" reminds us that the multigrid preconditioner is based on $A_k$, the negative of the discrete Laplacian, and not on the operator $A_k^* = \varepsilon^2 A_k + I$ and "sm" indicates that we smooth instead of solving exactly on the coarsest grid. Experimentally, we find that a reasonably good choice of $r$ and $\omega$ is $r = 2$ and $\omega = .8$ ($\omega = .8$ is optimal for the corresponding 2-grid multigrid solver, see [12]).

We first consider solving (4.2) with $F_k \equiv 1$. For $h = 1/64$ we show the dependence of the number of iterations required to reduce the norm of the residual by a factor of $10^{-6}$ on the choice of $\varepsilon$ and $h_1$. See Table 4.1. For given $\varepsilon$ and $h$, the number of iterations listed is the largest observed for various choices of $u_k^0$. Note, in particular, the cases where $h_1 = \varepsilon$.

Table 4.2 displays the number of iterations required to reduce the relative error by a factor of $10^{-6}$ for various choices of $h$ and $\varepsilon$, taking $h_1 = \varepsilon$. Here we used $F_k \equiv 1$. 

34
Finally, we compare the efficiency of PCCG($-\Delta,\text{sm}$) to other elliptic solvers. We take $h = 1/64$, $\varepsilon = 1/8$, $F_k \equiv 1$ and an initial guess consisting of a smooth and a rough component, namely:

$$u_0^k = 10 + 20 \cos(64\pi x) \cos(64\pi y).$$

We consider a symmetric V-cycle, which is a fast iterative solver for equation (4.1), where we solve exactly on the coarsest grid (we use a symmetric band solver to invert $\varepsilon^2 A_k + I$). We denote this algorithm by MULT. For comparison, an (extreme) choice of a preconditioner for the preconditioned conjugate gradient algorithm is considered, where the preconditioner is based on $A_k^* = A_k$ and we solve exactly on the coarsest grid. In other words, this preconditioner consists of one cycle of the solver, MULT, starting with initial guess of zero. This algorithm is called PCCG($-\varepsilon^2 \Delta + I, \text{so}$). Of course we expect the behaviour of this preconditioner to be better than that of the simpler ($-\Delta,\text{sm}$) preconditioner, but we have the added expense of a coarse grid solve and (slightly) more complicated operator. Of interest to us here is that PCCG($-\varepsilon^2 \Delta + I, \text{so}$) is not a significant improvement over PCCG($-\Delta,\text{sm}$) if the optimal choice of the number of grids is used.

In a conjugate gradient algorithm, the error reduction factor, $\frac{\|e_k\|}{\|e_{k-1}\|}$, typically decreases as $k$ increases, whereas for a multigrid algorithm the error reduction factor increases as $k$ increases. Therefore the preconditioned conjugate gradient routines will be more competitive when a large reduction in the relative residual is required and the multigrid algorithm is more competitive when a smaller reduction in the relative residual is required.

We also observe that increasing the number of smoothings per grid level will improve the performance of MULT more than it will improve the performance of the PCCG($-\Delta,\text{sm}$) algorithm. Similarly, optimizing the choice of the damping parameter, $\omega$, will improve MULT more than it will improve PCCG($-\Delta,\text{sm}$).

Furthermore, one should keep in mind that, though it is difficult to improve the behaviour of the multigrid preconditioner, it is quite obvious how to improve the multigrid solver. Using better smoothers, or using a full multigrid algorithm (FMG) will dramatically improve the convergence rate.

Our first comparison is made with parameters which should give the PCCG($-\Delta,\text{sm}$)
algorithm an advantage. We therefore consider a relatively inefficient choice of the damping parameter, \( \omega = .5 \), and require the norm of the residual to be reduced by a factor of \( 10^{-12} \). The total cpu time (seconds) is recorded in Figure 4.3, with the number of iterations given in parentheses next to the time. The PCCG(-\( \Delta \),sm) algorithm appears to be competitive with MULT, at least for this meshsize, \( h \). The PCCG(-\( \epsilon^2 \Delta + I \),so) algorithm is only slightly faster.

We then take a more reasonable value of \( \omega = .8 \) and require the norm of the residual to be reduced by a factor of \( 10^{-6} \). The total cpu time is recorded in Figure 4.4. The multigrid solver, MULT, is now the best choice.

All computations were done on a VAX 11/780.

We end this section with a few comments on the choice of using multigrid by itself as a solver, or using multigrid (based on a simpler operator) as a preconditioner:

- For the model problem (8.1), our experiments indicate that, for modest values of \( h \) and \( \epsilon \), a good multigrid algorithm is more efficient than a multigrid-preconditioned conjugate gradient algorithm.

- In a true variable coefficient problem, (1.1), the multigrid preconditioner has the advantage of being based on a constant coefficient operator. In this case, using multigrid as a preconditioner should be more competitive than in the model problem case. It is doubtful whether the multigrid preconditioner could outperform a good multigrid solver even in this case, but more testing would need to be done.

- In an indefinite problem, where multigrid solvers are more troublesome, one of the preconditioned conjugate gradient routines for indefinite problems might be preferable.
Table 4.1  Optimality of choosing $h_1 \approx \varepsilon$.

Largest (observed) # of iterations required for $\|r_k\|/\|r_k^0\| < 10^{-6}$.

$$F_k \equiv 1, \ \omega = .8, \ r = 2$$

<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$\varepsilon = 1/2$</th>
<th>$\varepsilon = 1/4$</th>
<th>$\varepsilon = 1/8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/32</td>
<td>&gt; 20</td>
<td>&gt; 20</td>
<td>20</td>
</tr>
<tr>
<td>1/16</td>
<td>12</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>1/8</td>
<td>9</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>1/4</td>
<td>7</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>1/2</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 4.2  Boundedness of condition number independent of $h$ and $\varepsilon$ taking $\varepsilon = h_1$.

Largest (observed) # of iterations required for $\|u_k - u_k^0\|/\|u_k - u_k^0\| < 10^{-6}$.

$$F_k \equiv 0, \ \omega = .8, \ r = 2$$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\varepsilon = 1/4$</th>
<th>$\varepsilon = 1/8$</th>
<th>$\varepsilon = 1/16$</th>
<th>$\varepsilon = 1/32$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/32</td>
<td>5</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>
Table 4.3  Experimental comparisons of approximate cpu time (sec).
Approximate cpu time (no. of iterations) required for \(\|\text{res}_k\|/\|\text{res}_0\| < 10^{-12}\).

\[
F_k \equiv 1, \quad \omega = 0.5, \quad r = 2
\]
\[
\epsilon = 1/8, \quad h = 1/64, \quad u_k^0 = 10 + 20 \cdot \cos 64\pi x \cos 64\pi y
\]

<table>
<thead>
<tr>
<th># of grids</th>
<th>MULT:V(2,2)</th>
<th>PCCG(-(\Delta),sm)</th>
<th>PCCG(-(\epsilon^2\Delta + I),so)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>61.3 (20)</td>
<td>- (&gt;20)</td>
<td>53.4 (10)</td>
</tr>
<tr>
<td>4</td>
<td>44.2 (21)</td>
<td>40.6 (11)</td>
<td>39.2 (10)</td>
</tr>
<tr>
<td>6</td>
<td>44.4 (21)</td>
<td>44.8 (12)</td>
<td>39.5 (10)</td>
</tr>
</tbody>
</table>

Table 4.4  Experimental comparisons of approximate cpu time (sec).
Approximate cpu time (no. of iterations) required for \(\|\text{res}_k\|/\|\text{res}_0\| < 10^{-6}\).

\[
F_k \equiv 1, \quad \omega = 0.8, \quad r = 2
\]
\[
\epsilon = 1/8, \quad h = 1/64, \quad u_k^0 = 10 + 20 \cdot \cos 64\pi x \cos 64\pi y
\]

<table>
<thead>
<tr>
<th># of grids</th>
<th>MULT:V(2,2)</th>
<th>PCCG(-(\Delta),sm)</th>
<th>PCCG(-(\epsilon^2\Delta + I),so)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>24.3 (6)</td>
<td>49.9 (14)</td>
<td>35.2 (5)</td>
</tr>
<tr>
<td>4</td>
<td>14.3 (6)</td>
<td>22.4 (5)</td>
<td>29.6 (5)</td>
</tr>
<tr>
<td>6</td>
<td>14.4 (6)</td>
<td>23.8 (6)</td>
<td>29.7 (5)</td>
</tr>
</tbody>
</table>
5.1 V-cycle Convergence Bounds

In this section we briefly describe the results of applying the same techniques, in particular Lemma 2.2, to obtain bounds on the asymptotic convergence rates for multigrid V-cycles used to solve the Dirichlet problem for Poisson's equation in the unit square. The analysis is simpler in this case because we don't need diagonal dominance. Instead, we numerically evaluate the \( \| \cdot \|_{\infty} \) norm of the appropriate matrix (i.e., the largest row sum of absolute values) which is a bound on the spectral radius. We present the details of this analysis in Section 5.2. We first define our basic multigrid V-cycle applied to the linear system

\[
B_k U_k = F_k
\]  \hspace{1cm} (5.1)

starting with initial guess, \( u_k^0 \), with auxiliary problems, \( B_p U_p = f_p \), \( p = 1, 2, \ldots, k - 1 \), corresponding to discretizations on the coarser grids.

1. Initialize:

\[
\begin{align*}
  f_k &\leftarrow F_k \\
  u_k &\leftarrow u_k^0
\end{align*}
\]

2. Update:

\[
\begin{align*}
  u_k &\leftarrow \tilde{u}_k
\end{align*}
\]

where each \( \tilde{u}_p \), \( p = 2, 3, \ldots, k \) is defined recursively by:

(a.) Smooth \( r \) times starting with initial guess \( u_p \):

\[
\tilde{u}_p = G_p^r(u_p, f_p)
\]  \hspace{1cm} (5.2a)

(b.) Compute the residual and transfer to the next coarser grid:

\[
\begin{align*}
  r_p &= f_p - B_p \tilde{u}_p, \quad f_{p-1} = I_p^{p-1} r_p
\end{align*}
\]  \hspace{1cm} (5.2b)

(c.) If \( p > 2 \) then return to (a.) to evaluate \( \tilde{u}_{p-1} \). If \( p = 2 \) then:

\[
\begin{align*}
  \tilde{u}_1 &= B_1^{-1} f_1
\end{align*}
\]  \hspace{1cm} (5.2c)

(d.) Add the coarse grid correction:

\[
\begin{align*}
  \tilde{u}_p &= \tilde{u}_p + I_p^{p-1} \tilde{u}_{p-1}
\end{align*}
\]  \hspace{1cm} (5.2d)
(e.) Smooth \( s \) times starting with initial guess \( \hat{u}_p \):

\[
\bar{u}_p = G_p^s(\hat{u}_p, f_p)
\]

(5.2e)

For the model problem analysis, we take \( \Omega_p, A_p, I_p^p, I_p^{p-1} \) and \( \tilde{G}_p \) as defined in Section 2.1.

5.2 Error Analysis

Bounds on the asymptotic convergence factors of the multigrid cycles \( M_{h,k,r,\omega} \) can be found in the following manner. Let \( \varepsilon_k = U_k - u_k \) be the initial error and \( \bar{\varepsilon}_k = \bar{U}_k - \bar{u}_k \) be the error after one multigrid cycle, where \( U_k \) satisfies \( A_k U_k = f_k \). In terms of the errors, definition (5.2) becomes:

(a) For \( p = k, k-1, \ldots, 2 \)

\[
\bar{\varepsilon}_p = G_p^{r} \varepsilon_p
\]

(b) For \( p = 1 \)

\[
\bar{\varepsilon}_1 = 0
\]

(c) For \( p = 2, \ldots, k \)

\[
\bar{\varepsilon}_p = \bar{\varepsilon}_p - I_p^{p-1}(\varepsilon_{p-1} - \bar{\varepsilon}_{p-1})
\]

Recall that \( \tilde{G}_p \) is the linear part of \( G_p \). If \( M^k \varepsilon_k = \bar{\varepsilon}_k \), then \( M^k \) is defined recursively by:

\[
M^p = \tilde{G}_p^r - I_p^{p-1}(I - M^{p-1})A_p^{-1}I_p^{p-1}G_p^r A_p, \quad 2 \leq p \leq k
\]

(5.3a)

\[
M^1 = 0
\]

(5.3b)

Note that the \( \alpha_j^{(k)} \) are eigenvectors of \( A_k \) and \( G_k \), but not of \( M^k \). Define

\[
S_i = \text{linearspan} \{ \alpha_j^{(k)} : j \sim i \}
\]

(5.4)

By formulas (2.12) and (2.13) we see that the \( S_i \) are orthogonal subspaces which are invariant under \( M^k \). Therefore a basis of eigenvectors, \( \{ v_{\mu} \} \), of \( M^k \) exists such that each \( v_{\mu} \) can be written as

\[
v_{\mu} = \sum_{j \sim i \sim 1} a_{j\mu} \alpha_j^{(k)}
\]

(5.5)
for some $i, |i| < N_k$, where $a_{jm} \in \mathbb{R}$. Since the $a_{i}^{(k)}$ are orthonormal with respect to the discrete $L_2$ inner product, then

$$\langle M^k v_\mu, v_\mu \rangle_k = \left( \sum_{j \sim i} (1) a_{jm} M^k a_j^{(k)} \right) \left( \sum_{m \sim i} (1) a_{nm} a_m^{(k)} \right)$$

$$= \sum_{m \sim i} (1) \sum_{j \sim i} (1) a_{jm} \langle M^k a_j^{(i)}, a_j^{(k)} \rangle_k = \lambda_\mu \sum_{n \sim i} (1) a_{nm}^2$$

(5.6)

where $\lambda_\mu$ is the eigenvalue of $M^k$ corresponding to $v_\mu$.

A bound on the $\lambda_\mu$'s will be a bound on the asymptotic convergence rate of the multigrid cycle. Let $M_i$ be the $4^{k-1} \times 4^{k-1}$ matrix with $(M_i)_{p,q} = (M^k a_{jp}^{(k)}, a_{jq}^{(k)})_k$ with $j_1, j_2, \cdots j_{4^{k-1}}$ some ordering of all the $j \sim i (1)$.

**Remark 5.1** Note that for some $i$'s, these $j_p$'s are not necessarily unique. For example, if $i = (N_k/2, 1)$ then $(N_k/2, 1) = (N_k - N_k/2, 1)$.

**Remark 5.2** The diagonal elements of $M_i$ are the Rayleigh quotients,

$$\frac{\langle M^k a_j^{(k)}, a_j^{(k)} \rangle_k}{\langle a_j^{(k)}, a_j^{(k)} \rangle_k}$$

and the off-diagonal elements are the contribution from the aliasing vectors.

By Gershgorin's theorem, any eigenvalue $\lambda$ of $M_i$ must satisfy

$$| \lambda - \langle M^k a_n^{(k)}, a_n^{(k)} \rangle_k | \leq \sum_{j \sim n (1) \neq n} | \langle M^k a_n^{(k)}, a_j^{(k)} \rangle_k |$$

(5.7)

for some $n \sim i (1)$. Therefore a bound on the asymptotic convergence rate, $\rho$, is given by

$$\rho \leq \max_{|i| < N_k} \left( \max_{n \sim i (1)} \sum_{j \sim n (1)} | \langle M^k a_n^{(k)}, a_j^{(k)} \rangle_k | \right)$$

$$= \max_{|i| < N_k} \sum_{j \sim i (1)} | \langle M^k a_i^{(k)}, a_j^{(k)} \rangle_k |$$

(5.8)

for the $k$-grid problem with meshsize $h_k = 1/N_k$ on the fine grid.

In section 5.3 we derive formulas for a bound on the righthand side of (5.8).
5.3 Derivation of Bounds on the Convergence Rate

For a fixed fine meshsize \( h \), a given number of grids \( k \), \( r \) smoothings and a damped Jacobi parameter \( \omega \), we derive formulas for a constant \( C_{k,r,h,\omega} < 1 \), independent of \( i \) which is a bound on the asymptotic convergence rate. In Section 5.4 we give values of these constants for various values of \( h \), \( k \) and \( r \) using a typical value of \( \omega \).

By (5.8) it is enough to bound \( \sum_{j \sim i} | \langle M^k \alpha^{(k)}_i, \alpha^{(k)}_j \rangle_k | \) independent of \( i \). Divide the sum into two parts,

\[
\sum_{j \sim i} | \langle M^k \alpha^{(k)}_i, \alpha^{(k)}_j \rangle_k | = | \langle M^k \alpha^{(k)}_i, \alpha^{(k)}_i \rangle_k | + \sum_{j \sim i (0)} | \langle M^k \alpha^{(k)}_i, \alpha^{(k)}_j \rangle_k | \\
=: D_i + J_i ,
\]

where \( D_i \) is the "diagonal part" and \( J_i \) is the "aliasing part" of the sum.

Let \( i = (i_1, i_2) \), \( k \), \( r \), \( h \) and \( \omega \) be fixed. Define

\[
\xi_p = \xi^{(p)}_{i_1} = \cos^2 \left( \frac{i_1 \pi h_p}{2} \right) , \quad (5.10a)
\]

\[
\eta_p = n^{(p)}_{i_2} = \cos^2 \left( \frac{i_2 \pi h_p}{2} \right) , \quad (5.10b)
\]

\[
g_p = \langle G^r \alpha^{(p)}_i, \alpha^{(p)}_i \rangle_p , \quad (5.10c)
\]

\[
e_p = \langle \left( \sum_{\sigma=0}^{r-1} G^\sigma \right) \alpha^{(p)}_i, \alpha^{(p)}_i \rangle_p, \quad (5.10d)
\]

and

\[
\nu_p = \nu^{(p)}_i, \quad (5.10e)
\]

where the \( i \), \( r \), \( h \) and \( \omega \) dependence has been suppressed in the notation and only the grid level is displayed.

We have the following theorem.
Theorem 5.1

\[ D_i = g_k - 2\omega c_k \nu_k \sum_{p=2}^{k-1} f_p \left( \prod_{m=p+1}^{k} 4g_m \xi_m \eta_m^2 \right) \frac{c_k \nu_k}{c_1 \nu_1} \left( \prod_{m=2}^{k} 4g_m \xi_m \eta_m^2 \right). \]  

(5.11a)

and

\[ J_i \leq 2\omega c_k \nu_k \sum_{p=2}^{k-1} f_p \left( 1 - \left( \prod_{m=p+1}^{k} \xi_m \eta_m \right) \right) \left( \prod_{m=p+1}^{k} 4 | g_m | \xi_m \eta_m \right) \]  

(5.11b)

\[ + \frac{c_k \nu_k}{c_1 \nu_1} \left( 1 - \left( \prod_{m=2}^{k} \xi_m \eta_m \right) \right) \left( \prod_{m=2}^{k} 4 | g_m | \xi_m \eta_m \right). \]

Remark 5.3 Theorem 5.1 allows us to obtain a bound on the asymptotic convergence rate that is no more complicated than the diagonal elements themselves.

Before proving Theorem 5.1 we find expressions for the inner products

\[ (M^k \alpha_i^{(k)}, \alpha_j^{(k)})_k. \]

Lemma 5.1

For any \( j \sim i \ (1) \),

\[ (M^k \alpha_i^{(k)}, \alpha_j^{(k)})_k = g_k \langle \alpha_i^{(k)}, \alpha_j^{(k)} \rangle_k - 2\omega c_k \nu_k \sum_{p=2}^{k-1} f_p \left( \prod_{m=p+1}^{k} 4g_m \xi_m \eta_m \right) \langle \alpha_i^{(p)}, I_k^{(p)} \alpha_j^{(k)} \rangle p 
\]

\[ - \frac{c_k \nu_k}{c_1 \nu_1} \left( \prod_{m=1}^{k} 4g_m \xi_m \eta_m \right) \langle \alpha_i^{(1)}, I_k^{(1)} \alpha_j^{(k)} \rangle_1. \]  

(5.12a)
Proof of Lemma 5.1

We prove by induction that for every $s \leq k$.

$$
\langle M^s \alpha_i^{(s)}, \alpha_j^{(s)} \rangle_s = g_s(\alpha_i^{(s)}, \alpha_j^{(s)})_s - 2\omega c_s \nu_s \sum_{p=2}^{s-1} \left( \prod_{m=p+1}^{s} 4g_m \xi_m \eta_m \right) \langle \alpha_i^{(p)}, \alpha_j^{(s)} \rangle_p
$$

$$
- \frac{c_s \nu_s}{c_1 \nu_1} \left( \prod_{m=1}^{s} 4g_m \xi_m \eta_m \right) \langle \alpha_i^{(1)}, \alpha_j^{(s)} \rangle_1. 
$$

(5.12b)

Taking $s = k$ gives (5.12a).

We start with $s = 2$. From (5.3), (5.10) and (2.12),

$$
\langle M^2 \alpha_i^{(2)}, \alpha_j^{(2)} \rangle_2 = \langle G_s^2 \alpha_i^{(1)}, \alpha_j^{(2)} \rangle_2 - \langle A_1^{-1} I_s^1 G_s^2 A_2 \alpha_i^{(2)}, \alpha_j^{(2)} \rangle_1
$$

$$
= g_2(\alpha_i^{(2)}, \alpha_j^{(2)})_2 - \frac{\nu_2}{\nu_1} g_2(\alpha_i^{(1)}, \alpha_j^{(2)})_1.
$$

(5.13)

Using $4c_2 = c_1$ gives us (5.12a) for $k = 2$.

Assume (5.12a) is true for $k = s - 1$ grids, $s \geq 3$. For the $s$-grid problem, (5.3), (5.10) and (2.12) give

$$
\langle M^s \alpha_i^{(s)}, \alpha_j^{(s)} \rangle_s = \langle G_s^s \alpha_i^{(s)}, \alpha_j^{(s)} \rangle_s - \langle (I - M^{s-1}) A_{s-1}^{-1} I_{s-1}^s G_s^s A_s \alpha_i^{(s)}, \alpha_j^{(s)} \rangle_{s-1}
$$

$$
= g_s - \frac{\nu_s}{\nu_{s-1}} \xi_s \eta_s g_s(\alpha_i^{(s-1)}, I_{s-1}^s \alpha_j^{(s)})_{s-1}
$$

$$
+ \frac{\nu_s}{\nu_{s-1}} \xi_s \eta_s g_s(M^{s-1} \alpha_i^{(s-1)}, I_{s-1}^s \alpha_j^{(s)})_{s-1}. 
$$

(5.14)

We factor $1 - g_{s-1} = 2\omega c_{s-1} \nu_{s-1} f_{s-1}$. Using the inductive hypothesis and using $4c_s = c_{s-1}$ finishes the proof.

Proof of Theorem 5.1

(a) Using Lemma 5.1 with $j = 1$, (5.10c) and (2.12) proves (5.11a).

(b) To prove (5.11b), split the grid levels by partitioning the $j \sim \tau(i)$, $j \neq \tau(i)$. See Figure 3.1 for a schematic illustration for $k = 3$. For each $n = 1, \ldots, k - 1$ consider the $j$'s such that $j \sim \tau(n)$ but $j \neq \tau(n + 1)$ Lemma 3.1, Lemma 2.1 and Lemma
2.3 show that

\[ J_i = \sum_{n=1}^{k-1} \sum_{j=1}^{n} | (M^k \alpha_i^{(k)}, \alpha_j^{(k)})_k | \]

\[ \leq 2 \omega c_k \nu_k \sum_{n=1}^{k-1} (1 - \xi_{n+1} \eta_{n+1}) \sum_{p=1}^{n} f_p \left( \prod_{m=p+1}^{n} \xi_m \eta_m \right) \left( \prod_{m=p+1}^{k} 4 | g_m | \xi_m \eta_m \right) \]

\[ + \frac{c_k \nu_k}{c_1 \nu_1} \left[ \sum_{n=1}^{k-1} (1 - \xi_{n+1} \eta_{n+1}) \left( \prod_{m=p+1}^{n} \xi_m \eta_m \right) \left( \prod_{m=p+1}^{k} 4 | g_m | \xi_m \eta_m \right) \right] \]

Changing the order of summation gives

\[ J_i \leq 2 \omega c_k \nu_k \sum_{p=2}^{k-1} f_p \left[ \sum_{n=p}^{k-1} (1 - \xi_{n+1}) \left( \prod_{m=p+1}^{n} \xi_m \eta_m \right) \right] \left( \prod_{m=p+1}^{k} 4 | g_m | \xi_m \eta_m \right) \]

\[ + \frac{c_k \nu_k}{c_1 \nu_1} \left[ \sum_{n=1}^{k-1} (1 - \xi_{n+1} \eta_{n+1}) \left( \prod_{m=p+1}^{n} \xi_m \eta_m \right) \right] \left( \prod_{m=p+1}^{k} 4 | g_m | \xi_m \eta_m \right) . \]

The quantities in the square brackets in (5.16) equal

\[ 1 - \prod_{m=p+1}^{k} \xi_m \eta_m \]

and

\[ 1 - \prod_{m=1}^{k} \xi_m \eta_m \]

respectively, and therefore (5.11b) has been proved.

We use this theorem to find bounds on the multigrid V-cycle asymptotic convergence rate for the k-grid problem with a given damped Jacobi parameter \( \omega \) and \( r \) iterations per smooth. The results are given in the next section.

### 5.4 Computed values of the asymptotic convergence bounds

Ideally, one would be able to compute k-grid convergence bounds independent of \( h \). The \( 4^{k-1} \times 4^{k-1} \) matrix, \( M_1 \), can be written as a \( 4^{k-1} \times 4^{k-1} \) matrix, \( M(\xi, \eta) \), with variable entries depending on the continuous variables \( \xi \) and \( \eta \in (0, 1) \) evaluated at \( \xi = \xi_i^{(k)} \) and \( \eta = \eta_i^{(k)} \).
In the two grid case one could get an analytic formula for the characteristic equation of \( \mathcal{M}(\xi, \eta) \) (a polynomial of degree 4 for fixed \( \xi, \eta \)), find analytic expressions for the eigenvalues and then find the supremum of these expressions over all \( \xi \) and \( \eta \) in the unit square. This would give an exact 2-grid asymptotic convergence rate independent of \( h \). In practice this is too much work even in the simple 2-grid case. Instead, one chooses a value of \( h = 1/N \) and computes the spectral radii of \( \mathcal{M}_i \) for each \( i \), \( |i| < N \), keeping track of the largest. One then repeats the procedure for different values of \( h \) and so constructs a table as in [12] see Table 5.1. From such tables one can predict the \( h \)-independent convergence rates.

In the \( k \) grid problem, \( k > 2 \) each \( \mathcal{M}_i \) is a \( 4^{k-1} \times 4^{k-1} \) matrix and therefore computing the spectral radius for each \( i \), \( |i| < N \) is expensive, especially for small \( h \). We therefore use Theorem 5.1 and Gershgorin's Theorem to compute a bound on the spectral radius of \( \mathcal{M}_i \), for each \( i \). This amounts to roughly twice the work of just evaluating the diagonal elements.

The sharpest bounds on the asymptotic convergence rates for the analysis of the V-cycle are obtained by these techniques when no smoothing is performed on the coarse-to-fine part of the cycle, i.e., \( s = 0 \) in step d. This is called an \( M \backslash \) cycle. The symmetric cycle, i.e., \( s = r \), is called an \( MG \) cycle. We consider two discretizations of the Laplacian, the five point discretization, \( B_p = A_p \), as given in Section 2, and a certain nine point discretization given by the following stencil:

\[
\hat{A}_p = \frac{1}{3h_p^2} \begin{bmatrix}
-1 & -1 & -1 \\
-1 & +8 & -1 \\
-1 & -1 & -1
\end{bmatrix}
\]  \hspace{2cm} (5.17)

The corresponding V-cycles will be denoted by, e.g., \( M_5 \backslash \) or \( MG_9 \), to indicated which discretization is being used.

We consider a \( M_5 \backslash \) algorithm and compare our theoretical bounds to the experimentally observed asymptotic convergence rates. In order to compare our two grid bounds to the exact two grid convergence rates obtained by the model problem analysis in [8], we consider a damped Jacobi parameter \( \omega = 4/5 \). Experimentally, this is a good choice, though its optimality depends on the number of smoothings and the number of grids. We take \( r = 1, 2, 3 \) or 4 smoothings (smoothing only from fine to coarse meshes). Tables 5.2-5.5 show the convergence bounds for commonly used meshsizes. Table 5.6 indicates the
limiting behaviour of these rates for very small $h$ and large number of grids. The experimentally observed asymptotic convergence rates are shown in Table 5.7 for $r = 1, 2, 3, 4$, $\omega = 4/5$ and $h = 1/64$. For exact two grid convergence rates, see Table 5.1.

In practice, as $k$ increases there is not as much degradation in the convergence rate as Tables 5.1-5.7 would indicate.

We compare our bounds to the finite element bounds of [8], using the $MG_9$ cycle given by taking $B_p = \tilde{A}_p$ and $s = r$. The comparison is possible because the operators $\tilde{A}_p$ satisfy:

$$\tilde{A}_{p-1} = I_p^{p-1} \tilde{A}_p I_p^{p-1} \text{ for } p = 1, 2, \ldots, k.$$  \hspace{1cm} (5.18)

Eigenvectors of $A_p$ are also eigenvectors of $\tilde{A}_p$. We also note that for a symmetric V-cycle, convergence bounds in the energy norm are equivalent to asymptotic convergence bounds given by the spectral radius. Our bounds are given in Table 5.8 for $\omega = 3/4$, $h = 1/64$, and $r = 1, 2, 3, 4$. In the next to the last column of Table 5.8 we show the bounds (which are independent of the number of grids used) obtained by the methods of [8]. We also calculate the exact two grid convergence rates for $MG_9$, as in [12]. These numbers are given in the last column of Table 5.8. In this symmetric case, at least for small $r$, our bounds are larger than the finite element bounds because in the Fourier analysis we essentially throw away the post smoothing factors in the off-diagonal terms in order to be able to apply Lemma 5.1.

---

Table 5.1  

<table>
<thead>
<tr>
<th></th>
<th>$r = 1$</th>
<th>$r = 2$</th>
<th>$r = 3$</th>
<th>$r = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/16</td>
<td>.592</td>
<td>.351</td>
<td>.208</td>
<td>.135</td>
</tr>
<tr>
<td>1/32</td>
<td>.598</td>
<td>.358</td>
<td>.214</td>
<td>.137</td>
</tr>
<tr>
<td>1/64</td>
<td>.600</td>
<td>.359</td>
<td>.216</td>
<td>.137</td>
</tr>
<tr>
<td>1/128</td>
<td>.600</td>
<td>.360</td>
<td>.216</td>
<td>.137</td>
</tr>
</tbody>
</table>
Table 5.2 \( M_5 \)\ Asymptotic convergence bounds \( \omega = .8 \), \( r = 1 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.615</td>
<td>.719</td>
<td>.715</td>
<td>.750</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.622</td>
<td>.749</td>
<td>.769</td>
<td>.800</td>
<td>.787</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.624</td>
<td>.758</td>
<td>.797</td>
<td>.826</td>
<td>.820</td>
<td>.815</td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.625</td>
<td>.760</td>
<td>.808</td>
<td>.835</td>
<td>.830</td>
<td>.828</td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.625</td>
<td>.761</td>
<td>.812</td>
<td>.835</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.3 \( M_5 \)\ Asymptotic convergence bounds \( \omega = .8 \), \( r = 2 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
<th>8 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.369</td>
<td>.454</td>
<td>.455</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.370</td>
<td>.460</td>
<td>.481</td>
<td>.481</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.370</td>
<td>.466</td>
<td>.490</td>
<td>.491</td>
<td>.491</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.370</td>
<td>.467</td>
<td>.495</td>
<td>.499</td>
<td>.500</td>
<td>.499</td>
<td></td>
</tr>
<tr>
<td>1/256</td>
<td>.370</td>
<td>.468</td>
<td>.495</td>
<td>.502</td>
<td>.505</td>
<td>.505</td>
<td>.504</td>
</tr>
</tbody>
</table>

Table 5.4 \( M_5 \)\ Asymptotic convergence bounds \( \omega = .8 \), \( r = 3 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.274</td>
<td>.348</td>
<td>.367</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.274</td>
<td>.348</td>
<td>.367</td>
<td>.372</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.275</td>
<td>.350</td>
<td>.370</td>
<td>.372</td>
<td>.373</td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.275</td>
<td>.350</td>
<td>.371</td>
<td>.376</td>
<td>.376</td>
<td>.376</td>
</tr>
</tbody>
</table>
Table 5.5 \( M_5 \)

Asymptotic convergence bounds \( \omega = .8, r = 4 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>7 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/16</td>
<td>.220</td>
<td>.284</td>
<td>.302</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/32</td>
<td>.221</td>
<td>.284</td>
<td>.302</td>
<td>.307</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/64</td>
<td>.221</td>
<td>.284</td>
<td>.302</td>
<td>.307</td>
<td>.308</td>
<td></td>
</tr>
<tr>
<td>1/128</td>
<td>.221</td>
<td>.284</td>
<td>.302</td>
<td>.307</td>
<td>.308</td>
<td>.309</td>
</tr>
</tbody>
</table>

Table 5.6 \( M_5 \)

Asymptotic convergence bounds for small \( h \)

\( \omega = .8 \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>( r = 1 )</th>
<th>( r = 2 )</th>
<th>( r = 3 )</th>
<th>( r = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2048</td>
<td>.843</td>
<td>.5105</td>
<td>.37779</td>
<td>.3087905</td>
</tr>
<tr>
<td>11 grids</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/4096</td>
<td>.846</td>
<td>.5111</td>
<td>.37777</td>
<td>.3087916</td>
</tr>
<tr>
<td>12 grids</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 5.7 \( M_5 \) Experimental asymptotic convergence rates

\[ \omega = .8, \quad h = 1/64 \]

<table>
<thead>
<tr>
<th>( r )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.600</td>
<td>.600</td>
<td>.600</td>
<td>.600</td>
<td>.600</td>
</tr>
<tr>
<td>2</td>
<td>.360</td>
<td>.360</td>
<td>.360</td>
<td>.360</td>
<td>.360</td>
</tr>
<tr>
<td>3</td>
<td>.216</td>
<td>.228</td>
<td>.233</td>
<td>.242</td>
<td>.246</td>
</tr>
<tr>
<td>4</td>
<td>.137</td>
<td>.158</td>
<td>.171</td>
<td>.181</td>
<td>.193</td>
</tr>
</tbody>
</table>

Table 5.8 \( MG_3 \) A comparison of the theoretical bounds

\[ \omega = .75, \quad h = 1/64 \]

<table>
<thead>
<tr>
<th>( r )</th>
<th>2 grids</th>
<th>3 grids</th>
<th>4 grids</th>
<th>5 grids</th>
<th>6 grids</th>
<th>bounds from [8]</th>
<th>exact 2 grid conv. rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.686</td>
<td>.717</td>
<td>.816</td>
<td>.860</td>
<td>.879</td>
<td>.40</td>
<td>.249</td>
</tr>
<tr>
<td>2</td>
<td>.275</td>
<td>.299</td>
<td>.348</td>
<td>.362</td>
<td>.364</td>
<td>.25</td>
<td>.067</td>
</tr>
<tr>
<td>3</td>
<td>.121</td>
<td>.147</td>
<td>.161</td>
<td>.162</td>
<td>.162</td>
<td>.18</td>
<td>.040</td>
</tr>
</tbody>
</table>
APPENDIX

A.1 Proof of Theorem 3.4

Fix \( i = (i_1, i_2), h, k, r \) and \( \omega \) as in Section 3.3. Define \( \xi, \eta, \gamma, e, \) and \( \nu \) as is (3.16a-e). As seen in the proof of Lemma 3.1,

\[
D_i^{(p)} = \langle M_p \alpha_i^{(p)}, \alpha_i^{(p)} \rangle_p = \langle (I - G^2 r) A_p^{-1} \alpha_i^{(p)}, \alpha_i^{(p)} \rangle_p + \langle M_{p-1} I_p^{-1} G^r \alpha_i^{(p)}, I_p^{-1} G^r \alpha_i^{(p)} \rangle_{p-1}.
\] (A.1)

Therefore a recursion formula for \( D_i^{(p)} \) is

\[
D_i^{(p)} = a_p + b_p D_i^{(p-1)} \quad p = 1, 2, \ldots, k
\] (A.2)

where \( a_p \) and \( b_p \) are given by:

\[
a_p = 2 \omega c_p e_p \quad p = 1, \ldots, k
\] (A.3a)

\[
b_p = g_p^2 c_p^2 \alpha_i^{(p-1)}(p-1), \alpha_i^{(p-1)}(p-1)_{p-1} \quad p = 1, \ldots, k
\] (A.3b)

\[
b_0 = 0.
\] (A.3c)

The following four lemmas are all proved by direct calculation.

Lemma A.1

For each \( p = 1, 2, \ldots, k \)

a.) \( a_p \leq 4 \omega c_p \) \hspace{1cm} (A.4a)

and b.) \( a_p \geq 4 \omega (1 - \omega) c_p \) \hspace{1cm} (A.4b)

Lemma A.2

For each \( p = 2, \ldots, k \)

a.) \( b_p \leq 1 \) \hspace{1cm} (A.5a)
and if $c_p \nu_p \leq 1/4$,

b.) $b_p \geq (1 - 4(1 + \omega)c_p \nu_p)$. \hfill (A.5b)

Lemma A.3

For each $p = 2, \ldots, k$

a.) $\nu_p/\nu_{p-1} \leq \frac{1}{\xi_p \eta_p}$ \hfill (A.6a)

and b.) $\nu_p/\nu_{p-1} \geq 1$. \hfill (A.6b)

Lemma A.4

If $\frac{\beta}{4^{\alpha+1}} \leq c_p \nu_p \leq \frac{\beta}{4^\alpha}$, $p = 2, \ldots, k$ and $\frac{1}{2} < \beta < 2$, then

a.) $c_{p-n} \nu_{p-n} \leq \frac{\beta}{4^{\alpha-n}}$ \hfill (A.7a)

and b.) $c_{p-n} \nu_{p-n} \geq \frac{\beta}{4^{\alpha-n+1}} \left(1 - \frac{2 \beta}{3 \cdot 4^{\alpha-n+1}}\right)$. \hfill (A.7b)

Proof of Lemma A.1

Inequality (A.4a) follows immediately from the inequality

$$1 - (1 - x)^{2r} \leq 2rx$$ \hfill (A.8)

since $|1 - x| \leq 1$ where $x = 2\omega c_p \nu_p$.

Using the inequality

$$1 - (1 - x)^{2r} \geq x(2 - x)$$ \hfill (A.9)

for all $x$ such that $|1 - x| \leq 1$,

it is clear that

$$a_p \geq 2\omega c_p (2 - 2\omega c_p \nu_p),$$ \hfill (A.10)

from which follows (A.4b) since $0 < c_p \nu_p < 1$. \hfill \blacksquare
Proof of Lemma A.2

Since

\[ c_p \nu_p = \frac{2 - \xi_p - \eta_p}{2}, \]  
(A.11)

where \(0 < \xi_p, \eta_p < 1\) and \(|g_p| = |1 - 2w_c p| < 1\), (A.5a) is obvious.

If \(c_p \nu_p \leq 1/4\), then \((1 - \xi_p)\) and \((1 - \eta_p) < 1/2\) and therefore

\[ \langle \alpha_i^{(p-1)}, \alpha_i^{(p-1)} \rangle_{p-1} = 1. \]

Moreover,

\[ \xi_p^2 \eta_p^2 \geq 1 - 4c_p \nu_p. \]  
(A.12)

It is also clear that \((1 - 2w_c p \nu_p)^{2r} \geq 1 - 4w_c c_p \nu_p\). Combining these inequalities gives

\[ b_p \geq (1 - 4w_c c_p \nu_p)(1 - 4c_p \nu_p) \]  
(A.13)

\[ \geq 1 - 4(1 + \omega)c_p \nu_p. \]

Proof of Lemma A.3

Since \(0 < \xi_p, \eta_p < 1\),

\[ -\xi_p (1 - \xi_p) (1 - \eta_p) - \eta_p (1 - \xi_p) (1 - \eta_p) \leq 0. \]  
(A.14)

Factoring the lefthand side gives

\[ \xi_p \eta_p (2 - \xi_p - \eta_p) \leq \xi_p (1 - \xi_p) + \eta_p (1 - \eta_p). \]  
(A.15)

Recall that

\[ \nu_p = \frac{4(2 - \xi_p - \eta_p)}{h_{p-1}^2} \]

and

\[ \nu_{p-1} = \frac{4(2 - \xi_{p-1} - \eta_{p-1})}{h_{p-1}^2} = \frac{4(\xi_p(1 - \xi_p) + \eta_p(1 - \eta_p))}{h_p^2}. \]  
(A.16)

Thus by (A.15)

\[ \frac{\nu_p}{\nu_{p-1}} \leq \frac{1}{\xi_p \eta_p}. \]

The second inequality, (A.6b), is clear since

\[ \frac{\nu_p}{\nu_{p-1}} = \frac{(1 - \xi_p) + (1 - \eta_p)}{\xi_p (1 - \xi_p) + \eta_p (1 - \eta_p)} > 1 \]  
(A.17)

and \(0 < \xi_p, \eta_p < 1\).
Proof of Lemma A.4

If
\[ \gamma \leq c_p \nu_p \leq \tau \leq 1/4 \]  \hspace{1cm} (A.18a)

then
\[ 4\gamma(1 - 2\gamma) \leq c_{p-1} \nu_{p-1} \leq 4\tau(1 - 2\tau). \]  \hspace{1cm} (A.18b)

Note that this is just the calculus problem: Find the maximum and minimum of \( f(x, y) = x(1-x) + y(1-y) \) in \( \Omega = \{(x, y): 2\gamma \leq x + y \leq 2\tau, x \geq 0, y \geq 0\} \), and the solution is straightforward.

By induction, it is easy to see that
\[
\frac{\beta}{4^{\alpha-n+1}} \prod_{j=0}^{n-1} \left(1 - \frac{2\beta}{4^{\alpha+1-j}}\right) \leq c_{p-n} \nu_{p-n} \leq \frac{\beta}{4^{\alpha-n}}. \]  \hspace{1cm} (A.19)

By (A.18) this is true for \( n = 1 \), i.e.
\[
\frac{\beta}{4^\alpha} \left(1 - \frac{2\beta}{4^{\alpha+1}}\right) \leq c_{p-1} \nu_{p-1} \leq \frac{\beta}{4^\alpha} \left(1 - \frac{2\beta}{4^\alpha}\right) \leq \frac{\beta}{4^{\alpha-1}}. \]  \hspace{1cm} (A.20)

Assume (A.19) is true for \( c_{p-n+1} \nu_{p-n+1} \), then
\[
c_{p-n} \nu_{p-n} \geq \frac{4^\alpha \beta}{4^{\alpha-n+2}} \prod_{j=0}^{n-2} \left(1 - \frac{2\beta}{4^{\alpha+1-j}}\right) \left[1 - \frac{2\beta}{4^{\alpha-n+2}} \prod_{j=0}^{n-2} \left(1 - \frac{2\beta}{4^{\alpha+1-j}}\right)\right]
\]
\[
\geq \frac{\beta}{4^{\alpha-n+1}} \left(\prod_{j=0}^{n-2} \left(1 - \frac{2\beta}{4^{\alpha+1-j}}\right)\right) \left(1 - \frac{2\beta}{4^{\alpha-n+2}}\right) \]  \hspace{1cm} (A.21)
\[= \frac{\beta}{4^{\alpha-n+1}} \prod_{j=0}^{n-1} \left(1 - \frac{2\beta}{4^{\alpha+1-j}}\right). \]

Using \( \prod_{j=0}^{n} (1 - x_j) \geq 1 - \sum_{j=0}^{n} x_j \) gives
\[
\prod_{j=0}^{n-1} \left(1 - \frac{2}{4^{\alpha+1-j}}\right) \geq 1 - \frac{2}{4^{\alpha-n+2}} \left(\sum_{j=0}^{n-1} \frac{1}{4^j}\right) \]  \hspace{1cm} (A.22)
\[
\geq 1 - \frac{2}{3} \cdot \frac{1}{4^{\alpha-n+1}}. \]
The upper bound for $c_{p-n}u_{p-n}$ is obvious.

Proof of Theorem 3.4 part a.

a.) \[ \frac{\omega(1 - \omega)}{\max(2, d(1 + r\omega))} h_1^2 \leq D_i^{(k)} \leq \frac{2r\omega}{3} h_1^2, \quad \text{for } v_k \leq \frac{d}{h_1^2}. \]

From lemmas A.1a and A.2a, $p = k, k - 1, \ldots, 2$,

\[ D_i^{(k)} \leq 4r\omega \left( \sum_{p=2}^{k} c_p \right) + D_i^{(1)}. \quad \text{(A.23)} \]

On the coarsest grid, $D_i^{(1)} \leq 4r\omega c_1$. Hence by the definition of the $c_p$'s (2.15)

\[ D_i^{(k)} \leq 4r\omega \left( \sum_{p=1}^{k} c_p \right) \leq \frac{16r\omega}{3} c_1. \quad \text{(A.24)} \]

Using $c_1 = h_1^2/8$ gives the upper bound

\[ D_i^{(k)} \leq \frac{2}{3} r\omega h_1^2. \quad \text{(A.25)} \]

To get the lower bound, use an induction argument. By lemma A.16,

\[ D_i^{(1)} \geq 4\omega (1 - \omega) c_1 = \frac{\omega(1 - \omega)h_1^2}{2}. \quad \text{(A.26)} \]

Let $1 \leq p < k$ and assume that

\[ D_i^{(p)} \geq \frac{\omega(1 - \omega)h_1^2}{\max(2, d(1 + r\omega))}. \quad \text{(A.27)} \]

By rearranging terms, using lemmas A.1b and A.2b (which can be used since $v_k < d/h_1^2$ implies $c_p\nu_p \leq d/(8 \cdot 4^p) \leq 1/4$ by lemma A.4a) it is seen that:

\[ D_i^{(p+1)} \geq \frac{\omega(1 - \omega)h_1^2}{\max(2, d(1 + r\omega))} + 4\omega(1 - \omega)c_{p+1} \left( 1 - \frac{(1 + r\omega)\nu_{p+1} h_1^2}{\max(2, d(1 + r\omega))} \right). \quad \text{(A.28)} \]

Lemma A.3b guarantees that $\nu_p \leq v_k \leq \frac{d}{h_1^2}$ and therefore the last term in (A.28) is positive and can be thrown out. This proves part a.) of Theorem 3.4.

55
Proof of Theorem 3.4 part b.

\[ \frac{d \omega (1 - \omega)}{8(1 + r \omega) \nu_k} \leq D_i^{(k)} \leq \frac{1}{\nu_k} \quad \text{for } \nu_i^{(k)} > \frac{d}{h_1^2}. \]

Using the definition of \( c_k, \nu_k > d/h_1^2 \) implies \( c_k \nu_k \geq d/(2 \cdot 4^{k+1}) \). For each \( p_1 \), \( \lambda_{i}^{(p_1)} \leq 1 \). To see this, first note that

\[ \lambda_{i}^{(1)} = 1 - (1 - 2 \omega \nu_1)^{2r} \leq 1. \] (A.29)

Lemma A.3a together with the definition of \( b_p \) imply

\[ b_p \leq \frac{(1 - 2 \omega \nu_p)^{2r} \nu_{p-1}}{\nu_p}. \] (A.30)

Combining (A.29) and (A.30) with the definition of \( a_p \), gives \( D_i^{(p)} \leq 1/\nu_p \).

For the lower bound, divide the argument into two cases. Define

\[ \gamma = \lfloor \log_4 2(1 + r \omega) \rfloor \] (A.31)

where \( \lfloor x \rfloor \) means the greatest integer in \( x \).

**case 1** \( c_k \nu_k \geq \frac{d}{2 \cdot 4^\gamma} \) (A.32)

**case 2** \( \frac{d}{2 \cdot 4^\alpha+1} \leq c_k \nu_k \leq 1 \) for some integer \( \alpha, \gamma \leq \alpha \leq k \). (A.33)

By the definition of \( \gamma \),

\[ \frac{1}{2}(1 + r \omega) \leq 4^\gamma \leq 2(1 + r \omega). \] (A.34)

For **case 1**, Lemma A.1b gives

\[ a_k \geq \frac{4 \omega (1 - \omega)}{\nu_k} \frac{d}{2 \cdot 4^\gamma} \geq \frac{d \omega (1 - \omega)}{(1 + r \omega) \nu_k}. \] (A.35)

For **case 2**, look at the finest grid on which the eigenvector is "rough enough." For \( \hat{p} = k - \alpha + \gamma \), lemma A.4 shows that

\[ \frac{d}{2 \cdot 4^\gamma+1} \left( 1 - \frac{d}{3 \cdot 4^\gamma+1} \right) \leq c_p \nu_p \leq \frac{d}{2 \cdot 4^\gamma} \] (A.36)

56
Therefore on $\Omega^\rho$,

$$D_i^{(\rho)} \geq a_\rho \geq 4\omega(1 - \omega)c_\rho \geq \frac{d(1 - \omega)\omega}{8(1 + r\omega)\nu_\rho}. \quad (A.37)$$

Now this information needs to get back to the fine grid, $\Omega^k$. On $\Omega^\rho$, for $p > \hat{p}$, lemma A.4 says

$$\frac{d}{2 \cdot 4^{a-k+p+1}} \left(1 - \frac{d}{3 \cdot 4^{a-k+p+1}}\right) \leq c_p \nu_p \leq \frac{d}{2 \cdot 4^{a-k+p}}. \quad (A.38)$$

Now using lemmas A.16, A.26 and A.36 and rearranging terms,

$$D_i^{(\rho)} \geq 4\omega(1 - \omega)c_p + (1 - 4(1 + r\omega)c_p\nu_p) \frac{\omega(1 - \omega)}{8(1 + r\omega)\nu_p}$$

$$= \frac{d(1 - \omega)\omega}{8(1 + r\omega)\nu_p} + 4\omega(1 - \omega)c_p \left[1 - \frac{d}{8}\right] \quad (A.39)$$

$$\geq \frac{d(1 - \omega)\omega}{8(1 + r\omega)\nu_p}.$$

Since this is true for all $p > \hat{p}$, take $p = k$. 

57
References


