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ON THE LEAST SQUARES ESTIMATOR IN MOVING AVERAGE MODELS OF ORDER ONE

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Keywords: Moving average model; Least squares estimator; Consistency

Language
Fortran 77

Description and purpose
Given a time series data, estimate $\theta$ in the moving average model

$$Y_t = e_t + \theta e_{t-1}, \quad t = 1, 2, 3, \ldots$$

of order one using the method of least squares.

Theory
Let $Y_t = e_t + \theta e_{t-1}, \quad t = 1, 2, 3, \ldots$ be a moving average process of order one, where $e_0 = 0$ and $e_1, e_2, \ldots$ is a sequence of independent identically distributed random variables with mean zero and variance $\sigma^2$. There are several methods of estimation available in the literature for estimating $\theta$ for a given time series data $y_1, y_2, \ldots, y_N$. McClave (1974) compared the performance of some five estimators of $\theta$ for moderate sample sizes by simulating the process $Y_t, \quad t = 1, 2, \ldots, 100$. All these methods use approximations of one kind or the other and involve selection of some indices for a good degree of approximation. Consequently, the execution of these methods looks complex and it is natural to search for a simple method which performs competitively well with these methods. The method of least square is intuitively very appealing, and recently, under some conditions, Macpherson and Fuller (1983) showed that the least squares
estimator is consistent for $\theta$ in $[-1,1]$. We have compared the performance of the least squares estimator for moderate sample sizes vis-a-vis with the five estimators examined by McClave (1974). See Niroomand Chapeh and Bhaskara Rao (1987). The least squares estimator comes out better than these five estimators. Additionally, as the present article demonstrates, the execution of the least squares method is much simpler than these five methods.

Following Macpherson and Fuller (1983), the least squares estimator of $\theta$ is that value of $\theta$ in $[-1,1]$ that minimizes

$$Q_N(\theta) = \sum_{t=1}^{N} [e_t(Y;\theta)]^2,$$

where $e_t(Y;\theta) = Y_t - \theta e_{t-1}(Y;\theta)$, $t = 1, 2, \ldots, N$ and $e_0(Y;\theta) = 0$. It works out that

$$e_k(Y;\theta) = Y_k - \theta Y_{k-1} + \theta^2 Y_{k-2} - \ldots + (-1)^{k-1} \theta^{k-1} Y_1,$$

$k = 1, 2, \ldots, N$. Consequently, $Q_N(\theta)$ is a polynomial in $\theta$ of degree $2N-2$. More explicitly, for $N$ even,

$$Q_N(\theta) = \sum_{i=1}^{N} Y_i^2 + \left(\sum_{i=1}^{N-1} Y_i^2 + 2 \sum_{i=1}^{N-2} Y_i Y_{i+2}\right)\theta^2 + \ldots + \left(\sum_{i=1}^{N/2+1} Y_i^2 + \sum_{i=1}^{N/2-1} Y_i Y_{i+2}\right)\theta^{N-2} + \left(\sum_{i=1}^{N/2} Y_i^2 + \sum_{i=1}^{N/2-2} Y_i Y_{i+2}\right)\theta^{N} + \left(\sum_{i=1}^{N/2-1} Y_i^2 + \sum_{i=1}^{N/2-2} Y_i Y_{i+2}\right)\theta^{N+2} + \ldots + \left(\sum_{i=1}^{N/2-1} Y_i^2 + \sum_{i=1}^{N/2-2} Y_i Y_{i+2}\right)\theta^{2N-2} - 2(\sum_{i=1}^{N-1} Y_i Y_{i+1})\theta.$$
-3-

\[ -2(\Sigma_{i=1}^{N-2} Y_1 Y_{i+1} + \Sigma_{i=1}^{N-3} Y_1 Y_{i+3}) \theta^3 - \ldots \]

\[ -2(\Sigma_{i=1}^{N/2} Y_1 Y_{i+1} + \Sigma_{i=1}^{N/2-1} Y_1 Y_{i+3} + \ldots + \Sigma_{i=1}^{1} Y_1 Y_{i+N-1}) \theta^{N-1} \]

\[ -2(\Sigma_{i=1}^{N/2-1} Y_1 Y_{i+1} + \Sigma_{i=1}^{N/2-2} Y_1 Y_{i+3} + \ldots + \Sigma_{i=1}^{1} Y_1 Y_{i+N-3}) \theta^{N+1} \]

\[ \ldots - 2Y_1 Y_2^3 \theta^{2N-3}. \]

The above expression looks formidable to include in a computer program. However, the above can be simplified as follows. Let \( A \) be the matrix of order \( N \times N \) defined by

\[
\begin{bmatrix}
Y_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
Y_2 & -Y_1 & 0 & 0 & \ldots & 0 & 0 \\
Y_3 & -Y_2 & Y_1 & 0 & \ldots & 0 & 0 \\
Y_4 & -Y_3 & Y_2 & -Y_1 & \ldots & 0 & 0 \\
& & & & & & \\
& & & & & & \\
Y_{N-1} & -Y_{N-2} & Y_{N-3} & -Y_{N-4} & \ldots & (-1)^{N-2} Y_1 & 0 \\
Y_N & -Y_{N-1} & Y_{N-2} & -Y_{N-3} & \ldots & (-1)^{N-2} Y_2 & (-1)^{N-1} Y_1 \\
\end{bmatrix}
\]

Let \( \theta \) be the column vector of order \( N \times 1 \) defined by

\[ \theta^T = [1, \theta, \theta^2, \theta^3, \ldots, \theta^{N-1}] \]

where \( T \) stands for operation transpose. It can be verified that

\[ Q_N(\theta) = \theta^T (A^T A) \theta. \]

We use this simple form of \( Q_N(\theta) \) in the computer program to estimate \( \theta \).
A case study

We used the method of least squares to fit a first order moving average model to the first order differences of the time series data "IBM Common Stock Closing Prices - Daily, 17th May, 1961 to 2nd November, 1962". See Box and Jenkins (1976, p.239 and p.526). The fitted model works out to be

$$7Y_t = e_t + 0.089e_{t-1},$$

$$\nabla Y_t = Y_t - Y_{t-1}$$ and $Y_t$'s are the original series. Box and Jenkins obtained the estimate of $\beta$ to be 0.09.

Structure

SUBROUTINE ZXMWD(FCN,M,NSIG,A1,BI,NSRCH,X,F,WORK,IWORK,IER)

Formal parameters

- M, Integer, Input: number of unknowns parameters
- NSIG, Integer, Input: number of digits of accuracy required in the parameter estimates
- A1, B1, Real arrays, Constraint vectors of length of M, Input: $X(I)$ is required to satisfy $-A1(I) \leq X(I) \leq B1(I)$
- NSRCH, Integer, Number of starting points to be generated, Input: suggested value $= \min(2^M+5,100)$
- X, Real array, Vector of length M containing the final parameter estimates (output)
- F, Real, Value of the function at the final parameter estimates (output)
- WORK, Real array, Work vector of length $M*(M+1)/2+11*M$
- IWORK, Integer array, Work vector of length M
- IER, Real, Error parameter (output)
Terminal error
IER=129 indicates that the algorithm has converged to a point which may only be a saddle point
IER=130 indicates that it was not possible to calculate the solution to NSIG digits (See remarks)
IER=131 indicates that iteration was terminated after 200*(N+1) function evaluations (See remarks)
IER=132 indicates that AI(I).GE.BI(I) for some I = 1,2,...,N. No attempt is made to find the minimum in this case.

Remarks
When IER is returned as 130 or 131, the parameter estimates in X may not be reliable. Further checking should be performed. Use of a larger NSRCH may produce more reliable parameter estimates.

Auxiliary algorithms
SUBROUTINE ZMFD will use the FCN(M,X,F) to minimize F
M Integer Input: number of unknown parameters
X Real array Vector of length M containing the final parameter estimates (output)
F Real Output: Value of the function at the final parameter estimates

SUBROUTINE VMULFF(AMATT,AMAT,L,KA,KB,IA,IB,C,IC,IER) will find the multiplication of two matrices AMATT and AMAT
Formal parameters
AMATT Real array Input: N by N matrix stored in full storage mode
AMAT Real array Input: N by N matrix stored in full storage mode
L Integer Input: number of rows in AMATT
KA Integer Input: number of columns in AMATT
KB Integer Input: number of columns in AMATT
LA Integer Input: row dimension of matrix AMATT
IB Integer Input: row dimension of matrix AMATT
C Real array Output: N by N matrix containing the product C = AMATT*AMAT
IC Integer Input: row dimension of matrix C
IER Integer Output: Error indicator
IER=129 indicates that AMATT or AMAT or C was dimensioned incorrectly

References
C THIS PROGRAM WILL FIT THE FIRST ORDER MOVING AVERAGE MODEL TO DATA

INTEGER IER,N,L,KA,KB,IA,IB,IC,NSIG,NSRCH,WORK(I),M,IV,N,
REAL X(I),AL,B1,WORK(11),F,AMAT(N,N),AMATT(N,N),C,N=1,
EXTERNAL FCN
COMMON N,C
FI=0.0
F2=0.0
OPEN(UNIT=100, FILE='DATA.DAT', STATUS='OLD')
OPEN(UNIT=99, FILE='RES.DAT', STATUS='NEW')
READ(100,*) IV
N=1
NSIG=I
NSRCH=7
AL=1.0
B1=1.0
N=
L=N
KA=N
KB=N
IA=N
IB=N
IC=N
C THE GIVEN DATA IS PUT INTO THE MATRIX A AS EXPLAINED IN THE THEORY
DO 2 J=1,N
DO 2 I=J,N
ABMAT 1.7 = A 0 ** I = *LIN=1=1.
CONTINUE
STOP
END

SUBROUTINE PCON, M, N, P
INTEGER M, I, J, N
REAL X, L, T, N, N=
COMMON N, X
PI=3.14
DO 100 I=1, N-1
SUM1=SUM1
DO 10 J=2, N-1
SUM1=SUM1+ X, J-1
100 PI=PI - X, 1 ** M * SUM1
PI=0.
DO 100 I=1, N-1
SUM1=SUM1+ X, M-1
100 PI=PI * N, M-1
DC 100 I=N-1, M,-1
SUM2=0.0
DO 30 I=J+2-N,N
30 SUM2=SUM2+C(I,J+2-I)
300 F2=F2+(X(1)**J)*SUM2
F=F1+F2
RETURN
END

Concluding remarks: This work is useful for drawing inferences for signal process models when the observations form a moving average model of order one. A manuscript is under preparation detailing applications of the results of this paper to signal processing.
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